MAT 2771 - Automne 2014
Devoir 8 : Solutionnaire

5.6-4 (4 points) A partir de la loi centrale limite

$$\frac{\bar{X}_n - 40}{\sqrt{8/32}} \approx N(0,1) \tag{1}$$

Donc

$$\begin{aligned} P(39.75 &\leq \bar{X}_n < 41.25) \quad (2) \\ &= P(\frac{39.75-40}{\sqrt{1/4}} \leq \frac{\bar{X}_n - 40}{\sqrt{8/32}} < \frac{41.25-40}{\sqrt{1/4}}) \quad (3) \\ &= P(-0.50 \leq Z < 2.5) = 0.6853 \quad (4) \end{aligned}$$

5.6-8 (3 points) a) $E\left(\bar{X}_n\right) = \mu = 24.43$
b) $Var\left(\bar{X}_n\right) = \frac{\sigma^2}{n} = \frac{2.20}{30} = 7.3333 \times 10^{-2}$
c) Donc

$$\begin{aligned} P(24.17 &\leq \bar{X}_n < 24.82) \quad (5) \\ &= P(\frac{24.17-24.43}{\sqrt{7.3333\times 10^{-2}}} \leq \frac{\bar{X}_n - 24.43}{\sqrt{7.3333\times 10^{-2}}} < \frac{24.82-24.43}{\sqrt{7.3333\times 10^{-2}}}) \quad (6) \\ &= P(-0.96 \leq Z < 1.44) = 0.7566 \quad (7) \end{aligned}$$

5.6-5 (extra)
a) La somme Y de n variables indépendantes qui suivent une loi de χ_1^2 aura une loi χ_n^2. Ici n=18.
b) En utilisant l'approximation fournie par la loi centrale limite on note que

$$\begin{aligned} P(\frac{Y - n\mu}{\sigma\sqrt{n}} &\leq \frac{9.390-18}{\sqrt{2}\sqrt{18}}) \quad (8) \\ &= \Phi(-1.435) = 0.075644 \quad (9) \end{aligned}$$

et

$$\begin{aligned} P(\frac{Y - n\mu}{\sigma\sqrt{n}} &\leq \frac{34.80-18}{\sqrt{2}\sqrt{18}}) \quad (10) \\ &= \Phi(2.8) = 0.997445 \quad (11) \end{aligned}$$

5.6-6 (extra) On peut calculer la fonction génératrice

$$\begin{aligned} M(t) &= \int_0^2 \left(1 - \frac{x}{2}\right) \exp(xt)\, dx \quad (12) \\ &= \frac{1}{2}\frac{e^{2t} - 2t - 1}{t^2} \quad (13) \\ &= 1 + \frac{2t}{3} + \frac{t^2}{3} + ... \quad (14) \end{aligned}$$

a) $\mu = M'(0) = \frac{2}{3}$
$\sigma^2 = M"(0) - [M'(0)]^2 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{9}$
b)

$$\begin{aligned} P(\frac{2}{3} &\leq \bar{X}_n < \frac{5}{6}) && (15) \\ &= P(0 \leq \frac{\bar{X}_n - \frac{2}{3}}{\sqrt{\frac{2}{9}}\sqrt{\frac{1}{18}}} < \frac{\frac{5}{6} - \frac{2}{3}}{\sqrt{\frac{1}{81}}}) && (16) \\ &= P(0 \leq Z < 1.5) = 0.4332 && (17) \end{aligned}$$

5.7-4 (2 points) X˜*binomiale* avec paramètres $n = 48, p = 0.75$.
Donc $np = 36, np(1-p) = 9$

$$\begin{aligned} P(35 \leq X \leq 40) &= P(34.5 \leq X \leq 40.5) && (18) \\ &= P\left(\frac{34.5 - 36}{3} \leq \frac{X - np}{\sqrt{np(1-p)}} \leq \frac{40.5 - 36}{3}\right) && (19) \\ &= P(-0.5 \leq Z \leq 1.5) = 0.6247 && (20) \end{aligned}$$

5.7-12 (6 points) a) $E(X) = 100(0.1) = 10.0, Var(X) = 9$

$$\begin{aligned} & P(11.5 < X < 14.5) \approx && (21) \\ &= \Phi\left(\frac{14.5 - 10}{3}\right) - \Phi\left(\frac{11.5 - 10}{3}\right) && (22) \\ &= 0.2417 && (23) \end{aligned}$$

b) Par contre, avec la loi Poisson, on calcule plus précisement

$$\begin{aligned} & P(X \leq 14) - P(X \leq 11) \\ &= 0.917 - 0.697 = 0.220 \end{aligned}$$

c) $\sum_{x=12}^{14} \binom{100}{x} (0.1)^x (0.9)^{100-x} = 0.224\,39$
5.7-14 (3 points) La moyenne et variance d'une variable aléatoire ayant une loi geomètrique sont données par $\mu = \frac{1}{p} = \frac{4}{3}, \sigma^2 = \frac{1-p}{p^2} = \frac{\frac{1}{4}}{\frac{9}{16}} = \frac{4}{9}$

a) Par la loi centrale limite

$$
\begin{aligned}
& P\left(45 \leq \sum_{i=1}^{36} X_i \leq 49\right) && (24)\\
= \; & P\left(45.5 \leq \sum_{i=1}^{36} X_i \leq 49.5\right) && (25)\\
= \; & P\left(\frac{45.5 - 36\left(\frac{4}{3}\right)}{\sqrt{\frac{4}{9}}\sqrt{36}} \leq \frac{\sum_{i=1}^{36} X_i - 36\mu}{\sqrt{n\sigma^2}} \leq \frac{49.5 - 36\left(\frac{4}{3}\right)}{\sqrt{\frac{4}{9}}\sqrt{36}}\right) && (26)\\
= \; & P\left(-0.625 \leq Z \leq 0.375\right) = 0.64615 - 0.26595 = 0.3802 && (27)
\end{aligned}
$$

b)

$$
\begin{aligned}
& P\left(1.25 \leq \bar{X} \leq 1.50\right) && (28)\\
= \; & P\left(45 \leq \sum_{i=1}^{36} X_i \leq 54\right) && (29)\\
= \; & P\left(44.5 \leq \sum_{i=1}^{36} X_i \leq 54.5\right) && (30)\\
= \; & P\left(\frac{44.5 - 36\left(\frac{4}{3}\right)}{\sqrt{\frac{4}{9}}\sqrt{36}} \leq \frac{\sum_{i=1}^{36} X_i - 36\mu}{\sqrt{n\sigma^2}} \leq \frac{54.5 - 36\left(\frac{4}{3}\right)}{\sqrt{\frac{4}{9}}\sqrt{36}}\right) && (31)\\
= \; & P\left(-0.875 \leq Z \leq 1.625\right) = 0.9479 - 0.1908 = 0.7571 && (32)
\end{aligned}
$$

5.8-2 (2 points) On calcule la variance $\sigma^2 = E\left(X^2\right) - \left(E\left(X\right)\right)^2 = 298 - 17^2 = 9$

a)

$$
\begin{aligned}
& P\left(10 < X < 43\right) && (33)\\
= \; & P\left(\frac{10-17}{3} < \frac{X-\mu}{\sigma} < \frac{24-17}{3}\right) && (34)\\
= \; & P\left(-2.333 < \frac{X-\mu}{\sigma} < 2.333\right) && (35)\\
= \; & P\left(\left|\frac{X-\mu}{\sigma}\right| < 2.333\right) && (36)\\
\geq \; & 1 - \frac{1}{2.333^2} = 0.816\,27 && (37)
\end{aligned}
$$

b)

$$\begin{aligned} & P\left(|X-17| \geq 16\right) && (38)\\ = \quad & P\left(\left|\frac{X-\mu}{\sigma}\right| \geq \frac{16}{3}\right) && (39)\\ \leq \quad & \left(\frac{3}{16}\right)^2 = 3.515\,6 \times 10^{-2} && (40) \end{aligned}$$

5.8-6 (2 points) La moyenne de $\bar{X}$ est égale à $\mu = 80$; la variance de $\bar{X}$ est égale à $\frac{\sigma^2}{n} = \frac{60}{15} = 4$

$$\begin{aligned} & P\left(75 < \bar{X} < 85\right) && (41)\\ = \quad & P\left(\frac{75-80}{2} < \frac{\sqrt{n}\left(\bar{X}-\mu\right)}{\sigma} < \frac{85-80}{2}\right) && (42)\\ = \quad & P\left(-2.5 < \frac{\sqrt{n}\left(\bar{X}-\mu\right)}{\sigma} < 2.5\right) && (43)\\ = \quad & P\left(\left|\frac{\sqrt{n}\left(\bar{X}-\mu\right)}{\sigma}\right| \leq 2.5\right) && (44)\\ \geq \quad & 1 - \frac{1}{2.5^2} = 0.84 && (45) \end{aligned}$$

Total 20