Brief overview on Grothendieck-Serre's conjecture The following conjecture was stated by A. Grothendieck in 60's:

Conjecture. *Let G be a smooth reductive group scheme over a local regular ring R and K be the field of fractions of R. Then the induced map*

$$H^1_{et}(R,G) \to H^1_{et}(K,G)$$

has trivial kernel.

Observe that the pointed set $H^1_{et}(R,G)$ classifies principal G-bundles over R, i.e., the conjecture says that any principal G-bundle which is rationally trivial is (Zariski-)locally trivial.

If R is of geometric type over a field k and G is defined over k, then the conjecture is known as Serre's Conjecture. If k is infinite, the latter was proven by Colliot-Thelene, Ojanguren [CO92] and Raghunathan [Ra94]. The case of a finite field k remains open till now.

The Grothendieck's conjecture was proven in the following cases:

- E. Nisnevich
  - $\dim R = 1$ or – R is Henselian of arbitrary dimension [Ni84]
  - $\dim R = 2$ and G is quasi-split over R [Ni89]
- J.-L. Colliot-Thelene, J.-J. Sansuc [CS87]
  - G is an arbitrary torus over R.
- M. Ojanguren
  - R is of geometric type and $G = O_q$ is the orthogonal group of a non-degenerate quadratic form q over R [Oj80].
  - $\dim R = 2$ and $G = O_q$ [Oj82].
  - $\dim R = 2$ and $G = SL_A$ is the special linear group associated to an Azumaya algebra A over R [Oj82].
- I. Panin, A. Suslin [PS97]
  - R is of geometric type and $G = SL_A$
- M. Ojanguren, I. Panin [OP01]
  - R contains a field of $char \neq 2$ and $G = U_{A,\sigma}$ is the unitary group of an Azumaya algebra A over R with involution σ of the second kind.
  - R contains a field of $char \neq 2$ and $G = O_{A,\sigma}$ is the orthogonal group of an Azumaya algebra with involution σ of the first kind.
- K. Zainoulline [Za01]

  - R contains an infinite field of $char \neq 2$ and $G = SU_{A,\sigma}$ is the special unitary group associated to an Azumaya algebra A over R with an involution σ of the second kind.
  - The case $R = SL_A$ for any R containing a field.

- M. Ojanguren, I. Panin, K. Zainoulline [OPZ04]
  - R contains an infinite field of $char \neq 2$ and $G = Spin_q$ is the Spinor group associated to a non-degenerate quadratic form over R.

- I. Panin [Pa05]
  - G is any adjoint group of classical type.

- V. Chernousov, I. Panin [CP07]
  - R contains a field of $char \neq 2$ and G is of type G_2 over R.

To finish the case of groups of classical types it remains to prove

- $G = Spin_{A,\sigma}$, where A is an Azumaya algebra over R with orthogonal involution σ, $ind(A) \geq 2$.

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