Witt groups of varieties and the purity problem.

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Abstract

We provide a general algorithm used to prove the purity for functors with transfers. As a basic example we consider the Witt group of an algebraic variety.

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1 The Witt ring of a field

Symmetric bilinear spaces Let k be a field of characteristic $\neq 2$. A *symmetric bilinear space* over k is a pair (V, b) consisting of a vector space V and an isomorphism $b\colon V \to V^\#$ into its dual $V^\# = Hom_k(V, k)$.

Observe that the map b defines a symmetric bilinear form $B\colon V \times V \to k$ via $B(x, y) := b(x)(y)$. Given a basis $\{e_1, e_2, \dots, e_n\}$ of V and a symmetric bilinear space there is a symmetric matrix

$$M_b := \Big(b(e_i)(e_j)\Big)_{i,j=1\dots n} = \Big(B(e_i, e_j)\Big)_{i,j=1\dots n}.$$

By definition the map $b\colon V \to V^\#$ is an isomorphism if and only if M_b is invertible.

Isometries Assume we are given two symmetric bilinear spaces (V, b) and (V', b'). An *isometry* $\phi\colon (V, b) \to (V', b')$ between them is an isomorphism $\phi\colon V \to V'$ of vector spaces such that the square

$$\begin{array}{ccc} V & \xrightarrow{\phi} & V' \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b'} \\ V^\# & \xleftarrow{\phi^\#} & V'^\# \end{array}$$

commutes. For the respective matrices M_b and $M_{b'}$ it means that

$$M_{b'} = C_\phi^t \cdot M_b \cdot C_\phi,$$

where C_ϕ is the transformation matrix corresponding to ϕ.

Hyperbolic spaces We define the hyperbolic space of V as

$$H(V) := (V \oplus V^\#, \begin{pmatrix} 0 & id \\ id & 0 \end{pmatrix})$$

A symmetric bilinear space isometric to $H(V)$ is called *hyperbolic*.

The Witt ring We define the *orthogonal sum* and the *tensor product* of symmetric bilinear spaces as

$$(V, b) \perp (V', b') := (V \oplus V', b \oplus b')$$
$$(V, b) \otimes (V', b') := (V \otimes V', b \otimes b').$$

By the very definition the orthogonal sum and the tensor product respect the isometries.

Let $KO(k)$ denote the *Grothendieck group* of isometry classes of symmetric bilinear spaces with respect to the orthogonal sum. Let H be the subgroup of $KO(k)$ generated by the classes of hyperbolic spaces. The *Witt group* of k is defined to be the quotient

$$W(k) := KO(k)/H.$$

The tensor product of spaces turns $W(k)$ into a commutative ring with a unit $1 = (k, id)$, where k is a vector space of rank one.

Properties

- Let l/k be a field extension, i.e. there is an inclusion $i\colon k \to l$. The base change induces a ring homomorphism $i^*\colon : W(k) \to W(l)$. Hence, the assignment $l/k \mapsto W(l)$ is a covariant functor from the category of field extensions to the category of commutative rings.

- Let l/k be a finite field extension and $s\colon l \to k$ be a non-trivial k-linear map. Then the composite $s \circ B$, where B is a bilinear form over l, defines a bilinear form over k and, moreover, induces a well-defined group homomorhism $s_*\colon W(l) \to W(k)$ called a Scharlau *transfer*.

- There is the *projection formula*:

  $$s_*(i^*(\alpha)\cdot\beta) = \alpha \cdot s_*(\beta),\ \alpha \in W(k),\ \beta \in W(l).$$

  In particular, the composite $s_* \circ i^*\colon W(k) \to W(k)$ is given by multiplication by $s_*(1)$.

Relations with quadratic forms By a *quadratic form* q over k we mean a homogeneous polynomial of degree 2 over k

$$q(x) = \sum_{i,j=1\dots n} a_{ij}x_ix_j, \quad a_{ij} \in k.$$

Assume we are given a symmetric bilinear space (V,b). Let M_b be the symmetric matrix corresponding to the space (V,b) and the chosen basis $\{e_1,e_2,\dots,e_n\}$. We can associate to M_b a quadratic form

$$q(x) = x^t \cdot M_b \cdot x.$$

In the opposite direction, let $q(x)$ be a quadratic form over k. Then we may define a symmetric matrix M_q as

$$M_q := \left(\tfrac{1}{2}(a_{ij}+a_{ji})\right)_{ij}$$

and, hence, a map $b\colon V \to V^{\#}$ by $b(e_i)(e_j) := (M_q)_{ij}$. If b is an isomorphism, then q is called *non-singular*.

We have just provided a bijection

$$\begin{matrix} \textit{isometry classes} \\ \textit{of nonsingular} \\ \textit{quadratic forms} \end{matrix} \leftrightarrow \begin{matrix} \textit{isometry classes of} \\ \textit{symmetric bilinear spaces} \end{matrix}$$

which is compatible with the orthogonal sum and the tensor product of spaces. Hence, to compute the Witt group we can use the language of quadratic forms. The following two properties turn to be very important for computations:

- Diagonalization: Any quadratic form q over k can be diagonalized. Namely, there exists an isometry such that q transforms into a diagonal form

  $$a_1x_1^2 + a_2x_2^2 + \dots a_nx_n^2, \text{ where } a_i \in k.$$

  We denote such a form as $\langle a_1,a_2,\dots,a_n\rangle$. In particular, a hyperbolic space corresponds to the form $\langle 1,-1,1,-1,\dots,1,-1\rangle$.

- Witt decomposition: Any nonsingular quadratic form q can be represented uniquely (up to an isometry) as an orthogonal sum of a maximal hyperbolic subspace H and the so called anisotropic part of q

$$q = q_{an} \perp H.$$

Examples

- $W(\mathbb{C}) = \mathbb{Z}/2$ or, more generally, $W(k) = \mathbb{Z}/2$, where k is quadratically closed, i.e. any element of k is a square. Indeed, in this case any form can be diagonalized to $\langle 1, 1, \dots, 1\rangle$
- $W(\mathbb{R}) = \mathbb{Z}$ (use the signature)
- $W(\mathbb{F}_p) = \begin{cases} \mathbb{Z}/2[k^*/k^{*2}], & p \equiv 1 \mod 4 \\ \mathbb{Z}/4, & p \equiv 3 \mod 4 \end{cases}$

2 The Witt ring of a variety

The affine case Let R be a commutative ring with unit, $\frac{1}{2} \in R$. The previous definition of the Witt ring perfectly works if one replaces

a field k by *a ring R*

a k-vector space V by *a finitely generated projective R-module P*

an isomorphism b by *an isomorphism of R-modules* $b\colon P \to P^{\#}$

In this way we obtain the definition of the Witt ring of a commutative ring R. Let $X = Spec(R)$ be the respective affine scheme. Then we define

$$W(X) := W(R).$$

The Euler trace We extend the notion of the Scharlau transfer to the affine case as follows.

Let T/S be a finite flat extension of commutative rings. Let $s\colon T \to S$ be a S-linear map such that the induced map

$$\lambda\colon T \to Hom_S(T, S)$$

defined by $\lambda(x)(y) := s(xy)$ is an isomorphism.

Then to every symmetric bilinear space (P, B) over T we associate the bilinear form $(P_S, s \circ B)$, where P_S denotes P considered as an S-module. This bilinear form gives rise to a symmetric bilinear space over S and, moreover, induces a generalized Scharlau transfer

$$s_*\colon W(T) \to W(S).$$

Example Let k be a field and T/S be a finite extension of smooth, purely d-dimensional k-algebras. Let Ω_S and Ω_T be the modules of Kähler differentials of S and T over k and let $\omega_S = \bigwedge^d \Omega_S$ and $\omega_T = \bigwedge^d \Omega_T$.

Assume ω_S and ω_T are trivial. Then there exists an isomorphism of T-modules

$$\lambda\colon T \xrightarrow{\cong} Hom_S(T, S)$$

which induces an S-linear map $\epsilon\colon T \to S$ via $\epsilon(x) := \lambda(1)(x)$, called an *Euler trace*. Observe that from ϵ we get back λ as $\lambda(x)(y) := \epsilon(xy)$.

The general case Let X be a scheme with structure sheaf $\mathcal{O}_X$, $\frac{1}{2} \in \Gamma(\mathcal{O}_X)$. A symmetric bilinear space over X is a pair $(\mathcal{E}, b)$, consisting of a vector bundle $\mathcal{E}$ and an isomorphism $b\colon \mathcal{E} \to \mathcal{E}^\#$, where $\mathcal{E}^\# = Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ such that $b^\# = b$ (we identify $\mathcal{E}$ with its double dual $\mathcal{E}^{\#\#}$).

An isometry $\phi\colon (\mathcal{E}, b) \to (\mathcal{E}', b')$ of symmetric bilinear spaces is an $\mathcal{O}_X$-linear isomorphism such that $\phi^\# \circ b' \circ \phi = b$. The orthogonal sum and the tensor product are defined in an obvious way.

If we now introduce hyperbolic spaces $H(\mathcal{E})$ in the same manner as before and then take the quotient, the result will NOT be the Witt group of X but its finer version. To obtain the correct definition we have to use the notion of a *metabolic* space instead of hyperbolic. Roughly speaking, the reason is that

$$\textit{metabolic} \;=\; \textit{locally hyperbolic}$$

Metabolic spaces Let $(\mathcal{E}, b)$ be a symmetric bilinear space over X and let $\mathcal{F}$ be a subbundle of $\mathcal{E}$ meaning that $\mathcal{F}$ is locally a direct summand of $\mathcal{E}$.

For a subbundle $\mathcal{F}$ of $\mathcal{E}$ we define its *orthogonal complement* $\mathcal{F}^\perp$ as the kernel of the composite $i^\# \circ b$, where $i^\#\colon \mathcal{E}^\# \to \mathcal{F}^\#$ is the dual of the inclusion $i\colon \mathcal{F} \to \mathcal{E}$. Clearly, $i^\#$ is an epimorphism and b induces an isomorphism $\mathcal{E}/\mathcal{F}^\perp \to \mathcal{F}^\#$. In particular, $\mathcal{F}^\perp$ is again a subbundle of $\mathcal{E}$.

A subbundle $\mathcal{F}$ of $\mathcal{E}$ is called a *lagrangian* of $\mathcal{E}$ if $\mathcal{F} = \mathcal{F}^\perp$. A symmetric bilinear space is called *metabolic* if it contains a lagrangian.

Observe that in the field case, i.e. $X = Spec(k)$, the orthogonal complement of a subspace U of (V, b) coincides with

$$U^\perp = \{x \in V \mid B(x, y) = 0 \text{ for all } y \in U\},$$

where B is the respective bilinear form.

The Witt group is defined to be the quotient of the Grothedieck group $KO(X)$ of isometry classes of symmetric bilinear spaces over X modulo the subgroup M generated by metabolic spaces, i.e.

$$W(X) := KO(X)/M.$$

As before, the tensor product turns $W(X)$ into a ring. Note that in the affine case this definition gives the old Witt ring.

Examples

- $W(\mathbb{A}^n_X) = W(X)$, where X is affine - Karoubi
- $W(\mathbb{P}^n_k) = W(k)$ - Arason
- for an irreducible smooth quasi-projective complex curve C

$$W(C) = \mathbb{Z}/2 \oplus Disc(X),$$

  where $Disc(X)$ is the group of isometry classes of symmetric bilinear spaces of rank 1.

- In particular, if C is projective of genus g, then

$$W(C) = (\mathbb{Z}/2)^{1+2g}.$$

- for an irreducible smooth quasi-projective complex surface X

$$W(X) = \mathbb{Z}/2 \oplus Disc(X) \oplus {}_2Br(X).$$

- Many examples of Witt groups of curves, surfaces and 3-folds were provided by Ojanguren, Parimala, Sridharan, Sujatha, Suresh,...

The key technical tool is the exact sequence

$$0 \to W(X) \to W(k(X)) \xrightarrow{\partial_x} \bigoplus_{x \in X^{(1)}} W(k(x)),$$

where X is an integral regular scheme over k, $k(X)$ is its quotient field, $k(x)$ is the residue field at a point x of codimension one and ∂_x is a second residue homomorphism. The exactness of this sequence at $W(k(X))$-term is called the PURITY and is the main subject of these lectures.

The purity for the Witt groups is known for the following cases

- X is a regular integral noetherian scheme of dimension at most 2 - by Colliot-Thélène and Sansuc
- X is a regular integral noetherian affine scheme of dimension 3 - by Ojanguren, Parimala, Sridharan, Suresh
- $X = Spec(R)$, where R is a local regular ring containing a field - by Ojanguren, Panin

Triangular Witt groups A vast generalization of the notion of the Witt group of a scheme was provided by Balmer. He introduced and studied the notion of the Witt group $\mathcal{W}$ of a *triangulated category with duality.* For triangular Witt groups the following computations were obtained:

- $\mathcal{W}(\mathbb{P}_X(\mathcal{E}))$, $\mathcal{W}$(quadric) by Nenashev, Walter
- $\mathcal{W}$(Grassmannian) by Balmer, Calmés

3 The Purity

Let A be a smooth algebra over an infinite field k and let K be its quotient field. Let $R = A_{\mathfrak{q}}$ be the localization of A at a prime ideal $\mathfrak{q} \in Spec(A)$. Such a ring R is called a *local regular ring of geometric type*.

Let $F\colon A\text{-}Alg \to Ab$ be a covariant functor from the category of A-algebras to the category of Abelian groups.

¿From now on

- the local regular ring of geometric type R

  and

- the covariant functor F

will be the main objects of our discussion.

Definition 1. For every prime ideal $\mathfrak{p} \in Spec(R)$ consider the group homomorphism $F(R_{\mathfrak{p}}) \to F(K)$ induced by the canonical inclusion of $R_{\mathfrak{p}}$ into its quotient field K.

We say that an element $\alpha \in F(K)$ is *unramified at a prime ideal* $\mathfrak{p}$ if it is in the image of $F(R_p)$. We say that an element $\alpha \in F(K)$ is *unramified over* R if it is unramified at every prime ideal $\mathfrak{p} \in Spec(R)$ of height 1.

Definition 2. We say that the PURITY holds for the functor F over R if every unramified element of $F(K)$ belongs to the image of $F(R)$ in $F(K)$. In other words, the following equality holds between subgroups of $F(K)$:

$$\bigcap_{ht\ \mathfrak{p}=1} im\{F(R_{\mathfrak{p}}) \to F(K)\} = im\{F(R) \to F(K)\}.$$

The (left) subgroup of unramified elements will be denoted by $F_{nr}(K)$.

Gersten conjecture and the purity The purity is a particular case of the following more general problem: Given a cohomological functor F over R and $U = Spec(R)$ to show that the Gersten complex

$$0 \to F(U) \to F(K) \to \bigoplus_{x \in U^{(1)}} F(k(x)) \to$$

$$\to \bigoplus_{x \in U^{(2)}} F(k(x)) \to \dots,$$

is exact. Observe that exactness at the $F(K)$-term gives the purity.

The injectivity $F(U) \to F(K)$ for $F = H^1_{et}(-, G)$, where G is a smooth reductive group scheme, is equivalent to the Grothendieck-Serre conjecture which is still open.

For the Witt groups the injectivity is due to Ojanguren and Pardon; the Gersten conjecture is due to Balmer, Walter and Rost, Schmid.

Our goal is to prove the following general fact

Theorem 1. *Let R be a local regular ring of geometric type obtained by localizing a smooth k-algebra A. Let $F\colon A\text{-}Alg \to Ab$ be a functor with transfers described below. Then the purity holds for F over R.*

As a corollary of the proof we obtain the following celebrated result

Theorem 2 (Ojanguren, Panin). *Let R be a local regular ring containing a field k, $char(k) \neq 2$. Then the purity holds for the Witt functor $F = W$ over R.*

- The proof of Theorem 1 uses the same techniques as the original proof by Panin-Ojanguren.
- The Witt group formally doesn't satisfy the condition of being a functor with transfers. Nevertheless, since for all varieties appearing in the proof the canonical sheaves ω turn to be trivial, the proof works after replacing transfers by the respective Euler traces.

Functors with transfers Let R be a local regular ring of geometric type obtained by localizing a smooth k-algebra A. Let $F\colon A\text{-}Alg \to Ab$ be a covaraint functor. We say F is a *functor with transfers* if it satisfies the following axioms:

(C) (Continuity) For any A-algebra S essentially smooth over k and for any multiplicative system M in S the canonical map

$$\lim_{g\in M} F(S_g) \to F(M^{-1}S)$$

is an isomorphism, where $M^{-1}S$ denotes the localization of S with respect to M.

Let $F_R\colon R\text{-}Alg \to Ab$ denote the restriction of the functor F to the category of R-algebras via the canonical inclusion $A \hookrightarrow R$.

(T) (Structure of transfer maps) For any finite étale R-algebra T there exist homomorphisms

$$Tr_R^T\colon F_R(T) \to F_R(R)$$
$$Tr_K^{T\otimes_R K}\colon F_R(T\otimes_R K) \to F_R(K)$$

called transfer maps which satisfy the following conditions:

(a) $Tr_R^R = id_R$ and for any finite étale R-algebras T_1 and T_2 the following relation holds

$$Tr_R^{T_1\times T_2}(x) = Tr_R^{T_1}(x_1) + Tr_R^{T_2}(x_2).$$

(b1) Let $R[t]$ denote a polynomial ring over R. For a finite $R[t]$-algebra S such that $S/(t)$ and $S/(t-1)$ are finite étale over R the following diagram commutes:

$$\begin{array}{ccc} F_R(S) & \longrightarrow & F_R(S/(t)) \\ \downarrow & & \downarrow{\scriptstyle Tr} \\ F_R(S/(t-1)) & \xrightarrow[Tr]{} & F_R(R) \end{array}$$

(b2) The transfer map $Tr_K^{T\otimes_R K}$ satisfies conditions (a) and (b1) above and the following diagram induced by extension of scalars via the canonical inclusion $R \hookrightarrow K$ commutes:

$$\begin{array}{ccc} F_R(T) & \longrightarrow & F_R(T\otimes_R K) \\ {\scriptstyle Tr}\downarrow & & \downarrow{\scriptstyle Tr} \\ F_R(R) & \longrightarrow & F_R(K) \end{array}$$

(E) (Finite monodromy) Holds automatically for the Witt group and for any functor defined over a field.

Examples Let S be an k-algebra

- $F = \mathbb{G}_m$, i.e. $F\colon S \mapsto S^*$. The purity is equivalent to the exact sequence

$$R^* \to K^* \stackrel{\oplus v_\mathfrak{p}}{\to} \bigoplus_{ht\mathfrak{p}=1} \mathbb{Z}$$

- For the K-theory $F = K_*$ the purity means the exact sequence

$$K_*(R) \to K_*(K) \stackrel{\partial_\mathfrak{p}}{\to} \bigoplus_{ht\mathfrak{p}=1} K_{*-1}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$$

- $F = coker(\mu)$, where $\mu\colon G \to \mathbb{G}_m$ is a surjective morphism of group schemes and G is a linear algebraic group over k, $char(k) = 0$, which is rational as a k-variety.

Let S be an R-algebra

- $F = H^*_{et}(-, \mathcal{C})$, where $\mathcal{C}$ is a locally constant sheaf with finite stalks of $\mathbb{Z}/n$-modules over R, $(n, char(k)) = 1$.
- $F = coker(Nrd)$, where $Nrd\colon GL_{1,\mathcal{A}} \to \mathbb{G}_m$ is the reduced norm of an Azumaya algebra $\mathcal{A}$ over R.
- $F = coker(Sn)$, where $Sn\colon SO_q \to H^1_{et}(-, \mu_2)$ is the spinor norm of a nonsingular quadratic form q defined over R.
- $F = coker(Nrd)$, where $Nrd\colon U(\mathcal{A}, \sigma) \to U(Z, \sigma|_Z)$ is the reduced norm from the unitary group of an Azumaya algebra $\mathcal{A}$ with involution of the second kind σ over R to the unitary group of its center Z.

We have also the torsion versions of the previous cases (here $d \geq 0$)

- $F\colon S \mapsto S^*/Nrd(\mathcal{A}^*_S) \cdot (S^*)^d$
- $F\colon S \mapsto U(Z, \sigma)/Nrd(U(\mathcal{A}, \sigma)) \cdot U(Z, \sigma)^d$

4 The proof of the Purity Theorem

First, we discuss two main geometric ingrediences of the proof: the geometric presentation lemma and the Quillen trick.

Lemma 1 (Geometric Presentation Lemma). *Let R be a local essentially smooth algebra over an infinite field k and let S be an essetially smooth k-algebra which is integral domain and is finite over the polynomial algebra $R[t]$. Suppose that $\varepsilon\colon S \to R$ is an R-augmentation and let $I = \ker \varepsilon$. Assume that $S/\mathfrak{m}S$ is smooth over the residue field $R/\mathfrak{m}$ at the maximal ideal $\varepsilon^{-1}(\mathfrak{m})/\mathfrak{m}S$. Then, given a regular function $f \in S$ such that $S/(f)$ is finite over R, we can find a $t' \in I$ such that*

- *S is finite over $R[t']$*

- *there is an ideal J comaximal with I such that $I \cap J = (t')$*
- *the ideals J and $(t'-1)$ are comaximal with the ideal (f)*
- *$S/(t')$ and $S/(t'-1)$ are étale over R*

In geometric terms we have to find a finite surjective projection π with certain conditions on the fibers (see the picture below)

Proof. We will show only how to construct this t' and prove the first and the part of the last property.

Replacing t by $t-\varepsilon(t)$ we may assume that $t \in I$. We denote by 'bar' the reduction modulo $\mathfrak{m}$. By the assumptions made on S the quotient $\bar{S} = S/\mathfrak{m}S$ is smooth over the residue field $\bar{R} = R/\mathfrak{m}$ at its maximal ideal $\bar{I} = \varepsilon^{-1}(\mathfrak{m})/\mathfrak{m}S$.

Choose an $\alpha \in S$ such that $\bar{\alpha}$ is a local parameter of the localization of $\bar{S}$ at $\bar{I}$. By the Chinese Remainder theorem we may assume that $\bar{\alpha}$ does not vanish at the zeros of $\bar{f}$ different from $\bar{I}$.

Without changing $\bar{\alpha}$ we may replace α by $\alpha - \varepsilon(\alpha)$ and assume that $\alpha \in I$. Since S is integral over $R[t]$, there exists a relation of integral dependence

$$\alpha^n + p_1(t)\alpha^{n-1} + \ldots + p_n(t) = 0. \tag{*}$$

For any $r \in k^\times$ and any N larger than the degree of each $p_i(t)$ we put

$$t' = \alpha - rt^N.$$

¿From the equation (*) t is integral over $R[t']$. Hence S, which is integral over $R[t]$, is integral over $R[t']$ and the first property is proven.

By the Bertini theorem we may choose α such that the algebra $\bar{S}/(\bar{\alpha})$ is étale over $\bar{R}$. Consider the fiber product diagram

$$\begin{array}{ccc} \bar{S}[u]/(\bar{\alpha} - u\bar{t}^N) & \longrightarrow & \bar{S}/(t') \\ {\scriptstyle \pi}\uparrow & & \uparrow \\ k[u] & \xrightarrow{u \mapsto r} & k \end{array}$$

Since the fiber of π at $u = 0$ is étale, the fibers of π at almost all rational points $u \in k^\times$ are étale. In other words, $\bar{S}/(\bar{t}')$ is étale over k and, hence, over $\bar{R}$ for almost all $r \in k^\times$.

By assumption S and $R[t']$ are smooth. Since S is finite over $R[t']$, S is finitely generated projective as an $R[t']$-module and, hence, $S/(t')$ is free as an R-module. In particular, $S/(t')$ is flat over R. Finally, the fact that $\bar{S}/(\bar{t}')$ is étale over $\bar{R}$ implies that $S/(t')$ is étale over R. □

Observe that in the original statement of Geometric Presentation Lemma by Panin-Ojanguren the last condition (that the fibers are étale) was missing. Hence, our lemma provides a bit stronger version. This difference becomes important when one looks for transfer maps related with certain group schemes (e.g. the spinor norm case).

The following fact which is due to Quillen can be viewed as a generalization of Noether normalization lemma

Lemma 2 (Quillen's trick). *Let A be a smooth finite type algebra of dimension d over a field k. Let $f \in A$ be a regular function. Let $\mathcal{I}$ be a finite subset of $Spec(A)$.*

Then there exist functions $x_1, \ldots, x_d$ in A algebraically independent over k such that if $\mathfrak{i}\colon W = k[x_1, \ldots, x_{d-1}] \to A$ denotes the inclusion, then

- *$A/(f)$ is finite over W*
- *A is smooth over W at the points of $\mathcal{I}$*
- *the inclusion $\mathfrak{i}$ factors as $\mathfrak{i}\colon W \hookrightarrow W[x_d] \to A$, where the last map is finite.*

The generalization of this lemma involving some support condition was proven recently by Panin and the author.

The proof of the Purity Theorem Let $R = A_{\mathfrak{q}}$ be a localization of a smooth k-algebra A at a prime ideal q and let K be its quotient field. We have to prove that any element $\alpha \in F_{nr}(K)$ belongs to the image of the canonical map $i_K^*\colon F(R) \to F(K)$.

The proof is based on the following three assertions:

For any $f \in A$ we have the following two commutative diagrams:

(i)
$$\begin{array}{ccccc} F(A_f) & \xrightarrow{\phi} & F(S_f) & \xrightarrow{i_C^*} & F(C_f) \\ \uparrow & & \Psi \downarrow & & \downarrow \Psi_K \\ \text{Use Quillen's Trick} & & F(R) & \xrightarrow{i_K^*} & F(K) \end{array} \quad \leftarrow \text{Use G.P.L.}$$

(ii)
$$\begin{array}{ccccc} F(A_f) & \xrightarrow{\phi} & F(S_f) & \xrightarrow{i_C^*} & F(C_f) \\ & \searrow & & & \downarrow \varepsilon_K^* \\ & & & & F(K) \end{array}$$

Use G.P.L. and the base change

and the relation

(iii) There exists a regular function $f \in \mathfrak{q}$ and an element $\alpha_f \in F(A_f)$ such that

- the image of α_f by means of the canonical map $i_K^*\colon F(A_f) \to F(K)$ coincides with α
- and we have the equality $\Psi_K(i_C^* \circ \phi(\alpha_f)) = \varepsilon_K^*(i_C^* \circ \phi(\alpha_f))$.

Assume that (i), (ii) and (iii) holds. Then the proof follows from the chain of equalities

$$i_K^*(\Psi \circ \phi(\alpha_f)) \overset{by\ (i)}{=} \Psi_K(i_C^* \circ \phi(\alpha_f)) \overset{by\ (iii)}{=} \varepsilon_K^*(i_C^* \circ \phi(\alpha_f)) \overset{by\ (ii)}{=} i_K^*(\alpha_f) = \alpha.$$

The construction of (i) We sketch here only the construction of the diagram (i).

First, we construct the ring S and the map $\phi\colon F(A_f) \to F(S_f)$. To do this we apply the Quillen trick to the algebra A, a function $f \in A$ and $\mathcal{I} = \{\mathfrak{q}\}$.

We define $S = A \otimes_W R$ and consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{i_S\colon a \mapsto a\otimes 1} & A \otimes_W R \\ \uparrow & & \uparrow{\scriptstyle r\mapsto 1\otimes r} \\ W & \xrightarrow{\quad i \quad} & R \end{array}$$

Define $\phi = i_S^*\colon F(A_f) \to F(S_f)$. It is easy to check that S/R and $f = f \otimes 1 \in S$ satisfy the hypothesis of the geometric presentation lemma. So we may apply the g.p.l. to construct $t' \in R[t]$ and the ideal J.

We define the map $\Psi\colon F(S_f) \to F(R)$ as

$$\Psi = Tr_1 \circ p_1^* - Tr_J \circ p_J^*,$$

where $p_1\colon S_f \to S/(t'-1)$, $p_J\colon S_f \to S/J$ are the quotient maps and Tr_1, Tr_J are the respective transfers.

Finally, the right hand side of the diagram (i) is obtained by the base change K/R. Here $C = S \otimes_R K$ and $\Psi_K = \Psi \otimes_R K$.

References

[**Books**]

[1] Baeza, R. Quadratic forms over semi-local rings, Lecture Notes in Mathematics, Vol. 655, Springer-Verlag, (1978).

[2] Knus, M.-A. Quadratic and Hermitian Forms over Rings. Grundlehren der Math. Wiss. 294. Springer-Verlag, Berlin, 1991.

[3] Knus, M.-A., Merkurjev, A., Rost, M.,Tignol, J.-P. The Book of Involutions. AMS Colloquium Publications, 44. AMS, Providence, RI, 1998.

[4] Lam, T.Y. The Algebraic Theory of Quadratic Forms. Mathematics lecture notes series, W.A.Benjamin Inc, 1980.

[5] Scharlau, W. Quadratic and Hermitian forms. Grundlehren der Math. Wiss. 270. Springer-Verlag, Berlin, 1985.

[**Research papers**]

[6] Colliot-Thélène, J.-L.; Hoobler, R.; Kahn, B. The Bloch-Ogus-Gabber Theorem. in Algebraic K-theory, 31-94, Fields Inst. Comm. 16, Amer. Math. Soc., Providence, RI, 1997.

[7] Ojanguren, M.; Panin, I. A purity theorem for the Witt group. Ann. Sci. École Norm. Sup. (4) 32 (1999), no. 1, 71–86.

[8] Ojanguren, M.; Panin, I.; Zainoulline, K. On the norm principle for quadratic forms. J.Ramanujan Math. Soc. 19 (2004), No.4, 1-12.

[9] Panin, I.; Zainoulline, K. Gersten resolution with support. Manuscripta Math. (2009), 13pp., DOI: 10.1007/s00229-008-0241-4

[10] Panin, I., Zainoulline, K. Variations on the Bloch-Ogus Theorem. Documenta Math. 8 (2003), 51-67

[11] Zainoulline, K. On Knebusch's Norm Principle for quadratic forms over semi-local rings. Math. Zeitschrift 251 (2005), no.2, 415-425.

[12] Zainoulline, K. The Purity Problem for Functors with Transfers, K-Theory J., 22(4), 2001, 303–333.

[13] Zainoulline, K. On Grothendieck's Conjecture about principal homogeneous spaces for some classical algebraic groups, St.Petersburg Math. J. Vol.12 No.1, 2001