MAT1341 C : Instructor Monica Nevins
Monday, March 6, 2017 : Test #3

Duration: 75 minutes

Family name: ______________________________

First name: _____SOLUTIONS___________

Student number : ____________________________

DGD Section number : ______________

Please read the following instructions carefully.

- You have 75 minutes to complete this exam.
- This is a closed book exam. No notes, calculators, cell phones or related devices of any kind are permitted. All such devices, including cell phones, must be stored in your bag under your desk for the duration of the exam.
- Read each question carefully — you will save yourself time and grief later on.
- Questions 1 and 2 are multiple choice, worth 1 point each. **Record your answers to the multiple choice questions in the boxes provided.**
- Questions 3–5 are long answer, with point values as indicated. **The correct answers here require justification written legibly and logically: you must convince the marker that you know why your solution is correct.**
- Question 6 is a bonus question, worth 3 points. **The bonus question is more difficult; do not attempt it until you are satisfied that you have completed the rest of the test to the best of your ability.**
- Where it is possible to check your work, do so.
- Good luck!

Faculté des sciences
Mathématiques et statistique

Faculty of Science
Mathematics and Statistics

613–562–5864
613–562–5776
www.uOttawa.ca

585 King Edward
Ottawa
ON K1N 6N5

Marker's use only:

Question	Marks
1 & 2 (/2)	
3 (/6)	
4 (/6)	
5 (/6)	
6 (/3) (bonus)	
Total (/20)	

uOttawa

1. (1 point) Which of the following are correct statements about sets of vectors in a vector space V?

(1) The set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent if and only if setting $a_1 = a_2 = a_3 = 0$ gives $a_1\vec{u} + a_2\vec{v} + a_3\vec{w} = \vec{0}$.

(2) The set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent if and only if the only solution to the equation $a_1\vec{u} + a_2\vec{v} + a_3\vec{w} = \vec{0}$ is $a_1 = a_2 = a_3 = 0$.

(3) If $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent then $\dim(V) = 3$.

(4) If $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent then it is a basis for $U = \mathsf{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

A. All are true.

B. Only (2) and (4) are true.

C. Only (1) and (3) are true.

D. Only (2), (3) and (4) are true.

E. Only (2) and (3) are true.

F. Only (3) and (4) are true.

Your answer: B : see other version

2. (1 point) Suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are vectors in $\mathbb{R}^5$ and let $U = \mathsf{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$. Suppose we also have two nonzero vectors $\vec{u}, \vec{w} \in U$ which are not collinear. Which of the following must ALWAYS be true?

A. $\dim(U) = 2$

B. $\dim(U) = 4$

C. $\dim(U) = 5$

D. $\dim(U) = 6$

E. $1 \leq \dim(U) < 4$

F. $2 \leq \dim(U) \leq 4$

Your answer: F: see other version

3. (6 points) Consider the following four vectors in $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix}1\\2\\0\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}3\\5\\1\end{bmatrix}, \vec{v}_4 = \begin{bmatrix}-1\\2\\-4\end{bmatrix}.$$

(a) Explain why $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent.

(b) Find at least one non-trivial linear combination of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ that results in the zero vector, by setting up and solving an appropriate linear system of equations using row reduction. Verify that your answer is correct.

(c) Express one of the vectors in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ as a linear combination of the rest.

(d) Let $W = \mathsf{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$. Give a spanning set for W which has fewer than 4 vectors, and explain why your answer is correct.

(a) These are four vectors in a three-dimensional space, so by a theorem from class, the vectors cannot be LI, hence are LD.

(b) We set up the system $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 + a_4\vec{v}_4 = \vec{0}$, which leads to the following augmented matrix:

$$\left[\begin{array}{cccc|c}1 & 1 & 3 & -1 & 0\\ 2 & 1 & 5 & 2 & 0\\ 0 & 1 & 1 & -4 & 0\end{array}\right]$$

which we row reduce as follows:

$$-2R1+R2 \left[\begin{array}{cccc|c}1 & 1 & 3 & -1 & 0\\ 0 & -1 & -1 & 4 & 0\\ 0 & 1 & 1 & -4 & 0\end{array}\right] -R2 \left[\begin{array}{cccc|c}1 & 1 & 3 & -1 & 0\\ 0 & 1 & 1 & -4 & 0\\ 0 & 1 & 1 & -4 & 0\end{array}\right]$$

$$-R2+R3, -R2+R1 \left[\begin{array}{cccc|c}1 & 0 & 2 & 3 & 0\\ 0 & 1 & 1 & -4 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right]$$

which is in RREF. Therefore the general solution is

$$a_1 + 2a_3 + 3a_4 = 0, \quad a_2 + a_3 - 4a_4 = 0, \quad a_3, a_4 \text{ are free.}$$

Take $a_3 = 1$ and $a_4 = 0$ (for example); this gives $a_1 = -2, a_2 = -1$, and this is the answer:

$$-2\begin{bmatrix}1\\2\\0\end{bmatrix} - 1\begin{bmatrix}1\\1\\1\end{bmatrix} + 1\begin{bmatrix}3\\5\\1\end{bmatrix} + 0\begin{bmatrix}-1\\2\\-4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},$$

which we can see is correct.

(c) Since $-2\vec{v}_1 - 1\vec{v}_2 + \vec{v}_3 = \vec{0}$, we can solve for $\vec{v}_3$ in terms of the rest:

$$\vec{v}_3 = 2\vec{v}_1 + \vec{v}_2$$

which we can check is correct. (Many correct answers were possible.)

(d) Since $\vec{v}_3 \in \mathsf{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$, by a theorem from class, this vector is redundant and

$$W = \mathsf{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}.$$

4. (6 points) Suppose that

$$U = \{p(x) \in P_2 \mid p(4) = 0\}$$

and consider the following two elements of U:

$$f(x) = x - 4, \quad g(x) = x^2 - 4x.$$

(a) Give a polynomial $p(x) \in P_2$ such that p is not in U.

For (b) and (c) you may use any of the following: that $\{1, x, x^2\}$ is linearly independent; that U is a subspace of P_2; that a polynomial has 4 as a root if and only if it has $(x - 4)$ as a factor; and/or theorems from class.

(b) Show that $\{f, g\}$ is linearly independent.

(c) Show that $\{f, g\}$ is a basis for U.

Now for (d) and (e) let $h(x) = x^2 - 16$, which is also in U.

(d) Express h as a linear combination of f and g.

(e) Find the coordinate vector of h with respect to the basis $\{f, g\}$.

(a) for example, $p(x) = x$ is not in U because $p(4) = 4 \neq 0$.

(b) If $af + bg = 0$ then we'd have $a(x-4) + b(x^2-4x) = bx^2 + (a-4b)x - 4a = 0$. Since $\{1, x, x^2\}$ is LI, the only solution is the trivial one, so $b = 0$ and $4a = 0$ and so $a = 0, b = 0$ is the only solution to $af + bg = 0$. Therefore $\{f, g\}$ is LI.

(c) Since $f, g \in U$ and $\{f, g\}$ is LI, by a theorem from class, $\dim(U) \geq 2$. On the other hand, since $U \subset P_2$ and $U \neq P_2$ and $\dim(P_2) = 3$, from a theorem in class we know that $\dim(U) \leq 2$. Therefore $\dim(U) = 2$. Therefore by a theorem from class, any LI set in U with 2 vectors is a basis. Therefore $\{f, g\}$ is a basis of U.

(d) We solve $h = af + bg$ so

$$x^2 - 16 = a(x - 4) + b(x^2 - 4x) = bx^2 + (a - 4b)x - 4a.$$

Therefore we need $b = 1$ and $-4a = -16$ so $a = 4$. Check:

$$4(x - 4) + (x^2 - 4x) = x^2 - 16$$

so yes, $h = 4f + g$.

(e) By (d), the coordinate vector of h with respect to the basis $B = \{f, g\}$ is

$$[h]_B = (4, 1).$$

5. (6 = 4 × 1.5 points) State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement. Then justify your answer:

- If you say the statement may be false, you must give an explicit counterexample (with numbers or functions!).
- If you say the statement is always true, then you CANNOT use an example to justify your response — you must give a clear general explanation using the theory from the course. If you use a theorem from class, you **must state the theorem**.

(a) If $\vec{u}, \vec{v} \in \mathbb{R}^2$ then $\{\vec{u}, \vec{v}\}$ is a basis for $\mathbb{R}^2$.

FALSE: if $\vec{u} = \vec{0}$, for example, then the set $\{\vec{u}, \vec{v}\}$ is LD so cannot be a basis.

(b) Let $U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\,2} \;\middle|\; a + 3b + d = 0 \text{ and } c = 0 \right\}$. Then $\dim(U) = 2$.

TRUE: since $c = 0$ and $a = -3b - d$ we have

$$U = \left\{ \begin{bmatrix} -3b-d & b \\ 0 & d \end{bmatrix} \in M_{2\,2} \;\middle|\; b, d \in \mathbb{R} \right\} = \left\{ b \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \;\middle|\; b, d \in \mathbb{R} \right\}$$

$$= \mathsf{span}\left\{ \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

These two vectors span U and we can see they are not multiples of one another, therefore they are LI and thus a basis for U. Therefore $\dim(U) = 2$.

(c) Suppose $\vec{u}, \vec{v}, \vec{w} \in M_{2\,2}$ and $U = \mathsf{span}\{\vec{u}, \vec{v}, \vec{w}\}$. Then any linearly independent set in U contains at least 3 vectors.

FALSE: Suppose $\vec{u}$ is not the zero vector; then $\{\vec{u}\}$ would be an LI set in U which contains fewer than 3 vectors.

(d) If $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent then $\{\vec{u}, \vec{w}\}$ must also be linearly independent.

TRUE: A theorem from class said that every subset of an LI set is also LI. (Alternately, you can prove this by noting that if $a\vec{u} + b\vec{w} = 0$, then this is an example of a dependence relation on $\{\vec{u}, \vec{v}, \vec{w}\}$ (with zero coefficient on $\vec{v}$); since the latter set is LI, all coefficients must be 0, whence $a = b = 0$.)

6. (Bonus, max 3 points) Let V be a vector space. Suppose $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a basis for a subspace U of V, and suppose $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is basis for a subspace W of V, such that the only vector that lies in both U and W is the zero vector. Carefully prove that $\dim(V) \geq 6$.

See other version for a solution.