Motives of torsor quotients via representations

Kirill Zainoulline (UOttawa)

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Goals

G a split semisimple linear algebraic group over a field k
E a G-torsor over k
E/P a variety of parabolic subgroups (twisted flag variety).
h an algebraic oriented cohomology theory over k

The purpose of the present talk is to relate:

$$\langle [E/P] \rangle_h$$

Tate subcategory generated by h-motives $[E/P]$, where P runs through all parabolic subgroups.

and

$$Proj\ D_E^h$$

Category of f.g. projective modules over certain Hecke-type algebra attached to h and E.

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- Describe the algebra D_E^h explicitly using generators and relations

Applications:

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Motivation

Motivic Galois group (Grothendieck, Deligne, ...):

Given a 'nice' category C find a group G so that

$$C = Reps\ G.$$

Applied to Tannakian categories (e.g. motivic with $\mathbb{Q}$-coefficients over a field of characteristic zero) to obtain G (the Galois group of C).

Unfortunately, we don't know how to apply it in our case as we work with $\mathbb{Z}$-coefficients and the category in question is not even Krull-Schmidt.

Remark: tensoring with $\mathbb{Q}$ kills all interesting (torsion) information about $h(E/P)$. Indeed, the motive $[E/P]$ with $\mathbb{Q}$-coefficients is just a direct sum of Tate motives. So the category $C = \langle [E/P] \rangle_h$ is equivalent to the category of Tate motives and $G = \mathbb{G}_m$.

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Reps G is the same as *Proj* $\mathbb{Z}[G]$.

We expect that with $\mathbb{Z}$-coefficients there is no G but rather a deformed version of $\mathbb{Z}[G]$

that is the Hecke-type algebra D_E^h we are looking for.

In general, it will be a bi-algebra but not the Hopf-algebra.

Key idea: To construct the algebra D_E^h use the Kostant-Kumar T-fixed point approach.

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The results/techniques mentioned in the talk can be found in

1. Calmès, Petrov, Z., Invariants, torsion indices and cohomology of complete flags (Ann. Sci. Ecole Norm. Sup. (4) 2013)
2. Hoffnung, Malagon-Lopez, Savage, Z., Formal Hecke algebras and algebraic oriented cohomology theories (Selecta Math. 2014)
3. Calmès, Z., Zhong, A coproduct structure on the formal affine Demazure algebra (Math. Zeitschrift 2015)
4. Calmès, Z., Zhong, Push-pull operators on the formal affine Demazure algebra and its dual (submitted)
5. Calmès, Z., Zhong, Equivariant oriented cohomology of flag varieties (Documenta Math. 2015)
6. Neshitov, Petrov, Semenov, Z., Motivic decompositions of twisted flag varieties and representations of Hecke-type algebras (submitted)
7. Calmès, Neshitov, Z. Relative equivariant oriented motivic categories (in progress)

The talk is dedicated to application of these techniques (especially of (6) and (7)) to the study of motives of twisted flag varieties.

Consider an algebraic G-equivariant oriented cohomology theory $h_G(-)$ in the sense of Levine-Morel.

Examples:

- equivariant Chow groups $CH_G(-)$ (Totaro, Edidin-Graham),
- equivariant K-theory (Thomason, Merkurjev),
- equivariant algebraic cobordism Ω (Heller, Malagon-Lopez)

Basic properties:

- Push-forwards for projective equivariant maps
- Localization: $h_G(X \setminus U) \to h_G(X) \to h_G(U) \to 0$
- Homotopy invariance: $h_G(E) \simeq h_G(X)$ for any torsor of a vector bundle $E \to X$
- Given a G-torsor $X \to X/G$ we have $h_G(X) \simeq h(X/G)$

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Fix $T \subset B \subset G$.
Define the category of relative equivariant motives $Mot_{G \to T}(k)$ by

- Taking the category of smooth projective G-varieties and defining the category of correspondences by

$$Mor(X, Y) := im(h_G(X \times Y) \to h_T(X \times Y))$$

  with the standard correspondence product
- Taking its pseudo-abelian completion: objects (X, p), $p \circ p = p$.

Remark: We deal here with the non-graded motives. However, if needed, one can put a grading.

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Versal torsors and equivariant motives

Let E be a versal torsor. Assume h satisfies dimension axiom (i.e. $h^m(X) = 0$ if $m > \dim X$).

Theorem [NPSZ]. There is a surjective homomorphism with nilpotent kernel

$$End_{Mot_{G\to T}}([G/B]) \to End_{usual\ h\text{-}motives}([E/B])$$

Theorem [CNZ]. There is a surjective homomorphism with nilpotent kernel

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Hence, motivic decompositions of $[E/P]$ are in 1-1 correspondence with relative equivariant motivic decompositions of $[G/P]$.

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Ioneda functor

We want to describe/compute $End_{Mot_{G\to T}}([G/P])$.

Fix a G-variety Z. Set

$$D^h = End_{Mot_{G\to T}}([Z]).$$

Define a functor

$$F_Z\colon Mot_{G\to T} \to D^h\text{-}Modules$$

via $M \mapsto Hom_{Mot_{G\to T}}([Z], M)$.

Idea: To show that F_Z becomes an equivalence for some specially chosen Z if restricted to some 'nice' subcategory of $Mot_{G\to T}$.

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Examples

Consider the subcategory Mot_{par} generated by motives of all G/P's (over all parabolic subgroups of G).

Lemma. If $Z = pt$, then F_Z is an equivalence if restricted to Mot_{par} of $Mot_{T \to T}$.
(here $Mot_{T \to T}$ is the usual category of T-equivariant motives)

Proof: Follows from the Künneth isomorphism, since all G/P's are T-equivariant cellular spaces (Bruhat decomposition).

Corollary. If $Z = G/B$, then F_Z is faithful if restricted to Mot_{par} of $Mot_{G \to T}$.

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Degenerate parabolic subgroups

Definition. We say that a parabolic subgroup P is h-degenerate if

$$h_P(pt) \to h_T(pt)^{W_P}$$

is surjective (here W_P is the Weyl group of the Levi part of P).

Remark: Observe that in general (e.g. for $h = CH$) it is neither surjective nor injective. It is an isomorphism rationally or if P is special (Edidin-Graham). In topology it is $H(BP) \to H(BT)^{W_P}$.

Definition. We say that two parabolic subgroups P and P' are h-degenerate to each other if $P_w = R_u P(P \cap {}^w P')$ is h-degenerate for all $w \in W_P \backslash W / W_P$. We say that a family of parabolic subgroups is h-degenerate if any two subgroups are h-degenerate to each other.

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Example

Fix an *h*-degenerate family of parabolic subgroups (e.g. take special subgroups).

Consider the subcategory Mot_{deg} generated by motives of all G/P's from that family.

Theorem [CNZ]. If $Z = G/B$, then F_Z is an equivalence if restricted to Mot_{deg} of Mot_{par}.

Proof: Is based on the Chernousov-Merkurjev G-equivariant cellular filtration for $G/P \times G/P'$ and the fact that

$$Mor_{Mot_{G\to T}}([G/P],[G/P']) = Mor_{Mot_{T\to T}}([G/P],[G/P'])^W.$$

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Endomorphisms

Recall that

$$F_{G/B} \colon Mot_{G\to T} \to D^h\text{-}modules$$

is faithful on all G/P's and it is an equivalence on a h-degenerate family of G/P's.

In particular,

$$End_{Mot_{G\to T}}([G/P]) \hookrightarrow End_{D^h}(h_T(G/P))$$

which turns into an isomorphism if $\{P\}$ is h-degenerate.

We want to understand the right hand side:

Theorem [NPSZ]. $D^h = (h_T(G/B), \circ)$ is the formal affine Demazure algebra and

$$End_{Mot_{G\to T}}([G/B]) \xrightarrow{\simeq} End_{D^h}(h_T(G/B)) = D^h.$$

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is faithful on all G/P's and it is an equivalence on a h-degenerate family of G/P's.
In particular,

$$End_{Mot_{G\to T}}([G/P]) \hookrightarrow End_{D^h}(h_T(G/P))$$

which turns into an isomorphism if $\{P\}$ is h-degenerate.
We want to understand the right hand side:

Theorem [NPSZ]. $D^h = (h_T(G/B), \circ)$ is the formal affine Demazure algebra and

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Examples

For CH and K-theory D^h is given by the same generators and relations as the affine Hecke algebra, except that the quadratic relation is replaced by $T_i^2 = 0$ (for CH) and by $T_i^2 = T_i$ (for K-theory). So

- for CH it is the nil affine Hecke algebra (Caution: nil affine Hecke is not the same as degenerate affine Hecke)
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Localized twisted group algebras

Set $R = h(pt)$, $S = h_T(pt)$, Q is S localized at all Chern classes of roots. Set

$$S_W = S \# R[W] \text{ and } Q_W = Q \# R[W].$$

Localization then gives
Theorem [CZZ].

$$End_{D^h}(h_T(G/P)) \hookrightarrow End_{Q_W}(Q^*_{W/W_P}),$$

i.e. direct sum decompositions of relative equivariant motive of G/P are determined by direct sum decompositions of the Q_W-module $Q^*_{W/W_P} = Hom_Q(Q_{W/W_P}, Q)$.

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Combining all these facts we obtain that for a versal E:

decompositions of $[E/P]$ are in 1-1 with decompositions of the relative equivariant motive of G/P, where the latter are determined by the decompositions of the parabolic affine Demazure algebra $h_T(G/P) = ((D^h)^*)^{W_P}$ and, by the Q_W-module Q^*_{W/W_P}.

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Examples

1. Take a generic maximal orthogonal Grassmanian for type B_n. Its Chow motive is indecomposable with $\mathbb{Z}/2\mathbb{Z}$-coefficients, so $Ind_{W_P}^{W}$ 1 is indecomposable with $\mathbb{Z}/2\mathbb{Z}$-coefficients.

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Define the rational version of D^h that is

$$D_E^h := im(h(E/P) \overset{res}{\to} h(G/P)) \otimes_{h_T(pt)} D^h$$

If E is versal, there is a surjective map with nilpotent kernel $D^h \to D_E^h$ induced by the characteristic map. If G is special, then $D_E^h = S^W \otimes_S D^h$ (here $S = h_T(pt)$).

Theorem [NPSZ]. Let E be an arbitrary G-torsor. There is a surjective map with nilpotent kernel $D_E^h \to End([E/B])$.

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