

The Grothendieck γ -filtration on projective homogeneous varieties

Kirill Zainoulline

Department of Mathematics and Statistics
University of Ottawa

2012

Introduction

Let G be a split semi-simple linear algebraic group over an arbitrary field k . Here split means Chevalley, so there is a root system, Weyl group W , etc.

Let $H^1(k, G)$ denote the pointed set of G -torsors/bundles.

One of the key problems in the theory of torsors and linear algebraic groups is to construct an invariant, i.e. a computable non-trivial map

$$\mathcal{F}: H^1(k, G) \longrightarrow \text{Algebraic objects (graded groups, rings)}$$

functorial with respect to a base change l/k .

Let G be a split semi-simple linear algebraic group over an arbitrary field k . Here split means Chevalley, so there is a root system, Weyl group W , etc.

Let $H^1(k, G)$ denote the pointed set of G -torsors/bundles.

One of the key problems in the theory of torsors and linear algebraic groups is to construct an invariant, i.e. a computable non-trivial map

$$\mathcal{F}: H^1(k, G) \longrightarrow \text{Algebraic objects (graded groups, rings)}$$

functorial with respect to a base change l/k .

- Cohomological invariants in the sense of Serre

$$\mathcal{F}: H^1(k, G) \rightarrow H_{\text{Gal}}^n(k, C) \text{ or } K_n^M(k)/p$$

- Usual cohomology rings: for every $\xi \in H^1(k, G)$

$$\mathcal{F}: \xi \mapsto h(\xi G) \text{ or } h(\xi \mathcal{B}),$$

where h is a cohomology theory, e.g. Chow groups, motivic cohomology, Grothendieck's K_0 , Levine-Morel's Ω^* etc.,

ξG is an algebraic group (non-split),

$\xi \mathcal{B} = \xi G/B$ is the associated variety of Borel subgroups.

- Motivic invariants: for every $\xi \in H^1(k, G)$

$$\mathcal{F}: \xi \mapsto M(\xi \mathcal{B}) \in h\text{-motives}/k$$

- Cohomological invariants in the sense of Serre

$$\mathcal{F}: H^1(k, G) \rightarrow H_{\text{Gal}}^n(k, \mathbb{C}) \text{ or } K_n^M(k)/p$$

- Usual cohomology rings: for every $\xi \in H^1(k, G)$

$$\mathcal{F}: \xi \mapsto h(\xi G) \text{ or } h(\xi \mathcal{B}),$$

where h is a cohomology theory, e.g. Chow groups, motivic cohomology, Grothendieck's K_0 , Levine-Morel's Ω^* etc.,
 ξG is an algebraic group (non-split),
 $\xi \mathcal{B} = \xi G/B$ is the associated variety of Borel subgroups.

- Motivic invariants: for every $\xi \in H^1(k, G)$

$$\mathcal{F}: \xi \mapsto M(\xi \mathcal{B}) \in h\text{-motives}/k$$

- Cohomological invariants in the sense of Serre

$$\mathcal{F}: H^1(k, G) \rightarrow H_{\text{Gal}}^n(k, C) \text{ or } K_n^M(k)/p$$

- Usual cohomology rings: for every $\xi \in H^1(k, G)$

$$\mathcal{F}: \xi \mapsto h(\xi G) \text{ or } h(\xi \mathcal{B}),$$

where h is a cohomology theory, e.g. Chow groups, motivic cohomology, Grothendieck's K_0 , Levine-Morel's Ω^* etc.,

ξG is an algebraic group (non-split),

$\xi \mathcal{B} = \xi G/B$ is the associated variety of Borel subgroups.

- Motivic invariants: for every $\xi \in H^1(k, G)$

$$\mathcal{F}: \xi \mapsto M(\xi \mathcal{B}) \in h\text{-motives}/k$$

- Cohomological invariants in the sense of Serre

$$\mathcal{F}: H^1(k, G) \rightarrow H_{\text{Gal}}^n(k, C) \text{ or } K_n^M(k)/p$$

- Usual cohomology rings: for every $\xi \in H^1(k, G)$

$$\mathcal{F}: \xi \mapsto h(\xi G) \text{ or } h(\xi \mathcal{B}),$$

where h is a cohomology theory, e.g. Chow groups, motivic cohomology, Grothendieck's K_0 , Levine-Morel's Ω^* etc.,
 ξG is an algebraic group (non-split),
 $\xi \mathcal{B} = \xi G/B$ is the associated variety of Borel subgroups.

- Motivic invariants: for every $\xi \in H^1(k, G)$

$$\mathcal{F}: \xi \mapsto M(\xi \mathcal{B}) \in h\text{-motives}/k$$

- Cohomological invariants in the sense of Serre

$$\mathcal{F}: H^1(k, G) \rightarrow H_{\text{Gal}}^n(k, C) \text{ or } K_n^M(k)/p$$

- Usual cohomology rings: for every $\xi \in H^1(k, G)$

$$\mathcal{F}: \xi \mapsto h(\xi G) \text{ or } h(\xi \mathcal{B}),$$

where h is a cohomology theory, e.g. Chow groups, motivic cohomology, Grothendieck's K_0 , Levine-Morel's Ω^* etc.,
 ξG is an algebraic group (non-split),
 $\xi \mathcal{B} = \xi G/B$ is the associated variety of Borel subgroups.

- Motivic invariants: for every $\xi \in H^1(k, G)$

$$\mathcal{F}: \xi \mapsto M(\xi \mathcal{B}) \in h\text{-motives}/k$$

In the present talk we will discuss the γ -invariant of a G -torsor, which is

$$\mathcal{F}: \xi \mapsto \gamma^*(\xi\mathcal{B}),$$

where γ^* is the graded commutative ring associated to the γ -filtration.

Observe that γ^* is not a cohomology theory in the usual sense (no push-forwards).

By the Riemann-Roch theorem $\gamma^*(-) \otimes \mathbb{Q} \simeq CH^*(-) \otimes \mathbb{Q}$ which is a free Abelian group in the case of $\xi G/B$, therefore, it is a question about the torsion part $\gamma_t^*(-) := \text{Tors } \gamma^*(-)$ only.

In the present talk we will discuss the γ -invariant of a G -torsor, which is

$$\mathcal{F}: \xi \mapsto \gamma^*(\xi\mathcal{B}),$$

where γ^* is the graded commutative ring associated to the γ -filtration.

Observe that γ^* is not a cohomology theory in the usual sense (no push-forwards).

By the Riemann-Roch theorem $\gamma^*(-) \otimes \mathbb{Q} \simeq CH^*(-) \otimes \mathbb{Q}$ which is a free Abelian group in the case of $\xi G/B$, therefore, it is a question about the torsion part $\gamma_t^*(-) := \text{Tors } \gamma^*(-)$ only.

In the present talk we will discuss the γ -invariant of a G -torsor, which is

$$\mathcal{F}: \xi \mapsto \gamma^*(\xi\mathcal{B}),$$

where γ^* is the graded commutative ring associated to the γ -filtration.

Observe that γ^* is not a cohomology theory in the usual sense (no push-forwards).

By the Riemann-Roch theorem $\gamma^*(-) \otimes \mathbb{Q} \simeq CH^*(-) \otimes \mathbb{Q}$ which is a free Abelian group in the case of ${}_{\xi}G/B$, therefore, it is a question about the torsion part $\gamma_t^*(-) := \text{Tors } \gamma^*(-)$ only.

Definition [SGA6, Manin]

Let X be a smooth projective variety over a field k . The i -th term of γ -filtration on X is defined to be an ideal generated by products

$$\gamma_{\geq i} := \left\{ \gamma_{n_1}(x_1)\gamma_{n_2}(x_2) \cdots \gamma_{n_m}(x_m) \mid \begin{array}{l} x_1, x_2, \dots, x_m \in K_0(X), \\ n_1 + n_2 + \dots + n_m \geq i \end{array} \right\},$$

where γ_n is the n -th characteristic class in K_0 which satisfies usual axioms, e.g. Whitney sum formula. For example, for a line bundle \mathcal{L} over X we have

$$\gamma_1([\mathcal{L}]) = 1 - [\mathcal{L}^\vee] \quad \text{and} \quad \gamma_2([\mathcal{L}]) = 0.$$

We define $\gamma^i(X) := \gamma_{\geq i} / \gamma_{\geq i+1}$ and $\gamma^*(X) := \bigoplus_{i \geq 0} \gamma^i(X)$.

Definition [SGA6, Manin]

Let X be a smooth projective variety over a field k . The i -th term of γ -filtration on X is defined to be an ideal generated by products

$$\gamma_{\geq i} := \left\{ \gamma_{n_1}(x_1)\gamma_{n_2}(x_2) \cdots \gamma_{n_m}(x_m) \mid \begin{array}{l} x_1, x_2, \dots, x_m \in K_0(X), \\ n_1 + n_2 + \dots + n_m \geq i \end{array} \right\},$$

where γ_n is the n -th characteristic class in K_0 which satisfies usual axioms, e.g. Whitney sum formula. For example, for a line bundle \mathcal{L} over X we have

$$\gamma_1([\mathcal{L}]) = 1 - [\mathcal{L}^\vee] \quad \text{and} \quad \gamma_2([\mathcal{L}]) = 0.$$

We define $\gamma^i(X) := \gamma_{\geq i} / \gamma_{\geq i+1}$ and $\gamma^*(X) := \bigoplus_{i \geq 0} \gamma^i(X)$.

Definition [SGA6, Manin]

Let X be a smooth projective variety over a field k . The i -th term of γ -filtration on X is defined to be an ideal generated by products

$$\gamma_{\geq i} := \left\{ \gamma_{n_1}(x_1) \gamma_{n_2}(x_2) \cdots \gamma_{n_m}(x_m) \mid \begin{array}{l} x_1, x_2, \dots, x_m \in K_0(X), \\ n_1 + n_2 + \dots + n_m \geq i \end{array} \right\},$$

where γ_n is the n -th characteristic class in K_0 which satisfies usual axioms, e.g. Whitney sum formula. For example, for a line bundle \mathcal{L} over X we have

$$\gamma_1([\mathcal{L}]) = 1 - [\mathcal{L}^\vee] \quad \text{and} \quad \gamma_2([\mathcal{L}]) = 0.$$

We define $\gamma^i(X) := \gamma_{\geq i} / \gamma_{\geq i+1}$ and $\gamma^*(X) := \bigoplus_{i \geq 0} \gamma^i(X)$.

Definition [SGA6, Manin]

Let X be a smooth projective variety over a field k . The i -th term of γ -filtration on X is defined to be an ideal generated by products

$$\gamma_{\geq i} := \left\{ \gamma_{n_1}(x_1)\gamma_{n_2}(x_2) \cdot \dots \cdot \gamma_{n_m}(x_m) \mid \begin{array}{l} x_1, x_2, \dots, x_m \in K_0(X), \\ n_1 + n_2 + \dots + n_m \geq i \end{array} \right\},$$

where γ_n is the n -th characteristic class in K_0 which satisfies usual axioms, e.g. Whitney sum formula. For example, for a line bundle \mathcal{L} over X we have

$$\gamma_1([\mathcal{L}]) = 1 - [\mathcal{L}^\vee] \quad \text{and} \quad \gamma_2([\mathcal{L}]) = 0.$$

We define $\gamma^i(X) := \gamma_{\geq i} / \gamma_{\geq i+1}$ and $\gamma^*(X) := \bigoplus_{i \geq 0} \gamma^i(X)$.

Definition [SGA6, Manin]

Let X be a smooth projective variety over a field k . The i -th term of γ -filtration on X is defined to be an ideal generated by products

$$\gamma_{\geq i} := \left\{ \gamma_{n_1}(x_1)\gamma_{n_2}(x_2) \cdots \gamma_{n_m}(x_m) \mid \begin{array}{l} x_1, x_2, \dots, x_m \in K_0(X), \\ n_1 + n_2 + \dots + n_m \geq i \end{array} \right\},$$

where γ_n is the n -th characteristic class in K_0 which satisfies usual axioms, e.g. Whitney sum formula. For example, for a line bundle \mathcal{L} over X we have

$$\gamma_1([\mathcal{L}]) = 1 - [\mathcal{L}^\vee] \quad \text{and} \quad \gamma_2([\mathcal{L}]) = 0.$$

We define $\gamma^i(X) := \gamma_{\geq i} / \gamma_{\geq i+1}$ and $\gamma^*(X) := \bigoplus_{i \geq 0} \gamma^i(X)$.

Definition [SGA6, Manin]

Let X be a smooth projective variety over a field k . The i -th term of γ -filtration on X is defined to be an ideal generated by products

$$\gamma_{\geq i} := \left\{ \gamma_{n_1}(x_1)\gamma_{n_2}(x_2) \cdots \gamma_{n_m}(x_m) \mid \begin{array}{l} x_1, x_2, \dots, x_m \in K_0(X), \\ n_1 + n_2 + \dots + n_m \geq i \end{array} \right\},$$

where γ_n is the n -th characteristic class in K_0 which satisfies usual axioms, e.g. Whitney sum formula. For example, for a line bundle \mathcal{L} over X we have

$$\gamma_1([\mathcal{L}]) = 1 - [\mathcal{L}^\vee] \quad \text{and} \quad \gamma_2([\mathcal{L}]) = 0.$$

We define $\gamma^i(X) := \gamma_{\geq i} / \gamma_{\geq i+1}$ and $\gamma^*(X) := \bigoplus_{i \geq 0} \gamma^i(X)$.

Examples of computations: upper bounds

We have $\gamma^1(X) = \text{Pic}(X)$. Hence, $\gamma_t^1(\xi\mathcal{B}) = 0$.

for groups of type A_n and C_n $\gamma_t^i(\mathcal{B}) = 0$;

for groups of type B_n and D_n there exists a non-trivial torsion in $\gamma_t^i(\mathcal{B})$ for some $i \geq 2$ [Grothendieck, SGA6, Exp.14].

$\gamma_t^2(SB(D))$ was computed by [Karpenko, 96].

Theorem [Garibaldi, Z., 2010] If G is simply connected, then

$$\gamma_t^2(\xi\mathcal{B}) \simeq \bigoplus_{i \text{ finite}} \mathbb{Z}/n_i\mathbb{Z}, \quad \text{and} \quad \gamma_t^3(\xi\mathcal{B}) \simeq \bigoplus_{i \text{ finite}} \mathbb{Z}/n_i\mathbb{Z}$$

where n_i divides the Dynkin index N of G . Moreover, for a simple G there exists a G -torsor ξ such that $\gamma_t^2(\xi G/B) \simeq \mathbb{Z}/N\mathbb{Z}$.

Examples of computations: upper bounds

We have $\gamma^1(X) = \text{Pic}(X)$. Hence, $\gamma_t^1(\xi\mathcal{B}) = 0$.

for groups of type A_n and C_n $\gamma_t^i(\mathcal{B}) = 0$;

for groups of type B_n and D_n there exists a non-trivial torsion in $\gamma_t^i(\mathcal{B})$ for some $i \geq 2$ [Grothendieck, SGA6, Exp.14].

$\gamma_t^2(SB(D))$ was computed by [Karpenko, 96].

Theorem [Garibaldi, Z., 2010] If G is simply connected, then

$$\gamma_t^2(\xi\mathcal{B}) \simeq \bigoplus_{i \text{ finite}} \mathbb{Z}/n_i\mathbb{Z}, \quad \text{and} \quad \gamma_t^3(\xi\mathcal{B}) \simeq \bigoplus_{i \text{ finite}} \mathbb{Z}/n_i\mathbb{Z}$$

where n_i divides the Dynkin index N of G . Moreover, for a simple G there exists a G -torsor ξ such that $\gamma_t^2(\xi G/B) \simeq \mathbb{Z}/N\mathbb{Z}$.

Examples of computations: upper bounds

We have $\gamma^1(X) = \text{Pic}(X)$. Hence, $\gamma_t^1(\xi\mathcal{B}) = 0$.

for groups of type A_n and C_n $\gamma_t^i(\mathcal{B}) = 0$;

for groups of type B_n and D_n there exists a non-trivial torsion in $\gamma_t^i(\mathcal{B})$ for some $i \geq 2$ [Grothendieck, SGA6, Exp.14].

$\gamma_t^2(SB(D))$ was computed by [Karpenko, 96].

Theorem [Garibaldi, Z., 2010] If G is simply connected, then

$$\gamma_t^2(\xi\mathcal{B}) \simeq \bigoplus_{i \text{ finite}} \mathbb{Z}/n_i\mathbb{Z}, \quad \text{and} \quad \gamma_t^3(\xi\mathcal{B}) \simeq \bigoplus_{i \text{ finite}} \mathbb{Z}/n_i\mathbb{Z}$$

where n_i divides the Dynkin index N of G . Moreover, for a simple G there exists a G -torsor ξ such that $\gamma_t^2(\xi G/B) \simeq \mathbb{Z}/N\mathbb{Z}$.

Examples of computations: upper bounds

We have $\gamma^1(X) = \text{Pic}(X)$. Hence, $\gamma_t^1(\xi\mathcal{B}) = 0$.

for groups of type A_n and C_n $\gamma_t^i(\mathcal{B}) = 0$;

for groups of type B_n and D_n there exists a non-trivial torsion in $\gamma_t^i(\mathcal{B})$ for some $i \geq 2$ [Grothendieck, SGA6, Exp.14].

$\gamma_t^2(SB(D))$ was computed by [Karpenko, 96].

Theorem [Garibaldi, Z., 2010] If G is simply connected, then

$$\gamma_t^2(\xi\mathcal{B}) \simeq \bigoplus_{i \text{ finite}} \mathbb{Z}/n_i\mathbb{Z}, \quad \text{and} \quad \gamma_t^3(\xi\mathcal{B}) \simeq \bigoplus_{i \text{ finite}} \mathbb{Z}/n_i\mathbb{Z}$$

where n_i divides the Dynkin index N of G . Moreover, for a simple G there exists a G -torsor ξ such that $\gamma_t^2(\xi G/B) \simeq \mathbb{Z}/N\mathbb{Z}$.

Examples of computations: upper bounds

We have $\gamma^1(X) = \text{Pic}(X)$. Hence, $\gamma_t^1(\xi\mathcal{B}) = 0$.

for groups of type A_n and C_n $\gamma_t^i(\mathcal{B}) = 0$;

for groups of type B_n and D_n there exists a non-trivial torsion in $\gamma_t^i(\mathcal{B})$ for some $i \geq 2$ [Grothendieck, SGA6, Exp.14].

$\gamma_t^2(SB(D))$ was computed by [Karpenko, 96].

Theorem [Garibaldi, Z., 2010] If G is simply connected, then

$$\gamma_t^2(\xi\mathcal{B}) \simeq \bigoplus_{i \text{ finite}} \mathbb{Z}/n_i\mathbb{Z}, \quad \text{and} \quad \gamma_t^3(\xi\mathcal{B}) \simeq \bigoplus_{i \text{ finite}} \mathbb{Z}/n_i\mathbb{Z}$$

where n_i divides the Dynkin index N of G . Moreover, for a simple G there exists a G -torsor ξ such that $\gamma_t^2(\xi G/B) \simeq \mathbb{Z}/N\mathbb{Z}$.

Theorem [Baek, Neher, Z., 2011] If G is simply connected, then
the integer $6N$ annihilates $\gamma_t^4(\xi\mathcal{B})$.

In general, we expect that for a simply connected G

the integer $(j - 1)! \cdot N$ annihilates $\gamma_t^j(\xi\mathcal{B})$.

Theorem [Baek, Neher, Z., 2011] If G is simply connected, then
the integer $6N$ annihilates $\gamma_t^4(\xi\mathcal{B})$.

In general, we expect that for a simply connected G

the integer $(j - 1)! \cdot N$ annihilates $\gamma_t^j(\xi\mathcal{B})$.

Essentially nothing is known about $\gamma_t^i(\xi\mathcal{B})$, $i \geq 2$, if G is not simply connected (we recall that $\xi \in H^1(k, G)$).

The following result provides a uniform low bound for $\gamma_t^i(\xi\mathcal{B})$:

Theorem [Z., 2011] For every $i \geq 0$ there is a surjective group homomorphism

$$\gamma_t^i(\xi\mathcal{B}) \twoheadrightarrow \gamma_\xi^i,$$

where the graded group γ_ξ^* is defined as follows:

Essentially nothing is known about $\gamma_t^i(\xi\mathcal{B})$, $i \geq 2$, if G is not simply connected (we recall that $\xi \in H^1(k, G)$).

The following result provides a uniform low bound for $\gamma_t^i(\xi\mathcal{B})$:

Theorem [Z., 2011] For every $i \geq 0$ there is a surjective group homomorphism

$$\gamma_t^i(\xi\mathcal{B}) \twoheadrightarrow \gamma_\xi^i,$$

where the graded group γ_ξ^* is defined as follows:

Essentially nothing is known about $\gamma_t^i(\xi\mathcal{B})$, $i \geq 2$, if G is not simply connected (we recall that $\xi \in H^1(k, G)$).

The following result provides a uniform low bound for $\gamma_t^i(\xi\mathcal{B})$:

Theorem [Z., 2011] For every $i \geq 0$ there is a surjective group homomorphism

$$\gamma_t^i(\xi\mathcal{B}) \twoheadrightarrow \gamma_\xi^i,$$

where the graded group γ_ξ^* is defined as follows:

Essentially nothing is known about $\gamma_t^i(\xi\mathcal{B})$, $i \geq 2$, if G is not simply connected (we recall that $\xi \in H^1(k, G)$).

The following result provides a uniform low bound for $\gamma_t^i(\xi\mathcal{B})$:

Theorem [Z., 2011] For every $i \geq 0$ there is a surjective group homomorphism

$$\gamma_t^i(\xi\mathcal{B}) \twoheadrightarrow \gamma_\xi^i,$$

where the graded group γ_ξ^* is defined as follows:

The twisted γ -filtration

Consider the canonical surjection

$$q: K_0(\mathcal{B}) \simeq \mathbb{Z}[\Lambda] \otimes_{\mathbb{Z}[\Lambda]^w} \mathbb{Z} \rightarrow \mathbb{Z}[C^*] \otimes_{\mathbb{Z}[\Lambda]^w} \mathbb{Z} \simeq K_0(G),$$

where Λ is the weight lattice, T^* is the group of characters of a split maximal torus and $C^* = \Lambda/T^*$ is the group of characters of the center of G .

Given a torsor ξ we define a twisted filtration on $K_0(G)$ to be the image of the γ -filtration on $K_0(\xi\mathcal{B})$ via the composite

$$K_0(\xi\mathcal{B}) \xrightarrow{\text{res}} K_0(\mathcal{B}) \xrightarrow{q} K_0(G),$$

i.e. $(\gamma_\xi)_{\geq i} := q(\text{res}(\gamma_{\geq i}))$. Then we set $\gamma_\xi^i := (\gamma_\xi)_{\geq i} / (\gamma_\xi)_{\geq i+1}$.

The twisted γ -filtration

Consider the canonical surjection

$$q: K_0(\mathcal{B}) \simeq \mathbb{Z}[\Lambda] \otimes_{\mathbb{Z}[\Lambda]^w} \mathbb{Z} \rightarrow \mathbb{Z}[C^*] \otimes_{\mathbb{Z}[\Lambda]^w} \mathbb{Z} \simeq K_0(G),$$

where Λ is the weight lattice, T^* is the group of characters of a split maximal torus and $C^* = \Lambda/T^*$ is the group of characters of the center of G .

Given a torsor ξ we define a twisted filtration on $K_0(G)$ to be the image of the γ -filtration on $K_0(\xi\mathcal{B})$ via the composite

$$K_0(\xi\mathcal{B}) \xrightarrow{\text{res}} K_0(\mathcal{B}) \xrightarrow{q} K_0(G),$$

i.e. $(\gamma_\xi)_{\geq i} := q(\text{res}(\gamma_{\geq i}))$. Then we set $\gamma_\xi^i := (\gamma_\xi)_{\geq i} / (\gamma_\xi)_{\geq i+1}$.

The twisted γ -filtration

Consider the canonical surjection

$$q: K_0(\mathcal{B}) \simeq \mathbb{Z}[\Lambda] \otimes_{\mathbb{Z}[\Lambda]^w} \mathbb{Z} \rightarrow \mathbb{Z}[C^*] \otimes_{\mathbb{Z}[\Lambda]^w} \mathbb{Z} \simeq K_0(G),$$

where Λ is the weight lattice, T^* is the group of characters of a split maximal torus and $C^* = \Lambda/T^*$ is the group of characters of the center of G .

Given a torsor ξ we define a twisted filtration on $K_0(G)$ to be the image of the γ -filtration on $K_0(\xi\mathcal{B})$ via the composite

$$K_0(\xi\mathcal{B}) \xrightarrow{\text{res}} K_0(\mathcal{B}) \xrightarrow{q} K_0(G),$$

i.e. $(\gamma_\xi)_{\geq i} := q(\text{res}(\gamma_{\geq i}))$. Then we set $\gamma_\xi^i := (\gamma_\xi)_{\geq i} / (\gamma_\xi)_{\geq i+1}$.

The twisted γ -filtration

Consider the canonical surjection

$$q: K_0(\mathcal{B}) \simeq \mathbb{Z}[\Lambda] \otimes_{\mathbb{Z}[\Lambda]^w} \mathbb{Z} \rightarrow \mathbb{Z}[C^*] \otimes_{\mathbb{Z}[\Lambda]^w} \mathbb{Z} \simeq K_0(G),$$

where Λ is the weight lattice, T^* is the group of characters of a split maximal torus and $C^* = \Lambda/T^*$ is the group of characters of the center of G .

Given a torsor ξ we define a twisted filtration on $K_0(G)$ to be the image of the γ -filtration on $K_0(\xi\mathcal{B})$ via the composite

$$K_0(\xi\mathcal{B}) \xrightarrow{\text{res}} K_0(\mathcal{B}) \xrightarrow{q} K_0(G),$$

i.e. $(\gamma_\xi)_{\geq i} := q(\text{res}(\gamma_{\geq i}))$. Then we set $\gamma_\xi^i := (\gamma_\xi)_{\geq i} / (\gamma_\xi)_{\geq i+1}$.

The twisted γ -filtration

Consider the canonical surjection

$$q: K_0(\mathcal{B}) \simeq \mathbb{Z}[\Lambda] \otimes_{\mathbb{Z}[\Lambda]^w} \mathbb{Z} \rightarrow \mathbb{Z}[C^*] \otimes_{\mathbb{Z}[\Lambda]^w} \mathbb{Z} \simeq K_0(G),$$

where Λ is the weight lattice, T^* is the group of characters of a split maximal torus and $C^* = \Lambda/T^*$ is the group of characters of the center of G .

Given a torsor ξ we define a twisted filtration on $K_0(G)$ to be the image of the γ -filtration on $K_0(\xi\mathcal{B})$ via the composite

$$K_0(\xi\mathcal{B}) \xrightarrow{\text{res}} K_0(\mathcal{B}) \xrightarrow{q} K_0(G),$$

i.e. $(\gamma_\xi)_{\geq i} := q(\text{res}(\gamma_{\geq i}))$. Then we set $\gamma_\xi^i := (\gamma_\xi)_{\geq i} / (\gamma_\xi)_{\geq i+1}$.

The twisted γ -filtration

Consider the canonical surjection

$$q: K_0(\mathcal{B}) \simeq \mathbb{Z}[\Lambda] \otimes_{\mathbb{Z}[\Lambda]^w} \mathbb{Z} \rightarrow \mathbb{Z}[C^*] \otimes_{\mathbb{Z}[\Lambda]^w} \mathbb{Z} \simeq K_0(G),$$

where Λ is the weight lattice, T^* is the group of characters of a split maximal torus and $C^* = \Lambda/T^*$ is the group of characters of the center of G .

Given a torsor ξ we define a twisted filtration on $K_0(G)$ to be the image of the γ -filtration on $K_0(\xi\mathcal{B})$ via the composite

$$K_0(\xi\mathcal{B}) \xrightarrow{\text{res}} K_0(\mathcal{B}) \xrightarrow{q} K_0(G),$$

i.e. $(\gamma_\xi)_{\geq i} := q(\text{res}(\gamma_{\geq i}))$. Then we set $\gamma_\xi^i := (\gamma_\xi)_{\geq i} / (\gamma_\xi)_{\geq i+1}$.

The twisted γ -filtration

Consider the canonical surjection

$$q: K_0(\mathcal{B}) \simeq \mathbb{Z}[\Lambda] \otimes_{\mathbb{Z}[\Lambda]^w} \mathbb{Z} \rightarrow \mathbb{Z}[C^*] \otimes_{\mathbb{Z}[\Lambda]^w} \mathbb{Z} \simeq K_0(G),$$

where Λ is the weight lattice, T^* is the group of characters of a split maximal torus and $C^* = \Lambda/T^*$ is the group of characters of the center of G .

Given a torsor ξ we define a twisted filtration on $K_0(G)$ to be the image of the γ -filtration on $K_0(\xi\mathcal{B})$ via the composite

$$K_0(\xi\mathcal{B}) \xrightarrow{\text{res}} K_0(\mathcal{B}) \xrightarrow{q} K_0(G),$$

i.e. $(\gamma_\xi)_{\geq i} := q(\text{res}(\gamma_{\geq i}))$. Then we set $\gamma_\xi^i := (\gamma_\xi)_{\geq i} / (\gamma_\xi)_{\geq i+1}$.

Computation of the twisted filtration

According to [Steinberg] $\mathbb{Z}[\Lambda]$ is a free $\mathbb{Z}[\Lambda]^W$ -module with a basis $\{e^{\rho_w}\}_{w \in W}$, where the weights ρ_w are defined as follows:

$$\rho_w := \sum_{\{i \in 1 \dots n \mid w^{-1}(\alpha_i) < 0\}} w^{-1}(\omega_i), \quad w \in W.$$

Then using Panin's computation of K -theory of twisted flag varieties, the twisted filtration can be computed as follows

$$(\gamma_\xi)_{\geq i} = \left\langle \prod_{j=1}^m \binom{\text{ind}(\beta(\bar{\rho}_{w_j}))}{n_j} (1 - e^{\bar{\rho}_{w_j}})^{n_j} \mid n_1 + \dots + n_m \geq i, w_j \in W \right\rangle$$

where $\bar{\rho}_{w_j}$ denotes the image of ρ_{w_j} in C^* and $\beta: C^* \rightarrow Br(k)$ is the Tits map.

Computation of the twisted filtration

According to [Steinberg] $\mathbb{Z}[\Lambda]$ is a free $\mathbb{Z}[\Lambda]^W$ -module with a basis $\{e^{\rho_w}\}_{w \in W}$, where the weights ρ_w are defined as follows:

$$\rho_w := \sum_{\{i \in 1 \dots n \mid w^{-1}(\alpha_i) < 0\}} w^{-1}(\omega_i), \quad w \in W.$$

Then using Panin's computation of K -theory of twisted flag varieties, the twisted filtration can be computed as follows

$$(\gamma_\xi)_{\geq i} = \left\langle \prod_{j=1}^m \binom{\text{ind}(\beta(\bar{\rho}_{w_j}))}{n_j} (1 - e^{\bar{\rho}_{w_j}})^{n_j} \mid n_1 + \dots + n_m \geq i, w_j \in W \right\rangle$$

where $\bar{\rho}_{w_j}$ denotes the image of ρ_{w_j} in C^* and $\beta: C^* \rightarrow Br(k)$ is the Tits map.

Computation of the twisted filtration

According to [Steinberg] $\mathbb{Z}[\Lambda]$ is a free $\mathbb{Z}[\Lambda]^W$ -module with a basis $\{e^{\rho_w}\}_{w \in W}$, where the weights ρ_w are defined as follows:

$$\rho_w := \sum_{\{i \in 1 \dots n \mid w^{-1}(\alpha_i) < 0\}} w^{-1}(\omega_i), \quad w \in W.$$

Then using Panin's computation of K -theory of twisted flag varieties, the twisted filtration can be computed as follows

$$(\gamma_\xi)_{\geq i} = \left\langle \prod_{j=1}^m \binom{\text{ind}(\beta(\bar{\rho}_{w_j}))}{n_j} (1 - e^{\bar{\rho}_{w_j}})^{n_j} \mid n_1 + \dots + n_m \geq i, w_j \in W \right\rangle$$

where $\bar{\rho}_{w_j}$ denotes the image of ρ_{w_j} in C^* and $\beta: C^* \rightarrow Br(k)$ is the Tits map.

Computation of the twisted filtration

According to [Steinberg] $\mathbb{Z}[\Lambda]$ is a free $\mathbb{Z}[\Lambda]^W$ -module with a basis $\{e^{\rho_w}\}_{w \in W}$, where the weights ρ_w are defined as follows:

$$\rho_w := \sum_{\{i \in 1 \dots n \mid w^{-1}(\alpha_i) < 0\}} w^{-1}(\omega_i), \quad w \in W.$$

Then using Panin's computation of K -theory of twisted flag varieties, the twisted filtration can be computed as follows

$$(\gamma_\xi)_{\geq i} = \left\langle \prod_{j=1}^m \binom{\text{ind}(\beta(\bar{\rho}_{w_j}))}{n_j} (1 - e^{\bar{\rho}_{w_j}})^{n_j} \mid n_1 + \dots + n_m \geq i, w_j \in W \right\rangle$$

where $\bar{\rho}_{w_j}$ denotes the image of ρ_{w_j} in C^* and $\beta: C^* \rightarrow Br(k)$ is the Tits map.

Computation of the twisted filtration

According to [Steinberg] $\mathbb{Z}[\Lambda]$ is a free $\mathbb{Z}[\Lambda]^W$ -module with a basis $\{e^{\rho_w}\}_{w \in W}$, where the weights ρ_w are defined as follows:

$$\rho_w := \sum_{\{i \in 1 \dots n \mid w^{-1}(\alpha_i) < 0\}} w^{-1}(\omega_i), \quad w \in W.$$

Then using Panin's computation of K -theory of twisted flag varieties, the twisted filtration can be computed as follows

$$(\gamma_\xi)_{\geq i} = \left\langle \prod_{j=1}^m \binom{\text{ind}(\beta(\bar{\rho}_{w_j}))}{n_j} (1 - e^{\bar{\rho}_{w_j}})^{n_j} \mid n_1 + \dots + n_m \geq i, w_j \in W \right\rangle$$

where $\bar{\rho}_{w_j}$ denotes the image of ρ_{w_j} in C^* and $\beta: C^* \rightarrow Br(k)$ is the Tits map.

Examples

Assume that $C^* = \langle \sigma \rangle$ is cyclic of order 2, e.g. G is a half-spin group.

Let d denote the g.c.d. of dimensions of fundamental representations corresponding to σ .

Let ξ be a G -torsor and let i_A denote the 2-adic valuation of the index of the Tits algebra $A = A_{\sigma, \xi}$.

$$\text{if } i_A = 1, \text{ then } \gamma_{\xi}^2 = \begin{cases} 0 & \text{if } v_2(d) \leq 1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } v_2(d) = 2 \\ \mathbb{Z}/4\mathbb{Z} & \text{if } v_2(d) \geq 3 \end{cases}$$

$$\text{if } i_A > 1, \text{ then } \gamma_{\xi}^2 = \begin{cases} 0 & \text{if } v_2(d) \leq i_A \\ \mathbb{Z}/2\mathbb{Z} & \text{if } v_2(d) > i_A \end{cases}$$

Examples

Assume that $C^* = \langle \sigma \rangle$ is cyclic of order 2, e.g. G is a half-spin group.

Let d denote the g.c.d. of dimensions of fundamental representations corresponding to σ .

Let ξ be a G -torsor and let i_A denote the 2-adic valuation of the index of the Tits algebra $A = A_{\sigma, \xi}$.

$$\text{if } i_A = 1, \text{ then } \gamma_{\xi}^2 = \begin{cases} 0 & \text{if } v_2(d) \leq 1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } v_2(d) = 2 \\ \mathbb{Z}/4\mathbb{Z} & \text{if } v_2(d) \geq 3 \end{cases}$$

$$\text{if } i_A > 1, \text{ then } \gamma_{\xi}^2 = \begin{cases} 0 & \text{if } v_2(d) \leq i_A \\ \mathbb{Z}/2\mathbb{Z} & \text{if } v_2(d) > i_A \end{cases}$$

Examples

Assume that $C^* = \langle \sigma \rangle$ is cyclic of order 2, e.g. G is a half-spin group.

Let d denote the g.c.d. of dimensions of fundamental representations corresponding to σ .

Let ξ be a G -torsor and let i_A denote the 2-adic valuation of the index of the Tits algebra $A = A_{\sigma, \xi}$.

$$\text{if } i_A = 1, \text{ then } \gamma_{\xi}^2 = \begin{cases} 0 & \text{if } v_2(d) \leq 1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } v_2(d) = 2 \\ \mathbb{Z}/4\mathbb{Z} & \text{if } v_2(d) \geq 3 \end{cases}$$

$$\text{if } i_A > 1, \text{ then } \gamma_{\xi}^2 = \begin{cases} 0 & \text{if } v_2(d) \leq i_A \\ \mathbb{Z}/2\mathbb{Z} & \text{if } v_2(d) > i_A \end{cases}$$

Examples

Assume that $C^* = \langle \sigma \rangle$ is cyclic of order 2, e.g. G is a half-spin group.

Let d denote the g.c.d. of dimensions of fundamental representations corresponding to σ .

Let ξ be a G -torsor and let i_A denote the 2-adic valuation of the index of the Tits algebra $A = A_{\sigma, \xi}$.

$$\text{if } i_A = 1, \text{ then } \gamma_{\xi}^2 = \begin{cases} 0 & \text{if } v_2(d) \leq 1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } v_2(d) = 2 \\ \mathbb{Z}/4\mathbb{Z} & \text{if } v_2(d) \geq 3 \end{cases}$$

$$\text{if } i_A > 1, \text{ then } \gamma_{\xi}^2 = \begin{cases} 0 & \text{if } v_2(d) \leq i_A \\ \mathbb{Z}/2\mathbb{Z} & \text{if } v_2(d) > i_A \end{cases}$$

Examples

Assume that $C^* = \langle \sigma \rangle$ is cyclic of order 2, e.g. G is a half-spin group.

Let d denote the g.c.d. of dimensions of fundamental representations corresponding to σ .

Let ξ be a G -torsor and let i_A denote the 2-adic valuation of the index of the Tits algebra $A = A_{\sigma, \xi}$.

$$\text{if } i_A = 1, \text{ then } \gamma_{\xi}^2 = \begin{cases} 0 & \text{if } v_2(d) \leq 1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } v_2(d) = 2 \\ \mathbb{Z}/4\mathbb{Z} & \text{if } v_2(d) \geq 3 \end{cases}$$

$$\text{if } i_A > 1, \text{ then } \gamma_{\xi}^2 = \begin{cases} 0 & \text{if } v_2(d) \leq i_A \\ \mathbb{Z}/2\mathbb{Z} & \text{if } v_2(d) > i_A \end{cases}$$

Examples

Assume that $C^* = \langle \sigma \rangle$ is cyclic of order 2, e.g. G is a half-spin group.

Let d denote the g.c.d. of dimensions of fundamental representations corresponding to σ .

Let ξ be a G -torsor and let i_A denote the 2-adic valuation of the index of the Tits algebra $A = A_{\sigma, \xi}$.

$$\text{if } i_A = 1, \text{ then } \gamma_{\xi}^2 = \begin{cases} 0 & \text{if } v_2(d) \leq 1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } v_2(d) = 2 \\ \mathbb{Z}/4\mathbb{Z} & \text{if } v_2(d) \geq 3 \end{cases}$$

$$\text{if } i_A > 1, \text{ then } \gamma_{\xi}^2 = \begin{cases} 0 & \text{if } v_2(d) \leq i_A \\ \mathbb{Z}/2\mathbb{Z} & \text{if } v_2(d) > i_A \end{cases}$$

Corollary (Z.) If $v_2(d) > i_A \geq 3$, then for every $\lambda \in \Lambda$ such that $\bar{\lambda} = \sigma$ there exists a non-trivial torsion element of order 2 in $\gamma^{2/3}(\xi\mathcal{B})$. Moreover, its image in $\gamma_\xi^{2/3} = \mathbb{Z}/2\mathbb{Z}$ is non-trivial and in $\gamma^{2/3}(\mathcal{B})$ (via *res*) is trivial.

Example. Let $G = HSpin_{2n}$. Then for any algebra with orthogonal involution (A, δ) where $8 \mid \text{ind}(A)$ and A is non-division, there exists a non-trivial torsion element of order 2 in $\gamma^{2/3}(\xi\mathcal{B})$ which vanishes over a splitting field of (A, δ) .

Corollary (Z.) If $v_2(d) > i_A \geq 3$, then for every $\lambda \in \Lambda$ such that $\bar{\lambda} = \sigma$ there exists a non-trivial torsion element of order 2 in $\gamma^{2/3}(\xi\mathcal{B})$. Moreover, its image in $\gamma_\xi^{2/3} = \mathbb{Z}/2\mathbb{Z}$ is non-trivial and in $\gamma^{2/3}(\mathcal{B})$ (via *res*) is trivial.

Example. Let $G = HSpin_{2n}$. Then for any algebra with orthogonal involution (A, δ) where $8 \mid \text{ind}(A)$ and A is non-division, there exists a non-trivial torsion element of order 2 in $\gamma^{2/3}(\xi\mathcal{B})$ which vanishes over a splitting field of (A, δ) .

Corollary (Z.) If $v_2(d) > i_A \geq 3$, then for every $\lambda \in \Lambda$ such that $\bar{\lambda} = \sigma$ there exists a non-trivial torsion element of order 2 in $\gamma^{2/3}(\xi\mathcal{B})$. Moreover, its image in $\gamma_\xi^{2/3} = \mathbb{Z}/2\mathbb{Z}$ is non-trivial and in $\gamma^{2/3}(\mathcal{B})$ (via *res*) is trivial.

Example. Let $G = HSpin_{2n}$. Then for any algebra with orthogonal involution (A, δ) where $8 \mid \text{ind}(A)$ and A is non-division, there exists a non-trivial torsion element of order 2 in $\gamma^{2/3}(\xi\mathcal{B})$ which vanishes over a splitting field of (A, δ) .

Corollary (Z.) If $v_2(d) > i_A \geq 3$, then for every $\lambda \in \Lambda$ such that $\bar{\lambda} = \sigma$ there exists a non-trivial torsion element of order 2 in $\gamma^{2/3}(\xi\mathcal{B})$. Moreover, its image in $\gamma_\xi^{2/3} = \mathbb{Z}/2\mathbb{Z}$ is non-trivial and in $\gamma^{2/3}(\mathcal{B})$ (via *res*) is trivial.

Example. Let $G = HSpin_{2n}$. Then for any algebra with orthogonal involution (A, δ) where $8 \mid \text{ind}(A)$ and A is non-division, there exists a non-trivial torsion element of order 2 in $\gamma^{2/3}(\xi\mathcal{B})$ which vanishes over a splitting field of (A, δ) .

Corollary (Z.) If $v_2(d) > i_A \geq 3$, then for every $\lambda \in \Lambda$ such that $\bar{\lambda} = \sigma$ there exists a non-trivial torsion element of order 2 in $\gamma^{2/3}(\xi\mathcal{B})$. Moreover, its image in $\gamma_\xi^{2/3} = \mathbb{Z}/2\mathbb{Z}$ is non-trivial and in $\gamma^{2/3}(\mathcal{B})$ (via *res*) is trivial.

Example. Let $G = HSpin_{2n}$. Then for any algebra with orthogonal involution (A, δ) where $8 \mid \text{ind}(A)$ and A is non-division, there exists a non-trivial torsion element of order 2 in $\gamma^{2/3}(\xi\mathcal{B})$ which vanishes over a splitting field of (A, δ) .

Corollary (Z.) If $v_2(d) > i_A \geq 3$, then for every $\lambda \in \Lambda$ such that $\bar{\lambda} = \sigma$ there exists a non-trivial torsion element of order 2 in $\gamma^{2/3}(\xi\mathcal{B})$. Moreover, its image in $\gamma_\xi^{2/3} = \mathbb{Z}/2\mathbb{Z}$ is non-trivial and in $\gamma^{2/3}(\mathcal{B})$ (via *res*) is trivial.

Example. Let $G = HSpin_{2n}$. Then for any algebra with orthogonal involution (A, δ) where $8 \mid \text{ind}(A)$ and A is non-division, there exists a non-trivial torsion element of order 2 in $\gamma^{2/3}(\xi\mathcal{B})$ which vanishes over a splitting field of (A, δ) .

Corollary (Z.) If $v_2(d) > i_A \geq 3$, then for every $\lambda \in \Lambda$ such that $\bar{\lambda} = \sigma$ there exists a non-trivial torsion element of order 2 in $\gamma^{2/3}(\xi\mathcal{B})$. Moreover, its image in $\gamma_\xi^{2/3} = \mathbb{Z}/2\mathbb{Z}$ is non-trivial and in $\gamma^{2/3}(\mathcal{B})$ (via *res*) is trivial.

Example. Let $G = HSpin_{2n}$. Then for any algebra with orthogonal involution (A, δ) where $8 \mid \text{ind}(A)$ and A is non-division, there exists a non-trivial torsion element of order 2 in $\gamma^{2/3}(\xi\mathcal{B})$ which vanishes over a splitting field of (A, δ) .

Corollary (Z.) If $v_2(d) > i_A \geq 3$, then for every $\lambda \in \Lambda$ such that $\bar{\lambda} = \sigma$ there exists a non-trivial torsion element of order 2 in $\gamma^{2/3}(\xi\mathcal{B})$. Moreover, its image in $\gamma_\xi^{2/3} = \mathbb{Z}/2\mathbb{Z}$ is non-trivial and in $\gamma^{2/3}(\mathcal{B})$ (via *res*) is trivial.

Example. Let $G = HSpin_{2n}$. Then for any algebra with orthogonal involution (A, δ) where $8 \mid \text{ind}(A)$ and A is non-division, there exists a non-trivial torsion element of order 2 in $\gamma^{2/3}(\xi\mathcal{B})$ which vanishes over a splitting field of (A, δ) .

Corollary (Z.) If $v_2(d) > i_A \geq 3$, then for every $\lambda \in \Lambda$ such that $\bar{\lambda} = \sigma$ there exists a non-trivial torsion element of order 2 in $\gamma^{2/3}(\xi\mathcal{B})$. Moreover, its image in $\gamma_\xi^{2/3} = \mathbb{Z}/2\mathbb{Z}$ is non-trivial and in $\gamma^{2/3}(\mathcal{B})$ (via *res*) is trivial.

Example. Let $G = HSpin_{2n}$. Then for any algebra with orthogonal involution (A, δ) where $8 \mid \text{ind}(A)$ and A is non-division, there exists a non-trivial torsion element of order 2 in $\gamma^{2/3}(\xi\mathcal{B})$ which vanishes over a splitting field of (A, δ) .

The group $\gamma_t^2(\xi\mathcal{B})$ describes/provides information about

- the torsion of $CH^2(\xi\mathcal{B})$, i.e. the torsion of the Chow group of a twisted flag variety [Garibaldi, Z., 2010].
- the torsion of the Grothendieck-Chow motive associated to ξ (generalized Rost motives) and, therefore, is related to the discrete motivic invariant of a torsor (the J -invariant) [Queguiner-Mathieu, Semenov, Z., 2011], [Junkins, 2011].
- Cohomological invariants in degree 3:
 - (a) the Rost invariant in the adjoint case
 - (b) the group of cohomological invariants
 - (c) low bounds for the essential dimension of G .

The group $\gamma_t^2(\xi\mathcal{B})$ describes/provides information about

- the torsion of $CH^2(\xi\mathcal{B})$, i.e. the torsion of the Chow group of a twisted flag variety [Garibaldi, Z., 2010].
- the torsion of the Grothendieck-Chow motive associated to ξ (generalized Rost motives) and, therefore, is related to the discrete motivic invariant of a torsor (the J -invariant) [Queguiner-Mathieu, Semenov, Z., 2011], [Junkins, 2011].
- Cohomological invariants in degree 3:
 - (a) the Rost invariant in the adjoint case
 - (b) the group of cohomological invariants
 - (c) low bounds for the essential dimension of G .

The group $\gamma_t^2(\xi\mathcal{B})$ describes/provides information about

- the torsion of $CH^2(\xi\mathcal{B})$, i.e. the torsion of the Chow group of a twisted flag variety [Garibaldi, Z., 2010].
- the torsion of the Grothendieck-Chow motive associated to ξ (generalized Rost motives) and, therefore, is related to the discrete motivic invariant of a torsor (the J -invariant) [Queguiner-Mathieu, Semenov, Z., 2011], [Junkins, 2011].
- Cohomological invariants in degree 3:
 - (a) the Rost invariant in the adjoint case
 - (b) the group of cohomological invariants
 - (c) low bounds for the essential dimension of G .

The group $\gamma_t^2(\xi\mathcal{B})$ describes/provides information about

- the torsion of $CH^2(\xi\mathcal{B})$, i.e. the torsion of the Chow group of a twisted flag variety [Garibaldi, Z., 2010].
- the torsion of the Grothendieck-Chow motive associated to ξ (generalized Rost motives) and, therefore, is related to the discrete motivic invariant of a torsor (the J -invariant) [Queguiner-Mathieu, Semenov, Z., 2011], [Junkins, 2011].
- Cohomological invariants in degree 3:
 - (a) the Rost invariant in the adjoint case
 - (b) the group of cohomological invariants
 - (c) low bounds for the essential dimension of G .

Related publications:

- Baek, S.; Neher, E.; Zainoulline, K. *Basic polynomial invariants, fundamental representations and the Chern class map*. arxiv.org:1106.4332 (2011), 13pp.
- Garibaldi, S.; Zainoulline, K. *The gamma-filtration and the Rost invariant*. arXiv:1007.3482 (2010), 19pp.
- Junkins, C. *The J-invariant and Tits indexes for groups of inner type E_6* . arXiv:1112.1454 (2011), 12pp.
- Queguiner-Mathieu A.; Semenov, N.; Zainoulline, K. *The J-invariant, Tits algebras and Triality*. arXiv:1104.1096 (2011), 28pp.
- Petrov, V.; Semenov, N.; Zainoulline, K. *J-invariant of linear algebraic groups*. Ann. Sci. Ec. Norm. Sup. (4) 41 (2008), no.6, 1023–1053.
- Zainoulline, K. *Twisted γ -filtration of a linear algebraic group to appear in Compositio Math.* (2012), 10pp.