Invariants of degree 3 and torsion in the Chow group of a versal flag

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Let $G$ be a split semisimple linear algebraic group over a field $F$.

The purpose of the present talk is to relate:

- the geometry of twisted $G$-flag varieties
- the theory of cohomological invariants of $G$
- the representation theory of $G$
Let $U/G$ be a **classifying space** of $G$ in the sense of Totaro, i.e. $U$ is an open $G$-invariant subset in some representation of $G$ with $U(F) \neq \emptyset$ and $U \to U/G$ is a $G$-torsor.

Consider the generic fiber $U^\text{gen}$ of $U$ over $U/G$. It is a $G$-torsor over the quotient field $K$ of $U/G$ called the **versal torsor**.

We denote by $X^\text{gen}$ the respective flag variety $U^\text{gen}/B$ over $K$, where $B$ is a Borel subgroup of $G$, and call it the **versal flag**.
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Consider the generic fiber $U_{\text{gen}}$ of $U$ over $U/G$. It is a $G$-torsor over the quotient field $K$ of $U/G$ called the **versal torsor**.

We denote by $X_{\text{gen}}$ the respective flag variety $U_{\text{gen}}/B$ over $K$, where $B$ is a Borel subgroup of $G$, and call it the **versal flag**.
The variety $X^{\text{gen}}$ can be viewed as

the ’most twisted’ form of the
‘most complicated’ $G$-flag variety.

Example: Take the variety of flags of ideals in a generic division algebra over $F$.

We want to understand its geometry via studying its

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The ring \( \text{CH}(X) \) of a \textbf{split flag} variety \( X \) is completely understood due to Grothendieck, Demazure, Bernstein-Gelfand-Gelfand using the Schubert calculus.

Moreover,

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\text{CH}(\text{twisted flag}) \otimes \mathbb{Q} \simeq \text{CH}(\text{split flag}) \otimes \mathbb{Q}
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The computation of $\text{Tors} \, CH(\text{twisted flag})$ has been pushed by the development of motivic cohomology theory in 90’s culminating in numerous results, e.g.

- anisotropic quadrics $\text{Tors} \, CH^i$, $i \leq 4$ [Karpenko, Merkurjev]
- group of zero-cycles $\text{Tors} \, CH_0(\text{twisted flag})$ [Chernousov, Krashen, Merkurjev, Parimala, Springer, Totaro]
- $\text{Tors} \, CH^2(\text{strongly inner twisted flag})$ [Baek, Merkurjev, Peyre, Z., Zhong]

In the talk we will concentrate on $\text{Tors}CH^2(X^{\text{gen}})$ for any $G$. 
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In the talk we will concentrate on $Tors\text{CH}^2(X_{\text{gen}})$ for any $G$. 
Has been mainly inspired by the works of J.-P. Serre and M. Rost.

Given a field extension $L/F$ and a positive integer $d$ we consider the Galois cohomology group $H^{d+1}(L, d) = H^{d+1}(L, \mathbb{Q}/\mathbb{Z}(d))$.

A degree $d$ cohomological invariant is a natural transformation of functors

$$a: H^1(\_, G) \to H^d(\_, d - 1)$$

on the category of field extensions over $F$. We denote the group of degree $d$ invariants by $\text{Inv}^d(G, d - 1)$. 
Cohomological Invariants

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An invariant $a$ is called **normalized** if it sends trivial torsor to zero. We denote the subgroup of normalized invariants by $\text{Inv}^d(G, d - 1)_{\text{norm}}$.

A normalized invariant $a$ is called **decomposable** if it is given by a cup-product with an invariant of degree 2 (class in the Brauer group). We denote the subgroup of decomposable invariants by $\text{Inv}^3(G, 2)_{\text{dec}}$.

The factor group $\text{Inv}^3(G, 2)_{\text{norm}}/\text{Inv}^3(G, 2)_{\text{dec}}$ is denoted by $\text{Inv}^3(G, 2)_{\text{ind}}$ and is called the group of **indecomposable** invariants.
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The group $\text{Inv}^3(G, 2)_{\text{ind}}$ has been studied by Garibaldi, Kahn, Levine, Rost, Serre and others in the simply-connected case and is closely related to the Rost invariant.

Recently, Merkurjev showed how to compute this group in general using new results on motivic cohomology. In particular, it was computed by him for all adjoint split groups and by Bermudez and Ruozzi for all split simple groups.
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Recall that a classical **character map** identifies the representation ring of $G$ with the subring $\mathbb{Z}[T^*]^W$ of $W$-invariant elements of the integral group ring $\mathbb{Z}[T^*]$, where $W$ is the Weyl group which acts naturally on the group of characters $T^*$ of a split maximal torus $T$ of $G$.

In particular, the ideal $(\tilde{I}^W)$ generated by augmented $W$-invariant elements in $\mathbb{Z}[\Lambda]$, where $\Lambda$ is the respective weight lattice, can be identified with the ideal generated by classes of augmented (i.e. virtual of dimension 0) representations of the simply-connected cover of $G$. 
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We introduce a subgroup of **semi-decomposable** invariants $\text{Inv}^3(G, 2)_\text{sdec}$ which consists of invariants $a \in \text{Inv}^3(G, 2)_{\text{norm}}$ such that for every field extension $L/F$ and a $G$-torsor $Y$ over $L$

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a(Y) = \sum_{i \text{ finite}} \phi_i \cup b_i(Y) \text{ for some } \phi_i \in L^\times \text{ and } b_i \in \text{Inv}^2(G, 1)_{\text{norm}}.
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Observe that by definition we have

$\text{Inv}^3(G, 2)_\text{dec} \subseteq \text{Inv}^3(G, 2)_\text{sdec} \subseteq \text{Inv}^3(G, 2)_{\text{norm}}$.

Roughly speaking,

**Semi-decomposable = Locally decomposable.**
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Main Theorem. Let $G$ be a split semisimple linear algebraic group over a field $F$ and let $X^{\text{gen}}$ denote the associated versal flag. There is a short exact sequence

$$0 \to \frac{\text{Inv}^3(G,2)_{\text{sdec}}}{\text{Inv}^3(G,2)_{\text{dec}}} \to \text{Inv}^3(G,2)_{\text{ind}} \to \text{CH}^2(X^{\text{gen}})_{\text{tors}} \to 0,$$

together with a group isomorphism

$$\frac{\text{Inv}^3(G,2)_{\text{sdec}}}{\text{Inv}^3(G,2)_{\text{dec}}} \cong \frac{c_2((\tilde{W}) \cap \mathbb{Z}[T^*])}{c_2(\mathbb{Z}[T^*]^W)},$$

where $c_2$ is the second Chern class map.

In addition, if $G$ is simple, then $\text{Inv}^3(G,2)_{\text{sdec}} = \text{Inv}^3(G,2)_{\text{dec}}$, so there is an isomorphism

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Example

If $G$ is not simple, then $\text{Inv}^3(G, 2)_{\text{sdec}}$ does not necessarily coincide with $\text{Inv}^3(G, 2)_{\text{dec}}$:

Consider a quadratic form $q$ of degree 4 with trivial discriminant (it corresponds to a $\text{SO}_4$-torsor). There is an invariant given by $q \mapsto \alpha \cup \beta \cup \gamma$, where $\alpha$ is represented by $q$ and $\langle \beta, \gamma \rangle = \langle \alpha \rangle q$ is the 2-Pfister form [Garibaldi-Merkurjev-Serre Ex. 20.3].

This invariant is semi-decomposable but not decomposable.
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This invariant is semi-decomposable but not decomposable.
Let $p$ be a prime integer and $G = \text{SL}_{ps}/\mu_{pr}$ for some integers $s \geq r > 0$. If $p$ is odd, we set $k = \min\{r, s - r\}$ and if $p = 2$ we assume that $s \geq r + 1$ and set $k = \min\{r, s - r - 1\}$.

It is shown by Bermudez-Ruozzi (2013) that the group $\text{Inv}^3(G, 2)_{\text{ind}}$ is cyclic of order $p^k$.

By Karpenko (1998) if $X$ is the Severi-Brauer variety of a generic algebra $A^\text{gen}$, then $\text{CH}^2(X)_{\text{tors}}$ is also a cyclic group of order $p^k$ (to show this Karpenko uses the Grothendieck $\gamma$-filtration).

The canonical morphism $X^\text{gen} \rightarrow X$ is an iterated projective bundle, hence, $\text{CH}^2(X^\text{gen})_{\text{tors}} \simeq \text{CH}^2(X)_{\text{tors}}$ is a cyclic group of order $p^k$. The exact sequence of the theorem implies that

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The canonical morphism $X^\text{gen} \rightarrow X$ is an iterated projective bundle, hence, $\text{CH}^2(X^\text{gen})_{\text{tors}} \cong \text{CH}^2(X)_{\text{tors}}$ is a cyclic group of order $p^k$. The exact sequence of the theorem implies that

$$\text{Inv}^3(G, 2)_{\text{sdec}} = \text{Inv}^3(G, 2)_{\text{dec}}$$
Consider $K_0(X^{\text{gen}})$. It can be shown that the Grothendieck $\gamma$-filtration on $X^{\text{gen}}$ coincides with the topological filtration. So that for simple groups

$$\gamma^{2/3}(X^{\text{gen}}) \simeq \tau^{2/3}(X^{\text{gen}}) \simeq \text{Tors \, CH}^2(X^{\text{gen}}) \simeq \text{Inv}^3(G, 2)_{\text{ind}}$$

The group $\gamma^{2/3}(X^{\text{gen}})$ can be computed using Panin’s theorem (this involves the Steinberg basis and indices of Tits algebras).

This gives a way to describe invariants via algebraic cycles (characteristic classes of bundles over $X^{\text{gen}}$).

How to construct non-trivial torsion elements in $\gamma^{2/3}(X^{\text{gen}})$ directly?

- for strongly inner and some inner $G$ (Garibaldi-Z., 2012)
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What about non-versal case? Namely, what is the relation between \( \gamma^{2/3}(X) \) and \( \text{Inv}^3(G, 2)_{\text{ind}} \), where \( X \) is a twisted form of \( G/B \) by means of an arbitrary \( G \)-torsor?

Does \( \gamma^{2/3}(X) \) correspond to a group of 'conditional invariants'?

Yes, for some \( \text{PGO}_{4n} \)-torsors (Junkins, 2013)

Here the non-trivial torsion elements of \( \gamma^{2/3}(X^\text{gen}) \) can be constructed using the twisted \( \gamma \)-filtration (Z., 2012).

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- for all versal flags
- for twisted flag varieties that share the same upper-motive as the versal flag (one uses here various motivic decomposition results), e.g.
- for (generic) maximal orthogonal Grassmannian,
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Non-triviality of invariants. Case $C_n$

Let $G = \text{PGSp}_{2n}$ be the split projective symplectic group. For a field extension $L/F$, the set $H^1(L, G)$ is identified with the set of isomorphism classes of central simple $L$-algebras $A$ of degree $2n$ with a symplectic involution $\sigma$.

A decomposable invariant of $G$ then takes an algebra with involution $(A, \sigma)$ to the cup-product $\phi \cup [A]$ for a fixed element $\phi \in F^\times$. In particular, decomposable invariants of $G$ are independent of the involution.
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Suppose that $4 \mid n$. It is known that the group of indecomposable invariants $\text{Inv}^3(G, 2)_{\text{ind}}$ is cyclic of order 2.

If $\text{char}(F) \neq 2$, Garibaldi, Parimala and Tignol constructed a degree 3 cohomological invariant $\Delta_{2n}$ of the group $G$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$. They showed that if $a \in A$ is a $\sigma$-symmetric element of $A^\times$ and $\sigma' = \text{Int}(a) \circ \sigma$, then

$$\Delta_{2n}(A, \sigma') = \Delta_{2n}(A, \sigma) + \text{Nrp}(a) \cup [A],$$

(1)

where Nrp is the pfaffian norm. In particular, $\Delta_{2n}$ does depend on the involution and, therefore, the invariant $\Delta_{2n}$ is not decomposable. Hence the class of $\Delta_{2n}$ in $\text{Inv}^3(G, 2)_{\text{ind}}$ is nontrivial.
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So the class $\Delta_{2n}(A) \in \frac{H^3(L,\mathbb{Z}/2\mathbb{Z})}{L \times \cup [A]}$ of $\Delta_{2n}(A, \sigma)$ depends only on the $L$-algebra $A$ of degree $2n$ and exponent 2 but not on the involution.

Since $\Delta_{2n}(A, \sigma)$ is not decomposable, it is not semi-decomposable by our main theorem. The latter implies that $\Delta_{2n}(A)$ is nontrivial generically,

i.e. there is a central simple algebra $A$ of degree $2n$ over a field extension of $F$ with exponent 2 such that $\Delta_{2n}(A) \neq 0$. 
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So the class \( \Delta_{2n}(A) \in \frac{H^3(L,\mathbb{Z}/2\mathbb{Z})}{L^\times \cup [A]} \) of \( \Delta_{2n}(A, \sigma) \) depends only on the \( L \)-algebra \( A \) of degree \( 2n \) and exponent 2 but not on the involution.

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Nontriviality of invariants. Case $A_n$

Let $G = \text{SL}_n / \mu_m$, where $n$ and $m$ are positive integers such that $n$ and $m$ have the same prime divisors and $m \mid n$.

Given a field extension $L/F$ the natural surjection $G \to \text{PGL}_n$ yields a map

$$\alpha : H^1(L, G) \to H^1(L, \text{PGL}_n) \subset \text{Br}(L)$$

taking a $G$-torsor $Y$ over $L$ to the class of a central simple algebra $A(Y)$ of degree $n$ and exponent dividing $m$.

By definition, a decomposable invariant of $G$ is of the form $Y \mapsto \phi \cup [A(Y)]$ for a fixed $\phi \in F^\times$. 
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The map $\text{SL}_m \rightarrow \text{SL}_n$ taking a matrix $M$ to the tensor product $M \otimes I_{n/m}$ with the identity matrix, gives rise to a group homomorphism $\text{PGL}_m \rightarrow G$.

The induced homomorphism

$$\varphi : \text{Inv}^3(G, 2)_{\text{norm}} \rightarrow \text{Inv}^3(\text{PGL}_m, 2)_{\text{norm}} = F^\times / F^\times m$$

is a splitting of the inclusion homomorphism

$$F^\times / F^\times m = \text{Inv}^3(G, 2)_{\text{dec}} \hookrightarrow \text{Inv}^3(G, 2)_{\text{norm}}.$$
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Collecting descriptions of $p$-primary components of $\text{Inv}^3(G, 2)_{\text{ind}}$ we get

$$\text{Inv}^3(G, 2)_{\text{ind}} \cong \frac{m\mathbb{Z}q}{m\mathbb{Z}q}, \quad \text{where } k = \begin{cases} \gcd\left(\frac{n}{m}, m\right), & \text{if } \frac{n}{m} \text{ is odd;} \\ \gcd\left(\frac{n}{2m}, m\right), & \text{if } \frac{n}{m} \text{ is even.} \end{cases}$$

(2)

Let $\Delta_{n,m}$ be a (unique) invariant in $\text{Inv}^3(G, 2)_{\text{norm}}$ such that its class in $\text{Inv}^3(G, 2)_{\text{ind}}$ corresponds to $\frac{m}{k}q + m\mathbb{Z}q$ and $\varphi(\Delta_{n,m}) = 0$.

Note that the order of $\Delta_{n,m}$ in $\text{Inv}^3(G, 2)_{\text{norm}}$ is equal to $k$. Therefore, $\Delta_{n,m}$ takes values in $H^3(-, \mathbb{Z}/k\mathbb{Z}(2)) \subset H^3(-, 2)$. 
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Nontriviality of invariants. Case $A_n$

Fix a $G$-torsor $Y$ over $F$ and consider the twists $^YG$ and $\text{SL}_1(A(Y))$ by $Y$ of the groups $G$ and $\text{SL}_n$ respectively.

By (2) the image of $\Delta_{n,m}$ under the natural composition

$$\text{Inv}^3(G, 2)_{\text{norm}} \simeq \text{Inv}^3(^YG, 2)_{\text{norm}} \rightarrow \text{Inv}^3(\text{SL}_1(A(Y)), 2)_{\text{norm}}$$

is a $\frac{m}{k}$-multiple of the Rost invariant.

Recall that the Rost invariant takes the class of $\phi$ in $F^\times / \text{Nrd}(A(Y)^\times) = H^1(F, \text{SL}_1(A(Y)))$ to the cup-product $\phi \cup [A(Y)] \in H^3(F, 2)$. So we get

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(Here the group $F^\times$ acts transitively on the fiber over $A(Y)$ of the map $\alpha$. If $\phi \in F^\times$, we write $\phi Y$ for the corresponding element in the fiber.)
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is a $\frac{m}{k}$-multiple of the Rost invariant.

Recall that the Rost invariant takes the class of $\phi$ in $F^\times / \text{Nrd}(A(Y)^\times) = H^1(F, \text{SL}_1(A(Y)))$ to the cup-product $\phi \cup [A(Y)] \in H^3(F, 2)$. So we get

\[ \Delta_{n,m}(\phi Y) - \Delta_{n,m}(Y) \in F^\times \cup \frac{m}{k}[A(Y)]. \quad (3) \]

(Here the group $F^\times$ acts transitively on the fiber over $A(Y)$ of the map $\alpha$. If $\phi \in F^\times$, we write $\phi Y$ for the corresponding element in the fiber.)
Nontriviality of invariants. Case $A_n$

Given a central simple $L$-algebra $A$ of degree $n$ and exponent dividing $m$, we define an element

$$\Delta_{n,m}(A) \in \frac{H^3(L,\mathbb{Z}/k\mathbb{Z}(2))}{L \times \bigcup \frac{m}{k}[A]}$$

as follows.

Choose a $G$-torsor $Y$ over $L$ with $A(Y) \cong A$ and set $\Delta_{n,m}(A)$ to be the class of $\Delta_{n,m}(Y)$ in the factor group.

It follows from (3) that $\Delta_{n,m}(A)$ is independent of the choice of $Y$. 
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\[
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It follows from (3) that \( \Delta_{n,m}(A) \) is independent of the choice of \( Y \).
**Proposition.** Let $A$ be a central simple $L$-algebra of degree $n$ and exponent dividing $m$. Then the order of $\Delta_{n,m}(A)$ divides $k$. If $A$ is a generic algebra, then the order of $\Delta_{n,m}(A)$ is equal to $k$.

Proof: If $k'$ is a proper divisor of $k$, then the multiple $k'\Delta_{n,m}$ is not decomposable. By our theorem $k'\Delta_{n,m}$ is not semi-decomposable and, hence, $k'\Delta_{n,m}(A) \neq 0$. 
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**Example.** Let $A$ be a central simple $F$-algebra of degree $2n$ divisible by 8 and exponent 2. Choose a symplectic involution $\sigma$ on $A$.

The group $\text{PGSp}_{2n}$ is a subgroup of $\text{SL}_{2n}/\mu_2$, hence, if $\text{char}(F) \neq 2$, the restriction of the invariant $\Delta_{2n,2}$ on $\text{PGSp}_{2n}$ is the invariant $\Delta_{2n}(A, \sigma)$ considered before. It follows that $\Delta_{2n,2}(A) = \Delta_{2n}(A)$ in the group $H^3(F, \mathbb{Z}/2\mathbb{Z})/(F^\times \cup \{[A]\})$. 
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Nontriviality of invariants. Case $A_n$

The class $\Delta_{n,m}$ is trivial on decomposable algebras:

**Proposition.** Let $n_1, n_2, m$ be positive integers such that $m$ divides $n_1$ and $n_2$. Let $A_1$ and $A_2$ be two central simple algebras over $F$ of degree $n_1$ and $n_2$ respectively and of exponent dividing $m$. Then $\Delta_{n_1n_2,m}(A_1 \otimes_F A_2) = 0$.

**Proof:** The tensor product homomorphism $\text{SL}_{n_1} \times \text{SL}_{n_2} \to \text{SL}_{n_1n_2}$ yields a homomorphism

$$\text{Sym}^2(T_{n_1n_2}^*) \to \text{Sym}^2(T_{n_1}^*) \oplus \text{Sym}^2(T_{n_2}^*),$$

where $T_{n_1}$, $T_{n_2}$ and $T_{n_1n_2}$ are maximal tori of respective groups. The image of the canonical Weyl-invariant generator $q_{n_1n_2}$ of $\text{Sym}^2(T_{n_1n_2}^*)$ is equal to $n_2q_{n_1} + n_1q_{n_2}$. Since $n_1$ and $n_2$ are divisible by $m$, the pull-back of the invariant $\Delta_{n_1n_2,m}$ under the homomorphism $(\text{SL}_{n_1}/\mu_m) \times (\text{SL}_{n_2}/\mu_m) \to \text{SL}_{n_1n_2}/\mu_m$ is trivial.
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Invariants vs. Representation Theory

How to show that $\text{Inv}^3(G, 2)_{\text{dec}} = \text{Inv}^3(G, 2)_{\text{sdec}}$ for simple groups of type $C_{4n}$?

We want to show that $c_2(x) \in 2\mathbb{Z}q$ for every element $x \in (\tilde{I}^W) \cap \mathbb{Z}[T^*]$. 

Given a weight $\chi \in \Lambda$ we denote by $W(\chi)$ its $W$-orbit and we define $\hat{e}^\chi := \sum_{\lambda \in W(\chi)} (1 - e^{-\lambda})$.

By definition, the ideal $(\tilde{I}^W)$ is generated by elements $\{\hat{e}^{\omega_i}\}_{i=1..4m}$ corresponding to the fundamental weights $\omega_i$.

An element $x$ can be written as

$$x = \sum_{i=1}^{4m} n_i \hat{e}^{\omega_i} + \delta_i \hat{e}^{\omega_i}, \quad \text{where } n_i \in \mathbb{Z} \text{ and } \delta_i \in \tilde{I}. \quad (4)$$
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Consider a ring homomorphism \( f : \mathbb{Z}[\Lambda] \to \mathbb{Z}[\Lambda/T^*] \) induced by taking the quotient \( \Lambda \to \Lambda/T^* = C^* \).

We have \( \Lambda/T^* \cong \mathbb{Z}/2\mathbb{Z} \) and \( \mathbb{Z}[\Lambda/T^*] = \mathbb{Z}[y]/(y^2 - 2y) \), where \( y = f(e^{\omega_1} - 1) \). Observe that \( C^* \) is \( W \)-invariant.

By definition, \( f(I) = 0 \), so \( f(x) = 0 \). Since \( \omega_i \in T^* \) for all even \( i \), \( f(e^{\omega_i}) = y \) for all odd \( i \) and \( f(\delta_i) \in f(I) = (y) \), we get

\[
0 = f(x) = \sum_{i \text{ is odd}} n_id_iy + m_id_iy^2 = (\sum_{i \text{ is odd}} n_i + 2m_i)d_iy,
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Dividing this sum by the g.c.d. of all \( d_i \)'s and taking the result modulo 2 (here one uses the fact \( \frac{n}{\text{g.c.d.}(n,k)} | \binom{n}{k} \)), we obtain that the coefficient \( n_1 \) in the presentation (4) has to be even.
Consider a ring homomorphism $f : \mathbb{Z}[\Lambda] \to \mathbb{Z}[\Lambda/T^*]$ induced by taking the quotient $\Lambda \to \Lambda/T^* = C^*$. We have $\Lambda/T^* \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}[\Lambda/T^*] = \mathbb{Z}[y]/(y^2 - 2y)$, where $y = f(e^{\omega_1} - 1)$. Observe that $C^*$ is $W$-invariant.

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We now compute $c_2(x)$. Let $\Lambda = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_{4m}$. The root lattice is given by $T^* = \{ \sum a_i e_i \mid \sum a_i \text{ is even} \}$ and

$$\omega_1 = e_1, \omega_2 = e_1 + e_2, \omega_3 = e_1 + e_2 + e_3, \ldots, \omega_{4m} = e_1 + \ldots + e_{4m}.$$ 

By Garibaldi-Z. we have $c_2(x) = \sum_{i=1}^{4m} n_i c_2(\hat{e}^{\omega_i})$ and $c_2(\hat{e}^{\omega_i}) = N(\hat{e}^{\omega_i})q$, where

$$N(\sum a_j e^{\lambda_j}) = \frac{1}{2} \sum a_j \langle \lambda_j, \alpha^\vee \rangle^2 \text{ for a fixed long root } \alpha.$$ 

If we set $\alpha = 2e_{4m}$, then $\langle \lambda, \alpha^\vee \rangle = (\lambda, e_{4m})$ and

$$N(\hat{e}^{\omega_i}) = \frac{1}{2} \sum_{\lambda \in W(\omega_i)} \langle \lambda, \alpha^\vee \rangle^2 = \frac{1}{2} \sum_{\lambda \in W(\omega_i)} (\lambda, e_{4m})^2 = 2^{i-1}(4m-1)_{i-1}$$

which is even for $i \geq 2$ (here we used the fact that the Weyl group acts by permutations and sign changes on $\{e_1, \ldots, e_{4m}\}$). Since $n_1$ is even, we get that $c_2(x) \in 2\mathbb{Z}q$. 


We now compute $c_2(x)$. Let $\Lambda = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_{4m}$. The root lattice is given by $T^* = \{\sum a_i e_i \mid \sum a_i \text{ is even}\}$ and

$$\omega_1 = e_1, \ \omega_2 = e_1 + e_2, \ \omega_3 = e_1 + e_2 + e_3, \ldots, \omega_{4m} = e_1 + \ldots + e_{4m}.$$ 

By Garibaldi-Z. we have $c_2(x) = \sum_{i=1}^{4m} n_i c_2(e^{\omega_i})$ and $c_2(e^{\omega_i}) = N(e^{\omega_i}) q$, where

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If we set $\alpha = 2e_{4m}$, then $\langle \lambda, \alpha^\vee \rangle = (\lambda, e_{4m})$ and

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By Garibaldi-Z. we have \( c_2(x) = \sum_{i=1}^{4m} n_i c_2(\hat{\omega}_i) \) and \( c_2(\hat{\omega}_i) = N(\hat{\omega}_i)q \), where

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