Motives of torsor quotients via representations

Kirill Zainoulline (UOttawa)

2015
Goals

\( G \) a split semisimple linear algebraic group over a field \( k \)
\( E \) a \( G \)-torsor over \( k \)
\( E/P \) a variety of parabolic subgroups (twisted flag variety).
\( h \) an algebraic oriented cohomology theory over \( k \)

The purpose of the present talk is to relate:

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\langle [E/P] \rangle_h
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Tate subcategory generated by \( h \)-motives \([E/P]\), where \( P \) runs through all parabolic subgroups.

and

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\text{Proj } D^h_E
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Category of f.g. projective modules over certain Hecke-type algebra attached to \( h \) and \( E \).
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Dreams/goals:

- Show that these two categories are equivalent
- Describe the algebra $D^h_E$ explicitly using generators and relations

Applications:

- Classification of motives of orthogonal Grassmannians, generalized Severi-Brauer varieties,... via representations
- New results in modular/integer representation theory of Hecke-type algebras... via motives
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Motivic Galois group (Grothendieck, Deligne, ...):
Given a 'nice' category $C$ find a group $G$ so that

$$C = \text{Reps } G.$$ 

Applied to Tannakian categories (e.g. motivic with $\mathbb{Q}$-coefficients over a field of characteristic zero) to obtain $G$ (the Galois group of $C$).

Unfortunately, we don’t know how to apply it in our case as we work with $\mathbb{Z}$-coefficients and the category in question is not even Krull-Schmidt.

Remark: tensoring with $\mathbb{Q}$ kills all interesting (torsion) information about $h(E/P)$. Indeed, the motive $[E/P]$ with $\mathbb{Q}$-coefficients is just a direct sum of Tate motives. So the category $C = \langle [E/P] \rangle_h$ is equivalent to the category of Tate motives and $G = \mathbb{G}_m$. 
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Motivation

Reps $G$ is the same as $Proj \mathbb{Z}[G]$.

We expect that with $\mathbb{Z}$-coefficients there is no $G$ but rather a deformed version of $\mathbb{Z}[G]$ that is the Hecke-type algebra $D^h_E$ we are looking for.

In general, it will be a bi-algebra but not the Hopf-algebra.

Key idea: To construct the algebra $D^h_E$ use the Kostant-Kumar $T$-fixed point approach.
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**Key idea:** To construct the algebra $D^h_E$ use the Kostant-Kumar $T$-fixed point approach.
The results/techniques mentioned in the talk can be found in

4. Calmès, Z., Zhong, Push-pull operators on the formal affine Demazure algebra and its dual (submitted)
5. Calmès, Z., Zhong, Equivariant oriented cohomology of flag varieties (Documenta Math. 2015)
6. Neshitov, Petrov, Semenov, Z., Motivic decompositions of twisted flag varieties and representations of Hecke-type algebras (submitted)
7. Calmès, Neshitov, Z. Relative equivariant oriented motivic categories (in progress)

The talk is dedicated to application of these techniques (especially of (6) and (7)) to the study of motives of twisted flag varieties.
Equivariant cohomology

Consider an algebraic $G$-equivariant oriented cohomology theory $h_G(-)$ in the sense of Levine-Morel.

Examples:

- equivariant Chow groups $CH_G(-)$ (Totaro, Edidin-Graham),
- equivariant $K$-theory (Thomason, Merkurjev),
- equivariant algebraic cobordism $\Omega$ (Heller, Malagon-Lopez)

Basic properties:

- Push-forwards for projective equivariant maps
- Localization: $h_G(X \setminus U) \to h_G(X) \to h_G(U) \to 0$
- Homotopy invariance: $h_G(E) \simeq h_G(X)$ for any torsor of a vector bundle $E \to X$
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Fix $T \subset B \subset G$.
Define the category of relative equivariant motives $Mot_{G \to T}(k)$ by

- Taking the category of smooth projective $G$-varieties and defining the category of correspondences by

$$\text{Mor}(X, Y) := \text{im}(h_G(X \times Y) \rightarrow h_T(X \times Y))$$

- Taking its pseudo-abelian completion: objects $(X, p)$,
  $$p \circ p = p.$$

Remark: We deal here with the non-graded motives. However, if needed, one can put a grading.
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Let $E$ be a versal torsor. Assume $h$ satisfies dimension axiom (i.e. $h^m(X) = 0$ if $m > \dim X$).

Theorem [NPSZ]. There is a surjective homomorphism with nilpotent kernel

$$\text{End}_{\text{Mot}_G \to T}([G/B]) \to \text{End}_{\text{usual } h\text{-motives}}([E/B])$$

Theorem [CNZ]. There is a surjective homomorphism with nilpotent kernel

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Hence, motivic decompositions of $[E/P]$ are in 1-1 correspondence with relative equivariant motivic decompositions of $[G/P]$. 
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Hence, motivic decompositions of $[E/P]$ are in 1-1 correspondence with relative equivariant motivic decompositions of $[G/P]$. 
We want to describe/compute $\text{End}_{\text{Mot}_{G\to T}}([G/P])$.

Fix a $G$-variety $Z$. Set

$$D^h = \text{End}_{\text{Mot}_{G\to T}}([Z]).$$

Define a functor

$$F_Z : \text{Mot}_{G\to T} \to D^h\text{-Modules}$$

via $M \mapsto \text{Hom}_{\text{Mot}_{G\to T}}([Z], M)$.

Idea: To show that $F_Z$ becomes an equivalence for some specially chosen $Z$ if restricted to some ‘nice’ subcategory of $\text{Mot}_{G\to T}$. 
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Consider the subcategory $Mot_{par}$ generated by motives of all $G/P$'s (over all parabolic subgroups of $G$).

Lemma. If $Z = pt$, then $F_Z$ is an equivalence if restricted to $Mot_{par}$ of $Mot_{T \rightarrow T}$.
(here $Mot_{T \rightarrow T}$ is the usual category of $T$-equivariant motives)

Proof: Follows from the Künneth isomorphism, since all $G/P$'s are $T$-equivariant cellular spaces (Bruhat decomposition).

Corollary. If $Z = G/B$, then $F_Z$ is faithful if restricted to $Mot_{par}$ of $Mot_{G \rightarrow T}$.
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Definition. We say that a parabolic subgroup $P$ is $h$-degenerate if

$$h_P(pt) \to h_T(pt)^{W_P}$$

is surjective (here $W_P$ is the Weyl group of the Levi part of $P$).

Remark: Observe that in general (e.g. for $h = CH$) it is neither surjective nor injective. It is an isomorphism rationally or if $P$ is special (Edidin-Graham). In topology it is $H(BP) \to H(BT)^{W_P}$.

Definition. We say that two parabolic subgroups $P$ and $P'$ are $h$-degenerate to each other if $P_w = R_u P(P \cap w P')$ is $h$-degenerate for all $w \in W_P \setminus W / W_P$. We say that a family of parabolic subgroups is $h$-degenerate if any two subgroups are $h$-degenerate to each other.
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Theorem [CNZ]. If $Z = G/B$, then $F_Z$ is an equivalence if restricted to $\text{Mot}_{\text{deg}}$ of $\text{Mot}_{\text{par}}$.

Proof: Is based on the Chernousov-Merkurjev $G$-equivariant cellular filtration for $G/P \times G/P'$ and the fact that

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\text{Mor}_{\text{Mot}_{G \rightarrow T}}([G/P], [G/P']) = \text{Mor}_{\text{Mot}_{T \rightarrow T}}([G/P], [G/P'])^W.
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Proof: Is based on the Chernousov-Merkurjev $G$-equivariant cellular filtration for $G/P \times G/P'$ and the fact that

$$\text{Mor}_{\text{Mot}_{G \rightarrow T}}([G/P], [G/P']) = \text{Mor}_{\text{Mot}_{T \rightarrow T}}([G/P], [G/P'])^W.$$
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\[ F_{G/B} : \text{Mot}_{G \to T} \to D^h\text{-modules} \]

is faithful on all \( G/P \)'s and it is an equivalence on a \( h \)-degenerate family of \( G/P \)'s.

In particular,

\[ \text{End}_{\text{Mot}_{G \to T}}([G/P]) \hookrightarrow \text{End}_{D^h}(h_T(G/P)) \]

which turns into an isomorphism if \( \{P\} \) is \( h \)-degenerate.

We want to understand the right hand side:

Theorem [NPSZ]. \( D^h = (h_T(G/B), \circ) \) is the formal affine Demazure algebra and

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For $CH$ and $K$-theory $D^h$ is given by the same generators and relations as the affine Hecke algebra, except that the quadratic relation is replaced by $T_i^2 = 0$ (for $CH$) and by $T_i^2 = T_i$ (for $K$-theory). So

- for $CH$ it is the nil affine Hecke algebra (Caution: nil affine Hecke is not the same as degenerate affine Hecke)
- for $K_0$ it is the 0-Hecke algebra

In the hyperbolic (generic singular elliptic) case $D^h$ contains the classical Iwahori-Hecke algebra as the constant part.
Examples

For \( CH \) and \( K \)-theory \( D^h \) is given by the same generators and relations as the affine Hecke algebra, except that the quadratic relation is replaced by \( T_i^2 = 0 \) (for \( CH \)) and by \( T_i^2 = T_i \) (for \( K \)-theory). So

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Localized twisted group algebras

Set $R = h(pt)$, $S = h_T(pt)$, $Q$ is $S$ localized at all Chern classes of roots. Set

$$S_W = S \# R[W] \text{ and } Q_W = Q \# R[W].$$

Localization then gives Theorem [CZZ].

$$\text{End}_{D^b}(h_T(G/P)) \hookrightarrow \text{End}_{Q_W}(Q^*_{W/W_P}),$$

i.e. direct sum decompositions of relative equivariant motive of $G/P$ are determined by direct sum decompositions of the $Q_W$-module $Q^*_{W/W_P} = \text{Hom}_Q(Q_{W/W_P}, Q)$.

Here the action of $Q_W$ is given by

$$q\delta_w \circ pf_v = qw(p)f_{wv}.$$
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Here $S_W^*/W_P$ can be think of as $\bigoplus h_T(pt)$ taken over all $T$-fixed points of $G/P$.

Theorem [CNZ]. Restricting to degree 0 endomorphisms (idempotents anyway sit in degree 0) gives an embedding

$$\text{End}^{(0)}_{S_W}(S_W^*/W_P) \hookrightarrow \text{End}^{(0)}_{D_h}(h_T(G/P)).$$

In the case of Chow groups the left hand side coincides with $\text{End}_{R[W]}(R[W/W_P])$, which reduces to the study of decompositions of $\text{Ind}_{W_P}^W 1$ into irreducible $W$-submodules.
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For the symmetric group $W = S_n$ and $W_P = S_{n-1}$ (the case of a projective space $P^{n-1}$) there are no proper irreducible submodules of $\text{Ind}_{S_{n-1}}^{S_n} 1$ over $\mathbb{Z}$

Two proofs:
1. using symmetric polynomials and Schur functions
2. follows from the fact that the motive of a generic Severi-Brauer variety is indecomposable (Karpenko).
Examples

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Combining all these facts we obtain that for a versal $E$:

decompositions of $[E/P]$ are in 1-1 with decompositions of the relative equivariant motive of $G/P$, where the latter are determined by the decompositions of the parabolic affine Demazure algebra $h_T(G/P) = ((D^h)^*)^{WP}$ and, by the $Q_W$-module $Q^{*}_{W/W_P}$.

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1. Take a generic maximal orthogonal Grassmanian for type $B_n$. Its Chow motive is indecomposable with $\mathbb{Z}/2\mathbb{Z}$-coefficients, so $\text{Ind}_{WP}^W 1$ is indecomposable with $\mathbb{Z}/2\mathbb{Z}$-coefficients.

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Define the rational version of $D^h$ that is

$$D^h_E := \text{im}(h(E/P) \xrightarrow{\text{res}} h(G/P)) \otimes_{hT(pt)} D^h$$

If $E$ is versal, there is a surjective map with nilpotent kernel $D^h \to D^h_E$ induced by the characteristic map. If $G$ is special, then $D^h_E = S^W \otimes_S D^h$ (here $S = hT(pt)$).

Theorem [NPSZ]. Let $E$ be an arbitrary $G$-torsor. There is a surjective map with nilpotent kernel $D^h_E \to \text{End}([E/B])$.

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Thank You!