Invariants of degree 3 and torsion in the Chow group of a versal flag

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Let G be a split semisimple linear algebraic group over a field F.

The purpose of the present talk is to relate:

the *geometry* of twisted G-flag varieties

the theory of *cohomological invariants* of G

the representation theory of G

Let U/G be a **classifying space** of G in the sense of Totaro, i.e. U is an open G-invariant subset in some representation of G with $U(F) \neq \emptyset$ and $U \to U/G$ is a G-torsor.

Consider the generic fiber U^{gen} of U over U/G. It is a G-torsor over the quotient field K of U/G called the **versal torsor**.

We denote by X^{gen} the respective flag variety U^{gen}/B over K, where B is a Borel subgroup of G, and call it the **versal flag**.

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The variety X^{gen} can be viewed as

the 'most twisted' form of the
'most complicated' G-flag variety.

Example: Take the variety of flags of ideals in a generic division algebra over F.

We want to understand its geometry via studying its

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Moreover,

$$\mathrm{CH}(\text{twisted flag}) \otimes \mathbb{Q} \simeq \mathrm{CH}(\text{split flag}) \otimes \mathbb{Q}$$

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Geometry of twisted G-flag varieties

The computation of *Tors* CH(twisted flag) has been pushed by the development of motivic cohomology theory in 90's culminating in numerous results, e.g.

- anisotropic quadrics *Tors* CH^i, $i \leq 4$ [Karpenko, Merkurjev]
- group of zero-cycles *Tors* CH_0(twisted flag) [Chernousov, Krashen, Merkurjev, Parimala, Springer, Totaro]
- *Tors* CH^2(strongly inner twisted flag) [Baek, Merkurjev, Peyre, Z., Zhong]

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Cohomological Invariants

Has been mainly inspired by the works of J.-P. Serre and M. Rost.

Given a field extension L/F and a positive integer d we consider the Galois cohomology group $H^{d+1}(L, d) = H^{d+1}(L, \mathbb{Q}/\mathbb{Z}(d))$.

A degree d *cohomological invariant* is a natural transformation of functors

$$a\colon H^1(—, G) \to H^d(—, d-1)$$

on the category of field extensions over F. We denote the group of degree d invariants by $\mathrm{Inv}^d(G, d-1)$.

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An invariant a is called **normalized** if it sends trivial torsor to zero. We denote the subgroup of normalized invariants by $\mathsf{Inv}^d(G, d-1)_{\mathrm{norm}}$.

A normalized invariant a is called **decomposable** if it is given by a cup-product with an invariant of degree 2 (class in the Brauer group). We denote the subgroup of decomposable invariants by $\mathsf{Inv}^3(G, 2)_{\mathrm{dec}}$.

The factor group $\mathsf{Inv}^3(G, 2)_{\mathrm{norm}}/\mathsf{Inv}^3(G, 2)_{\mathrm{dec}}$ is denoted by $\mathsf{Inv}^3(G, 2)_{\mathrm{ind}}$ and is called the group of **indecomposable** invariants.

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Cohomological Invariants

The group $\mathrm{Inv}^3(G, 2)_{\mathrm{ind}}$ has been studied by Garibaldi, Kahn, Levine, Rost, Serre and others in the simply-connected case and is closely related to the Rost invariant.

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Recall that a classical **character map** identifies the representation ring of G with the subring $\mathbb{Z}[T^*]^W$ of W-invariant elements of the integral group ring $\mathbb{Z}[T^*]$, where W is the Weyl group which acts naturally on the group of characters T^* of a split maximal torus T of G.

In particular, the ideal $(\widetilde{I}^W)$ generated by augmented W-invariant elements in $\mathbb{Z}[\Lambda]$, where Λ is the respective weight lattice, can be identified with the ideal generated by classes of augmented (i.e. virtual of dimension 0) representations of the simply-connected cover of G.

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Semidecomposable Invariants

We introduce a subgroup of **semi-decomposable** invariants $\mathrm{Inv}^3(G,2)_{\mathrm{sdec}}$ which consists of invariants $a \in \mathrm{Inv}^3(G,2)_{\mathrm{norm}}$ such that for every field extension L/F and a G-torsor Y over L

$$a(Y) = \sum_{i \text{ finite}} \phi_i \cup b_i(Y) \text{ for some } \phi_i \in L^\times \text{ and } b_i \in \mathrm{Inv}^2(G,1)_{\mathrm{norm}}.$$

Observe that by definition we have
$\mathrm{Inv}^3(G,2)_{\mathrm{dec}} \subseteq \mathrm{Inv}^3(G,2)_{\mathrm{sdec}} \subseteq \mathrm{Inv}^3(G,2)_{\mathrm{norm}}$.

Roughly speaking,

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Main Theorem. Let G be a split **semisimple** linear algebraic group over a field F and let X^{gen} denote the associated versal flag. There is a short exact sequence

$$0 \to \frac{\mathrm{Inv}^3(G,2)_{\mathrm{sdec}}}{\mathrm{Inv}^3(G,2)_{\mathrm{dec}}} \to \mathrm{Inv}^3(G,2)_{\mathrm{ind}} \to \mathrm{CH}^2(X^{\mathrm{gen}})_{\mathrm{tors}} \to 0,$$

together with a group isomorphism

$$\frac{\mathrm{Inv}^3(G,2)_{\mathrm{sdec}}}{\mathrm{Inv}^3(G,2)_{\mathrm{dec}}} \simeq \frac{c_2((\widetilde{I}^W)\cap\mathbb{Z}[T^*])}{c_2(\mathbb{Z}[T^*]^W)},$$

where c_2 is the second *Chern class* map.

In addition, if G is **simple**, then $\mathrm{Inv}^3(G,2)_{\mathrm{sdec}} = \mathrm{Inv}^3(G,2)_{\mathrm{dec}}$, so there is an isomorphism

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Example

If G is not simple, then $\mathrm{Inv}^3(G,2)_{\mathrm{sdec}}$ does not necessarily coincide with $\mathrm{Inv}^3(G,2)_{\mathrm{dec}}$:

Consider a quadratic form q of degree 4 with trivial discriminant (it corresponds to a $\mathbf{SO}_4$-torsor). There is an invariant given by $q \mapsto \alpha \cup \beta \cup \gamma$, where α is represented by q and $\langle\langle \beta, \gamma \rangle\rangle = \langle \alpha \rangle q$ is the 2-Pfister form [Garibaldi-Merkurjev-Serre Ex. 20.3].

This invariant is semi-decomposable but not decomposable.

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Invariants vs. Geometry

Let p be a prime integer and $G = \mathbf{SL}_{p^s}/\boldsymbol{\mu}_{p^r}$ for some integers $s \geq r > 0$. If p is odd, we set $k = \min\{r, s-r\}$ and if $p = 2$ we assume that $s \geq r+1$ and set $k = \min\{r, s-r-1\}$.
It is shown by Bermudez-Ruozzi (2013) that the group $\mathrm{Inv}^3(G,2)_{\mathrm{ind}}$ is cyclic of order p^k.

By Karpenko (1998) if X is the Severi-Brauer variety of a generic algebra A^{gen}, then $\mathrm{CH}^2(X)_{\mathrm{tors}}$ is also a cyclic group of order p^k (to show this Karpenko uses the Grothendieck γ-filtration).

The canonical morphism $X^{\mathrm{gen}} \to X$ is an iterated projective bundle, hence, $\mathrm{CH}^2(X^{\mathrm{gen}})_{\mathrm{tors}} \simeq \mathrm{CH}^2(X)_{\mathrm{tors}}$ is a cyclic group of order p^k. The exact sequence of the theorem implies that

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Consider $K_0(X^{\mathrm{gen}})$. It can be shown that the Grothendieck γ-filtration on X^{gen} coincides with the topological filtration. So that for simple groups

$$\gamma^{2/3}(X^{\mathrm{gen}}) \simeq \tau^{2/3}(X^{\mathrm{gen}}) \simeq \mathit{Tors}\,\mathrm{CH}^2(X^{\mathrm{gen}}) \simeq \mathrm{Inv}^3(G,2)_{\mathrm{ind}}$$

The group $\gamma^{2/3}(X^{\mathrm{gen}})$ can be computed using Panin's theorem (this involves the Steinberg basis and indices of Tits algebras).

This gives a way to describe invariants via algebraic cycles (characteristic classes of bundles over X^{gen}).

How to construct non-trivial torsion elements in $\gamma^{2/3}(X^{\mathrm{gen}})$ directly ?

- for strongly inner and some inner G (Garibaldi-Z., 2012)
- is open for a general G.

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where X is a twisted form of G/B by means of an arbitrary G-torsor ?

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Yes, for some $\mathbf{PGO}_{4n}$-torsors (Junkins, 2013)

Here the non-trivial torsion elements of $\gamma^{2/3}(X^{\mathrm{gen}})$ can be constructed using the twisted γ-filtration (Z., 2012).
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As $\mathrm{Inv}^3(G,2)_{\mathrm{ind}}$ has been computed for all simple groups, we immediately compute $\mathit{Tors}\,\mathrm{CH}^2$

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Non-triviality of invariants. Case C_n

Let $G = \mathbf{PGSp}_{2n}$ be the split projective symplectic group. For a field extension L/F, the set $H^1(L, G)$ is identified with the set of isomorphism classes of central simple L-algebras A of degree $2n$ with a symplectic involution σ.

A decomposable invariant of G then takes an algebra with involution (A, σ) to the cup-product $\phi \cup [A]$ for a fixed element $\phi \in F^\times$. In particular, decomposable invariants of G are independent of the involution.

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Suppose that $4 \mid n$. It is known that the group of indecomposable invariants $\mathrm{Inv}^3(G,2)_{\mathrm{ind}}$ is cyclic of order 2.

If $\mathrm{char}(F) \neq 2$, Garibaldi, Parimala and Tignol constructed a degree 3 cohomological invariant Δ_{2n} of the group G with coefficients in $\mathbb{Z}/2\mathbb{Z}$. They showed that if $a \in A$ is a σ-symmetric element of $A^\times$ and $\sigma' = \mathrm{Int}(a) \circ \sigma$, then

$$\Delta_{2n}(A,\sigma') = \Delta_{2n}(A,\sigma) + \mathrm{Nrp}(a) \cup [A], \qquad (1)$$

where Nrp is the pfaffian norm.
In particular, Δ_{2n} does depend on the involution and, therefore, the invariant Δ_{2n} is not decomposable. Hence the class of Δ_{2n} in $\mathrm{Inv}^3(G,2)_{\mathrm{ind}}$ is nontrivial.

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So the class $\Delta_{2n}(A) \in \frac{H^3(L,\mathbb{Z}/2\mathbb{Z})}{L^\times \cup [A]}$ of $\Delta_{2n}(A,\sigma)$ depends only on the L-algebra A of degree $2n$ and exponent 2 but not on the involution.

Since $\Delta_{2n}(A,\sigma)$ is not decomposable, it is not semi-decomposable by our main theorem. The latter implies that $\Delta_{2n}(A)$ is *nontrivial generically*,

i.e. there is a central simple algebra A of degree $2n$ over a field extension of F with exponent 2 such that $\Delta_{2n}(A) \neq 0$.

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Let $G = \mathbf{SL}_n/\boldsymbol{\mu}_m$, where n and m are positive integers such that n and m have the same prime divisors and $m \mid n$.
Given a field extension L/F the natural surjection $G \to \mathbf{PGL}_n$ yields a map

$$\alpha : H^1(L, G) \to H^1(L, \mathbf{PGL}_n) \subset \mathrm{Br}(L)$$

taking a G-torsor Y over L to the class of a central simple algebra $A(Y)$ of degree n and exponent dividing m.

By definition, a decomposable invariant of G is of the form $Y \mapsto \phi \cup [A(Y)]$ for a fixed $\phi \in F^\times$.

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The map $\mathbf{SL}_m \to \mathbf{SL}_n$ taking a matrix M to the tensor product $M \otimes I_{n/m}$ with the identity matrix, gives rise to a group homomorphism $\mathbf{PGL}_m \to G$.

The induced homomorphism

$$\varphi : \mathsf{Inv}^3(G, 2)_{\mathrm{norm}} \to \mathsf{Inv}^3(\mathbf{PGL}_m, 2)_{\mathrm{norm}} = F^\times / F^{\times m}$$

is a splitting of the inclusion homomorphism

$$F^\times / F^{\times m} = \mathsf{Inv}^3(G, 2)_{\mathrm{dec}} \hookrightarrow \mathsf{Inv}^3(G, 2)_{\mathrm{norm}}.$$

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Collecting descriptions of p-primary components of $\mathrm{Inv}^3(G,2)_{\mathrm{ind}}$ we get

$$\mathrm{Inv}^3(G,2)_{\mathrm{ind}} \simeq \tfrac{m}{k}\mathbb{Z}q/m\mathbb{Z}q, \qquad \text{where } k = \begin{cases} \gcd(\frac{n}{m}, m), & \text{if } \frac{n}{m} \text{ is odd;} \\ \gcd(\frac{n}{2m}, m), & \text{if } \frac{n}{m} \text{ is even.} \end{cases} \tag{2}$$

Let $\Delta_{n,m}$ be a (unique) invariant in $\mathrm{Inv}^3(G,2)_{\mathrm{norm}}$ such that its class in $\mathrm{Inv}^3(G,2)_{\mathrm{ind}}$ corresponds to $\frac{m}{k}q + m\mathbb{Z}q$ and $\varphi(\Delta_{n,m}) = 0$.

Note that the order of $\Delta_{n,m}$ in $\mathrm{Inv}^3(G,2)_{\mathrm{norm}}$ is equal to k. Therefore, $\Delta_{n,m}$ takes values in $H^3(-,\mathbb{Z}/k\mathbb{Z}(2)) \subset H^3(-,2)$.

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Fix a G-torsor Y over F and consider the twists ${}^{Y}G$ and $\mathbf{SL}_1(A(Y))$ by Y of the groups G and $\mathbf{SL}_n$ respectively.
By (2) the image of $\Delta_{n,m}$ under the natural composition

$$\operatorname{Inv}^3(G,2)_{\mathrm{norm}} \simeq \operatorname{Inv}^3({}^{Y}G,2)_{\mathrm{norm}} \longrightarrow \operatorname{Inv}^3(\mathbf{SL}_1(A(Y)),2)_{\mathrm{norm}}$$

is a $\frac{m}{k}$-multiple of the Rost invariant.

Recall that the Rost invariant takes the class of ϕ in $F^\times/\operatorname{Nrd}(A(Y)^\times) = H^1(F,\mathbf{SL}_1(A(Y)))$ to the cup-product $\phi \cup [A(Y)] \in H^3(F,2)$. So we get

$$\Delta_{n,m}({}^{\phi}Y) - \Delta_{n,m}(Y) \in F^\times \cup \tfrac{m}{k}[A(Y)]. \tag{3}$$

(Here the group $F^\times$ acts transitively on the fiber over $A(Y)$ of the map α. If $\phi \in F^\times$, we write ${}^{\phi}Y$ for the corresponding element in the fiber.)

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Given a central simple L-algebra A of degree n and exponent dividing m, we define an element

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as follows.

Choose a G-torsor Y over L with $A(Y) \simeq A$ and set $\Delta_{n,m}(A)$ to be the class of $\Delta_{n,m}(Y)$ in the factor group.

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Proposition. Let A be a central simple L-algebra of degree n and exponent dividing m. Then the order of $\Delta_{n,m}(A)$ divides k. If A is a generic algebra, then the order of $\Delta_{n,m}(A)$ is equal to k.

Proof: If k' is a proper divisor of k, then the multiple $k'\Delta_{n,m}$ is not decomposable. By our theorem $k'\Delta_{n,m}$ is not semi-decomposable and, hence, $k'\Delta_{n,m}(A) \neq 0$.

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Proof: If k' is a proper divisor of k, then the multiple $k'\Delta_{n,m}$ is not decomposable. By our theorem $k'\Delta_{n,m}$ is not semi-decomposable and, hence, $k'\Delta_{n,m}(A) \neq 0$.

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Example. Let A be a central simple F-algebra of degree $2n$ divisible by 8 and exponent 2. Choose a symplectic involution σ on A.

The group $\mathbf{PGSp}_{2n}$ is a subgroup of $\mathbf{SL}_{2n}/\boldsymbol{\mu}_2$, hence, if $\mathrm{char}(F) \neq 2$, the restriction of the invariant $\Delta_{2n,2}$ on $\mathbf{PGSp}_{2n}$ is the invariant $\Delta_{2n}(A, \sigma)$ considered before.
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The class $\Delta_{n,m}$ is trivial on decomposable algebras:

Proposition. Let n_1, n_2, m be positive integers such that m divides n_1 and n_2. Let A_1 and A_2 be two central simple algebras over F of degree n_1 and n_2 respectively and of exponent dividing m.
Then $\Delta_{n_1 n_2, m}(A_1 \otimes_F A_2) = 0$.

Proof: The tensor product homomorphism $\mathbf{SL}_{n_1} \times \mathbf{SL}_{n_2} \to \mathbf{SL}_{n_1 n_2}$ yields a homomorphism

$$\mathrm{Sym}^2(T^*_{n_1 n_2}) \to \mathrm{Sym}^2(T^*_{n_1}) \oplus \mathrm{Sym}^2(T^*_{n_2}),$$

where T_{n_1}, T_{n_2} and $T_{n_1 n_2}$ are maximal tori of respective groups. The image of the canonical Weyl-invariant generator $q_{n_1 n_2}$ of $\mathrm{Sym}^2(T^*_{n_1 n_2})$ is equal to $n_2 q_{n_1} + n_1 q_{n_2}$. Since n_1 and n_2 are divisible by m, the pull-back of the invariant $\Delta_{n_1 n_2, m}$ under the homomorphism $(\mathbf{SL}_{n_1}/\boldsymbol{\mu}_m) \times (\mathbf{SL}_{n_2}/\boldsymbol{\mu}_m) \to \mathbf{SL}_{n_1 n_2}/\boldsymbol{\mu}_m$ is trivial.

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Invariants vs. Representation Theory

How to show that $\mathrm{Inv}^3(G,2)_{\mathrm{dec}} = \mathrm{Inv}^3(G,2)_{\mathrm{sdec}}$ for simple groups of type C_{4n} ?

We want to show that $\mathfrak{c}_2(x) \in 2\mathbb{Z}q$ for every element $x \in (\widetilde{I}^W) \cap \mathbb{Z}[T^*]$.

Given a weight $\chi \in \Lambda$ we denote by $W(\chi)$ its W-orbit and we define $\widehat{e^\chi} := \sum_{\lambda \in W(\chi)} (1 - e^{-\lambda})$.

By definition, the ideal $(\widetilde{I}^W)$ is generated by elements $\{\widehat{e^{\omega_i}}\}_{i=1..4m}$ corresponding to the fundamental weights ω_i.

An element x can be written as

$$x = \sum_{i=1}^{4m} n_i \widehat{e^{\omega_i}} + \delta_i \widehat{e^{\omega_i}}, \quad \text{where } n_i \in \mathbb{Z} \text{ and } \delta_i \in \widetilde{I}. \tag{4}$$

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By definition, $f(I) = 0$, so $f(x) = 0$. Since $\omega_i \in T^*$ for all even i, $f(\widehat{e^{\omega_i}}) = y$ for all odd i and $f(\delta_i) \in f(\widetilde{I}) = (y)$, we get

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Dividing this sum by the g.c.d. of all d_i's and taking the result modulo 2 (here one uses the fact $\frac{n}{g.c.d.(n,k)} \mid \binom{n}{k}$), we obtain that the coefficient n_1 in the presentation (4) has to be even.

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We now compute $\mathfrak{c}_2(x)$. Let $\Lambda = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_{4m}$. The root lattice is given by $T^* = \{\sum a_i e_i \mid \sum a_i \text{ is even}\}$ and

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$$\omega_1 = e_1,\ \omega_2 = e_1 + e_2,\ \omega_3 = e_1 + e_2 + e_3, \ldots, \omega_{4m} = e_1 + \ldots + e_{4m}.$$

By Garibaldi-Z. we have $\mathfrak{c}_2(x) = \sum_{i=1}^{4m} n_i \mathfrak{c}_2(\widehat{e^{\omega_i}})$ and $\mathfrak{c}_2(\widehat{e^{\omega_i}}) = N(\widehat{e^{\omega_i}})q$, where

$$N(\sum a_j e^{\lambda_j}) = \tfrac{1}{2} \sum a_j \langle \lambda_j, \alpha^\vee \rangle^2 \text{ for a fixed long root } \alpha.$$

If we set $\alpha = 2e_{4m}$, then $\langle \lambda, \alpha^\vee \rangle = (\lambda, e_{4m})$ and

$$N(\widehat{e^{\omega_i}}) = \tfrac{1}{2} \sum_{\lambda \in W(\omega_i)} \langle \lambda, \alpha^\vee \rangle^2 = \tfrac{1}{2} \sum_{\lambda \in W(\omega_i)} (\lambda, e_{4m})^2 = 2^{i-1} \binom{4m-1}{i-1}$$

which is even for $i \geq 2$ (here we used the fact that the Weyl group acts by permutations and sign changes on $\{e_1, \ldots, e_{4m}\}$).
Since n_1 is even, we get that $\mathfrak{c}_2(x) \in 2\mathbb{Z}q$.