MAT1341 C : Instructor Monica Nevins
Monday, February 13, 2017 (4pm) : Test #2

Duration: 75 minutes

Family name: ______Solutions______________________

First name: ________________________________

Student number : ____________________________

DGD Section number : ______________

Please read the following instructions carefully.

- You have 75 minutes to complete this exam.
- This is a closed book exam. No notes, calculators, cell phones or related devices of any kind are permitted. All such devices, including cell phones, must be stored in your bag under your desk for the duration of the exam.
- Read each question carefully — you will save yourself time and grief later on.
- Questions 1–4 are multiple choice, worth 1 point each. **Record your answers to the multiple choice questions into the table below.**
- Questions 5–7 are long answer, with point values as indicated. **The correct answers here require justificaiton written legibly and logically: you must convince the marker that you know why your solution is correct.**
- Question 8 is a bonus question, worth 3 points. **The bonus question is more difficult; do not attempt it until you are satisfied that you have completed the rest of the test to the best of your ability.**
- Where it is possible to check your work, do so.
- Good luck!

Question	Your answer to the Multiple choice question
1	
2	
3	
4	

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Marker's use only:

Question	Marks
1–4 (/4)	
5 (/3)	
6 (/6)	
7 (/6)	
8 (/3) (bonus)	
Total (/19)	

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1. Which of the following are **axioms of a vector space** V?

(1) If $u, v \in V$ then $uv \in V$.

(2) If $u \in V$ and $c, d \in \mathbb{R}$ then $(c+d)u = cu + du$.

(3) If $u, v, w \in V$ then $u + (v + w) = (u + v) + w$.

(4) If $c \in \mathbb{R}$ and $u \in V$ then $c + u \in V$.

(5) For every $u \in V$, $\|u\| = \sqrt{u \cdot u}$.

(6) If $u, v, w \in V$ then $u \in \operatorname{span}\{v, w\}$.

A. (4) and (6)

B. (2) and (3) correct

C. (2) and (6)

D. (3) and (6)

E. (3) and (5)

F. (4) only

See other version for explanations

2. Which of the following are subspaces of $\mathbb{R}^3$?

$$U = \{(1,2,3) + s(1,0,1) \in \mathbb{R}^3 \mid s \in \mathbb{R}\} \qquad V = \{(x,y,z) \in \mathbb{R}^3 \mid x = y + 3z\}$$
$$X = \{r(1,2,3) + s(1,0,1) \in \mathbb{R}^3 \mid r, s \in \mathbb{R}\} \qquad Y = \{(x,y,z) \in \mathbb{R}^3 \mid x + 3 = y + 3z\}$$

A. U and V

B. U and X

C. X and V correct

D. X and Y

E. U and Y

F. V and Y

3. Let $U = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \in \mathbf{M}_{22} \;\middle|\; abc = 0 \right\}$ be a subset of the vector space of 2×2 matrices. Which of the following is/are true?

(1) U contains $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(2) U is closed under vector addition

(3) U is closed under scalar multiplication

A. (1) and (2) only
B. (1) and (3) only correct
C. (2) and (3) only
D. (1) only
E. (1), (2) and (3)
F. none of them

4. For which value(s) of t is $(8, 4, 5, t)$ in the span of $(2, 2, 1, 3)$ and $(-1, 1, -1, 4)$?

A. $t = 0$ and $t = 1$ only
B. $t = 2$ only
C. $t = 1$ only correct
D. $t \geq 0$ only
E. all t
F. $t = 1$ and $t = 7$ only

5. (3 = 2 × 1.5 points) State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement. Then justify your answer:

- If you say the statement may be false, you must give an explicit counterexample (with numbers or functions!).
- If you say the statement is always true, you CANNOT use an example to justify your response — you must give a clear explanation that works in all cases.

(a) The set $W = \{c + 2bx + (b + c)x^2 \mid b, c \in \mathbb{R}\}$ is a subspace of $F(\mathbb{R})$.

True or false:

Justification:

True: $W = span\{2x + x^2, 1 + x^2\}$ so it is a subspace.

(b) If U is a subspace of $\mathbb{R}^3$, and $v, w \in \mathbb{R}^3$ are two vectors such that

$$v + w \in U \quad \text{and} \quad v - w \in U,$$

then v and w are in U.

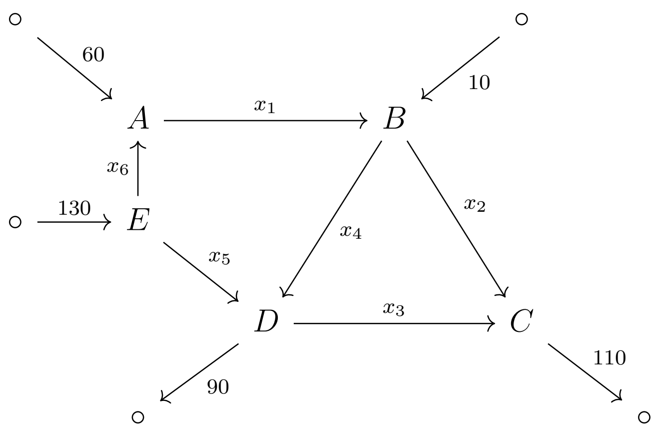
True or false:

Justification:

True: since $v = \frac{1}{2}(v + w) + \frac{1}{2}(v - w)$ and $w = \frac{1}{2}(v + w) - \frac{1}{2}(v - w)$, both vectors are linear combinations of elements of U, and so lie in the subspace U.

6. (6 = 2+2+2 points) Consider the following network of storm-water pipes in a city, which meet at intersections A, B, C, D and E below. Civil engineers designed valves at each of the intersections to allow operators to regulate the flow of water. The arrows indicate the direction of the flow along each pipe (corresponding to downhill).

The numbers refer to the constant flow rate, measured in thousands of litres per hour, along the given sections of pipe, at a given time. Each x_i in the diagram denotes the unknown flow rate along that section of pipe during the same time period.



(a) Write down the linear system which describes the water flow, **together with the constraints** on the variables x_i, for $i \in \{1, 2, 3, 4, 5, 6\}$. **Do not solve this linear system; this is done for you in part (b).** Note: no credit is given for copying the equations implicit in (b).

By comparing flow in to flow out (Kirchoff's laws) we have

$$\begin{aligned} 60 + x_6 &= x_1 \\ x_1 + 10 &= x_2 + x_4 \\ x_2 + x_3 &= 110 \\ x_4 + x_5 &= x_3 + 90 \\ 130 &= x_5 + x_6 \end{aligned}$$

Moreover, all of the variables must satisfy $x_i \geq 0$, since water should flow in the given direction.

(b) The reduced row-echelon form of the augmented matrix from part (a) is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -1 & 60 \\ 0 & 1 & 0 & 1 & 0 & -1 & 70 \\ 0 & 0 & 1 & -1 & 0 & 1 & 40 \\ 0 & 0 & 0 & 0 & 1 & 1 & 130 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Find the general solution. (Ignore the constraints at this point.)

Reading the augmented matrix gives

$$\begin{aligned} x_1 - x_6 &= 60 \\ x_2 + x_4 - x_6 &= 70 \\ x_3 - x_4 + x_6 &= 40 \\ x_5 + x_6 &= 130 \end{aligned}$$

We see that x_4 and x_6 are non-leading variables, so we replace them by parameters s and t respectively. This gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 60 + t \\ 70 - s + t \\ 40 + s - t \\ s \\ 130 - t \\ t \end{bmatrix}$$

(c) Now incorporating the constraints, suppose EA were closed due to blockage. Find the minimum flow along DC, **using your results from (b)**.

If EA is closed then $x_6 = t = 0$. Therefore the flow along DC is $x_3 = 40 + s - 0 \geq 40$ since $s \geq 0$.

7. (6 points). Let $\mathbf{M}_{2\,2}$ denote the vector space of 2 by 2 matrices with real entries, and define

$$U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{2\,2} \;\middle|\; a - b + 7c = 0, \text{ and } d = 0 \right\}.$$

(a) Either check that U is closed under addition, or express U in another form so you can simply state a theorem that guarantees that U is a subspace of $\mathbf{M}_{2\,2}$.
(For parts (b) and (c) you may assume that U is a subspace of $\mathbf{M}_{2\,2}$.)

(b) Find a spanning set for U.

(c) Give a matrix $A \in \mathbf{M}_{2\,2}$ such that $A \notin U$.

We note that $a = b - 7c$ and $d = 0$ so

$$\begin{aligned} U &= \left\{ \begin{bmatrix} b-7c & b \\ c & 0 \end{bmatrix} \middle| c, d \in \mathbb{R} \right\} \\ &= \left\{ b \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} -7 & 0 \\ 1 & 0 \end{bmatrix} \middle| c, d \in \mathbb{R} \right\} \\ &= \mathit{span} \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -7 & 0 \\ 1 & 0 \end{bmatrix} \right\} \end{aligned}$$

and thus U is the span of two vectors, whence it is a subspace.
Alternatively, to show that U is closed under addition, we begin with two elements of U:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in U$$

which means that $a - b + 7c = 0$ and $d = 0$ and $a' - b' + 7c' = 0$ and $d' = 0$. Now consider their sum:

$$\begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}.$$

Is this in U? We compute

$$(a+a') - (b+b') + 7(c+c') = (a - b + 7c) + (a' - b' + 7c') = 0 + 0 = 0$$

and

$$d + d' = 0 + 0 = 0.$$

Therefore the sum satisfies the two conditions, therefore lies in U.
(b) We found a spanning set above.
(c) We just need to choose a matrix which does not satisfy the given conditions. So for example,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \notin U$$

since $a - b + 7c = 7 \neq 0$.

8. (bonus, max 3 points) Do not attempt this question until you have completed the rest of the exam to the best of your ability.

Consider the set

$$V = \{(x^3, x) \in \mathbb{R}^2 \mid x > 0\}$$

equipped with the following non-standard operations: vector addition is given by the formula

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 x_2, y_1 y_2)$$

and scalar multiplication is given by the formula, for each $c \in \mathbb{R}$,

$$c * (x, y) = (x^c, y^c).$$

(a) Show that V is closed under under this scalar multiplication $*$. typo in question

(b) Identify the "zero vector" of V, and show that it satisfies the axiom of being a zero vector.

For $\oplus$, see other versions.

Let $(x^3, x) \in V$ and let $c \in \mathbb{R}$. Then $x > 0$. We calculate $c * (x^3, x) = ((x^3)^c, x^c) = ((x^c)^3, x^c)$ and since $x^c > 0$, this is an element of V. So V is closed under $*$.

Consider $0 * (4, 2) = (4^0, 2^0) = (1, 1)$; so our candidate for the "zero vector" is $(1, 1)$. We verify axiom 3. Let $(x^3, x) \in V$. Calculate

$$(1, 1) \oplus (x^3, x) = (1(x^3), 1(x)) = (x^3, x)$$

which is what we needed to show.