Introduction to motives and algebraic cycles on projective homogeneous varieties

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Introduction

To be written

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Algebraic groups and homogeneous varieties

2.1 Linear algebraic groups

2.1.1. Fix a base field F. An *affine group scheme* over F is an affine algebraic variety G over F endowed with a group structure, i.e. endowed with an identity element and two algebraic morphisms over F

$$\mathrm{mult}\colon G \times G \to G \quad \text{and} \quad \mathrm{inv}\colon G \to G$$

defining the multiplication and the inverse respectively and satisfying the usual group axioms. Alternatively, using the language of *functors of points* an affine group scheme G can be identified with a *functor*

$$\mathcal{G}\colon \mathfrak{Alg}_F \longrightarrow \mathfrak{Gr}$$

from the category of commutative associative F-algebras to the category of groups *represented* by a commutative finitely generated *Hopf algebra* H over F. This identification is given by $G = \operatorname{Spec} H$ and $\mathcal{G}(-) = Hom_F(-, G)$.

2.1.2. Here are basic examples of affine group schemes

- The additive group $\mathbb{G}_a = \operatorname{Spec} F[t]$;
- The multiplicative group $\mathbb{G}_m = \operatorname{Spec} F[t, t^{-1}]$.
- The group of n-th roots of unity $\mu_n = \operatorname{Spec} F[t]/(t^n - 1)$. This is a closed subgroup of $\mathbb{G}_m$.
- The general linear group $\mathrm{GL}_n = \operatorname{Spec} F[t_{ij}, D^{-1}]$, where $1 \le i, j \le n$ and D is the determinant of the $n \times n$ matrix (t_{ij}).

2.1.3. One can show that G is an affine group scheme over F if and only if it is a closed subgroup of a general linear group GL_n over F. To stress this fact we call G by a *linear algebraic group* over F.

2.1.4. If the base field F has characteristic 0 any linear algebraic group G is *smooth* as a variety over F. In the language of functors of points it means that for any commutative F-algebra R and any nilpotent ideal I of R the induced map $\mathcal{G}(R) \to \mathcal{G}(R/I)$ is surjective or, equivalently, the Hopf algebra representing $\mathcal{G}$ is reduced. For instance, the group $G = \mu_n$ is smooth if and only if n is invertible in F.

2.1.5. A linear algebraic group G over F is called *connected*, if it is irreducible as an algebraic variety over F. For example, the orthogonal group O_n of isometry classes of the quadratic form $x_1^2 + \ldots + x_n^2$ is not connected. It consists of two connected components, where the connected component of the identity is isomorphic to the special orthogonal group O_n^+.

2.1.6. A connected non-trivial linear algebraic group G is called *simple* (resp. *semisimple*) if it does not have any non-trivial closed (resp. solvable) connected normal subgroups over the algebraic closure F_a of F. For instance, the product of two copies of $\mathbb{G}_m$ is not semisimple, since it contains a copy of $\mathbb{G}_m$ as the diagonal. The group SL_n or O_n^+, $n > 1$ provides an example of a simple linear algebraic group. A product of simple groups provides an example of a semisimple group.

2.1.7. A linear algebraic group $\mathbb{T}$ over F is called a *torus*, if over the algebraic closure F_a it becomes isomorphic to a product of several copies of $\mathbb{G}_m$, i.e.

$$\mathbb{T}_{F_a} \simeq \mathbb{G}_{m,F_a} \times \ldots \times \mathbb{G}_{m,F_a}.$$

If this isomorphisms is defined already over F, then $\mathbb{T}$ is called a split torus.

2.1.8 Example. Let F'/F be a finite separable field extension. Consider the functor

$$\mathcal{G}\colon R \to (R \otimes_F F')^\times$$

which maps a commutative F-algebra R to the group of units of its base change. One can show that $\mathcal{G}$ is representable by a Hopf algebra and, therefore, defines a linear algebraic group over F denoted by $\mathcal{R}_{F'/F}(\mathbb{G}_m)$. The usual norm map induces a morphism of group schemes $\mathcal{R}_{F'/F}(\mathbb{G}_m) \to \mathbb{G}_m$ over F. Its kernel provides an example of a non-split torus over F.

2.1.9. A semisimple linear algebraic group G over F is called *split*, if it contains a split maximal torus (maximal with respect to the inclusion). In particular, if the field F is algebraically closed, then all semisimple algebraic groups over F are split.

In the present notes we will mainly study *non-split* simple groups over arbitrary fields. Here is a typical example

2.1.10 Example. Let A be a central division algebra over F. Its group of automorphisms is a non-split simple linear algebraic group over F called a *projective general linear group* of A and denoted by PGL_A.

2.2 Root systems and Dynkin diagrams.

In the present section we introduce a combinatorial language which will be used to classify all split semisimple linear algebraic groups.

2.2.1. Let V be a non-trivial $\mathbb{R}$-vector space together with an Euclidean scalar product $(-,-)\colon V\otimes V \to \mathbb{R}$. Given a vector $v \in V$ we denote by s_v the *reflection* orthogonal to v with respect to this product. By definition s_v is a linear map given by

$$s_v\colon w \mapsto w - 2\frac{(w,v)}{(v,v)}\cdot v, \quad w \in V.$$

A finite set Φ of non-zero vectors of V is called a *root system* if

- The set Φ spans the $\mathbb{R}$-vector space V;
- For all $\alpha \in \Phi$ we have $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$;
- If $\alpha, \beta \in \Phi$, then $s_\alpha(\beta) \in \Phi$;
- For all $\alpha, \beta \in \Phi$ we have $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

The elements of Φ are called *roots*. The group generated by reflections

$$W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi \rangle$$

is a finite group and is called the *Weyl group* of Φ.

2.2.2. Let $\Phi \subset V$ be a root system. Then there exists a subset $\Pi \subset \Phi$ of linear independent vectors, called a *basis* of Φ, such that any root $\alpha \in \Phi$ is a linear combination of basis vectors with integer coefficients, where either all coefficients are non-negative or non-positive. In the first case α is called a *positive* root and in the second case it is called a *negative* root.

The number of elements of Π is called the *rank* of Φ and is denoted by n. Observe that $n = \operatorname{rk} V$. The elements of $\Pi = \{\alpha_1, \dots, \alpha_n\}$ are called *simple roots*. The corresponding generators $s_i := s_{\alpha_i}$ of the Weyl group $W(\Phi)$ are called *simple reflections*.

2.2.3. Chosen a basis $\Pi = \{\alpha_1, \dots, \alpha_n\}$ of the root system Φ we define an oriented graph called a *Dynkin diagram* of Φ as follows.

- The vertices are in one-to-one correspondence with elements of Π.
- Given two different vertices $\alpha_i, \alpha_j \in \Pi$ we connect them by the following number of edges:

number of edges	angle between α_i and α_j
1	120°
2	135°
3	150°
0	otherwise

- If $(\alpha_i, \alpha_i) \neq (\alpha_j, \alpha_j)$, then we orient the edges between α_i and α_j as follows:

$$\begin{array}{ll} \text{toward } \alpha_i & \text{if } (\alpha_i, \alpha_i) < (\alpha_j, \alpha_j) \\ \text{toward } \alpha_j & \text{otherwise} \end{array}$$

2.2.4. It can be shown that the Dynkin diagram of Φ does not depend on the choice of the basis $\Pi \subset \Phi$ and, moreover, uniquely determines Φ. The Dynkin diagram corresponding to a root system Φ will be denoted by $\mathcal{D}(\Phi)$. In particular, the Weyl group $W(\Phi)$ can be restored from the Dynkin diagram $\mathcal{D}(\Phi)$ as follows:

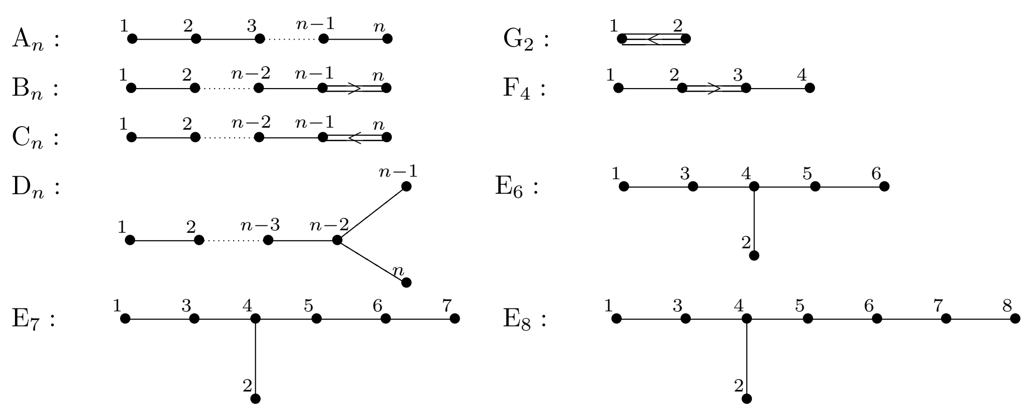
Let $n_{ij} \in \{0, 1, 2, 3\}$ denote the number of edges between two different vertices α_i, $\alpha_j \in \Pi$. Then

$$W(\Phi) \simeq \langle s_1, \ldots, s_n \mid s_i^2 = 1,\ (s_i s_j)^{n_{ij}+2} = 1 \text{ if } n_{ij} < 3 \text{ and } (s_i s_j)^6 = 1 \text{ if } n_{ij} = 3 \rangle$$

2.2.5. A root system $\Phi \subset V$ is called *irreducible*, if V can not be decomposed as $V = V_1 \perp V_2$ such that $V_i \cap \Phi$ is a root system in V_i, $i = 1, 2$.

2.2.6. The key result concerning root systems says that all possible Dynkin diagrams of irreducible root systems are classified and can be subdivided into four classical types A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 2)$, D_n $(n \geq 3)$ and five exceptional types E_6, E_7, E_8, F_4 and G_2. Here the lower index means the rank of the respective root system. In the sequel we say that a root system Φ has type $\mathcal{D}$ if and only if the respective Dynkin diagram has type $\mathcal{D}$.

2.2.7. Our enumeration of vertices of Dynkin diagrams follows Bourbaki and looks as follows:



2.2.8. Let Φ be a root system in V. For each $\alpha \in \Phi$ there is a linear map

$$\alpha^\vee \colon V \to \mathbb{R} \quad \text{ given by } v \mapsto 2\,\frac{(v, \alpha)}{(\alpha, \alpha)}$$

called the *coroot* of α. The set of coroots $\Phi^\vee$ is again a root system in the dual space $V^* := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. It is called the *dual* root system of Φ.

There are two important lattices in V associated to the root system Φ. The first lattice Λ_r is given by the $\mathbb{Z}$-linear span of Φ and is called the *root lattice* of Φ. The second lattice Λ_w is given by

$$\Lambda_w := \{ v \in V \mid \alpha^\vee(v) \in \mathbb{Z} \ \forall \alpha \in \Phi \}$$

and is called the *weight lattice* of Φ. We have $\Lambda_r \subset \Lambda_w$ and the quotient Λ_w/Λ_r is a finite abelian group.

2.3 The root system of a split group

Let G be a split group over a field F. We associate a root system to G as follows

2.3.1. Choose a split maximal torus $\mathbb{T}$ inside G. We denote the character and the cocharacter groups of $\mathbb{T}$ by $\mathbb{T}^* = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ and $\mathbb{T}_* = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$, respectively. The abelian groups $\mathbb{T}^*$ and $\mathbb{T}_*$ are isomorphic to $\mathbb{Z}^n$ for some number n, called the rank of G, and there is a perfect pairing

$$\langle -, - \rangle \,:\, T^* \times T_* \longrightarrow \mathbb{Z} = \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)\,,$$

which is defined on points by the equation

$$\chi(\lambda(x)) = x^{\langle \chi, \lambda \rangle} \text{ for all } \chi \in \mathbb{T}^*,\ \lambda \in \mathbb{T}_* \text{ and } x \in k^\times.$$

2.3.2. Consider the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(L)$ of the affine group scheme G on the associated Lie algebra $\mathcal{L} = \mathcal{L}(G)$ over F. A character $\alpha \in \mathbb{T}^*$ is called a *weight* of the adjoint representation if the respective weight subspace $\mathcal{L}_\alpha$ defined on F-points by

$$\mathcal{L}_\alpha(F) \;:=\; \{\, v \in \mathcal{L}(F) \,|\, \mathrm{Ad}(t)v = \alpha(t)v \ \ \forall t \in \mathbb{T}(F) \,\}$$

is non-trivial. Note that all non-trivial weight subspaces are one dimensional and there is a direct sum decomposition $\mathcal{L} = \bigoplus_{\text{weights } \alpha} \mathcal{L}_\alpha$ over F.

We denote the set of all non-zero weights by $\Phi(G, \mathbb{T})$.

2.3.3. We define the Weyl group $W(G)$ of G to be the quotient

$$W(G) = N_G(\mathbb{T})/\mathbb{T},$$

where $N_G(\mathbb{T})$ is the normalizer of $\mathbb{T}$ in G. It is a *constant* finite group scheme which acts on $\mathbb{T}$ by the conjugation, i.e. $w(t) := \bar{w} \cdot t \cdot \bar{w}^{-1}$, where $\bar{w}$ is a representative of w in $N_G(\mathbb{T})$. And it also acts on $\mathbb{T}^*$ and $\mathbb{T}_*$ by

$$w(\chi)(t) \;:=\; \chi(w^{-1}(t)) \text{ and } w(\lambda)(\xi) \;:=\; w(\lambda(\xi)) \text{ , where } w \in W, \chi \in \mathbb{T}^*, \lambda \in \mathbb{T}_*.$$

By definition $W(G)$ permutes elements of $\Phi(G, \mathbb{T}) \subset \mathbb{T}^*$ and preserves the pairing $\langle -, - \rangle$, i.e. $\langle w(\chi), w(\lambda) \rangle \;=\; \langle \chi, \lambda \rangle$.

2.3.4. Consider a vector space $\mathbb{T}^*_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{T}^*$ with a scalar product ρ. Define a new scalar product $(-,-)$ as

$$(x,y) := \sum_{w \in W(G)} \rho(w(x), w(y)),$$

where the action of $W(G)$ on $\mathbb{T}^*$ is as above. Then the set $\Phi(G,\mathbb{T})$ forms a root system in $\mathbb{T}^*_{\mathbb{R}}$ with respect to $(-,-)$. It can be shown that $\Phi(G,\mathbb{T})$ does not depend on the choice of the split torus $\mathbb{T}$. This justifies to call $\Phi(G) := \Phi(G,\mathbb{T})$ a *root system* of G.

2.3.5. One of the main observations concerning $W(G)$ is that it is isomorphic to the Weyl group of the root system $\Phi(G)$ of G, i.e.

$$W(G) \simeq W(\Phi(G)).$$

2.3.6. As for lattices, consider the root lattice Λ_r and the weight lattice Λ_w of the root system $\Phi(G)$. Then we have $\Lambda_r \subseteq \mathbb{T}^* \subseteq \Lambda_w$ and the respective dual root system $\Phi(G)^\vee$ can be identified with a subset of $\mathbb{T}_*$ in such a way that $\alpha^\vee(\beta) = \langle \beta, \alpha^\vee \rangle$.

2.4 Classification of split groups

It turns out that a root system together with an intermediate lattice subgroup classifies all split groups over a field F.

2.4.1. Namely, for any root system Φ and any (additive) subgroup Λ, $\Lambda_r \subseteq \Lambda \subseteq \Lambda_w$, there exists a split group G over F such that $(\Phi(G), \mathbb{T}^*) \simeq (\Phi, \Lambda)$. Moreover, two split groups G_1 and G_2 are isomorphic over F if and only if $(\Phi(G_1), \mathbb{T}_1^*)$ and $(\Phi(G_2), \mathbb{T}_2^*)$ are isomorphic.

2.4.2. Observe that a split group G is simple over F if and only if the respective root system $\Phi(G)$ is irreducible. If $\mathbb{T}^* = \Lambda_r$ then G is called *adjoint*, and if $\mathbb{T}^* = \Lambda_w$ then G is called *simply connected*.

2.4.3. In general, a surjective morphism $\varphi\colon H \to G$ of group schemes over F is called a (central) *isogeny* if for any F-algebra R the kernel of $\varphi(R)\colon H(R) \to G(R)$ is finite and central in $H(R)$. Two group schemes G, G' over F are called *isogeneous* if there is a group scheme H over F and two isogenies $\varphi\colon H \to G$ and $\varphi'\colon H \to G'$.

Note that two split algebraic groups G, G' over F are isogeneous if and only if their root systems are isomorphic, i.e. they have the same Dynkin diagrams. Moreover, all isogenies $G \to G'$ with fixed G are uniquely determined by subgroups of the quotient $\mathbb{T}^*/\Lambda_r$. In particular, for any group G there exists a simply connected group G^{sc} and an adjoint group G^{ad} together with two isogenies $G^{sc} \to G \to G^{ad}$.

2.4.4 Example. We now provide a list of all split simple groups of classical types:

A_n**:** Here $\Lambda_w/\Lambda_r \simeq \mathbb{Z}/(n+1)\mathbb{Z}$ and $G = \mathrm{SL}_{n+1}/\mu_l$, where l divides $n+1$. For $l = n+1$ we obtain the projective linear group $G^{ad} = \mathrm{PGL}_{n+1}$ (adjoint), and for $l = 1$ the special linear group $G^{sc} = \mathrm{SL}_{n+1}$ (simply connected).

B_n**:** In this case $\Lambda_w/\Lambda_r \simeq \mathbb{Z}/2\mathbb{Z}$ has only two subgroups. The trivial subgroup corresponds to the special orthogonal group O^+_{2n+1} (adjoint) while the whole group corresponds to the spinor group Spin_{2n+1} (simply connected).

C_n**:** As in the previous case $\Lambda_w/\Lambda_r \simeq \mathbb{Z}/2\mathbb{Z}$ and, therefore, there are two isogenies: the projective symplectic group PGSp_{2n} (adjoint) and the symplectic group Sp_{2n} (simply connected).

D_n**:** In this case Λ_w/Λ_r depends on n.

- If n is odd, then $\Lambda_w/\Lambda_r \simeq \mathbb{Z}/4\mathbb{Z}$ and the respective isogenies are given by the projective orthogonal group PGO^+_{2n} (adjoint), the special orthogonal group O^+_{2n} (corresponds to the subgroup $\mathbb{Z}/2\mathbb{Z}$) and the spinor group Spin_{2n} (simply connected).
- If n is even, then $\Lambda_w/\Lambda_r \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the respective isogenies are given by PGO^+_{2n} (adjoint), two semispinor groups $Ss^{\pm}_{2n}$ (corresponding to the subgroups $\mathbb{Z}/2\mathbb{Z}\times 1$ and $1\times\mathbb{Z}/2\mathbb{Z}$ respectively), the special orthogonal group O^+_{2n} (the diagonal subgroup $\mathbb{Z}/2\mathbb{Z}$) and the spinor group Spin_{2n} (simply connected).

2.4.5. A celebrated result due to Chevalley says that given an irreducible root system Φ and a lattice subgroup $\Lambda_r \subset \Lambda \subset \Lambda_w$ there exists a group scheme $\hat{G}$ defined over $\operatorname{Spec}\mathbb{Z}$ called a *Chevalley group* such that for any split simple linear algebraic group G over F with $(\Phi(G), \mathbb{T}^*) \simeq (\Phi, \Lambda)$ there is an isomorphism $G \simeq \hat{G} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} F$.

2.5 Projective homogeneous varieties

In the present section we study projective homogeneous G-varieties, where G is a split group over a field F. We start with the following general definition.

2.5.1. Let G be a group scheme over a field F. A variety X over F is called a *homogeneous* G-variety if there is a morphism $\rho\colon G \times X \to X$ of varieties over F such that

$$\rho(g \cdot f, x) = f(g(x)) \text{ for all } f\ , g \in G(F_a),\ x \in X(F_a),$$

where F_a denotes the algebraic closure of F, and the action of $G(F_a)$ on $X(F_a)$ is *transitive*, i.e. for any x, $y \in X(F_a)$ there exists $g \in G(F_a)$ such that $g\cdot x = y$, where $g \cdot x$ denotes $\rho(g, x)$.

From now on let G be a split group over F. We fix a split maximal torus $\mathbb{T} \subset G$ and the respective set of simple roots Π.

2.5.2. It is known that the set of isomorphism classes of *projective* homogeneous G-varieties is in one-to-one correspondence with the set of conjugacy classes of stabilizer subgroups of G. Namely, an F-point x of a projective homogeneous variety X corresponds to its stabilizer subgroup G_x and for any two F-points x and y of X we have $G_y = g^{-1}G_x g$, where $g \in G(F)$ is such that $g \cdot x = y$.

On the other hand this set of conjugacy classes is in one-to-one correspondence with the set of subsets of Π. Namely, given a subset $\Theta \subset \Pi$ we define the representative P_Θ of a conjugacy class as follows: We set P_Θ to be the subgroup of G generated by $\mathbb{T}$ and all unipotent subgroups corresponding to all positive roots and to all roots in the linear span of Θ with no Θ terms.

Here by the unipotent subgroup corresponding to α we mean the image of an embedding $x_\alpha\colon \mathbb{G}_a \hookrightarrow G$ such that $tx_\alpha(\xi)t^{-1} = x_\alpha(\alpha(t)\xi)$ for all $t \in \mathbb{T}(F)$, $\xi \in \mathbb{G}_m(F)$.

2.5.3. The representative P_Θ is called the *standard parabolic subgroup* of G. In particular, if $\Theta = \emptyset$, the respective $P_\emptyset$ is called the *Borel subgroup* of G. We have $\mathbb{T} \subset P_\emptyset \subset G$.

By $P_{i_1,\ldots,i_r}$ we denote the standard parabolic subgroup $P_{\Theta'}$ of the complementary subset $\Theta' = \Theta \setminus \{\alpha_{i_1}, \ldots, \alpha_{i_r}\}$.

2.5.4. Observe that if we choose another $\mathbb{T}'$ inside G, then the respective representatives P_Θ and P'_Θ will be conjugate to each other. Therefore, the correspondence above doesn't depend on the choice of a split maximal torus $\mathbb{T}$. Moreover, since any two isogeneous split groups have the same root systems, it doesn't depend on the isogeny class of G. In particular, when dealing with projective homogeneous G-varieties we may always assume that G is adjoint.

2.5.5. Summarizing the above discussion, all projective homogeneous G-varieties are classified (up to an isomorphism) by subsets Θ of the set of vertices of the Dynkin diagram of G. An isomorphism class of projective homogeneous G-varieties corresponding to a subset Θ will be denoted by $\mathcal{D}/P_\Theta$, where $\mathcal{D}$ is the type of the root system of G.

2.5.6 Example. We now provide basic examples of projective homogeneous G-varieties, where G is a split simple linear algebraic group over F (enumeration of roots follows 2.2.7):

A_n**:** We have $\mathrm{A}_n/P_1 \simeq \mathrm{A}_n/P_n \simeq \mathbb{P}^n$ and, more generally, $\mathrm{A}_n/P_i \simeq \mathrm{A}_n/P_{n-i} \simeq \mathrm{Gr}(i, n+1)$, where $\mathrm{Gr}(i, n+1)$ is the *Grassmannian* of i-dimensional linear subspaces in $\mathbb{A}^{n+1}$.

The variety $\mathrm{A}_n/P_{1,n}$ is called the *incidence variety*. A point on this variety is given by the pair (l, H), where l is a line and H is a hyperplane in $\mathbb{A}^{n+1}$ such that $l \subset H$. In geometric terms it is given by the equation $\sum_{i=0}^n x_i y_i = 0$ in $\mathbb{P}^n \times \mathbb{P}^n$, where x_i (resp. y_i) are the projective coordinates of the first (resp. second) factor.

Finally, $\mathrm{A}_n/P_\emptyset$ is isomorphic to the *variety of complete flags*: points are given by n-tuples of linear subspaces $(l_1, l_2, \ldots, l_n)$ in $\mathbb{A}^{n+1}$ such that $\dim l_i = i$ and $l_1 \subset l_2 \subset \ldots \subset l_n$.

B_n, D_n: The variety B_n/P_1 (resp. D_n/P_1) is isomorphic to a smooth projective *quadric* of dimension $2n-1$ (resp. $2n-2$).

The variety B_n/P_n (resp. D_n/P_n or D_n/P_{n-1}) is isomorphic to a (resp. a connected component of) *maximal orthogonal Grassmannian* that is a variety of maximal totally-isotropic linear subspaces in the quadratic space of rank $2n+1$ (resp. $2n$).

G_2, F_4 and E_6: The variety G_2/P_2 is isomorphic to a 5-dimensional smooth projective quadric and the variety G_2/P_1 is a 5-dimensional Fano variety. The variety E_6/P_6 is isomorphic to the so called *Cayley plane* $\mathbb{OP}^2$ that is the octonionic projective plane of dimension 16 (see [IM05]). The variety F_4/P_4 can be identified with a hyperplane section of E_6/P_6.

2.6 Twisted forms

In the present section we recall the notions of twisted forms of Chevalley groups and study the respective projective homogeneous varieties.

2.6.1. A group scheme G over F is called a *twisted form* of a linear algebraic group G' if there is an isomorphism of group schemes $G\times_F F_a \simeq G'\times_F F_a$ over the algebraic closure F_a. The general descent theory provides an isomorphism of pointed sets

$$\begin{array}{ccc} H^1_{et}(F,\mathrm{Aut}_F(G')) & \simeq & Twist_F(G') \\ \xi\in Z^1(F,\mathrm{Aut}_F(G')) & \mapsto & G={}_\xi G' \end{array}$$

between the set of cohomology classes of the first étale cohomology group of the automorphism group scheme $\mathrm{Aut}_F(G')$ and the set of twisted forms of G'.

Since any simple linear algebraic group G becomes split over F_a, it can be viewed as a twisted form of the respective Chevalley group $\hat{G}$, i.e. as an element of $H^1_{et}(F,\mathrm{Aut}_F(\hat{G}))$.

2.6.2. Consider an action of the adjoint group $\hat{G}^{ad}$ on $\hat{G}$ by means of inner automorphisms $\hat{G}^{ad}\to \mathrm{Aut}_F(\hat{G})$. It induces the map

$$Inn\colon H^1_{et}(F,\hat{G}^{ad})\to H^1_{et}(F,\mathrm{Aut}_F(\hat{G})).$$

A linear algebraic group G is called an *inner form* of $\hat{G}$ if it is a twisted form of $\hat{G}$ such that the corresponding cohomology class lies in the image of this map.

The isogeny $\hat{G}^{sc}\to\hat{G}^{ad}$, where $\hat{G}_{sc}$ is the simply connected group, induces the map

$$s\colon H^1_{et}(F,\hat{G}^{sc})\to H^1_{et}(F,\hat{G}^{ad}).$$

Twisted forms that correspond to cohomology classes in the image of the composition $Inn\circ s$ are called *strongly inner forms* of $\hat{G}$.

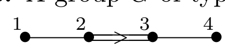
2.6.3. A twisted form of $\hat{G}$ is called an *outer form*, if the respective cohomology class is not in the image of the map Inn. Such cohomology classes correspond to non-trivial graph-automorphisms of the Dynkin diagram of $\hat{G}$. In particular, if $\mathcal{D}(\hat{G})$ has no non-trivial automorphisms, then the map Inn is surjective. This implies that outer forms exist only for groups of types A_n, D_n and E_6.

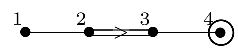
We will write the order of the respective graph-automorphism of $\mathcal{D}(\hat{G})$ as an upper-left index. For example, the notation ${}^1\mathrm{A}_n$ means an inner form of a group of type A_n, and ${}^6\mathrm{D}_4$ means an outer form of a group of type D_4 corresponding to an automorphism of order 6.

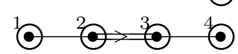
In general a twisted form G of a Chevalley group $\hat{G}$ is not necessary a split group. To measure how far is G from being split we use the Tits index of G.

2.6.4. Let G be an inner form of $\hat{G}$. The Tits index of G is a set of Dynkin digrams of $\hat{G}$ where certain vertices are circled. Each circled vertex corresponds to the copy of a split one dimensional torus $\mathbb{G}_m$ sitting inside G. For instance, if all vertices are circled then the group G is split. If non of them are circled, then the group G is called *anisotropic*. If at least one of the vertices is circled, then the group G is called *isotropic*. For each type of $\hat{G}$ the list of all possible Tits (diagrams) indices can be found in the table (see [Ti66]).

2.6.5 Example. A group G of type F_4 can have three possible Tits indices:

(i) 

(ii) 

(iii) 

where the index (i) corresponds to the anisotropic case, (ii) to the non-split isotropic case and (iii) to the split case.

2.6.6. Consider a projective homogeneous G-variety X over F. By the very definition the variety X becomes isomorphic over F_a to a projective homogeneous $\hat{G}$-variety corresponding to a standard parabolic subgroup of $\hat{G}$ and, therefore, to a subset Θ of the Dynkin diagram of $\hat{G}$. To stress this fact we will denote X as ${}_\xi\mathcal{D}/P_\Theta$, where $G = {}_\xi\hat{G}$ and $\mathcal{D}$ is the type of $\hat{G}$.

2.6.7. The projective homogeneous G-variety $X = {}_\xi\mathcal{D}/P_\Theta$ has a rational point if and only if all non-circled vertices of the Tits index of G belong to the subset Θ. Moreover, a projective homogeneous variety is rational if and only if it has a rational point.

2.6.8 Example. We now provide a list of examples of projective homogeneous G-varieties over F, where G is a twisted form of an adjoint split group $\hat{G}$.

A_n**:** Let $\hat{G} = \mathrm{PGL}_{n+1}$ be the projective linear group. The pointed set $H^1_{et}(F, \mathrm{PGL}_{n+1})$ is isomorphic to the set of isomorphism classes of central simple algebras A of degree $n+1$ over F. Moreover, cohomology classes in the image of

$$H^1_{et}(F, \mathrm{SL}_{n+1}/\mu_r) \to H^1_{et}(F, \mathrm{PGL}_{n+1}), \quad r \mid n+1,$$

correspond to central simple algebras A of index r. In particular, PGL_{n+1} has no non-trivial strongly inner forms.

For an algebra A the respective inner twisted form G is given by the automorphism group $\mathrm{PGL}_A = \mathrm{Aut}_F\, A$ of A (see Example 2.1.10). The respective projective homogeneous G-variety X can be identified with the variety of flags of (right) ideals in A. For instance, if $X = {}_A(\mathrm{A}_n/P_i)$, then $X \simeq \mathrm{SB}_i(A)$ is the generalized *Severi-Brauer variety* of ideals of reduced dimension i in A.

The set of outer forms of PGL_{n+1} is in one-to-one correspondence with the set of isomorphism classes of central simple algebras with unitary involutions. In this case any projective homogeneous varieties becomes isomorphic over F_a to the variety $\mathrm{A}_n/P_{i_1,\ldots,i_m}$, where $i_j = n+1-i_{m-j+1}$ for all $j = 1\ldots m$.

$\mathrm{B}_n, \mathrm{D}_n$**:** Let $\hat{G}$ be the projective orthogonal group. Depending on the type $\hat{G}$ is either O^+_{2n+1} or PGO^+_{2n}. We assume $char(F) \neq 2$.

In the first case (B_n) the pointed set $H^1_{et}(F, \mathrm{O}^+_{2n+1})$ is isomorphic to the set of isometry classes of quadratic forms q of rank $2n+1$. The respective inner form G is given by the special orthogonal group $\mathrm{O}^+(q)$ of q and the respective projective homogeneous G-variety $X = {}_q(\mathrm{B}_n/P_1)$ is the *projective quadric* given by the equation $q = 0$. Observe that for O^+_{2n+1} all twisted forms are inner.

In the second case (D_n) the pointed set $H^1_{et}(F, \mathrm{PGO}^+_{2n})$ is parametrised by the set of isomorphism classes of certain central simple algebras A with orthogonal involutions σ. For an inner form $G = {}_{(A,\sigma)}\mathrm{PGO}^+_{2n}$ the projective homogeneous G-variety $X = {}_{(A,\sigma)}\mathrm{D}_n/P_1$ is a *projective quadric* if the algebra A is split and is an *involution variety* if A is non-split. More precisely, if A is split, then the variety X is given by the equation $q = 0$, where q is the quadratic form of rank $2n$ associated to the involution σ. All outer forms of PGO^+_{2n} correspond to algebras with involutions which have non-trivial discriminant.

Pfister case: An important example of a twisted inner form of the projective orthogonal group $\mathrm{PGO}^+_{2^n}$ (resp. $\mathrm{O}^+_{2^n+1}$) is given by the so called *Pfister* quadratic form ϕ (resp. its maximal neighbor). By an n-fold Pfister form ϕ we call the tensor product of n binary quadratic forms $\phi = \bigotimes_{i=1}^n \langle 1, -a_i\rangle$, $a_i \in F$, where the notation $\langle 1, -a_i\rangle$ stands for the form $x^2 - a_i y^2$.

Chapter 3

Chow motives and Rost nilpotence

3.1 Chow groups

In the present section we introduce the notion of a Chow ring of an algebraic variety X and provide some of its properties. We follow the notation from the book [Fu98].

3.1.1. By an algebraic variety over a field F we mean a reduced scheme of finite type over a field F. By an algebraic cycle of codimension m on a variety X we mean a finite formal sum $\sum_V n_V[V]$ of irreducible subvarieties of codimension m of X taken with integral coefficients. The group (with respect to the addition) of algebraic cycles of codimension m on X modulo the rational equivalence relation is called the Chow group of codimension m cycles on X and is denoted by $\mathrm{CH}^m(X)$. We will also use the dimension (low index) notation meaning that $\mathrm{CH}^m(X_i) = \mathrm{CH}_{\dim X_i - m}(X_i)$ for each irreducible component X_i of X. Replacing the integral coefficient ring by a commutative ring Λ we obtain the Chow group with coefficients in Λ denoted by $\mathrm{CH}^m(X;\Lambda)$.

An equivalent definition of $\mathrm{CH}(X)$ can be given using the Gersten complex for the K-theory. Namely, we define $\mathrm{CH}^m(X)$ to be the cokernel of the residue map d_X

$$\mathrm{CH}^m(X) = \mathrm{coker}\Big(\coprod_{x\in X^{(m-1)}} K_1(F(x)) \xrightarrow{d_X} \coprod_{x\in X^{(m)}} K_0(F(x))\Big),$$

where $x \in X^{(m)}$ denotes a point of codimension m in X, $F(x)$ denotes its residue field, $K_1(F(x))$ is equal to the group of invertible elements $F(x)^\times$ and $K_0(F(x)) = \mathbb{Z}$.

3.1.2. The assignment $X \mapsto \mathrm{CH}^m(X)$ satisfies the following two important properties:

- Given a projective morphism between two varieties $f\colon Y \to X$ there is an induced group homomorphism which preserves the dimension of cycles $f_*\colon \mathrm{CH}_m(Y) \to \mathrm{CH}_m(X)$ called the push-forward.

- Given a flat morphism $f\colon Y \to X$ there is an induced group homomorphism in the opposite direction which preserves the codimension of cycles $f^*\colon \mathrm{CH}^m(X) \to \mathrm{CH}^m(Y)$ called the pull-back. Moreover, according to [Fu98, §6] if X and Y are smooth then the pull-back exists for any morphism f.

The latter means that the assignment $X \mapsto \mathrm{CH}^m(X)$ can be viewed as a contravariant functor with respect to pull-backs from the category of smooth varieties over a field F to the category of graded abelian groups.

3.1.3. By [Fu98] the Chow group $\mathrm{CH}(X)$ has a product structure, which turns $\mathrm{CH}(X)$ into a graded commutative ring. This (intersection) product has the following properties:

- It is preserved by pull-backs, i.e. f^* is a ring homomorhism;

- There is the projection formula

$$f_*(f^*(\alpha)\cdot\beta) = \alpha \cdot f_*(\beta), \text{ where } \alpha \in \mathrm{CH}(X),\ \beta \in \mathrm{CH}(Y).$$

3.1.4. Let $i\colon Z \hookrightarrow X$ be a closed subvariety and $j\colon U = X \setminus Z \subset X$, be its open complement. Then there is an exact sequence of Chow groups preserving the low indices

$$\mathrm{CH}_m(Z) \xrightarrow{i_*} \mathrm{CH}_m(X) \xrightarrow{j^*} \mathrm{CH}_m(U) \to 0$$

called the localization sequence (see [Fu98, Prop. 1.8]). In particular, the induced pull-back $f^*\colon \mathrm{CH}(X) \to \mathrm{CH}(\operatorname{Spec} F(X))$ is surjective, where $\operatorname{Spec} F(X)$ is the generic point of X.

Let Y be a vector bundle over X with the projection $f\colon Y \to X$. Then the induced pull-back $f^*\colon \mathrm{CH}(X) \xrightarrow{\cong} \mathrm{CH}(Y)$ is an isomorphism (see [Fu98, Thm. 3.3.(a)]). This property is usually called the homotopy invariance for Chow groups.

3.1.5. Assume that the Chow group $\mathrm{CH}(X)$ is a free $\mathbb{Z}$-module. Then there is the Künneth decomposition $\mathrm{CH}(X \times X) = \mathrm{CH}(X) \otimes_{\mathbb{Z}} \mathrm{CH}(X)$ and the Poincaré duality (see [KM06, Rem. 5.6]). The latter means that for a given $\mathbb{Z}$-basis of $\mathrm{CH}(X)$ there is a dual one with respect to the pairing $\alpha \otimes \beta \mapsto \deg(\alpha \cdot \beta)$, where $\deg\colon \mathrm{CH}(X) \to \mathbb{Z}$ denotes the push-forward p_* induced by the structure morphism $p\colon X \to \operatorname{Spec} F$ and is called the degree map.

3.1.6. Consider the Chern character $ch\colon K^o(X) \to \mathrm{CH}(X) \otimes \mathbb{Q}$, where $K^o(X)$ is the Grothendieck group of vector bundles on X (see [Fu98, (15.1.2)]). By definition it respects pull-backs and by the Riemann-Roch theorem (see [Fu98, Example 15.2.16.(b)]) it induces an isomorphism $K^o(X) \otimes \mathbb{Q} \to \mathrm{CH}(X) \otimes \mathbb{Q}$.

3.2 Chow motives with coefficients

Let X be a variety over a field F. We say X is *essentially smooth* over F if it is an inverse limit of smooth varieties X_i over F taken with respect to open embeddings. Let $\mathrm{CH}^m(X;\Lambda) = \mathrm{CH}^m(X) \otimes_{\mathbb{Z}} \Lambda$ denote the Chow group of codimension m cycles on X with coefficients in a commutative ring Λ. If X is essentially smooth, then $\mathrm{CH}^m(X;\Lambda) = \varinjlim \mathrm{CH}^m(X_i;\Lambda)$, where the limit is taken with respect to the pull-backs induced by open embeddings.

In the present section we introduce the category of *Chow motives* over an essentially smooth variety X with Λ-coefficients. Our arguments follow the papers [Ma68] and [VZ08].

3.2.1. First, we define the *category of correspondences* $\mathcal{C}(X;\Lambda)$. The objects of $\mathcal{C}(X;\Lambda)$ are smooth projective maps $Y \to X$. The morphisms are given by

$$\mathrm{Hom}([Y \to X],[Z \to X]) = \oplus_i \mathrm{CH}^{\dim(Z_i/X)}(Y \times_X Z_i;\Lambda),$$

where the sum is taken over all irreducible components Z_i of Z of relative dimensions $\dim(Z_i/X)$. The composition of two morphisms is given by the usual correspondence product

$$\psi \circ \phi = (p_{Y,T})_*\big((p_{Y,Z})^*(\phi) \cdot (p_{Z,T})^*(\psi)\big),$$

where $\phi \in \mathrm{Hom}([Y \to X],[Z \to X])$, $\psi \in \mathrm{Hom}([Z \to X],[T \to X])$ and $p_{Y,T}$, $p_{Y,Z}$, $p_{Z,T}$ are projections $Y \times_X Z \times_X T \to Y \times_X T$, $Y \times_X Z$, $Z \times_X T$. The category $\mathcal{C}(X;\Lambda)$ is a tensor additive category, where the direct sum is given by $[Y \to X] \oplus [Z \to X] := [Y \coprod Z \to X]$ and the tensor product by $[Y \to X] \otimes [Z \to X] := [Y \times_X Z \to X]$ (cf. [Ma68, §2-4]). As usual we denote by $\phi^t \in \mathrm{CH}(Z \times_X Y;\Lambda)$ the transposition of a cycle $\phi \in \mathrm{CH}(Y \times_X Z;\Lambda)$.

3.2.2 Example. Assume we have the Künneth decomposition, i.e., the map

$$\mathrm{CH}(Y;\Lambda) \otimes_{\mathrm{CH}(X;\Lambda)} \mathrm{CH}(Z;\Lambda) \xrightarrow{p_Y^*(-)\cdot p_Z^*(-)} \mathrm{CH}(Y \times_X Z;\Lambda)$$

is an isomorphism. Then the respective correspondence product in $\mathcal{C}(X;\Lambda)$ is given by the formula

$$(\alpha_1 \otimes \beta_1) \circ (\alpha_2 \otimes \beta_2) = (p_Z)_*(\alpha_1\beta_2) \cdot (\alpha_2 \otimes \beta_1).$$

3.2.3. The category of *effective* Chow motives $\mathit{Chow}^{eff}(X;\Lambda)$ can be defined as the pseudo-abelian completion of $\mathcal{C}(X;\Lambda)$. Namely, the objects are pairs (U,ρ), where U is an object of $\mathcal{C}(X;\Lambda)$ and $\rho \in \mathrm{End}_{\mathcal{C}(X;\Lambda)}(U)$ is a projector, i.e. $\rho \circ \rho = \rho$. The morphisms between (U_1,ρ_1) and (U_2,ρ_2) are given by the group $\rho_2 \circ \mathrm{Hom}_{\mathcal{C}(X;\Lambda)}(U_1,U_2) \circ \rho_1$. The composition of morphisms is induced by the correspondence product. In the case $X = \mathrm{Spec}(F)$ and $\Lambda = \mathbb{Z}$ we obtain the usual category of effective Chow motives over F with integral coefficients (cf. [Ma68, §5]).

3.2.4 Example. Consider the projective line $\mathbb{P}^1$ over F. The projector $\rho = [\mathrm{Spec}(F) \times \mathbb{P}^1] \in \mathrm{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1)$ defines an object $(\mathbb{P}^1,\rho)$ in $\mathit{Chow}^{eff}(F,\mathbb{Z})$ called the *Tate motive* over F and denoted by $\mathbb{Z}\{1\}$ (cf. [Ma68, §6]).

3.2.5. We have two types of restriction functors.

1) For any morphism $f\colon X_1 \to X_2$ of essentially smooth varieties we have a tensor additive functor

$$res_{X_2/X_1}\colon \mathcal{C}(X_2;\Lambda) \to \mathcal{C}(X_1;\Lambda)$$

given on the objects by $[Y_2 \to X_2] \mapsto [Y_2 \times_{X_2} X_1 \to X_1]$ and on the morphisms by $\phi \mapsto (\mathrm{id} \times f)^*(\phi)$, where $\mathrm{id} \times f : (Y_2 \times_{X_2} Z_2) \times_{X_2} X_1 \to Y_2 \times_{X_2} Z_2$ is the natural map. It induces a functor on pseudo-abelian completions

$$res_{X_2/X_1}\colon \mathit{Chow}^{eff}(X_2;\Lambda) \to \mathit{Chow}^{eff}(X_1;\Lambda).$$

2) For any homomorphism of commutative rings $h\colon \Lambda \to \Lambda'$ we have a tensor additive functor

$$res_{\Lambda'/\Lambda}\colon \mathcal{C}(X,\Lambda) \to \mathcal{C}(X;\Lambda')$$

which is identical on objects and is given by $\mathrm{id}\otimes h\colon\ \mathrm{CH}(Y\times_X Z;\Lambda) \to \mathrm{CH}(Y\times_X Z;\Lambda')$ on morphisms. Again, it induces a functor on pseudo-abelian completions

$$res_{\Lambda'/\Lambda}\colon \mathit{Chow}^{eff}(X;\Lambda) \to \mathit{Chow}^{eff}(X;\Lambda').$$

Observe that the functor $res_{\Lambda'/\Lambda}$ commutes with res_{X_2/X_1}. We denote by $res_{X_2/X_1,\Lambda'/\Lambda}$ the composite $res_{X_2/X_1} \circ res_{\Lambda'/\Lambda}$. To simplify the notation we ommit X_1 (resp. Λ), if $X_1 = \operatorname{Spec} F$ (resp. $\Lambda = \mathbb{Z}$).

3.2.6. Let $f\colon X \to \operatorname{Spec} F$ and $h\colon \mathbb{Z} \to \Lambda$ be the structure maps. Then $res_{X,\Lambda}\colon \mathit{Chow}^{eff}(F;\mathbb{Z}) \to \mathit{Chow}^{eff}(X;\Lambda)$. Given a motive N over F we denote by $N_{X,\Lambda}$ its image $res_{X,\Lambda}(N)$ in $\mathit{Chow}^{eff}(X;\Lambda)$. The image $\mathbb{Z}\{1\}_{X,\Lambda}$ of the Tate motive is denoted by T and is called the *Tate motive over* X. Let M be a motive from $\mathit{Chow}^{eff}(X;\Lambda)$ and $l \geq 0$ be an integer. The tensor product $M \otimes T^{\otimes l}$ is denoted by $M\{l\}$ and is called the *twist* of M.

The same arguments as in the proof of [Ma68, Lemma of §8] show that for any motives U and V from $\mathit{Chow}^{eff}(X;\Lambda)$ and $l \geq 0$ the natural map

$$\mathrm{Hom}_{\mathit{Chow}^{eff}(X;\Lambda)}(U,V) \to \mathrm{Hom}_{\mathit{Chow}^{eff}(X;\Lambda)}(U\{l\},V\{l\}) \tag{3.1}$$

given by $\phi \mapsto \phi \otimes \mathrm{id}_T$ is an isomorphism.

3.2.7. We define the category $\mathit{Chow}(X;\Lambda)$ of Chow motives over X with Λ-coefficients as follows. The objects are pairs (U,l), where U is an object of $\mathit{Chow}^{eff}(X;\Lambda)$ and l is an integer. The morphisms are given by

$$\mathrm{Hom}((U,l),(V,m)) := \lim_{N\to+\infty} \mathrm{Hom}_{\mathit{Chow}^{eff}(X;A)}(U\{N+l\},V\{N+m\}).$$

This is again a tensor additive category, where the sum and the product are given by

$$(U,l)\oplus(V,m) := (U\{l-n\}\oplus V\{m-n\},n),\ \text{where } n=\min(l,m),$$

$$(U,l)\otimes(V,m):=(U\otimes V,l+m).$$

Observe that the Tate motive T is isomorphic to $([\mathrm{id}:X\to X],1)$ and, hence, it is invertible in $(\mathit{Chow}(X;\Lambda),\otimes)$. Moreover, we can say that $\mathit{Chow}(X;\Lambda)$ is obtained from $\mathit{Chow}^{eff}(X;\Lambda)$ by inverting T (cf. [Ma68, §8]).

According to (3.1) the natural functor $\mathit{Chow}^{eff}(X;\Lambda)\to\mathit{Chow}(X;\Lambda)$ given by $U\mapsto(U,0)$ is fully faithful and the restriction $\mathit{res}_{X,\Lambda}$ descend to the respective functor $\mathit{res}_{X,\Lambda}\colon \mathit{Chow}(F)\to\mathit{Chow}(X;\Lambda)$.

3.2.8. For a smooth projective morphism $Y\to X$ we denote by $M(Y\to X)$ its effective motive $([Y\to X],\mathrm{id})$ considered as an object of $\mathit{Chow}(X;\Lambda)$. If $X=\operatorname{Spec}F$, then we denote the motive $M(Y\to X)$ simply by $M(Y;\Lambda)$. If $\Lambda=\mathbb{Z}$, then we denote $M(Y;\mathbb{Z})$ by $M(Y)$ having in mind the integer coefficients. By definition there is a natural identification

$$\mathrm{Hom}_{\mathit{Chow}(X;\Lambda)}(M(Y\to X)\{i\},M(Z\to X)\{j\})=\mathrm{CH}^{\dim(Z/X)+j-i}(Y\times_X Z;\Lambda).$$

3.2.9. Let M be an object of $\mathit{Chow}(X;\Lambda)$. We define the Chow group with low index $\mathrm{CH}_m(M)$ of M as

$$\mathrm{CH}_m(M):=\mathrm{Hom}_{\mathit{Chow}(X;\Lambda)}(T\{m\},M)$$

and the Chow group with upper index $\mathrm{CH}^m(M)$ as

$$\mathrm{CH}^m(M):=\mathrm{Hom}_{\mathit{Chow}(X;\Lambda)}(M,T\{m\}).$$

Observe that if $M=M(Y\to X)$, then we obtain the usual Chow groups $\mathrm{CH}^{\dim(Y/X)-m}(Y;\Lambda)$ and $\mathrm{CH}^m(Y;\Lambda)$ of a variety Y. A composite with a morphism $f:M\to N$ induces a homomorphism between the Chow groups $R_m(f)\colon \mathrm{CH}_m(M)\to\mathrm{CH}_m(N)$ and $R^m(f)\colon \mathrm{CH}^m(N)\to\mathrm{CH}^m(M)$ called the *realization map*.

3.2.10. Assume that a motive M is a direct sum of twisted Tate motives. In this case its Chow group $\mathrm{CH}(M)$ is a free abelian group. We define its *Poincaré polynomial* as

$$P(M,t)=\sum_{i\geq 0}a_it^i,$$

where a_i is the rank of $\mathrm{CH}_i(M)$.

3.3 Splitting fields and rational cycles

3.3.1 Definition. Let M be an object of $\mathit{Chow}(F,\Lambda)$. We say that M is Λ-*split* over a field extension L/F if $M_L=\mathit{res}_L(M)$ is isomorphic to a direct sum of twisted Tate motives over L. Such a field L will be called a Λ-*splitting* field of M. In particular, we say that a smooth projective variety X is Λ-split over a field extension L/F if and only if the motive $M(X;\Lambda)$ is split over L.

Clearly, if M is $\mathbb{Z}$-split over L, then it is Λ-split over F for any commutative ring Λ. We will say that M is split over L (resp. L is a splitting field) if it is $\mathbb{Z}$-split over L (resp. L is a $\mathbb{Z}$-splitting field). We will denote a motive M (resp. a variety X) considered over its Λ-splitting field by $\overline{M}$ (resp. by $\overline{X}$).

3.3.2 Definition. Given a motive M over a field F and a field extension L/F we say a cycle in $\mathrm{CH}(M_L)$ is *rational* if it is in the image of the restriction map $res_L\colon \mathrm{CH}(M) \to \mathrm{CH}(M_L)$.

Observe that the rationality of cycles is preserved by push-forward and pull-back maps. It also respects addition, intersection and correspondence product of cycles.

3.3.3. Assume that a motive $M \in \mathit{Chow}(F, \Lambda)$ has a Λ-splitting field L. Then the subgroup of rational cycles in $\mathrm{CH}(M_L)$ will be denoted by $\overline{\mathrm{CH}}(M)$ (cf. [KM06, 1.2]). If L' is another splitting field of X, then there is a chain of canonical isomorphisms $\mathrm{CH}(M_L) \simeq \mathrm{CH}(M_{LL'}) \simeq \mathrm{CH}(M_{L'})$, where LL' is the composite of L and L'. Hence, the groups $\overline{\mathrm{CH}}(M)$ and $\mathrm{CH}(\overline{M})$ don't depend on the choice of L.

3.3.4 Lemma. *Let X and Y be smooth projective varieties over a field F such that Y is irreducible, $F(Y)$ is a Λ-splitting field of X and Y has a Λ-splitting field. For any r consider the projection in the Künneth decomposition*

$$\mathrm{pr}_0\colon \mathrm{CH}^r(\overline{X} \times \overline{Y}; \Lambda) = \bigoplus_{i=0}^{r} \mathrm{CH}^{r-i}(\overline{X}; \Lambda) \otimes \mathrm{CH}^i(\overline{Y}; \Lambda) \to \mathrm{CH}^r(\overline{X}; \Lambda).$$

Then for any $\rho \in \mathrm{CH}^r(\overline{X}; \Lambda)$ we have $\mathrm{pr}_0^{-1}(\rho) \cap \overline{\mathrm{CH}^r}(X \times Y; \Lambda) \neq \emptyset$.

Proof. Let L be a common Λ-splitting field of X and Y. The lemma follows from the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^r(X \times_F Y; \Lambda) \arrow[r, "res_{L/F}"] \arrow[d, twoheadrightarrow] & \mathrm{CH}^r(X_L \times_L Y_L; \Lambda) \arrow[d, twoheadrightarrow] \arrow[dr, "\mathrm{pr}_0"] & \\
\mathrm{CH}^r(X_{F(Y)}; \Lambda) \arrow[r, "\simeq"] & \mathrm{CH}^r((X_L)_{L(Y_L)}; \Lambda) & \mathrm{CH}^r(X_L; \Lambda) \arrow[l, "\simeq"']
\end{tikzcd}$$

where the left square is obtained by taking the generic fiber of the base change morphism $X_L \to X$; the vertical arrows are taken from the localization sequence for Chow groups and, hence, are surjective; and the bottom horizontal maps are isomorphisms since L is a Λ-splitting field. □

3.3.5 Definition. We say a motive M is *generically* Λ*-split* if there exists a smooth projective variety X over F and an integer l such that M is a direct summand of the twisted motive $M(X; \Lambda)\{l\}$ of X and M is split over $F(X)$. In particular, a smooth projective variety X is called *generically* Λ*-split* if its Chow motive $M(X; \Lambda)$ is split over $F(X)$.

3.3.6 Example. The classical examples of generically ($\mathbb{Z}$-)split varieties are Severi-Brauer varieties, Pfister quadrics and (connected components of) maximal orthogonal Grassmannians.

3.3.7 Example. More generally, let P_Θ be the standard parabolic subgroup of a split simple group $\hat{G}$ corresponding to a subset Θ of the respective Dynkin diagram $\mathcal{D}$. Consider a twisted inner form $G = {}_\xi\hat{G}$, where $\xi \in Z^1(F,\hat{G})$, and let $X = {}_\xi(\hat{G}/P_\Theta)$ be the respective projective homogeneous G-variety. Let q denote the degree of a splitting field of G and let d be the index of the associated Tits algebra (see [Ti66, Table II]). For groups of type D_n, we set d to be the index of the Tits algebra associated with the vector representation.

Analyzing Tits indices of G we see that G becomes split over $F(X)$ and, hence, X is generically split, if the subset $\mathcal{D} \setminus \Theta$ contains one of the following vertices k (cf. [KR94, §7]):

$\hat{G}$	${}^1\mathrm{A}_n$	B_n	C_n	${}^1\mathrm{D}_n$	G_2
k	$\gcd(k,d)=1$	$k=n$; any k in the Pfister case	k is odd;	$k=n-1,n$ if $d=1$; any k in the Pfister case	any

$\hat{G}$	F_4	${}^1\mathrm{E}_6$	E_7	E_8
k	$k=1,2,3$; any k if $q=3$	$k=3,5$; $k=2,4$ if $d=1$; $k=1,6$ if q is odd	$k=2,5$; $k=3,4$ if $d=1$; $k\neq 7$ if $q=3$	$k=2,3,4,5$; any k if $q=5$

(here by the Pfister case we mean the case when the cocycle ξ corresponds to a Pfister form or its maximal neighbor)

3.4 Rost nilpotence

We will extensively use the following version of the Rost nilpotence (cf. [Ro98, Prop. 9] and [VZ08, Prop. 3.1])

3.4.1 Proposition. *Let N be a generically Λ-split motive over a field F. Then for any field extension E/F the kernel of the restriction*

$$res_{E/F} : \mathrm{End}_F(N) \to \mathrm{End}_E(N_E)$$

consists of nilpotents.

To simplify the notation we denote by $\mathrm{End}_X(M)$ the endomorphism group $\mathrm{Hom}_{Chow(X;\Lambda)}(M,M)$, where M is a motive over a variety X.

Proof. Recall that a motive N over F is generically split if there exists a smooth projective variety X and $l \in \mathbb{Z}$ such that N is a direct summand of $M(X;\Lambda)\{l\}$ and $N_K = res_{K/F}(N)$ is split, where $K = F(X)$ denotes the function field of X.

We may assume that N is a direct summand of $M(X;\Lambda)$ (that is, $l = 0$). Since for a split motive M and a field extension E/L, the map $\mathrm{End}_L(M_L) \to \mathrm{End}_E(M_E)$ is an isomorphism, we may assume that $E = K$.

Consider the composite of ring homomorphisms

$$res_{K/F}\colon\ \mathrm{End}_F(N) \xrightarrow{res_{X/F}} \mathrm{End}_X(N_X) \xrightarrow{res_{K/X}} \mathrm{End}_K(N_K),$$

where the last map is induced by passing to the generic point $\operatorname{Spec} K \to X$.

Observe that $\operatorname{End}_K(N_K) = \varinjlim \operatorname{End}_U(N_U)$, where the limit is taken over all open subvarieties $U \subset X$. Then $\ker(res_{K/X}) = \cup_U \ker(res_{U/X})$ and by Lemma 3.4.2 the kernel of $res_{K/X}$ consists of nilpotents.

On the other hand, the map $res_{X/F}$ is injective. Indeed, since N is a direct summand of $M(X;\Lambda)$, $\operatorname{End}_F(N)$ is a subring of $\operatorname{End}_F(M(X;\Lambda))$ and $\operatorname{End}_X(N_X)$ is a subring of $\operatorname{End}_X(M(X;\Lambda)_X)$. So, it is sufficient to prove the injectivity for the case $N = M(X;\Lambda)$. The restriction $res_{X/F}\colon \operatorname{End}_F(M(X;\Lambda)) \to \operatorname{End}_X(M(X;\Lambda)_X)$ coincides with the pull-back $\pi_{1,2}^*\colon \operatorname{CH}(X\times X;\Lambda) \to \operatorname{CH}(X\times X\times X;\Lambda)$ induced by the projection on the first two coordinates. And $\pi_{1,2}^*$ splits by $(\mathrm{id}_X \times \Delta_X)^*\colon \operatorname{CH}(X\times X\times X;\Lambda) \to \operatorname{CH}(X\times X;\Lambda)$, where $\Delta_X\colon X \to X\times X$ is the diagonal. The proposition is proven. □

3.4.2 Lemma. *Let X be a smooth projective variety over F and Λ be a commutative ring. Let $U \subset X$ be an open embedding. Then for any motive M from $Chow(X;\Lambda)$ the kernel of the restriction map*

$$res_{U/X} : \operatorname{End}_X(M) \to \operatorname{End}_U(M_U)$$

consists of nilpotents.

Proof. If M is a direct summand of $[Y \to X]\{i\}$, then $\operatorname{End}_X(M)$ is a subring of $\operatorname{End}_X(M(Y \to X))$ and it is sufficient to study the case $M = M(Y \to X)$. Recall that $\operatorname{End}_X(M(Y \to X)) = \operatorname{CH}^{\dim(Y)-\dim(X)}(Y \times_X Y;\Lambda)$.

Let ϕ be an element from the kernel of $res_{U/X}$. Let $j\colon Z \to X$ be the reduced closed complement to U in X. Then by the localization sequence for Chow groups the cycle ϕ belongs to the image of the induced push-forward

$$(\mathrm{id}_{(Y\times_X Y)} \times j)_*\colon \operatorname{CH}((Y\times_X Y)\times_X Z;\Lambda) \to \operatorname{CH}(Y\times_X Y;\Lambda).$$

By additivity of $Chow(X;\Lambda)$ we may assume Z is irreducible.

Set $\operatorname{codim}(Z) := \dim(X) - \dim(Z)$ and $d := [\frac{\dim(X)}{\operatorname{codim}(Z)}]+1$. We claim that the d-th power $\phi^{\circ d}$ of ϕ taken with respect to the correspondence product is trivial. Indeed, $\phi^{\circ d} = (\pi_{1,d+1})_*(\phi_1\cdot\phi_2\cdot\ldots\cdot\phi_d)$, where $\phi_i = \pi_{i,i+1}^*(\phi)$ and the map $\pi_{i,i'}\colon Y^{\times(d+1)} \to Y\times_X Y$ is the projection on the i-th and i'-th components. Since $\pi_{i,i'}^* \circ (\mathrm{id}_{(Y\times_X Y)} \times j)_*$ coincides with $(\mathrm{id}_{Y^{\times(d+1)}} \times j)_* \circ (\pi_{i,i'} \times \mathrm{id}_Z)^*$, all cycles ϕ_i belong to the image of the push-forward

$$(\mathrm{id}_{Y^{\times(d+1)}} \times j)_*\colon \operatorname{CH}(Y^{\times(d+1)}\times_X Z) \to \operatorname{CH}(Y^{\times(d+1)}).$$

By [VZ08, Prop. 6.1] applied to the projection $Y^{\times(d+1)} \to X$ and the closed embedding $j\colon Z \hookrightarrow X$ we obtain that the product

$$\phi_1\cdot\ldots\cdot\phi_d \in \big((\mathrm{id}_{Y^{\times(d+1)}} \times j)_* \operatorname{CH}(Y^{\times(d+1)}\times_X Z)\big)^d$$

is trivial. Therefore, $\phi^{\circ d}$ is trivial as well. □

3.5 Lifting of coefficients

This section is devoted to lifting of idempotents and isomorphisms. In the exposition we follow [PSZ, §2]. First, we treat the case of general graded algebras. The main results here are Lemma 3.5.5 and Proposition 3.5.6. Then, assuming *Rost Nilpotence* 3.5.8 we provide conditions to lift motivic decompositions and isomorphisms (Theorem 3.5.19).

3.5.1. Let A^* be a $\mathbb{Z}$-graded ring. Assume we are given two orthogonal idempotents ϕ_i and ϕ_j in A^0 that is $\phi_i\phi_j = \phi_j\phi_i = 0$. We say an element θ_{ij} provides an *isomorphism* of degree d between idempotents ϕ_i and ϕ_j if $\theta_{ij} \in \phi_j A^{-d}\phi_i$ and there exists $\theta_{ji} \in \phi_i A^d \phi_j$ such that $\theta_{ij}\theta_{ji} = \phi_j$ and $\theta_{ji}\theta_{ij} = \phi_i$.

3.5.2 Example. Let X be a smooth projective irreducible variety over a field F and let $\mathrm{CH}^*(X \times X; \Lambda)$ be the Chow ring with coefficients in a commutative ring Λ. Set $A^* = \mathrm{End}^*(M(X; \Lambda))$, where

$$\mathrm{End}^{-i}(M(X;\Lambda)) = \mathrm{CH}^{\dim X - i}(X \times X; \Lambda) = \mathrm{CH}_{\dim X + i}(X \times X; \Lambda), \quad i \in \mathbb{Z}$$

and the multiplication is given by the correspondence product. By definition $\mathrm{End}^0(M(X;\Lambda))$ is the ring of endomorphisms of the motive $M(X;\Lambda)$ (see 3.2.8). Note that a direct summand of $M(X;\Lambda)$ can be identified with a pair (X, ϕ_i), where ϕ_i is an idempotent. Then an isomorphism θ_{ij} of degree d between ϕ_i and ϕ_j can be identified with an isomorphism between the motives (X, ϕ_i) and $(X, \phi_j)(d)$.

3.5.3. Let $f\colon A^* \to B^*$ be a homomorphism of $\mathbb{Z}$-graded rings. We say that *f lifts decompositions* if given a family $\phi_i \in B^0$ of pair-wise orthogonal idempotents such that $\sum_i \phi_i = 1_B$, there exists a family of pair-wise orthogonal idempotents $\varphi_i \in A^0$ such that $\sum_i \varphi_i = 1_A$ and each $f(\varphi_i)$ is isomorphic to ϕ_i by means of an isomorphism of degree 0. We say *f lifts decompositions strictly* if, moreover, one can choose φ_i such that $f(\varphi_i) = \phi_i$.

We say *f lifts isomorphisms* if for any idempotents φ_1 and φ_2 in A^0 and any isomorphism θ_{12} of degree d between idempotents $f(\varphi_1)$ and $f(\varphi_2)$ in B^0 there exists an isomorphism ϑ_{12} of degree d between φ_1 and φ_2. We say *f lifts isomorphisms strictly* if, moreover, one can choose ϑ_{12} such that $f(\vartheta_{12}) = \theta_{12}$.

3.5.4. By definition we have the following properties of morphisms which lift decompositions and isomorphisms (strictly):

(i) Let $f\colon A^* \to B^*$ and $g\colon B^* \to C^*$ be two morphisms. If both f and g lift decompositions or isomorphisms (strictly), then so does the composite $g \circ f$.

(ii) If $g \circ f$ lifts decompositions (resp. isomorphisms) and g lifts isomorphisms, then f lifts decompositions (resp. isomorphisms).

(iii) Assume we are given a commutative diagram with $\ker f' \subset \mathrm{im} h$

$$\begin{tikzcd} A^* \arrow[r, two heads, "f"] \arrow[d, hook, "h"'] & B^* \arrow[d, hook, "h'"] \\ A'^* \arrow[r, two heads, "f'"] & B'^*. \end{tikzcd}$$

If f' lifts *decompositions strictly* (resp. *isomorphisms strictly*), then so does f.

We will use the following technical fact (for the proof see [PSZ, Lemma 2.5])

3.5.5 Lemma. *Let A, B be two rings, A^0, B^0 their subrings, $f^0\colon A^0 \to B^0$ a ring homomorphism and $f\colon A \to B$ a map of sets satisfying the following conditions:*

- *$f(\alpha)f(\beta)$ equals either $f(\alpha\beta)$ or 0 for all $\alpha, \beta \in A$;*
- *$f^0(\alpha)$ equals $f(\alpha)$ if $f(\alpha) \in B^0$ or 0 otherwise;*
- *$\ker f^0$ consists of nilpotent elements.*

Let φ_1 and φ_2 be two idempotents in A^0 and let ψ_{12}, ψ_{21} be elements in A such that $\psi_{12}A^0\psi_{21} \subset A^0$, $\psi_{21}A^0\psi_{12} \subset A^0$, $f(\psi_{21})f(\psi_{12}) = f(\varphi_1)$, $f(\psi_{12})f(\psi_{21}) = f(\varphi_2)$.

Then there exist elements $\vartheta_{12} \in \varphi_2 A^0\psi_{12}A^0\varphi_1$ and $\vartheta_{21} \in \varphi_1 A^0\psi_{21}A^0\varphi_2$ such that $\vartheta_{21}\vartheta_{12} = \varphi_1$, $\vartheta_{12}\vartheta_{21} = \varphi_2$, $f(\vartheta_{12}) = f(\varphi_2)f(\psi_{12}) = f(\psi_{12})f(\varphi_1)$, $f(\vartheta_{21}) = f(\varphi_1)f(\psi_{21}) = f(\psi_{21})f(\varphi_2)$.

3.5.6 Proposition. *Let $f\colon A^* \to B^*$ be a surjective homomorphism such that the kernel of the restriction of f to A^0 consists of nilpotent elements. Then f lifts decompositions and isomorphisms strictly.*

Proof. The fact that f lifts *decompositions strictly* follows from [AF92, Proposition 27.4].

Let φ_1 and φ_2 be two idempotents in A^0 and let θ_{12} be an isomorphism between $f(\varphi_1)$ and $f(\varphi_2)$. Let $\psi_{12} \in A$ (resp. ψ_{21}) be a homogeneous lifting of θ_{12} (resp. θ_{21}). The proposition follows now from Lemma 3.5.5. □

3.5.7 Corollary. *Let m be an integer and let $m = p_1^{n_1} \dots p_l^{n_l}$ be its prime factorization. Then the product of reduction maps*

$$\mathrm{End}^*(M(X;\mathbb{Z}/m)) \twoheadrightarrow \prod_{i=1}^{l} \mathrm{End}^*(M(X;\mathbb{Z}/p_i))$$

lifts decompositions and isomorphisms strictly.

Proof. We apply Proposition 3.5.6 to the case $A^* = \mathrm{End}^*(M(X;\mathbb{Z}/p_i^{n_i}))$, $B^* = \mathrm{End}^*(M(X;\mathbb{Z}/p_i))$ and the reduction map $f_i\colon A^* \twoheadrightarrow B^*$. We obtain that f_i lifts decompositions and isomorphisms strictly for each i. To finish the proof observe that by the Chinese remainder theorem $\mathrm{End}^*(M(X;\mathbb{Z}/m)) \simeq \prod_{i=1}^{l} \mathrm{End}^*(M(X;\mathbb{Z}/p_i^{n_i}))$. □

3.5.8. Let X be a smooth projective variety over a field F. Assume that X has a splitting field (see 3.3.1). We say that *Rost Nilpotence* holds for X if the kernel of the restriction map

$$res_E\colon \mathrm{End}^*(M(X_E;\Lambda)) \to \mathrm{End}^*(M(\overline{X};\Lambda))$$

consists of nilpotent elements for all field extensions E/F and all rings of coefficients Λ.

3.5.9 Example. Let X be a smooth projective variety which splits over any field over which it has a rational point. Then Rost Nilpotence holds for X.

Indeed, by [EKM, Theorem 67.1] if α is in the kernel of the restriction map res_E then $\alpha^{\circ(\dim X+1)} = 0$.

3.5.10. We say that a field extension E/F is *rank preserving* with respect to X if the restriction map $res_{E/F}\colon \mathrm{CH}(X) \to \mathrm{CH}(X_E)$ becomes an isomorphism after tensoring with $\mathbb{Q}$.

3.5.11 Lemma. *Assume X has a ($\mathbb{Z}$-)splitting field. Then for any rank preserving finite field extension E/F we have $[E:F]\cdot\overline{\mathrm{CH}}(X_E) \subset \overline{\mathrm{CH}}(X)$.*

Proof. Let L be a splitting field containing E. Let γ be any element in $\overline{\mathrm{CH}}(X_E)$. By definition there exists $\alpha \in \mathrm{CH}(X_E)$ such that $\gamma = res_{L/E}(\alpha)$. Since $res_{E/F}\otimes\mathbb{Q}$ is an isomorphism, there exists an element $\beta \in \mathrm{CH}(X)$ and a non-zero integer n such that $res_{E/F}(\beta) = n\alpha$. By the projection formula

$$n\cdot cores_{E/F}(\alpha) = cores_{E/F}(res_{E/F}(\beta)) = [E:F]\cdot\beta.$$

Applying $res_{L/E}$ to the both sides of the identity we obtain

$$n(res_{L/E}(cores_{E/F}(\alpha))) = n[E:F]\cdot\gamma.$$

Therefore, $res_{L/E}(cores_{E/F}(\alpha)) = [E:F]\cdot\gamma$. □

3.5.12 Example. Let G be an inner form of a split group over a field F and let X be a projective homogeneous G-variety. Then any field extension E/F is rank preserving with respect to X and $X\times X$.

Indeed, by [Pa94, Theorem 2.2 and 4.2] the restriction map $K_0(X) \to K_0(X_E)$ becomes an isomorphism after tensoring with $\mathbb{Q}$. Since the Chern character $ch\colon K_0(X)\otimes\mathbb{Q} \to \mathrm{CH}(X)\otimes\mathbb{Q}$ is an isomorphism and respects pullbacks, E is rank preserving with respect to X. It remains to note that $X\times X$ is a homogeneous $G\times G$-variety.

Observe that for outer forms this fails in general. Indeed, for even dimensional quadrics with non-trivial discriminant the restriction map $K_0(X) \to K_0(\overline{X})$ is not surjective.

3.5.13 Lemma. *Assume that Rost Nilpotence holds for X. Then for any field extension E/F the restriction $res_E\colon \mathrm{End}^*(M(X_E;\Lambda)) \twoheadrightarrow \mathrm{im}(res_E)$ onto the image lifts decompositions and isomorphisms strictly.*

Proof. Apply Proposition 3.5.6 to the homomorphism $res_E\colon A^* \twoheadrightarrow B^*$ between the graded rings $A^* = \mathrm{End}^*(M(X_E;\Lambda))$ and $B^* = \mathrm{im}(res_E)$. □

3.5.14 Corollary. *Assume that Rost Nilpotence holds for X. Let m be an integer and let E/F be a field extension of degree coprime to m which is rank preserving with respect to $X \times X$ (see 3.5.10). Then the restriction map*

$$res_{E/F}\colon \mathrm{End}^*(M(X;\mathbb{Z}/m)) \to \mathrm{End}^*(M(X_E;\mathbb{Z}/m))$$

lifts decompositions and isomorphisms.

Proof. By Lemma 3.5.11 we have $\mathrm{im}(res_E) = \mathrm{im}(res_F)$. We apply now Lemma 3.5.13 and 3.5.4(ii) with $A^* = \mathrm{End}^*(M(X;\mathbb{Z}/m))$, $B^* = \mathrm{End}^*(M(X_E;\mathbb{Z}/m))$ and $C^* = \mathrm{im}(res_E)$. □

3.5.15. Let V^* be a free graded Λ-module of finite rank and let $A^* = \mathrm{End}^*(V^*)$ be its ring of endomorphisms, where $\mathrm{End}^{-d}(V^*)$, $d \in \mathbb{Z}$, is the group of endomorphisms of V^* decreasing the degree by d. Assume we are given a direct sum decomposition of $V^* = \bigoplus_i \mathrm{im}\phi_i$ by means of idempotents ϕ_i in A^*. We say that this decomposition is Λ-*free* if all graded components of $\mathrm{im}\phi_i$ are free Λ-modules. Observe that if $\Lambda = \mathbb{Z}$ or $\mathbb{Z}/p$, where p is prime, then any decomposition is Λ-free.

3.5.16 Example. Assume X has a splitting field. Define $V^* = \mathrm{CH}^*(\overline{X})$. Then by Poincaré duality and by the Künneth decomposition we have $\mathrm{End}^*(V^*) = \mathrm{End}^*(M(\overline{X}))$ (see Example 3.5.2).

3.5.17 Lemma. *The map $\mathrm{SL}_l(\mathbb{Z}) \to \mathrm{SL}_l(\mathbb{Z}/m)$ induced by the reduction modulo m is surjective.*

Proof. Since $\mathbb{Z}/m$ is a semi-local ring, the group $\mathrm{SL}_l(\mathbb{Z}/m)$ is generated by elementary matrices (see [HOM, Theorem 4.3.9]). □

3.5.18 Proposition. *Consider a free graded $\mathbb{Z}$-module V^* of finite rank and the reduction map $f\colon \mathrm{End}^*(V^*) \twoheadrightarrow \mathrm{End}^*(V^* \otimes_{\mathbb{Z}} \mathbb{Z}/m)$. Then f lifts $\mathbb{Z}/m$-free decompositions strictly. Moreover, if $(\mathbb{Z}/m)^\times = \{\pm 1\}$, then f lifts isomorphisms of $\mathbb{Z}/m$-free decompositions strictly.*

Proof. We are given a decomposition $V^k \otimes_{\mathbb{Z}} \mathbb{Z}/m = \oplus_i W_i^k$, where W_i^k is the k-graded component of $\mathrm{im}\phi_i$. Present V^k as a direct sum $V^k = \bigoplus_i V_i^k$ of free $\mathbb{Z}$-modules such that $\mathrm{rk}_{\mathbb{Z}} V_i^k = \mathrm{rk}_{\mathbb{Z}/m} W_i^k$. Fix a $\mathbb{Z}$-basis $\{v_{ij}^k\}_j$ of V_i^k. For each W_i^k choose a basis $\{w_{ij}^k\}_j$ such that the linear transformation D^k of $V^k \otimes_{\mathbb{Z}} \mathbb{Z}/m$ sending each $v_{ij}^k \otimes 1$ to w_{ij}^k has determinant 1. By Lemma 3.5.17 there is a lifting $\tilde{D}^k$ of D^k to a linear transformation of V^k. So we obtain $V^k = \bigoplus_i \tilde{W}_i^k$, where $\tilde{W}_i^k = \tilde{D}^k(V_i^k)$ satisfies $\tilde{W}_i^k \otimes_{\mathbb{Z}} \mathbb{Z}/m = W_i^k$. It remains to define φ_i on each V^k to be the projection onto $\tilde{W}_i^k$.

Now let φ_1, φ_2 be two idempotents in $\mathrm{End}^*(V^*)$. Let V_i^k denote the k-graded component of $\mathrm{im}\varphi_i$. An isomorphism θ_{12} between $\varphi_1 \otimes 1$ and $\varphi_2 \otimes 1$ of degree d can be identified with a family of isomorphisms $\theta_{12}^k\colon V_1^k \otimes \mathbb{Z}/m \to V_2^{k-d} \otimes \mathbb{Z}/m$. In the case $(\mathbb{Z}/m)^\times = \{\pm 1\}$ all these isomorphisms are given by matrices with determinants $\{\pm 1\}$ and, hence, can be lifted to isomorphisms $\vartheta_{12}^k\colon V_1^k \to V_2^{k-d}$ by Lemma 3.5.17. □

Now we are ready to state and to prove the main result of this section.

3.5.19 Theorem. *Let X be a smooth projective irreducible variety over a field F. Assume that X has a splitting field of degree m which is rank preserving with respect to $X \times X$. Assume that Rost Nilpotence holds for X. Consider only decompositions of $M(X;\mathbb{Z}/m)$ which become $\mathbb{Z}/m$-free over the splitting field. Then the reduction map*

$$f\colon \mathrm{End}^*(M(X;\mathbb{Z})) \twoheadrightarrow \mathrm{End}^*(M(X;\mathbb{Z}/m))$$

lifts such decompositions. If additionally $(\mathbb{Z}/m)^\times = \{\pm 1\}$, then this map lifts isomorphisms of such decompositions.

Proof. Consider the diagram

$$\begin{tikzcd}
\mathrm{End}^*(M(X;\mathbb{Z})) \arrow[r, two heads] \arrow[d, "f"] & \mathrm{im}(res_F) \arrow[r, hook, "h"] \arrow[d, "\bar{f}"] & \mathrm{End}^*(M(\overline{X};\mathbb{Z})) \arrow[d, "f'"] \\
\mathrm{End}^*(M(X;\mathbb{Z}/m)) \arrow[r, two heads] & \mathrm{im}(res_F) \arrow[r, hook] & \mathrm{End}^*(M(\overline{X};\mathbb{Z}/m))
\end{tikzcd}$$

Recall that we can identify $\mathrm{End}^{-d}(M(\overline{X}))$ with the group of endomorphisms of $\mathrm{CH}^*(\overline{X})$ which decrease the grading by d (see 3.5.16). Applying Proposition 3.5.18 to the case $V^* = \mathrm{CH}^*(\overline{X})$ we obtain that the map f' lifts decompositions strictly. Moreover, if $(\mathbb{Z}/m)^\times = \{\pm 1\}$ then f' lifts isomorphisms strictly.

By Lemma 3.5.11 $\ker f' \subset \mathrm{im}h$ and, therefore, applying 3.5.4(iii) we obtain that $\bar{f}$ lifts decompositions strictly and, moreover, $\bar{f}$ lifts isomorphisms strictly if $(\mathbb{Z}/m)^\times = \{\pm 1\}$.

Now by Lemma 3.5.13 the horizontal arrows of the left square lift decompositions and isomorphisms strictly. It remains to apply 3.5.4(i) and (ii). □

3.5.20 Example. Let Q be an odd dimensional quadric or an even dimensional quadric with trivial discriminant. Then the reduction map $\mathrm{End}(M(Q;\mathbb{Z})) \twoheadrightarrow \mathrm{End}(M(Q;\mathbb{Z}/2))$ lifts decompositions and isomorphisms strictly.

3.6 Rost type motivic decompositions

In the present section we provide motivic decompositions which are similar to those obtained by Rost and Voevodsky. The main object is a *generically split* smooth projective variety X over a field F with the property that the greatest common divisor of degrees of all closed points on X is equal to a prime integer p. The main result is Theorem 3.6.1 which provides a motivic decomposition with $\mathbb{Z}/p$-coefficients of such a variety.

To simplify the notation we set $A^* = \mathrm{Ch}^*(\overline{X})$, where $\mathrm{Ch}(X) = \mathrm{CH}(X;\mathbb{Z}/p)$ is the Chow ring with $\mathbb{Z}/p$-coefficients. The subring of rational cycles in A^* will be denoted by A^*_{rat}.

3.6.1 Theorem. *Let X be a generically split variety over a field F such that the greatest common divisor of degrees of all closed points on X is equal to a prime integer p. Assume there exists an element $\rho \in A^r$ for some r such that A^* is generated by elements $\{1, \rho, \dots, \rho^{p-1}\}$ as a graded module over A^*_{rat}. Then the Chow motive of X with $\mathbb{Z}/p$-coefficients is isomorphic to the direct sum*

$$M(X;\mathbb{Z}/p) \simeq \bigoplus_{i\geq 0} \mathcal{R}_p\{i\}^{\oplus c_i},$$

where the motive $\mathcal{R}_p$ satisfies the following properties:

- *it is indecomposable*
- *over a splitting field it becomes isomorphic to a direct sum of twisted Lefschetz motives*
$$\overline{\mathcal{R}}_p \simeq \bigoplus_{j=0}^{p-1} (\mathbb{Z}/p)\{jr\}$$
- *the integers c_i are the coefficients of the polynomial*
$$\sum c_i t^i = P(A^*, t) \cdot \frac{1-t^r}{1-t^{rp}},$$
where $P(A^*, t)$ is the Poincare polynomial.

Assume we are under the hypothesis of Theorem 3.6.1.

3.6.2 Lemma. *There exists $\varepsilon \in (\mathbb{Z}/p)^\times$ such that the cycle $\rho\otimes 1 + \varepsilon(1\otimes\rho)$ is rational.*

Proof. Since X is generically split by Lemma 3.3.4 applied to the projection $\mathrm{pr}_0\colon \mathrm{Ch}^r(\overline{X}\times\overline{X}) \to \mathrm{Ch}^r(\overline{X})$ there exists a rational preimage of $\rho\otimes 1$ of the form

$$\rho\otimes 1 + \delta(1\otimes\rho) + \sum_k \mu_k\otimes\nu_k + 1\otimes\gamma,$$

where $\delta\in\mathbb{Z}/p$, codimensions of μ_k and ν_k are strictly less than r and the cycle γ is rational.

Since $\{1, \rho, \dots, \rho^{p-1}\}$ generates A^* as an A^*_{rat}-module and $\operatorname{codim}\rho = r$, the cycles μ_k and ν_k are rational. Therefore $\tau := \rho\otimes 1 + \delta(1\otimes\rho)$ is rational. If $\delta\in(\mathbb{Z}/p)^\times$, then we take $\varepsilon=\delta$. Otherwise, the cycle $\tau+\tau^t = (1+\delta)(\rho\otimes 1+1\otimes\rho)$ is rational and $1+\delta$ is invertible. Therefore, the cycle $\rho\otimes 1+1\otimes\rho$ is rational and we take $\varepsilon = 1$. $\square$

3.6.3 Lemma. *Let $\alpha, \beta \in A^*_{rat}$ be two cycles with the property $p \nmid \deg(\rho^l \alpha\beta)$. Assume that the following cycle is rational*

$$\xi = \sum_{j=0}^{l} x_j(\rho^j \otimes \rho^{l-j}) \cdot (\alpha \otimes \beta), \quad \text{where } x_j \in \mathbb{Z}/p.$$

Then p divides the number of invertible coefficients x_j.

Proof. Consider the rational cycle

$$\xi^{\circ(p-2)} \cdot \xi^t = (\sum_{j=0}^{l} x_j^{p-1})\rho^l\alpha\beta \otimes \rho^l\alpha\beta.$$

Taking the push-forward induced by the first projection we obtain a rational zero-cycle $(\sum_{j=0}^{l} x_j^{p-1})\rho^l\alpha\beta$. Since p divides the degree of any rational zero-cycle on X and $p \nmid \deg(\rho^l\alpha\beta)$, we have $p \mid \sum_{j=0}^{l} x_j^{p-1}$. Finally, observe that $x^{p-1} \equiv 1 \mod p$ if $x \in (\mathbb{Z}/p)^\times$ and $x^{p-1} = 0 \mod p$ otherwise. □

3.6.4 Lemma. *For all $l = 0, \ldots, p-2$ and homogeneous $\mu \in A^*_{rat}$ we have $p \mid \deg(\rho^l\mu)$.*

Proof. Assume $p \nmid \deg(\rho^l\mu)$. Consider the rational cycle

$$\xi = (\rho \otimes 1 + \varepsilon(1 \otimes \rho))^l(\mu \otimes 1) = \sum_{j=0}^{l} \varepsilon^j \binom{l}{j} (\rho^{l-j} \otimes \rho^j)(\mu \otimes 1).$$

Note that all coefficients $\varepsilon^j\binom{l}{j}$ on the right hand side are invertible. Since the number of coefficients is $l+1 < p$, we obtain the contradiction by Lemma 3.6.3 applied to $\alpha = \mu$ and $\beta = 1$. □

3.6.5 Lemma. *We have a perfect $\mathbb{Z}/p$-linear pairing*

$$A^k_{rat} \times A^{\dim X - r(p-1)-k}_{rat} \to \mathbb{Z}/p$$

$$\mu \times \nu \mapsto \deg(\rho^{p-1}\mu\nu) \mod p.$$

Proof. It suffices to show that for an element $\mu \in A^k_{rat}$ there exists a cycle $\nu \in A^{\dim X - r(p-1)-k}_{rat}$ such that $\deg(\rho^{p-1}\mu\nu) \equiv 1 \mod p$. By the Poincaré duality there exists a cycle $\mu^\vee \in A^{\dim X - k}$ such that $\deg(\mu\mu^\vee) = 1 \mod p$. Expand $\mu^\vee$ as $\mu^\vee = \sum_{l=0}^{p-1} \rho^l\alpha_l$ for some $\alpha_l \in A^*_{rat}$. By Lemma 3.6.4 we have $p \mid \deg(\rho^l\mu\alpha_l)$ for all $l = 0 \ldots p-2$. Therefore, $\deg(\rho^{p-1}\mu\alpha_{p-1}) = \deg(\mu\mu^\vee) = 1 \mod p$. To finish the proof we set $\nu = \alpha_{p-1}$. □

3.6.6 Lemma. *Let α and $\beta \in A^*_{rat}$ be two homogeneous cycles such that $p \nmid \deg(\rho^{p-1}\alpha\beta)$. Then the cycle $\sigma \cdot (\alpha \otimes \beta)$, where $\sigma = \sum_{j=0}^{p-1} \rho^j \otimes \rho^{p-1-j}$, is rational.*

Proof. Consider the rational cycle

$$\lambda = (\rho \otimes 1 + \varepsilon(1 \otimes \rho))^{p-1}(\alpha \otimes \beta) = \sum_{j=0}^{p-1} \varepsilon^j \binom{p-1}{j} (\rho^{p-1-j} \otimes \rho^j)(\alpha \otimes \beta).$$

Since all coefficients $\varepsilon^j \binom{p-1}{j}$ on the right hand side are invertible, we have the equality $\lambda^{\circ p-1} = \sigma \cdot (\alpha \otimes \beta)$. □

3.6.7 Lemma. *For all $l = p, p+1, \dots, 2(p-1)$ and homogeneous $\mu \in A_{rat}$ we have $p \mid \deg(\rho^l \mu)$.*

Proof. Assume that $p \nmid \deg(\rho^l \mu)$. By Lemma 3.6.5 there exists a rational cycle α such that $\deg(\rho^{p-1}\alpha) = 1 \mod p$. Consider the composite of two rational cycles:

$$\Big((\rho \otimes 1 + \varepsilon(1 \otimes \rho))^l (\mu \otimes 1)\Big) \circ \Big(\sigma \cdot (\alpha \otimes 1)\Big),$$

where $\sigma = \sum_{j=0}^{p-1} \rho^j \otimes \rho^{p-1-j}$ is the cycle from Lemma 3.6.6. It is equal to

$$\Big(\sum_{j=0}^{l} \varepsilon^j \binom{l}{j} (\rho^{l-j}\mu \otimes \rho^j)\Big) \circ \Big(\sum_{j=0}^{p-1} \rho^j \alpha \otimes \rho^{p-1-j}\Big) = \sum_{j=0}^{p-1} \varepsilon^j \binom{l}{j} \rho^{p-1-j}\alpha \otimes \rho^j.$$

Note that the coefficient $\varepsilon^j \binom{l}{j}$ at the summand $j = 0$ is equal to 1 and all other coefficients are divisible by p. By Lemma 3.6.3 we obtain a contradiction. □

3.6.8 Lemma. *Let $\{\alpha_i\}_{i \in \mathcal{I}}$ be a homogeneous $\mathbb{Z}/p$-basis of A^*_{rat}. Then there exists a family $\{\beta_i\}_{i \in \mathcal{I}}$ of homogeneous elements of A^*_{rat} such that for all $l = 0, 1, \dots, 2(p-1)$ we have*

$$\deg(\rho^l \alpha_i \beta_j) = \begin{cases} 1, & \textit{if } l = p-1 \textit{ and } i = j \\ 0, & \textit{otherwise.} \end{cases} \tag{*}$$

Moreover, the family $\{\rho^l \alpha_i \mid i \in \mathcal{I},\, l = 0, \dots, p-1\}$ forms a $\mathbb{Z}/p$-basis of A^*.

Proof. By Lemma 3.6.5 there exists a family of homogeneous elements $\{\beta_i\}_{i \in \mathcal{I}}$ in A^*_{rat} such that

$$\deg(\rho^{p-1}\alpha_i \beta'_j) \mod p = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 3.6.4 and Lemma 3.6.7 the family $\{\beta_i\}_{i \in \mathcal{I}}$ satisfies the condition $(*)$.

The family $\{\rho^l \alpha_i \mid i \in \mathcal{I},\, l = 0, \dots p-1\}$ generates A^*, since the family $\{\rho^l, l = 0, \dots, p-1\}$ generates A^* as an A^*_{rat}-module. Assume we have a linear relation $\sum a_{il} \rho^l \alpha_i = 0$. Multiplying by $\rho^{p-1-l}\beta_i$ and taking the degree map we see that $a_{il} = 0$. Hence, the family $\{\rho^l \alpha_i\}$ is linear independent. □

Proof of Theorem 3.6.1. Choose a homogeneous $\mathbb{Z}/p$-basis $\{\alpha_i\}_{i\in\mathcal{I}}$ in A^*_{rat}. Let $\{\beta_i\}_{i\in\mathcal{I}}$ be the family of elements from Lemma 3.6.8. Consider the cycles $\phi_i = \sigma\cdot(\alpha_i\otimes\beta_i)$, where $i\in\mathcal{I}$ and $\sigma=\sum_{j=0}^{p-1}\rho^j\otimes\rho^{p-1-j}$. They are rational by Lemma 3.6.6 and they form a family of pair-wise orthogonal idempotents by $(*)$. Moreover, their sum

$$\sum_{i\in\mathcal{I}}\phi_i=\sum_{l=0}^{p-1}\sum_{i\in\mathcal{I}}\rho^l\alpha_i\otimes\rho^{q-1-l}\beta_i$$

is the diagonal. The cycle $\theta_{ij}=\sigma\cdot(\alpha_j\otimes\beta_i)$ provides a rational isomorphism between $(\bar{X},\phi_i)$ and $(\bar{X},\phi_j)$. We apply now Proposition 3.4.1 and Lemma 3.5.13 and obtain the desired motivic decomposition

$$M(X;\mathbb{Z}/p)\simeq\bigoplus_{i\in\mathcal{I}}\mathcal{R}_p(\operatorname{codim}\alpha_i),$$

where $\mathcal{R}_p=(X,\varphi_{i_0})$ with $\operatorname{codim}\alpha_{i_0}=0$.

It remains to show that $\mathcal{R}_p$ is indecomposable. Observe that its ring of endomorphisms $\mathrm{CH}^{\dim X}(\bar{X}\times\bar{X})$ is additively generated by $\rho^l\alpha_{i_0}\otimes\rho^{p-1-l}\beta_{i_0}$, $l=0,\dots,p-1$. Consider a nonzero rational idempotent $\phi=\sum_{l=0}^{p-1}a_l(\rho^l\alpha_{i_0}\otimes\rho^{p-1-l}\beta_{i_0})$. We have $a_l=0$ or 1 for all l. By Lemma 3.6.3 $a_l=1$ for all l. So $\phi=\Delta_{\overline{X}}$ is the identity. □

3.6.9 Proposition. *Let X_1 and X_2 be varieties satisfying the hypothesis of Theorem 3.6.1 with the respective $\rho_1\in\mathrm{Ch}^r(\overline{X}_1)$ and $\rho_2\in\mathrm{Ch}^r(\overline{X}_2)$. Assume that $F(X_2)$ is a splitting field of X_1 and $F(X_1)$ is a splitting field of X_2. Then the motives $\mathcal{R}_{X_1}$ and $\mathcal{R}_{X_2}$ appearing in the decomposition of $M(X_1;\mathbb{Z}/p)$ and $M(X_2;\mathbb{Z}/p)$ respectively are isomorphic.*

Proof. Following the proof of Lemma 3.6.2 we show that there exists $\varepsilon\in(\mathbb{Z}/p)^\times$ such that the cycle $\rho_1\otimes 1+\varepsilon(1\otimes\rho_2)$ is rational. Let $\{\alpha_i\}_{i\in\mathcal{I}}$ and $\{\beta_i\}_{i\in\mathcal{I}}$ be the familes of elements in $\overline{\mathrm{Ch}}^*(X_1)$, let $\{\alpha_j\}_{j\in\mathcal{J}}$, $\{\beta_j\}_{j\in\mathcal{J}}$ be the families in $\overline{\mathrm{Ch}}^*(X_2)$ chosen as in Lemma 3.6.8. The cycle

$$\theta_{i_0,j_0}=(\rho_1\otimes 1+\varepsilon(1\otimes\rho_2))^{p-1}\cdot(\alpha_{i_0}\otimes\beta_{j_0}).$$

is rational and produces an isomorphism between $(\overline{X}_1,\phi_{i_0})$ and $(\overline{X}_2,\phi_{j_0})$. It remains to apply Proposition 3.4.1 and Lemma 3.5.13. □

3.7 Integral decompositions

to be filled

Chapter 4

Motives of isotropic homogeneous varieties

4.1 Relative cellular spaces

4.1.1. A scheme E together with a flat morphism $p : E \to X$ is called an *affine fibration* of relative dimension r if, for every point $x \in X$, there is a Zariski open neighborhood $U \subset X$ such that $p^{-1}(U) \cong U \times \mathbb{A}^r$ over U (see [NZ06, §4]).

4.1.2. (see [EKM, §66]) Let X be a smooth projective variety over a field F. We call X a *relative cellular space* over F if there exists a finite decreasing filtration by closed proper subvarieties

$$X = X_0 \supset X_1 \supset \ldots \supset X_n \supset \emptyset$$

such that each complement $E_i = X_i \smallsetminus X_{i+1}$ is an affine fibration of a relative dimension r_i over a smooth projective variety Y_i (we assume $Y_n = X_n$). We denote by $p_i : E_i \to Y_i$ the projection morphism.

In particular, if the base Y_i consists of points, i.e., each E_i is isomorphic to a disjoint union of affine spaces $\mathbb{A}_F^{r_i}$, then X is called an (absolute) *cellular space* over F.

4.1.3 Example. The basic example of an absolute cellular filtration is given by the descreasing filtration consisting of projective linear subspaces

$$\mathbb{P}^n \supset \mathbb{P}^{n-1} \supset \ldots \supset \mathbb{P}^1 \supset pt,$$

where on each step $\mathbb{P}^i \smallsetminus \mathbb{P}^{i-1} \simeq \mathbb{A}^i$.

4.1.4 Example (Rost filtration)**.** Let Q be an isotropic smooth projective quadric of dimension $2d-1$. It can be identified with a hypersurface given by the equation

$$Q = \{[x_0 : x_1 : \ldots : x_{2d}] \in \mathbb{P}^{2d} \mid x_0x_1 + q(x_2, \ldots, x_{2d}) = 0\},$$

where q is a quadratic form. Set $Q = X_0$ and consider the closed subvariety X_1 of Q given by the equation $x_0 = 0$. Then $X_0 \smallsetminus X_1$ is given by two equations $x_0 \neq 0$ and $x_0x_1 + q(x_2, \dots, x_{2d})$ in $\mathbb{P}^{2d}$. Dividing by x_0 we see that $X_0 \smallsetminus X_1$ can be identified with the affine subvariety $\{(y_1, \dots, y_{2d}) \in \mathbb{A}^{2d} \mid y_1 + q(y_2, \dots, y_{2d}) = 0\}$, where $y_i = x_i/x_0$, which is isomorphic to the affine space $\mathbb{A}^{2d-1}$.

Let $X_2 = pt$ be the point on X_1 given by $x_1 = 1$ and $x_i = 0$ for all $i \neq 1$. Then the map forgetting the first two coordinates induces a well-defined morphism $\pi\colon X_1 \smallsetminus X_2 \to Q_1$, where Q_1 is the projective quadric given by the equation

$$Q_1 = \{[x_2 : \dots : x_{2d}] \in \mathbb{P}^{2d-2} \mid q(x_2, \dots, x_{2d}) = 0\}.$$

It is easy to see that π is, indeed, an affine fibration with fibers isomorphic to $\mathbb{A}^1$.

Hence, Q is a relative cellular space with the bases $Y_0 = pt$, $Y_1 = Q_1$ and $Y_2 = X_2 = pt$. Observe that X_1 is not necessary smooth at the point X_2.

4.1.5 Example. (cf. [NZ06, §7.1]) Let Q be a split projective quadric of dimension $2d$. Then Q can be thought of as the hypersurface given by

$$Q = \{[x_0 : y_0 : \dots : x_d : y_d] \in \mathbb{P}^{2d+1} \mid \sum_{i=0,d} x_iy_i = 0\}.$$

We claim that Q has a smooth filtration defined as follows.

Consider two linear subspaces of $\mathbb{P}^{2d+1}$ of dimension d defined by the equations

$$Z_1 = \{[x_0 : y_0 : \dots : x_d : y_d] \in \mathbb{P}^{2d+1} \mid x_0 = x_1 = \dots = x_d = 0\} \quad \text{and}$$

$$Z_2 = \{[x_0 : y_0 : \dots : x_d : y_d] \in \mathbb{P}^{2d+1} \mid y_0 = y_1 = \dots = y_d = 0\}.$$

Clearly $Z_1 \cong Z_2 \cong \mathbb{P}^d$ and $Z_1 \cap Z_2 = \emptyset$. Consider the linear map $\pi_1 : \mathbb{P}^{2d+1} \smallsetminus Z_2 \to Z_1$ given by the projection to the y-coordinates. Both subspaces Z_1 and Z_2 lie in Q. It can be easily seen that the restriction $\pi_1 : Q \smallsetminus Z_2 \to Z_1$ naturally has a structure of a vector bundle of rank d. Hence, we get a smooth filtration on Q:

$$X_0 = Q \supset X_1 = Z_2, \qquad Y_0 = Z_1, Y_1 = Z_2.$$

4.1.6 Example. (cf. [NZ06, §7.3]) Let $\mathrm{Gr}(d, n)$ be the Grassmannian of d-dimensional linear subspaces in affine space $\mathbb{A}^n$. Then $\mathrm{Gr}(d, n)$ has a smooth filtration defined as follows.

Fix a $(n-1)$-dimensional linear subspace V of $\mathbb{A}^n$. Then there is the induced closed embedding $\mathrm{Gr}(d, n-1) \hookrightarrow \mathrm{Gr}(d, n)$ of smooth projective varieties, where $\mathrm{Gr}(d, n-1)$ is identified with the variety of d-planes in V. Consider the complement $E = \mathrm{Gr}(d, n) \smallsetminus \mathrm{Gr}(d, n-1)$. We claim that this is a vector bundle over $\mathrm{Gr}(d-1, n-1)$. Indeed, an element of E is a d-plane that intersects V along a linear subspace of dimension $d-1$, i.e., it gives a point of $\mathrm{Gr}(d-1, n-1)$. Moreover, the set of all elements of E that intersect V along a fixed linear subspace of dimension $d-1$ can be identified with the affine space of dimension $n-d$. This gives the structure of an affine fibration.

Applying this procedure recursively we obtain a smooth cellular filtration

$$X_0 = \mathrm{Gr}(d,n) \supset X_1 = \mathrm{Gr}(d,n-1) \supset \ldots \supset X_{n-d} = \mathrm{Gr}(d,d) = pt$$

with the base $Y_0 = \mathrm{Gr}(d-1,n-1), Y_1 = \mathrm{Gr}(d-1,n-2), \ldots, Y_{n-d} = pt$.

The following two facts (Lemmas 4.1.7 and 4.1.8) play an important role in computations of motives of relative cellular spaces:

4.1.7 Lemma. *(cf. [NZ06, Thm. 6.5]) Let X be a relative cellular space over a field F. Then we have an isomorphism of motives in $Chow(F,\mathbb{Z})$*

$$M(X) \simeq \bigoplus_{i\geq 0} M(Y_i)\{r_i\},$$

where Y_i is the base and r_i is the rank of the respective fibration from 4.1.2.

To formulate the next fact we use the following notion

4.1.8. Let X be a smooth projective variety over a field F. We say a smooth projective morphism $f\colon Y \to X$ is a *cellular fibration* if it is a locally trivial fibration whose fiber $\mathcal{F}$ is an absolute cellular space has a decomposition into affine cells.

4.1.9 Lemma. *(cf. [PSZ, Lemma 3.2]) Let $f\colon Y \to X$ be a cellular fibration. Then $M(Y)$ is isomorphic to $M(X)\otimes M(\mathcal{F})$.*

Proof. We follow the proof of [EG97, Proposition 1]. Define the morphism

$$\varphi\colon \bigoplus_{i\in\mathcal{I}} M(X)\{\mathrm{codim}\, B_i\} \to M(Y)$$

to be the direct sum $\varphi = \bigoplus_{i\in\mathcal{I}} \varphi_i$, where each φ_i is given by the cycle $[\mathrm{pr}_Y^*(B_i)\cdot \Gamma_f] \in \mathrm{CH}(X\times Y)$ produced from the graph cycle Γ_f and the chosen (non-canonical) basis $\{B_i\}_{i\in\mathcal{I}}$ of $\mathrm{CH}(Y)$ over $\mathrm{CH}(X)$. The realization of φ coincides exactly with the isomorphism of abelian groups $\mathrm{CH}(X)\otimes\mathrm{CH}(\mathcal{F}) \to \mathrm{CH}(Y)$ constructed in [EG97, Proposition 1]. Then by Manin's identity principle (see [Ma68, §3]) φ is an isomorphism. □

4.2 Bruhat decomposition

In the present section we will see that any projective homogeneous G-variety, where G is a split linear algebraic group, admits an absolute cellular filtration.

4.2.1. Let G be a split group of type $\mathcal{D}$ over a field F. Choose a split maximal torus $\mathbb{T}$ and a Borel subgroup $B = P_\emptyset$. Let W be the Weyl group of G. The *Bruhat decomposition* of G is the following disjoint union of double cosets

$$G = \coprod_{w\in W} B\cdot w'\cdot B,$$

where $w' \in N_G(\mathbb{T})$ is a representative of w.

4.2.2. Consider the variety X of Borel subgroups of G, i.e. the projective homogeneous variety $\mathcal{D}/P_\emptyset$. Let x be an F-rational point on X with the stabilizer subgroup B (see 2.5.2). Then by Bruhat decomposition

$$X = \coprod_{w \in W} B \cdot w' \cdot x.$$

4.2.3. Let $\Phi \subset V$ be a root system and $W = W(\Phi)$ be its Weyl group. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a basis of Φ. Denote by $s_i = s_{\alpha_i} \in W$ the corresponding generators (simple reflections). We define the *length* $\ell(w)$ of an element $w \in W$ as

$$\ell(w) := \min\{m \mid w = s_{i_1} \cdot s_{i_2} \cdot \ldots \cdot s_{i_m}\}.$$

4.2.4. Let Θ be a subset of Π and let $W_\Theta \subset W$ be the subgroup generated by all s_i with $\alpha_i \in \Theta$. Define $W^\Theta \subset W$ to be the set of the *representatives of minimal length* in the left cosets W/W_Θ. Observe that for each coset there exists only one such representative. Moreover, one can show that

$$W^\Theta = \{\, w \in W \mid \forall\, i \in \Theta \quad \ell(ws_i) = \ell(w) + 1 \,\}.$$

4.2.5. Let X be a projective homogeneous G-variety of dimension d corresponding to a subset $\Theta \subset \mathcal{D}$. By the Bruhat decomposition we have

$$X = \coprod_{w \in W^\Theta} B \cdot w' \cdot x,$$

where x is an F-rational point on X. We define the cellular filtration as

$$X_i = \coprod_{\substack{w \in W^\Theta, \\ l(w) \le d-i}} B \cdot w' \cdot x.$$

We have

$$X = X_0, \qquad X_i \setminus X_{i-1} \simeq \coprod_{\substack{w \in W^\Theta, \\ l(w) = d-i}} \mathbb{A}_F^{l(w)} \quad \text{and } X_d = x,$$

i.e. there are exactly $\#\{w \in W^\Theta \mid l(w) = j\}$ affine cells of dimension j.

Observe that for a projective space A_n/P_1 this cellular filtration coincides with the one from Example 4.1.3

4.3 Hasse diagram

The cellular structure of a projective homogeneous variety for a split group can be naturally described using the Hasse diagram. The latter is a purely combinatorial object depending only on a root system and a subset of vertices of the respective Dynkin diagram.

4.3.1. Given a root system Φ, its basis Π and a subset $\Theta \subset \Pi$ we define a labeled graph called the *Hasse diagram* as follows

- The vertices are given by elements of W^{Θ};
- We draw an edge from w_1 toward w_2 and mark it by a number i if and only if $\ell(w_2) = \ell(w_1) + 1$ and $w_2 = s_i w_1$.

It can be shown that this diagram (graph) doesn't depend on the choice of Π in Φ, hence, it depend only on the type $\mathcal{D}$ of Φ and the subset Θ of the set of vertices of the Dynkin diagram $\mathcal{D}$.

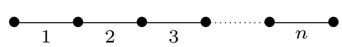
4.3.2. By a distance between two vertices v_1 and v_2 we call a minimal number of edges needed to connect v_1 and v_2. Hence, the distance between a vertex v and the starting vertex $v = 1$ of the Hasse diagram is equal to the length $l(v)$. We make the convention to draw all vertices of the same length on one vertical line.

4.3.3. Let G be a split group of type $\mathcal{D}$. The relation between the Hasse diagram for the subset Θ of $\mathcal{D}$ and the cellular decomposition of the variety $\mathcal{D}/P_{\Theta}$ is given as follows:

There is an obvious one-to-one correspondence between the vertices and the cells due to the fact that they are parametrized by the same finite set W^{Θ}. Moreover, two vertices $v_1, v_2 \in W^{\Theta}$ with $l(v_1) + 1 = l(v_2)$ are connected by an edge if and only if the cell corresponding to v_1 lies in the closure of the cell corresponding to v_2.

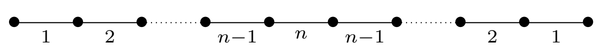
4.3.4 Example. For $\Theta = \{\alpha_2, \dots, \alpha_n\}$ and $\mathcal{D} = \mathrm{A}_n$, B_n or D_n we have

A_n :



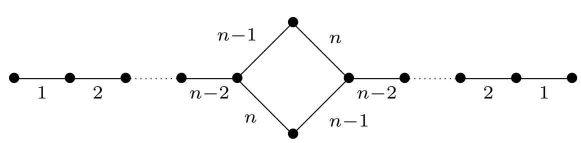
projective space

B_n :



quadric

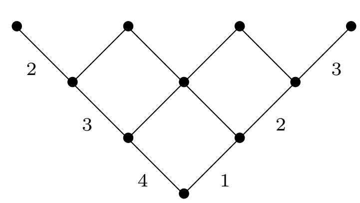
D_n :



quadric

where on the right hand side we identify the respective projective homogeneous variety $\mathcal{D}/P_{\Theta}$.

4.3.5 Example. For $\Theta = \{\alpha_1, \alpha_3, \alpha_4\}$ and $\mathcal{D} = \mathrm{A}_4$, i.e. for the variety $\mathrm{Gr}(2,5)$, we have



where all other labels are obtained by parallel translations along the edges.

4.4 The case of an isotropic group

We show that if G is an isotropic group, then a projective homogeneous G-variety X admits a relative cellular filtration. In view of Lemma 4.1.7 this fact reduces the problem of computing the motivic decomposition of X to the case of anisotropic groups.

4.4.1. Let G be an isotropic group of type $\mathcal{D}$ over a field F. Let X be a projective homogeneous G-variety which (over the algebraic closure F_a) corresponds to a subset $\Theta \subset \mathcal{D}$. Let $\Xi \subset \mathcal{D}$ be a subset of circled vertices of the Tits index of G. Since G is isotropic, this subset is non-emty. The main result of [CGM] and [Br05] says that the variety X is a relative cellular space where the base varieties Y_i are projective homogeneous varieties for anisotropic groups.

4.4.2. To identify the base spaces Y_i we may use the following procedure. Consider the Hasse diagram for $\Theta \subset \mathcal{D}$, i.e. for the variety X. Erase all edges which are labeled by the roots from Ξ. Then the Hasse diagram for X will split into several connected components. Each connected component turns out to be the Hasse diagram for a base space Y_i such that the rank r_i of the fibration $X_i \setminus X_{i+1} \to Y_i$ coinsides with the distance between the very left vertex of the connected component and the left hand side of the Hasse diagram and the codimension of X_i in X is equal to the distance between the very right vertex of the connected component and the right hand side of the diagram.

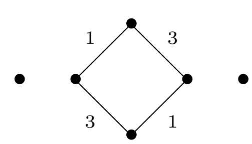
4.4.3 Example. Let G be an isotropic group of type A_3 with the set of circled vertices $\Xi = \{\alpha_2\}$.

Let X be a projective homogeneous G-variety with $\Theta = \{\alpha_2, \alpha_3\}$, i.e. X is a Severi-Brauer variety of a central simple algebra A of degree 4 and index 2. Then the respective Hasse diagram (see Example 4.3.4) splits into two connected components

each corresponding to the diagram of the conic $\mathrm{SB}(D)$, where D is division and $A = M_2(D)$. Here Y_0 is the right component and Y_1 is the left component, the ranks are $r_0 = 2$ and $r_1 = 0$ and the codimensions are $\operatorname{codim} X_0 = 0$ and $\operatorname{codim} X_1 = 2$. By Lemma 4.1.7 we obtain the following motivic decomposition

$$M(X) \simeq M(\mathrm{SB}(D)) \oplus M(\mathrm{SB}(D))\{2\}.$$

Let X be a projective homogeneous G-variety with $\Theta = \{\alpha_1, \alpha_3\}$, i.e. X is a 4-dimensional quadric. In this case the diagram splits as



where the middle component corresponds to a 2-dimensional anisotropic quadric Q_2 and isolated vertices to rational points. Here $Y_0 = pt$, $Y_1 = Q_2$, $Y_2 = pt$, the ranks $r_0 = 4$, $r_1 = 1$, $r_2 = 0$ and the codimensions $\operatorname{codim} X_0 = 0$, $\operatorname{codim} X_1 = 1$, $\operatorname{codim} X_2 = 4$. Observe that the obtained relative cellular filtration coincides with the Rost filtration from Example 4.1.4 and the motivic decomposition coincides with the Rost decomposition:

$$M(X) \simeq \mathbb{Z} \oplus M(Q_2)\{1\} \oplus \mathbb{Z}\{4\}.$$

4.5 Motives of fibered spaces

In the present section we discuss motives of cellular fibrations. We essentially follow [PSZ, §3]. The main result (Theorem 4.5.4) generalizes and uniformizes the proofs of paper [CPSZ].

4.5.1 Lemma. *Let G be a linear algebraic group over a field F and let X be a projective homogeneous G-variety. Let Y be a G-variety and let $f\colon Y \to X$ be a G-equivariant projective morphism. Assume that the fiber of f over $F(X)$ is isomorphic to $\mathcal{F}_{F(X)}$ for some variety $\mathcal{F}$ over F. Then f is a locally trivial fibration with fiber $\mathcal{F}$.*

Proof. By the assumptions, we have $Y \times_X \operatorname{Spec} F(X) \simeq (\mathcal{F} \times X) \times_X \operatorname{Spec} F(X)$ as schemes over $F(X)$. Since $F(X)$ is a direct limit of $\mathcal{O}(U)$ taken over all non-empty affine open subsets U of X, by [EGA IV, Corollaire 8.8.2.5] there exists U such that $f^{-1}(U) = Y \times_X U$ is isomorphic to $(\mathcal{F} \times X) \times_X U \simeq \mathcal{F} \times U$ as a scheme over U. Since G acts transitively on X and f is G-equivariant, the map f is a locally trivial fibration. □

4.5.2 Corollary. *Let X be a projective homogeneous G-variety and let Y be a projective variety such that $Y_{F(X)} \simeq \mathcal{F}_{F(X)}$ for some variety $\mathcal{F}$. Then the projection map $X \times Y \to X$ is a locally trivial fibration with fiber $\mathcal{F}$. Moreover, if $\mathcal{F}$ is cellular, then $M(X \times Y) \simeq M(X) \otimes M(\mathcal{F})$.*

Proof. We apply Lemma 4.5.1 to the projection map $X \times Y \to X$ and use Lemma 4.1.9. □

4.5.3 Lemma. *Let G be a semisimple linear algebraic group over F. Let X and Y be projective homogeneous G-varieties corresponding to parabolic subgroups P and Q of the split form $\hat{G}$, $Q \subseteq P$. Let $f\colon Y \to X$ be the map induced by the quotient map. If G splits over $F(X)$, then f is a cellular fibration with fiber $\mathcal{F} = P/Q$.*

Proof. Since G splits over $F(X)$, the fiber of f over $F(X)$ is isomorphic to $(P/Q)_{F(X)} = \mathcal{F}_{F(X)}$. Now apply Lemma 4.5.1 and note that $\mathcal{F}$ is cellular. □

Case-by-case arguments from the paper [CPSZ] show that under certain conditions the Chow motive of a twisted flag variety X can be expressed in terms of the motive of a minimal flag. These conditions cover almost all twisted flag varieties corresponding to groups of types A_n and B_n together with some examples of types C_n, G_2 and F_4. The following theorem together with the table of Example 3.3.7 provides a uniform proof of these results and extends them to some other types.

4.5.4 Theorem. *Let Y and X be taken as in Lemma 4.5.3. Then the Chow motive $M(Y)$ of Y is isomorphic to a direct sum of twisted copies of the motive $M(X)$, i.e.*

$$M(Y) \simeq \bigoplus_{i\geq 0} M(X)\{i\}^{\oplus c_i},$$

where $\sum c_i t^i = P(M(\overline{Y}), t)/P(M(\overline{X}), t)$ is the quotient of the respective Poincaré polynomials.

Proof. Apply Lemmas 4.5.3 and 4.1.9. □

4.5.5 Remark. An explicit formula for $P(M(\overline{X}), t)$ involves degrees of the basic polynomial invariants of $\hat{G}$ and is provided in [Hi82, Ch. IV, Cor. 4.5].

Chapter 5

Motives of generically split homogeneous varieties.

5.1 The J-invariant

Fix a prime integer p. We denote the Chow ring of a variety X with $\mathbb{Z}/p$-coefficients by $\mathrm{Ch}(X)$ and the image of the restriction map $\mathrm{CH}(X;\mathbb{Z}/p) \to \mathrm{CH}(\overline{X};\mathbb{Z}/p)$ by $\overline{\mathrm{Ch}}(X)$.

5.1.1. Let $\hat{G}$ be a split group over a field F with a split maximal torus T and a Borel subgroup B containing T. Let $G = {}_{\xi}\hat{G}$ be a twisted form of $\hat{G}$ given by a cocycle $\xi \in Z^1(F, \hat{G})$. Let $\mathcal{X} = {}_{\xi}(\hat{G}/B)$ be the corresponding *variety of complete flags*. Observe that the group G splits over any field K over which $\mathcal{X}$ has a rational point, in particular, over the function field $F(\mathcal{X})$. According to [De74] the Chow ring $\mathrm{Ch}(\overline{\mathcal{X}})$ can be expressed in purely combinatorial terms and, therefore, depends only on the type of G but not on the base field F.

5.1.2. Let $\hat{T}$ be the group of characters of T and let $S(\hat{T})$ be the symmetric algebra. We denote by R the image of the characteristic map $c\colon S(\hat{T}) \to \mathrm{Ch}(\overline{\mathcal{X}})$ (see [Gr58, (4.1)]). According to [KM06, Thm.6.4] there is an embedding

$$R \subseteq \overline{\mathrm{Ch}}(\mathcal{X}), \tag{5.1}$$

where the equality holds if the cocycle ξ corresponds to a generic torsor.

5.1.3. Let $\mathrm{Ch}(\overline{G})$ denote the Chow ring with $\mathbb{Z}/p$-coefficients of the group $\hat{G}$ over a splitting field of $\overline{\mathcal{X}}$. Consider the pull-back induced by the quotient map

$$\pi\colon\ \mathrm{Ch}(\overline{\mathcal{X}}) \to \mathrm{Ch}(\overline{G})$$

According to [Gr58, p. 21, Rem. 2°] π is surjective with its kernel generated by R^+, where R^+ stands for the subgroup of the non-constant elements of R.

5.1.4. An explicit presentation of $\mathrm{Ch}(\overline{G})$ is known for all types of G and all torsion primes p of G (in the sense of [De73, Proposition 6.(a)]) provided in Table 5.2.5. Namely, by [Kc85, Theorem 3] it is a quotient of the polynomial ring in r variables $x_1,\ldots,x_r$ of codimensions $d_1\le d_2\le\ldots\le d_r$ coprime to p, modulo an ideal generated by certain p-powers $x_1^{p^{k_1}},\ldots,x_r^{p^{k_r}}$ ($k_i\ge 0$, $i=1,\ldots,r$)

$$\mathrm{Ch}^*(\overline{G})=(\mathbb{Z}/p)[x_1,\ldots,x_r]/(x_1^{p^{k_1}},\ldots,x_r^{p^{k_r}}). \tag{5.2}$$

In the case where p is not a torsion prime of G we have $\mathrm{Ch}^*(\overline{G})=\mathbb{Z}/p$, i.e. $r=0$.

Note that a complete list of numbers $\{d_ip^{k_i}\}_{i=1\ldots r}$ called *p-exceptional degrees* of $\hat{G}$ is provided in [Kc85, Table II]. Taking the p-primary and p-coprimary parts of each p-exceptional degree from this table one restores the respective k_i and d_i.

5.1.5. We introduce two orders on the set of additive generators of $\mathrm{Ch}(\overline{G})$, i.e. on the monomials $x_1^{m_1}\ldots x_r^{m_r}$. To simplify the notation, we will denote the monomial $x_1^{m_1}\ldots x_r^{m_r}$ by x^M, where M is an r-tuple of integers $(m_1,\ldots,m_r)$. The codimension of x^M will be denoted by $|M|$. Observe that $|M|=\sum_{i=1}^r d_im_i$.

- Given two r-tuples $M=(m_1,\ldots,m_r)$ and $N=(n_1,\ldots,n_r)$ we say $x^M\preccurlyeq x^N$ (or equivalently $M\preccurlyeq N$) if $m_i\le n_i$ for all i. This gives a partial ordering on the set of all monomials (r-tuples).

- Given two r-tuples $M=(m_1,\ldots,m_r)$ and $N=(n_1,\ldots,n_r)$ we say $x^M\le x^N$ (or equivalently $M\le N$) if either $|M|<|N|$, or $|M|=|N|$ and $m_i\le n_i$ for the greatest i such that $m_i\ne n_i$. This gives a well-ordering on the set of all monomials (r-tuples) known as the *DegLex* order.

5.1.6. Let $G={}_\xi\hat{G}$ be the twisted form of a split group $\hat{G}$ over a field F by means of a cocycle $\xi\in Z^1(F,\hat{G})$ and let $\mathcal{X}={}_\xi(\hat{G}/B)$ be the respective variety of complete flags. Let $\mathfrak{Ch}(G)$ denote the image of the composite

$$\mathrm{Ch}(\mathcal{X})\xrightarrow{res}\mathrm{Ch}(\overline{\mathcal{X}})\xrightarrow{\pi}\mathrm{Ch}(\overline{G})$$

Since both maps are ring homomorphisms, $\mathfrak{Ch}(G)$ is a subring of $\mathrm{Ch}(\overline{G})$.

For each $1\le i\le r$ set j_i to be the smallest non-negative integer such that the subring $\mathfrak{Ch}(G)$ contains an element a with the greatest monomial $x_i^{p^{j_i}}$ with respect to the *DegLex* order on $\mathrm{Ch}(\overline{G})$, i.e. of the form

$$a=x_i^{p^{j_i}}+\sum_{x^M\lneq x_i^{p^{j_i}}}c_Mx^M,\quad c_M\in\mathbb{Z}/p.$$

The r-tuple of integers $(j_1,\ldots,j_r)$ will be called the *J-invariant of G modulo p* and will be denoted by $J_p(G)$.

Observe that if the Chow ring $\mathrm{Ch}(\overline{G})$ has only one generator, i.e. $r=1$, then the J-invariant is equal to the smallest non-negative integer j_1 such that $x_1^{p^{j_1}} \in \mathfrak{Ch}(G)$. Moreover, if $j_1=1$, then the preimage of x in $\mathrm{Ch}(\overline{\mathcal{X}})$ satisfies the hypotheses for an element $\rho \in \mathrm{Ch}^{d_1}(\overline{\mathcal{X}})$ from Theorem 3.6.1.

5.1.7 Example. By definition it follows that $J_p(G_E) \preccurlyeq J_p(G)$ for any field extension E/F. Moreover, $J_p(G) \preccurlyeq (k_1,\dots,k_r)$ by (5.2).

According to (5.1) the J-invariant takes its maximal possible value $J_p(G) = (k_1,\dots,k_r)$ if the cocycle ξ corresponds to a generic torsor. Later on (see Corollary 7.2.6) it will be shown that the J-invariant takes its minimal value $J_p(G) = (0,\dots,0)$ if and only if the group G splits by a finite field extension of degree coprime to p.

The next example explains the terminology 'J-invariant'.

5.1.8 Example. Let ϕ be a quadratic form with trivial discriminant. In [Vi05, Definition 5.11] A. Vishik introduced the notion of the J-invariant of ϕ, a set of integers which describes the subgroup of rational cycles on the respective maximal orthogonal Grassmannian. This invariant provides an important tool for study of algebraic cycles on quadrics. In particular, it was one of the main ingredients used by A. Vishik in his significant progress on the solution of Kaplansky's Problem. More precisely, in the notation of paper [Vi06] the J-invariant of a quadric corresponds to the upper row of its *elementary discrete invariant* (see [Vi06, Definition 2.2]).

An equivalent but slightly different definition of the J-invariant of ϕ can be found in [EKM, § 88]. Namely, if $J(\phi)$ denotes the J-invariant of ϕ from [EKM], then Vishik's J-invariant is equal to $\{0,1,\dots,n\}\setminus J(\phi)$ in the case $\dim\phi = 2n$ and $\{1,\dots,n\}\setminus J(\phi)$ in the case $\dim\phi = 2n-1$.

Using Theorem 4.5.4 one can show that $J(\phi)$ introduced in [EKM] can be expressed in terms of $J_2(\mathrm{O}(\phi)) = (j_1,\dots,j_r)$ as follows:

$$J(\phi) = \{2^l d_i \mid i=1,\dots,r,\ 0 \le l \le j_i - 1\}.$$

Since all d_i are odd, $J_2(\mathrm{O}(\phi))$ is uniquely determined by $J(\phi)$.

5.2 Properties of the J-invariant

We have the following reduction formula (cf. [EKM, Cor. 88.6] in the case of quadrics).

5.2.1 Proposition. *Let G be a semisimple group of inner type over a field F and let $\mathcal{X}$ be the variety of complete G-flags. Let Y be a projective variety such that the map $\mathrm{Ch}^l(Y) \to \mathrm{Ch}^l(Y_{F(x)})$ is surjective for all $x \in X$ and $l \le n$. Then $j_i(G) = j_i(G_{F(Y)})$ for all i such that $d_i p^{j_i(G_{F(Y)})} \le n$.*

Proof. By [EKM, Lemma 88.5] the map $\mathrm{Ch}^l(X) \to \mathrm{Ch}^l(X_{F(Y)})$ is surjective for all $l \le n$ and, therefore $j_i(G) \le j_i(G_{F(Y)})$. The converse inequality is obvious. □

5.2.2 Corollary. $J_p(G) = J_p(G_{F(t)})$.

Proof. Take $Y = \mathbb{P}^1$ and apply Proposition 5.2.1. □

5.2.3. To find restrictions on the possible values of $J_p(G)$ we use Steenrod p-th power operations introduced by P. Brosnan. Recall (see [Br03]) that if the characteristic of the base field F is different from p then one can construct *Steenrod p-th power operations*

$$S^l\colon \operatorname{Ch}^*(X) \to \operatorname{Ch}^{*+l(p-1)}(X), \qquad l \geq 0$$

such that $S^0 = \mathrm{id}$, the restriction $S^l|_{\operatorname{Ch}^l(X)}$ coincides with taking to the p-th power, $S^l|_{\operatorname{Ch}^i(X)} = 0$ for $l > i$, and the total operation $S^\bullet = \sum_{l\geq 0} S^l$ is a homomorphism of $\mathbb{Z}/p$-algebras compatible with pull-backs. In particular, Steenrod operations preserve rationality of cycles.

In the case of projective homogeneous varieties over the field of complex numbers S^l is compatible with its topological counterparts: the reduced power operation $\mathcal{P}^l$ if $p \neq 2$ and the Steenrod square Sq^{2l} if $p = 2$ (over complex numbers $\operatorname{Ch}^*(\overline{\mathcal{X}})$ can be viewed as a subring of the singular cohomology $H^{2*}_{sing}(\overline{\mathcal{X}}, \mathbb{Z}/p)$). Moreover, $\operatorname{Ch}^*(\overline{G})$ may be identified with the image of the pull-back map $H^{2*}_{sing}(\overline{\mathcal{X}}, \mathbb{Z}/p) \to H^{2*}_{sing}(\overline{G}, \mathbb{Z}/p)$. An explicit description of this image and formulae describing the action of $\mathcal{P}^l$ and Sq^{2l} on $H^*_{sing}(\overline{G}, \mathbb{Z}/p)$ can be found in [MT91] for exceptional groups and in [EKM] for classical groups.

The action of the Steenrod operations on $\operatorname{Ch}(\overline{\mathcal{X}})$ and on $\operatorname{Ch}(\overline{G})$ can be described in purely combinatorial terms (see [DZ07]) and, hence, doesn't depend on the choice of a base field F.

The following lemma provides an important technical tool for computing possible values of the J-invariant of G.

5.2.4 Lemma. *Assume that in $\operatorname{Ch}^*(\overline{G})$ we have $S^l(x_i) = x_m^{p^s}$, and for any $i' < i$ $S^l(x_{i'}) < x_m^{p^s}$ with respect to the DegLex order. Then $j_m \leq j_i + s$.*

Proof. By definition there exists a cycle $\alpha \in \overline{\operatorname{Ch}}(\mathcal{X})$ such that the leading term of $\pi(\alpha)$ is $x_i^{p^{j_i}}$. For the total operation we have

$$S(x_i^{p^{j_i}}) = S(x_i)^{p^{j_i}} = S^0(x_i)^{p^{j_i}} + S^1(x_i)^{p^{j_i}} + \ldots + S^{d_i}(x_i)^{p^{j_i}}.$$

In particular, $S^{lp^{j_i}}(x_i^{p^{j_i}}) = S^l(x_i)^{p^{j_i}}$. Applying $S^{lp^{j_i}}$ to α we obtain a rational cycle whose image under π has the leading term $x_m^{p^{j_i+s}}$. □

5.2.5. We summarize information about restrictions on the J-invariant into the following table (numbers r, d_i and k_i are taken from [Kc85, Table II]). For groups we use the notation from [INV].

Recall that r is the number of generators of $\operatorname{Ch}^*(\overline{G})$, d_i are their codimensions, k_i define the p-power relations, $J_p(G) = (j_1, \ldots, j_r)$, $0 \leq j_i \leq k_i$ for all i, and $G = {}_\xi(\hat{G})$ for some $\xi \in H^1(F, \hat{G})$. If p is not in the table, then $J_p(G) = ()$ is empty.

$\hat{G}$	p	r	$k_i,\ i=1..r$	$d_i,\ i=1..r$	restrictions on j_i
SL_n/μ_m, $m\vert n$	$p\vert m$	1	$p^{k_1}\Vert n$	1	
PGSp_n, $2\vert n$	2	1	$2^{k_1}\Vert n$	1	
O_n^+, $n>2$	2	$[\frac{n+1}{4}]$	$[\log_2 \frac{n-1}{2i-1}]$	$2i-1$	$j_i \geq j_{i+l}$ if $2 \nmid \binom{i-1}{l}$, $j_i \leq j_{2i-1}+1$
Spin_n, $n>2$	2	$[\frac{n-3}{4}]$	$[\log_2 \frac{n-1}{2i+1}]$	$2i+1$	$j_i \geq j_{i+l}$ if $2 \nmid \binom{i}{l}$, $j_i \leq j_{2i}+1$
PGO_{2n}^+, $n>1$	2	$[\frac{n+2}{2}]$	$2^{k_1}\Vert n$, $[\log_2 \frac{2n-1}{2i-3}]$	$1,\ i=1$; $2i-3,\ i\geq 2$	$j_i \geq j_{i+l}$ if $2 \nmid \binom{i-2}{l}$, $j_i \leq j_{2i-2}+1$
$\mathrm{Spin}_{2n}^{\pm}$, $2\vert n$	2	$\frac{n}{2}$	$2^{k_1}\Vert n$, $[\log_2 \frac{2n-1}{2i-1}]$	$1,\ i=1$; $2i-1,\ i\geq 2$	$j_i \geq j_{i+l}$ if $2 \nmid \binom{i-1}{l}$, $j_i \leq j_{2i-1}+1$
G_2, F_4, E_6	2	1	1	3	
F_4, E_6^{sc}, E_7	3	1	1	4	
E_6^{ad}	3	2	2, 1	1, 4	
E_7^{sc}	2	3	1, 1, 1	3, 5, 9	$j_1 \geq j_2 \geq j_3$
E_7^{ad}	2	4	1, 1, 1, 1	1, 3, 5, 9	$j_2 \geq j_3 \geq j_4$
E_8	2	4	3, 2, 1, 1	3, 5, 9, 15	$j_1 \geq j_2 \geq j_3$, $j_1 \leq j_2+1$, $j_2 \leq j_3+1$.
E_8	3	2	1, 1	4, 10	$j_1 \geq j_2$
E_8	5	1	1	6	

The upper indices ^{sc} and ^{ad} denote simply-connected and adjoint groups respectively. The last column of the table follows from Lemma 5.2.4 and, hence, requires $\mathrm{char}\,(F) \neq p$ restriction. All other columns are taken directly from [Kc85, Table II] and are independent on the characteristic of the base field.

5.3 Motives of complete flags

In the present section we describe a basis of the subring of rational cycles of $\mathrm{Ch}(\overline{\mathcal{X}} \times \overline{\mathcal{X}})$, where $\mathcal{X}$ is the variety of complete flags. The key results here are Propositions 5.3.3 and 5.3.10. As a consequence, we obtain a motivic decomposition of $\mathcal{X}$ (Theorem 5.3.13) in terms of certain motive $\mathcal{R}_p(G)$.

5.3.1. We use the notation of the previous section. Let G be a semisimple group of inner type over F and let $\mathcal{X}$ be the respective variety of complete flags. Let $R \subseteq \overline{\mathrm{Ch}}(\mathcal{X})$ be the image of the characteristic map. Consider the quotient map $\pi\colon \mathrm{Ch}(\overline{\mathcal{X}}) \to \mathrm{Ch}(\overline{G})$. Fix preimages e_i of x_i in $\mathrm{Ch}(\overline{\mathcal{X}})$. For an r-tuple $M = (m_1, \ldots, m_r)$ set $e^M = \prod_{i=1}^r e_i^{m_i}$. Set $N = (p^{k_1}-1, \ldots, p^{k_r}-1)$ and $d = \dim \mathcal{X} - |N|$.

5.3.2 Lemma. *The Chow ring* $\mathrm{Ch}(\overline{\mathcal{X}})$ *is a free* R*-module with a basis* $\{e^M\}$, $M \preccurlyeq N$.

Proof. Note that the subgroup R^+ of the non-constant elements of R is a nilpotent ideal in R. Applying the Nakayama Lemma we obtain that $\{e^M\}$ generates

$\mathrm{Ch}(\overline{\mathcal{X}})$. By [Kc85, (2)] $\mathrm{Ch}(\overline{\mathcal{X}})$ is a free R-module, hence, for the Poincaré polynomials we have

$$P(\mathrm{Ch}^*(\overline{\mathcal{X}}), t) = P(\mathrm{Ch}^*(\overline{G}), t) \cdot P(R^*, t).$$

Substituting $t = 1$ we obtain that

$$\mathrm{rk}\, \mathrm{Ch}(\overline{\mathcal{X}}) = \mathrm{rk}\, \mathrm{Ch}(\overline{G}) \cdot \mathrm{rk}\, R.$$

To finish the proof observe that $\mathrm{rk}\, \mathrm{Ch}(\overline{G})$ coincides with the number of generators $\{e^M\}$. □

5.3.3 Proposition. *The pairing $R \times R \to \mathbb{Z}/p$ given by $(\alpha, \beta) \mapsto \deg(e^N \alpha\beta)$ is non-degenerated, i.e. for any non-zero element $\alpha \in R$ there exists β such that $\deg(e^N \alpha\beta) \neq 0$.*

Proof. Choose a homogeneous basis of $\mathrm{Ch}(\overline{\mathcal{X}})$. Let $\alpha^\vee$ be the Poincaré dual of α with respect to this basis. By Lemma 5.3.2 $\mathrm{Ch}(\overline{\mathcal{X}})$ is a free R-module with the basis $\{e^M\}$, hence, expanding $\alpha^\vee$ we obtain

$$\alpha^\vee = \sum_{M \preccurlyeq N} e^M \beta_M, \text{ where } \beta_M \in R.$$

Note that if $M \neq N$ then $\mathrm{codim}\, \alpha\beta_M > d$, therefore, $\alpha\beta_M = 0$. So we can set $\beta = \beta_N$. □

We fix a homogeneous $\mathbb{Z}/p$-basis $\{\alpha_i\}$ of R and the dual basis $\{\alpha_i^\#\}$ with respect to the pairing introduced in Proposition 5.3.3.

5.3.4 Corollary. *For $|M| \leq |N|$ we have*

$$\deg(e^M \alpha_i \alpha_j^\#) = \begin{cases} 1, & M = N \text{ and } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. If $M = N$, then it follows from the definition of the dual basis. Assume $|M| < |N|$. If $\deg(e^M \alpha_i \alpha_j^\#) \neq 0$, then $\mathrm{codim}(\alpha_i \alpha_j^\#) > d$, in contradiction with the fact that $\alpha_i \alpha_j^\# \in R$. Hence, we are reduced to the case $M \neq N$ and $|M| = |N|$. Since $|M| = |N|$, $\mathrm{codim}(\alpha_i \alpha_j^\#) = d$ and, hence, $R^+ \alpha_i \alpha_j^\# = 0$. On the other hand there exists i such that $m_i \geq p^{k_i}$ and $e^{p^{k_i}} \in \mathrm{Ch}(\overline{\mathcal{X}}) \cdot R^+$. Hence, $e^M \alpha_i \alpha_j^\# = 0$. □

5.3.5 Definition. Given two pairs (l, L) and (m, M), where L, M are r-tuples and l, m are integers, we say $(l, L) \leq (m, M)$ if either $l < m$, or in the case $l = m$ we have $L \leq M$. We introduce a filtration on the ring $\mathrm{Ch}(\overline{\mathcal{X}})$ as follows:

> The (m, M)-th term $\mathrm{Ch}(\overline{\mathcal{X}})_{m,M}$ of the filtration is the $\mathbb{Z}/p$-subspace spanned by the elements $e^I \alpha$ with $I \leq M$, $\alpha \in R$ homogeneous, $|I| + \mathrm{codim}\, \alpha \leq m$.

Define the *associated graded ring* as follows:

$$A^{*,*} = \bigoplus_{(m,M)} A^{m,M}, \text{ where } A^{m,M} = \mathrm{Ch}(\overline{\mathcal{X}})_{m,M} / \bigcup_{(l,L) \precneqq (m,M)} \mathrm{Ch}(\overline{\mathcal{X}})_{l,L}.$$

By Lemma 5.3.2 if $M \preccurlyeq N$ the graded component $A^{m,M}$ consists of the classes of elements $e^M\alpha$ with $\alpha \in R$ and $\operatorname{codim}\alpha = m - |M|$. In particular, $\operatorname{rk} A^{m,M} = \operatorname{rk} R^{m-|M|}$. Comparing the ranks we see that $A^{m,M}$ is trivial when $M \not\preccurlyeq N$.

Consider the subring $\overline{\mathrm{Ch}}(\mathcal{X})$ of rational cycles with the induced filtration. The associated graded subring will be denoted by $A^{*,*}_{rat}$. From the definition of the J-invariant it follows that the elements $e_i^{p^{j_i}}$, $i = 1, \ldots, r$, belong to $A^{*,*}_{rat}$.

Similarly, we introduce a filtration on the ring $\mathrm{Ch}(\overline{\mathcal{X}} \times \overline{\mathcal{X}})$ as follows:

> The (m, M)-th term of the filtration is the $\mathbb{Z}/p$-subspace spanned by the elements $e^I\alpha \times e^L\beta$ with $I + L \leq M$, $\alpha, \beta \in R$ homogeneous and $|I| + |L| + \operatorname{codim}\alpha + \operatorname{codim}\beta \leq m$.

The associated graded ring will be denoted by $B^{*,*}$. By definition $B^{*,*}$ is isomorphic to the tensor product of graded rings $A^{*,*} \otimes_{\mathbb{Z}/p} A^{*,*}$. The graded subring associated to $\overline{\mathrm{Ch}}(\mathcal{X} \times \mathcal{X})$ will be denoted by $B^{*,*}_{rat}$.

5.3.6. The key observation is that due to Corollary 5.3.4 we have

$$\mathrm{Ch}(\overline{\mathcal{X}} \times \overline{\mathcal{X}})_{m,M} \circ \mathrm{Ch}(\overline{\mathcal{X}} \times \overline{\mathcal{X}})_{l,L} \subset \mathrm{Ch}(\overline{\mathcal{X}} \times \overline{\mathcal{X}})_{m+l-\dim \overline{\mathcal{X}}, M+L-N} \text{ and}$$

$$(\mathrm{Ch}(\overline{\mathcal{X}} \times \overline{\mathcal{X}})_{m,M})_\star(\mathrm{Ch}(\overline{\mathcal{X}})_{l,L}) \subset \mathrm{Ch}(\overline{\mathcal{X}})_{m+l-\dim \overline{\mathcal{X}}, M+L-N}$$

and, therefore, we have a correctly defined composition law

$$\circ\colon B^{M,m} \times B^{L,l} \to B^{m+l-\dim \overline{\mathcal{X}}, M+L-N}$$

and the realization map

$$\star\colon B^{M,m} \times A^{L,l} \to A^{m+l-\dim \overline{\mathcal{X}}, M+L-N}$$

In particular, $B^{\dim \overline{\mathcal{X}}+*, N+*}$ can be viewed as a graded ring with respect to the composition and $(\alpha \circ \beta)_\star = \alpha_\star \circ \beta_\star$. Note also that both operations preserve rationality of cycles.

The proof of the following result is based on the fact that the group G splits over $F(\mathcal{X})$.

5.3.7 Lemma. *The classes of the elements $e_i \times 1 - 1 \times e_i$ in $B^{*,*}$, $i = 1, \ldots, r$, belong to $B^{*,*}_{rat}$.*

Proof. Fix an i. Since G splits over $F(\mathcal{X})$, $F(\mathcal{X})$ is a splitting field of $\mathcal{X}$ and by Lemma 3.3.4 there exists a cycle in $\overline{\mathrm{Ch}^{d_i}}(\mathcal{X} \times \mathcal{X})$ of the form

$$\xi = e_i \times 1 + \sum_s \mu_s \times \nu_s + 1 \times \mu,$$

where $\operatorname{codim}\mu_s, \operatorname{codim}\nu_s < d_i$. Then the cycle

$$\mathrm{pr}_{13}^*(\xi) - \mathrm{pr}_{23}^*(\xi) = (e_i \times 1 - 1 \times e_i) \times 1 + \sum_s (\mu_s \times 1 - 1 \times \mu_s) \times \nu_s$$

belongs to $\overline{\mathrm{Ch}}(\mathcal{X} \times \mathcal{X} \times \mathcal{X})$, where pr_{ij} denotes the projection on the product of the i-th and j-th factors. Applying Corollary 4.5.2 to the projection $\mathrm{pr}_{12}\colon \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ we conclude that there exists a (non-canonical) $\mathrm{Ch}(\mathcal{X} \times \mathcal{X})$-linear isomorphism $\mathrm{Ch}(\mathcal{X}\times\mathcal{X}\times\mathcal{X}) \simeq \mathrm{Ch}(\mathcal{X}\times\mathcal{X})\otimes\mathrm{Ch}(\overline{\mathcal{X}})$, where $\mathrm{Ch}(\mathcal{X}\times\mathcal{X})$ acts on the left-hand side via pr_{12}^*. This gives rise to a $\mathrm{Ch}(\mathcal{X} \times \mathcal{X})$-linear retraction δ to the pull-back map pr_{12}^*. Since the construction of the retraction preserves base change, it preserves rationality of cycles. Hence, passing to a splitting field we obtain a rational cycle

$$\bar{\delta}(\mathrm{pr}_{13}^*(\xi) - \mathrm{pr}_{23}^*(\xi)) = e_i \times 1 - 1 \times e_i + \sum_s (\mu_s \times 1 - 1 \times \mu_s)\bar{\delta}(1 \times 1 \times \nu_s)$$

whose image in $B_{rat}^{*,*}$ is $e_i \times 1 - 1 \times e_i$. $\square$

We will write $(e \times 1 - 1 \times e)^M$ for the product $\prod_{i=1}^r (e_i \times 1 - 1 \times e_i)^{m_i}$ and $\binom{M}{L}$ for the product of binomial coefficients $\prod_{i=1}^r \binom{m_i}{l_i}$. We assume that $\binom{m_i}{l_i} = 0$ if $l_i > m_i$. In the computations we will extensively use the following two formulae (the first follows directly from Corollary 5.3.4 and the second one is a well-known binomial identity).

5.3.8. Let α be an element of R^* and $\alpha^\#$ be its dual with respect to the non-degenerate pairing from 5.3.3, i.e. $\deg(e^N \alpha\alpha^\#) = 1$. Then we have

$$((e \times 1 - 1 \times e)^M (\alpha^\# \times 1))_\star (e^L \alpha) = \binom{M}{M+L-N}(-1)^{M+L-N} e^{M+L-N}.$$

Indeed, expanding the brackets in the left-hand side, we obtain

$$\Big(\sum_{I \preccurlyeq M} (-1)^I \binom{M}{I} e^{M-I}\alpha^\# \times e^I\Big)_\star (e^L \alpha),$$

and it remains to apply Corollary 5.3.4.

5.3.9 (Lucas' Theorem)**.** The following identity holds

$$\binom{n}{m} \equiv \prod_{i \geq 0} \binom{n_i}{m_i} \mod p,$$

where $m = \sum_{i\geq 0} m_i p^i$ and $n = \sum_{i \geq 0} n_i p^i$ are the base p presentations of m and n.

Let $J = J_p(G) = (j_1, \ldots, j_r)$ be the J-invariant of G (see Definition 5.1.6). Set $K = (k_1, \ldots, k_r)$.

5.3.10 Proposition. *Let $\{\alpha_i\}$ be a homogeneous $\mathbb{Z}/p$-basis of R. Then the set of elements $\mathcal{B} = \{e^{p^J L}\alpha_i \mid L \preccurlyeq p^{K-J}-1\}$ forms a $\mathbb{Z}/p$-basis of $A^{*,*}_{rat}$.*

Proof. According to Lemma 5.3.2 the elements from $\mathcal{B}$ are linearly independent. Assume $\mathcal{B}$ does not generate $A^{*,*}_{rat}$. Choose an element $\omega \in A^{m,M}_{rat}$ of the smallest index (m, M) which is not in the linear span of $\mathcal{B}$. By definition of $A^{m,M}$ (see Definition 5.3.5) ω can be written as $\omega = e^M\alpha$, where $M \preccurlyeq N$, $\alpha \in R^{m-|M|}$ and M can not be presented as $M = p^J L'$ for an r-tuple L'. The latter means that in the decomposition of M into p-primary and p-coprimary components $M = p^S L$, where $M = (m_1,\dots,m_r)$, $S = (s_1,\dots,s_r)$, $L = (l_1,\dots,l_r)$ and $p \nmid l_k$ for $k = 1,\dots,r$, we have $J \not\preccurlyeq S$. Choose an i such that $s_i < j_i$. Denote $M_i = (0,\dots,0,m_i,0,\dots,0)$ and $S_i = (0,\dots,0,s_i,0,\dots,0)$, where m_i and s_i stand at the i-th place.

Set $T = N - M + M_i$. By Lemma 5.3.7 and 5.3.8 together with observation 5.3.6 the element

$$((e\times 1 - 1\times e)^T(\alpha^{\#}\times 1))_\star(e^M\alpha) = \binom{p^{k_i}-1}{m_i}(-1)^{m_i}e^{m_i}$$

belongs to $A^{|M_i|,M_i}_{rat}$. By 5.3.9 we have $p \nmid \binom{p^{k_i}-1}{m_i}$ and, therefore, this element is non-trivial. Moreover, since $s_i < j_i$, this element is not in the span of $\mathcal{B}$. Since (m, M) was chosen to be the smallest index and $(|M_i|, M_i) \le (m, M)$ we obtain that $(m, M) = (|M_i|, M_i)$. Repeating the same arguments for $T = N - M_i + p^{S_i}$ we obtain that $M_i = p^{S_i}$, i.e., $l_i = 1$.

Now let γ be a representative of $\omega = e_i^{p^{s_i}}$ in $\overline{\mathrm{Ch}}(\mathcal{X})$. Then its image $\pi(\gamma)$ in $\mathfrak{Ch}(G)$ has the leading term $x_i^{p^{s_i}}$ with $s_i < j_i$. This contradicts the definition of the J-invariant. $\square$

5.3.11 Corollary. *The elements*

$$\{(e\times 1 - 1\times e)^S(e^{p^J L}\alpha_i \times e^{p^J(p^{K-J}-1-M)}\alpha_j^{\#}) \mid L, M \preccurlyeq p^{K-J}-1,\ S \preccurlyeq p^J - 1\}$$

form a $\mathbb{Z}/p$-basis of $B^{*,*}_{rat}$. In particular, the ones such that $S = p^J - 1$ and $L = M$ form a basis of $B^{N,d}_{rat}$.

Proof. According to Lemma 5.3.2 these elements are linearly independent and their number is $p^{|2K-J|}(\operatorname{rk} R)^2$. They are rational by Definition 5.3.5 and Lemma 5.3.7. Applying Corollary 4.5.2 to the projection $\overline{\mathcal{X}} \times \overline{\mathcal{X}} \to \overline{\mathcal{X}}$ we obtain that

$$\operatorname{rk} B^{*,*}_{rat} = \operatorname{rk}\overline{\mathrm{Ch}}(\mathcal{X}\times\mathcal{X}) = \operatorname{rk}\overline{\mathrm{Ch}}(\mathcal{X})\cdot \operatorname{rk}\mathrm{Ch}(\overline{\mathcal{X}}),$$

where the latter coincides with $\operatorname{rk} A^{*,*}_{rat}\cdot p^{|K|}\operatorname{rk} R = p^{|2K-J|}(\operatorname{rk} R)^2$ by Lemma 5.3.2 and Proposition 5.3.10. $\square$

5.3.12 Lemma. *The elements*

$$\theta_{L,M,i,j} = (e\times 1 - 1\times e)^{p^J-1}(e^{p^J L}\alpha_i \times e^{p^J(p^{K-J}-1-M)}\alpha_j^{\#}),\ L, M \preccurlyeq p^{K-J}-1,$$

belong to $B^{*,*}_{rat}$ and satisfy the relations $\theta_{L,M,i,j}\circ\theta_{L',M',i',j'} = \delta_{LM'}\delta_{ij'}\theta_{L',M,i',j}$ and $\sum_{L,i}\theta_{L,L,i,i} = \Delta_{\overline{\mathcal{X}}}$.

Proof. Expanding the brackets and using the identity $\binom{p^j-1}{i} \equiv (-1)^i \mod p$, we see that

$$\theta_{L,M,i,j} = \sum_{I \preccurlyeq p^J-1} e^{p^J L+I}\alpha_i \times e^{N-p^J M-I}\alpha_j^{\#},$$

and the composition relation follows from Corollary 5.3.4. By definition we have

$$\sum_{L,i} \theta_{L,L,i,i} = \sum_{I \preccurlyeq N,i} e^I \alpha_i \times e^{N-I}\alpha_i^{\#}.$$

By Corollary 5.3.4 the latter sum acts trivially on all basis elements of $\mathrm{Ch}(\overline{\mathcal{X}})$ and, hence, coincides with the diagonal. $\square$

We are now ready to provide a motivic decomposition of the variety of complete flags.

5.3.13 Theorem. *Let G be a semisimple linear algebraic group of inner type over a field F and $\mathcal{X}$ be the variety of complete G-flags. Let p be a prime. Assume that $J_p(G) = (j_1, \ldots, j_r)$. Then the motive of $\mathcal{X}$ is isomorphic to the direct sum*

$$M(\mathcal{X}; \mathbb{Z}/p) \simeq \bigoplus_{i \geq 0} \mathcal{R}_p(G)(i)^{\oplus c_i},$$

where the motive $\mathcal{R}_p(G)$ is indecomposable, its Poincaré polynomial over a splitting field is given by

$$P(\overline{\mathcal{R}_p(G)}, t) = \prod_{i=1}^{r} \frac{1 - t^{d_i p^{j_i}}}{1 - t^{d_i}}, \tag{5.3}$$

and the integers c_i are the coefficients of the quotient

$$\sum_{i \geq 0} c_i t^i = P(\mathrm{Ch}^*(\overline{\mathcal{X}}), t)/P(\overline{\mathcal{R}_p(G)}, t).$$

Proof. Consider the projection map

$$f^0\colon\ \overline{\mathrm{Ch}}(\mathcal{X} \times \mathcal{X})_{\dim \mathcal{X}, N} \to B_{rat}^{\dim \mathcal{X}, N}.$$

Observe that the kernel of f^0 is nilpotent. Indeed, any element ξ from $\ker f^0$ belongs to $\mathrm{Ch}(\overline{\mathcal{X}} \times \overline{\mathcal{X}})_{m,M}$ for some $(m, M) \lneq (\dim \mathcal{X}, N)$ which depends on ξ. Then by 5.3.6 its i-th composition power $\xi^{\circ i}$ belongs to the graded component $\mathrm{Ch}(\overline{\mathcal{X}} \times \overline{\mathcal{X}})_{im-(i-1)\dim \mathcal{X}, iM-(i-1)N}$, and, therefore, becomes trivial for i big enough.

By Lemma 5.3.12 the elements $\theta_{L,L,i,j}$ form a family of pairwise-orthogonal idempotents whose sum is the identity. Therefore, by Proposition 3.5.6 there exist pair-wise orthogonal idempotents $\varphi_{L,i}$ in $\overline{\mathrm{Ch}}(\mathcal{X} \times \mathcal{X})$ which are mapped to $\theta_{L,L,i,i}$ and whose sum is the identity.

Recall that given two correspondences ϕ and ψ in $\overline{\mathrm{Ch}}(\mathcal{X} \times \mathcal{X})$ of degrees c and c' respectively its composite $\phi \circ \psi$ has degree $c + c'$. Using this fact we conclude that the homogeneous components of $\varphi_{L,i}$ of codimension $\dim\mathcal{X}$ are pair-wise orthogonal idempotents whose sum is the identity. Hence, we may assume that $\varphi_{L,i}$ belong to $\overline{\mathrm{Ch}}^{\dim\mathcal{X}}(\mathcal{X}\times\mathcal{X})$.

We now show that $\varphi_{L,i}$ are indecomposable. By Corollary 5.3.11 and Lemma 5.3.12 the ring $(B_{rat}^{\dim\mathcal{X},N},\circ)$ can be identified with a product of matrix rings over $\mathbb{Z}/p$

$$B_{rat}^{\dim\mathcal{X},N}\simeq\prod_{s=0}^{d}\mathrm{End}((\mathbb{Z}/p)^{p^{|K-J|}\,\mathrm{rk}\,R^s}).$$

By means of this identification $\theta_{L,L,i,i}\colon e^{p^J M}\alpha_j\mapsto\delta_{L,M}\delta_{i,j}e^{p^J L}\alpha_i$ is an idempotent of rank 1 and, therefore, is indecomposable. Since the kernel of f^0 is nilpotent, the $\varphi_{L,i}$ are indecomposable as well.

Next we show that $\varphi_{L,i}$ is isomorphic to $\varphi_{M,j}$. In the ring $B_{rat}^{*,*}$ mutually inverse isomorphisms between them are given by $\theta_{L,M,i,j}$ and $\theta_{M,L,j,i}$. Let

$$f\colon\ \overline{\mathrm{Ch}}(\mathcal{X}\times\mathcal{X})\to B_{rat}^{*,*}$$

be the *leading term* map; it means that for any $\gamma\in\overline{\mathrm{Ch}}(\mathcal{X}\times\mathcal{X})$ we find the smallest degree (s,I) such that γ belongs to $\overline{\mathrm{Ch}}(\mathcal{X}\times\mathcal{X})_{s,I}$ and set $f(\gamma)$ to be the image of γ in $B_{rat}^{s,I}$. Note that f is not a homomorphism but satisfies the condition that $f(\xi)\circ f(\eta)$ equals either $f(\xi\circ\eta)$ or 0. Choose preimages $\psi_{L,M,i,j}$ and $\psi_{M,L,j,i}$ of $\theta_{L,M,i,j}$ and $\theta_{M,L,j,i}$ by means of f. Applying Lemma 3.5.5 we obtain mutually inverse isomorphisms $\vartheta_{L,M,i,j}$ and $\vartheta_{M,L,j,i}$ between $\varphi_{L,i}$ and $\varphi_{M,j}$. By the definition of f it remains to take their homogeneous components of the appropriate degrees.

Applying now Lemma 3.5.9 and Corollary 3.5.13 to the restriction map

$$res_F\colon\ \mathrm{End}(M(\mathcal{X};\mathbb{Z}/p))\to\overline{\mathrm{End}}(M(\mathcal{X};\mathbb{Z}/p))$$

and the family of idempotents $\varphi_{L,i}$ we obtain a family of pair-wise orthogonal idempotents $\phi_{L,i}\in\mathrm{End}(M(\mathcal{X};\mathbb{Z}/p))$ such that

$$\Delta_{\mathcal{X}}=\sum_{L,i}\phi_{L,i}.$$

Since $res_{F_s/F}$ lifts isomorphisms, for the respective motives we have $(\mathcal{X},\phi_{L,i})\simeq(\mathcal{X},\phi_{0,0})(|L|+\mathrm{codim}\,\alpha_i)$ for all L and i (see 3.5.2). The twists $|L|+\mathrm{codim}\,\alpha_i$ can be easily recovered from the explicit formula for $\theta_{L,L,i,i}$ (see Lemma 5.3.12). Denoting $\mathcal{R}_p(G)=(\mathcal{X},\phi_{0,0})$ we obtain the desired motivic decomposition.

Finally, consider the motive $\mathcal{R}_p(G)$ over a splitting field. The idempotent $\theta_{0,0,0,0}$ splits into the sum of pair-wise orthogonal (non-rational) idempotents $e^I\times e^{N-I}1^{\#}$, $I\preccurlyeq p^J-1$. The motive corresponding to each summand is isomorphic to $(\mathbb{Z}/p)(|I|)$. Therefore, we obtain the decomposition into Tate motives

$$\overline{\mathcal{R}_p(G)}\simeq\bigoplus_{I\preccurlyeq p^J-1}(\mathbb{Z}/p)(|I|),$$

which gives formula (5.3) for the Poincaré polynomial. □

As a direct consequence of the proof we obtain

5.3.14 Corollary. *Any direct summand of $M(\mathcal{X};\mathbb{Z}/p)$ is isomorphic to a direct sum of twisted copies of $\mathcal{R}_p(G)$.*

Proof. Indeed, in the ring $B^{N,d}_{rat}$ any idempotent is isomorphic to a sum of idempotents $\theta_{L,L,i,i}$, and the map f^0 lifts isomorphisms. □

5.3.15 Remark. Corollary 5.3.14 can be viewed as a particular case of the Krull-Schmidt Theorem proven by V. Chernousov and A. Merkurjev (see [CM06, Corollary 9.7]).

5.4 Motives of generically split flags

Using motivic decompositions of cellular fibrations (see Theorem 4.5.4) we generalize Theorem 5.3.13 to arbitrary generically split projective homogeneous varieties. At the end we discuss some properties of the motive $\mathcal{R}_p(G)$.

5.4.1 Definition. Let G be a linear algebraic group over a field F and let X be a projective homogeneous G-variety. We say X is *generically split* if the group G splits over the generic point of X.

The main result of the present section is the following

5.4.2 Theorem. *Let G be a semisimple linear algebraic group of inner type over a field F and let p be a prime integer. Let X be a generically split projective homogeneous G-variety. Then the motive of X with $\mathbb{Z}/p$-coefficients is isomorphic to the direct sum*

$$M(X;\mathbb{Z}/p) \simeq \bigoplus_{i\geq 0} \mathcal{R}_p(G)(i)^{\oplus a_i},$$

where $\mathcal{R}_p(G)$ is an indecomposable motive; Poincaré polynomial $P(\overline{\mathcal{R}_p(G)},t)$ is given by (5.3) and, hence, only depends on the J-invariant of G; the a_i's are the coefficients of the quotient polynomial

$$\sum_{i\geq 0} a_i t^i = P(\mathrm{CH}^*(\overline{X}),t)/P(\overline{\mathcal{R}_p(G)},t).$$

Proof. Let $\mathcal{X}$ be the variety of complete G-flags. According to Theorem 4.5.4 the motive of $Y=\mathcal{X}$ is isomorphic to a direct sum of twisted copies of the motive of X. To finish the proof we apply Theorem 5.3.13 and Corollary 5.3.14. □

We now provide several properties of $\mathcal{R}_p(G)$ which will be extensively used in the applications.

5.4.3 Proposition. *Let G, G' be two semisimple algebraic groups of inner type over F and let $\mathcal{X}$, $\mathcal{X}'$ be the corresponding varieties of complete flags.*

(i) **(base change)** *For any field extension E/F we have*

$$\mathcal{R}_p(G)_E \simeq \bigoplus_{i\geq 0} \mathcal{R}_p(G_E)(i)^{\oplus a_i},$$

where $\sum a_i t^i = P(\overline{\mathcal{R}_p(G)}, t)/P(\overline{\mathcal{R}_p(G_E)}, t)$.

(ii) **(transfer argument)** *If E/F is a field extension of degree coprime to p then $J_p(G_E) = J_p(G)$ and $\mathcal{R}_p(G_E) = \mathcal{R}_p(G)_E$. Moreover, if $\mathcal{R}_p(G_E) \simeq \mathcal{R}_p(G'_E)$ then $\mathcal{R}_p(G) \simeq \mathcal{R}_p(G')$.*

(iii) **(comparison lemma)** *If G splits over $F(\mathcal{X}')$ and G' splits over $F(\mathcal{X})$ then $\mathcal{R}_p(G) \simeq \mathcal{R}_p(G')$.*

Proof. The first claim follows from Theorem 5.3.13 and Corollary 5.3.14. To prove the second claim note that E is rank preserving with respect to $\mathcal{X}$ and $\mathcal{X} \times \mathcal{X}$ by Lemma 3.5.12. Now $J_p(G_E) = J_p(G)$ by Lemma 3.5.11, and hence $\mathcal{R}_p(G_E) = \mathcal{R}_p(G)_E$ by the first claim. The remaining part of the claim follows from Corollary 3.5.14 applied to the variety $\mathcal{X} \coprod \mathcal{X}'$.

We now prove the last claim. The variety $\mathcal{X} \times \mathcal{X}'$ is the variety of complete $G \times G'$-flags. By Corollary 4.5.2 applied to the projections $\mathcal{X} \times \mathcal{X}' \to \mathcal{X}$ and $\mathcal{X} \times \mathcal{X}' \to \mathcal{X}'$ we can express $M(\mathcal{X} \times \mathcal{X}'; \mathbb{Z}/p)$ in terms of $M(\mathcal{X}; \mathbb{Z}/p)$ and $M(\mathcal{X}'; \mathbb{Z}/p)$. The latter motives can be expressed in terms of $\mathcal{R}_p(G)$ and $\mathcal{R}_p(G')$. The claim now follows from the Krull-Schmidt theorem (see Corollary 5.3.14). $\square$

5.4.4 Corollary. *We have $\mathcal{R}_p(G) \simeq \mathcal{R}_p(G_{an})$, where G_{an} is the semisimple anisotropic kernel of G.*

Finally, we provide conditions which allow one to lift a motivic decomposition of a generically split homogeneous variety with $\mathbb{Z}/m$-coefficients to a decomposition with $\mathbb{Z}$-coefficients.

5.4.5. Let m be a positive integer. We say a polynomial $g(t)$ is *m-positive*, if $g \neq 0$, $P(\overline{\mathcal{R}_p(G)}, t) \mid g(t)$ and the quotient polynomial $g(t)/P(\overline{\mathcal{R}_p(G)}, t)$ has non-negative coefficients for all primes p dividing m.

5.4.6 Proposition. *Let G be a semisimple linear algebraic group of inner type over a field F and let X be a generically split projective homogeneous G-variety. Assume that X splits by a field extension of degree m. Let $f(t)$ be an m-positive polynomial dividing $P(M(\overline{X}), t)$ which can not be presented as a sum of two m-positive polynomials. Then the motive of X with integer coefficients splits as a direct sum*

$$M(X; \mathbb{Z}) \simeq \bigoplus_i \mathcal{R}_i(c_i), \quad c_i \in \mathbb{Z},$$

where $\mathcal{R}_i$ are indecomposable and $P(\overline{\mathcal{R}}_i, t) = f(t)$ for all i. Moreover, if $m = 2, 3, 4$ or 6, then all motives $\mathcal{R}_i$ are isomorphic up to twists.

Proof. First, we apply Corollary 3.5.7 to obtain a decomposition with $\mathbb{Z}/m$-coefficients. By Lemma 3.5.12 our field extension is rank preserving so we can apply Theorem 3.5.19 to lift the decomposition to the category of motives with $\mathbb{Z}$-coefficients. $\square$

Chapter 6

Splitting properties of linear algebraic groups

6.1 Tits indices

To be filled

6.2 Rational cycles and rational bundles

To be filled

6.3 The J-invariant in degree one and the Tits algebras

To be filled

6.4 Higher Tits indices

To be filled

Chapter 7

Applications of motivic decompositions

7.1 Canonical dimensions

The notion of the *canonical dimension* $\mathrm{cd}(G)$ of a linear algebraic group G appears naturally in the study of the splitting properties of G-torsors. It was introduced by Berhuy and Reichstein in [BR05] and developed further by Karpenko and Merkurjev [KM06]. In the present section we discuss the relations between the J-invariant and the values of $\mathrm{cd}(G)$. In particular, we provide computations of canonical p-dimensions $cd_p(G)$ for all split groups G and primes p. This generalizes previously known results by Karpenko, Merkurjev and Reichstein.

7.1.1. According to [KM06, §4] the *canonical dimension* $\mathrm{cd}(X)$ of an irreducible projective variety X over a field F is defined to be the minimum of dimensions of irreducible closed subvarieties Y such that $Y_{F(X)}$ has a rational point. Assuming that $Y_{F(X)}$ has a closed point of degree coprime to p, where p is a fixed prime, we obtain the definition of a canonical p-dimension $\mathrm{cd}_p(X)$ of X. Observe that $\mathrm{cd}_p(X) \leq \mathrm{cd}(X) \leq \dim X$. The canonical ($p$-)dimension of a split group G is equal to the supremum of canonical (p-)dimensions of all twisted forms of the variety of complete G-flags taken over all field extensions of F.

Let G be an inner form of a split group $\hat{G}$. Let $\mathcal{X}$ be the twisted form of the variety of complete $\hat{G}$-flags. We obtain the following formula for the *canonical p-dimension* of $\mathcal{X}$

7.1.2 Proposition. *In the notation of Theorem 5.3.13 we have*

$$\mathrm{cd}_p(\mathcal{X}) = \sum_{i=1}^{r} d_i(p^{j_i} - 1),$$

where $J_p(G) = (j_1, \ldots, j_r)$ is the J-invariant of G.

Proof. By [KM06, Theorem 5.8] we have

$$\mathrm{cd}_p(G) = \min\{i \mid \overline{\mathrm{Ch}}_i(\mathcal{X}) \neq 0\}.$$

The proposition then follows from Proposition 5.3.10. □

7.1.3 Example. (cf. [Za07, Corollary 5]) Applying the formula of the proposition for the maximal value of the J-invariant, i.e. for $j_i = k_i$, by Table 5.2.5 we obtain the following values for the canonical p-dimensions of split groups of types G_2, D_4, F_4, E_6, E_7 and E_8

$\mathrm{cd}_2\,\mathrm{G}_2 = 3,$	$\mathrm{cd}_2\,\mathrm{D}_4^{sc} = 3,$	$\mathrm{cd}_2\,\mathrm{D}_4^{ad} = 9,$
$\mathrm{cd}_2\,\mathrm{F}_4 = 3,$	$\mathrm{cd}_3\,\mathrm{F}_4 = 8,$	
$\mathrm{cd}_2\,\mathrm{E}_6 = 3,$	$\mathrm{cd}_3\,\mathrm{E}_6^{sc} = 8,$	$\mathrm{cd}_3\,\mathrm{E}^{ad} = 16$
$\mathrm{cd}_2\,\mathrm{E}_7^{sc} = 17,$	$\mathrm{cd}_2\,\mathrm{E}_7^{ad} = 18$	$\mathrm{cd}_3\,\mathrm{E}_7 = 8$
$\mathrm{cd}_2\,\mathrm{E}_8 = 60,$	$\mathrm{cd}_3\,\mathrm{E}_8 = 28$	$\mathrm{cd}_5\,\mathrm{E}_8 = 24$

7.2 Degrees of splitting fields

Let X be a smooth projective variety which has a splitting field.

7.2.1 Lemma. *For any $\phi, \psi \in \mathrm{CH}^*(\overline{X} \times \overline{X})$ one has*

$$\deg((\mathrm{pr}_2)_*(\phi \cdot \psi^t)) = \mathrm{tr}((\phi \circ \psi)_\star).$$

Proof. Choose a homogeneous basis $\{e_i\}$ of $\mathrm{CH}^*(\overline{X})$. Let $\{e_i^\vee\}$ be its Poincaré dual. Since both sides of the relation under proof are bilinear, it suffices to check the assertion for $\phi = e_i \times e_j^\vee$ and $\psi = e_k \times e_l^\vee$. In this case both sides of the relation are equal to $\delta_{il}\delta_{jk}$. □

We denote the greatest common divisor of degrees of all zero cycles on X by $d(X)$ and its p-primary component by $d_p(X)$.

7.2.2 Corollary. *Let m be an integer. For any $\phi \in \overline{\mathrm{CH}}(X \times X; \mathbb{Z}/m)$ we have*

$$\gcd(d(X), m) \mid \mathrm{tr}(\phi_\star).$$

Proof. Set $\psi = \Delta_{\overline{X}}$ and apply Lemma 7.2.1. □

7.2.3 Corollary. *Assume that $M(X; \mathbb{Z}/p)$ has a direct summand M. Then*

1. *$d_p(X) \mid P(\overline{M}, 1)$;*

2. *if $d_p(X) = P(\overline{M}, 1)$ and the kernel of the restriction $\mathrm{End}(M(X)) \to \mathrm{End}(M(\overline{X}))$ consists of nilpotents, then M is indecomposable.*

Proof. Set $q = d_p(X)$ for brevity. Let $M = (X, \phi)$. By Corollary 3.5.7 there exists an idempotent $\varphi \in \mathrm{End}(M(X); \mathbb{Z}/q)$ such that $\varphi \mod p = \phi$. Then $res(\varphi) \in \mathrm{End}(M(\overline{X}); \mathbb{Z}/q)$ is a rational idempotent. Since every projective module over $\mathbb{Z}/q$ is free, we have

$$\mathrm{tr}(res(\varphi)_\star) = \mathrm{rk}_{\mathbb{Z}/q}(res(\varphi)_\star) = \mathrm{rk}_{\mathbb{Z}/p}(res(\phi)_\star) = P(\overline{M}, 1) \mod q,$$

and the first claim follows from Corollary 7.2.2. The second claim follows from the first one, since the second assumption implies that for any non-trivial direct summand M' of M we have $P(\overline{M'}, 1) < P(\overline{M}, 1)$. □

7.2.4. Let $n(G)$ denote the greatest common divisor of degrees of all finite splitting fields of G and let $n_p(G)$ be its p-primary component. Note that $n(G) = d(\mathcal{X})$ and $n_p(G) = d_p(\mathcal{X})$.

We obtain the following estimate on $n_p(G)$ in terms of the J-invariant (cf. [EKM, Prop. 88.11] in the case of quadrics).

7.2.5 Proposition. *Let G be a semisimple linear algebraic group of inner type with $J_p(G) = (j_1, \ldots, j_r)$. Then*

$$n_p(G) \leq p^{\sum_i j_i}.$$

Proof. Follows from Theorem 5.3.13 and Corollary 7.2.3. □

7.2.6 Corollary. *The following statements are equivalent:*

- $J_p(G) = (0, \ldots, 0)$*;*
- $n_p(G) = 1$*;*
- $\mathcal{R}_p(G) = \mathbb{Z}/p$*.*

Proof. If $J_p(G) = (0, \ldots, 0)$ then $n_p(G) = 1$ by Proposition 7.2.5. If $n_p(G) = 1$ then there exists a splitting field L of degree m prime to p and, therefore, $\mathcal{R}_p(G) = \mathbb{Z}/p$ by transfer argument 5.4.3(ii). The remaining implication is obvious. □

7.3 Examples of decompositions

In the present section we provide examples of motivic decompositions of projective homogeneous G-varieties obtained using Theorem 5.4.2, where G is a simple group of inner type over a field F. We identify certain groups of small ranks via exceptional isomorphisms, e.g. $\mathrm{B}_2 = \mathrm{C}_2$ (see [INV, §15]).

The case $r = d_1 = 1$. According to Table 5.2.5 $J_p(G) = (j_1)$ for some $j_1 \geq 0$ if and only if $\hat{G}$ is a split group of type A_n or C_n. In this case $p|(n+1)$ or $p = 2$ respectively.

Let $G = {}_\xi\hat{G}$ by means of $\xi \in Z^1(F, \hat{G})$. Let D be a central division algebra of degree d corresponding to the class of the image of ξ in $H^1(F, \mathrm{Aut}\,\hat{G})$ (see [INV, p.396 and p.404]). Observe that $p \mid d$.

Let X_Θ be a projective homogeneous G-variety given by a subset Θ of vertices of the respective Dynkin diagram $\mathcal{D}$. By [Ti66, p.55-56] X_Θ is generically split if $\mathcal{D} \setminus \Theta$ contains at least one r-th vertex with $p \nmid r$ (cf. Example 3.3.7). Observe that the converse is also true by the Index Reduction Formula (see [MPW, Appendix I, III]).

Then by Theorem 5.4.2 we obtain that for a generically split X_Θ

$$M(X_\Theta; \mathbb{Z}/p) \simeq \bigoplus_{i \geq 0} \mathcal{R}_p(G)(i)^{\oplus a_i}, \tag{7.1}$$

where $\mathcal{R}_p(G)$ is indecomposable and

$$\overline{\mathcal{R}_p(G)} \simeq \bigoplus_{i=0}^{p^{j_1}-1} (\mathbb{Z}/p)(i).$$

We now identify $\mathcal{R}_p(G)$. Using the comparison lemma (see Proposition 5.4.3) we conclude that $\mathcal{R}_p(G)$ only depends on D, so $p^{j_1} \mid d$. On the other hand by Proposition 7.2.5 we have $n_p(G) \leq p^{j_1}$. Since $n_p(G)$ is the p-primary part of d, it coincides with p^{j_1} (in the C_n-case it coincides with d).

We have $D \simeq D_p \otimes_F D'$, where $p^{j_1} = \mathrm{ind}(D_p)$ and $p \nmid \mathrm{ind}(D')$. Passing to a splitting field of D' of degree prime to p and using Proposition 5.4.3 we conclude that the motives of X_Θ and $\mathrm{SB}(D_p)$ are direct sums of twisted $\mathcal{R}_p(G)$. Comparing the Poincaré polynomials we conclude that

7.3.1 Lemma. $M(\mathrm{SB}(D_p); \mathbb{Z}/p) \simeq \mathcal{R}_p(G)$.

7.3.2 Corollary. *The motive of* $\mathrm{SB}(D)$ *with integer coefficients is indecomposable.*

Proof. We apply Proposition 5.4.6 to $X = \mathrm{SB}(D)$ and compare the Poincaré polynomials of $M(X)$ and $\mathcal{R}_i$. □

7.3.3 Remark. Indeed, we provided a uniform proof of the results of paper [Ka96]. Namely, the decomposition of $M(\mathrm{SB}(M_m(D)); \mathbb{Z}/p)$ (see [Ka96, Cor. 1.3.2]) and indecomposability of $M(\mathrm{SB}(D); \mathbb{Z})$ (see [Ka96, Thm. 2.2.1]).

The case $r = 1$ and $d_1 > 1$. According to Table 5.2.5 this holds if and only if $\hat{G}$ is a split group of type (here ^{sc} denotes a simply-connected group)

$p = 2$: G_2, F_4, E_6, B_3^{sc}, B_4^{sc}, D_4^{sc}, D_5^{sc};

$p = 3$: F_4, E_7, E_6^{sc};

$p = 5$: E_8.

Moreover, it follows from Table 5.2.5 that in all these cases $k_1 = 1$.

Let $G = {}_\xi\hat{G}$, where $\xi \in Z^1(F, \hat{G})$. Let X be a generically split projective homogeneous G-variety (cf. Example 3.3.7). By Theorem 5.4.2 we obtain the decomposition

$$M(X; \mathbb{Z}/p) \simeq \bigoplus_{i \geq 0} \mathcal{R}_p(G)(i)^{\oplus a_i}, \tag{7.2}$$

where the motive $\mathcal{R}_p(G)$ is indecomposable and (cf. [Vo03, (5.4-5.5)])

$$\overline{\mathcal{R}_p(G)} \simeq \bigoplus_{i=0}^{p-1} (\mathbb{Z}/p)(i \cdot (p+1)).$$

We now identify $\mathcal{R}_p(G)$.

Observe that for any such group G there exists a finite field extension E/F of degree coprime to p such that $G_E = {}_\xi G_{0,E}$, where $\xi \in Z^1(E, \hat{G}^{sc})$. Indeed, for E_6 ($p = 2$) and E_7 ($p = 3$) it follows from the fact that the center of E_6^{sc} is μ_3 and the center of E_7^{sc} is μ_2. For all other groups it is obvious.

Let $\mathfrak{r}$ be the Rost invariant as defined in [Me03]. According to [GP07, Lemma 2.1] if a group is given by a cocycle with values in a simply-connected group, the invariant $\mathfrak{r}$ depends only on the isomorphism class of the group but not on the particular choice of the cocycle. Therefore, the invariant $\mathfrak{r}(G) = \mathfrak{r}(\xi)$ is well-defined over E.

Let $\mathfrak{r}_p$ be the p-component of $\mathfrak{r}$. Observe that for the group G over E the invariant $\mathfrak{r}_p$ takes values in $H^3(E, \mu_p^{\otimes 2})$. Define $\mathfrak{r}_p(G) = \frac{1}{[E:F]} \, cores_{E/F}\, \mathfrak{r}_p(G_E)$. It is easy to see that $\mathfrak{r}_p(G)$ does not depend on the choice of E. Indeed, if E and E' are two such field extensions, we can pass to the composite field $E \cdot E'$ and use the functoriality of $\mathfrak{r}_p$.

7.3.4 Lemma. *Let G be a simple linear algebraic group of inner type over F satisfying $r = 1$ and $d_1 > 1$ and let p be a prime. Then $\mathfrak{r}_p(G)$ is trivial iff $\mathcal{R}_p(G) \simeq \mathbb{Z}/p$.*

Proof. According to [Ga01, Theorem 0.5], [Ch94] and [Gi00, Theoreme 10] the invariant $\mathfrak{r}_p(G)$ is trivial iff the group G splits over a p-primary closure of F. By Corollary 7.2.6 the latter is equivalent to the fact that $\mathcal{R}_p(G) \simeq \mathbb{Z}/p$. □

7.3.5 Lemma. *Let G and G' be simple linear algebraic groups of inner type over F satisfying $r = 1$ and $d_1 > 1$ (observe that in this case $k = 1$).*

If $\mathfrak{r}_p(G) = c \cdot \mathfrak{r}_p(G')$ for some $c \in (\mathbb{Z}/p)^\times$, then $\mathcal{R}_p(G) \simeq \mathcal{R}_p(G')$.

Proof. By transfer arguments (see Proposition 5.4.3) it is enough to prove this over a p-primary closure of F. Let $\mathcal{X}$ and $\mathcal{X}'$ be the respective varieties of complete flags. Observe that the invariant $\mathfrak{r}_p(G)$ becomes trivial over the function field $F(\mathcal{X})$. Since $\mathfrak{r}_p(G) = c \cdot \mathfrak{r}_p(G')$, the invariant becomes trivial over $F(\mathcal{X}')$ as well. By Lemma 7.3.4 $\mathcal{X}$ splits over $F(\mathcal{X}')$. Similarly, $\mathcal{X}'$ splits over $F(\mathcal{X})$.

Therefore, by Lemma 3.3.4 there exists a rational cycle ϕ in $\mathrm{Ch}_{\dim \mathcal{X}}(\overline{\mathcal{X}} \times \overline{\mathcal{X}}')$ of the form $\phi = 1 \times pt + \sum_{\operatorname{codim} \alpha_i > 0} \alpha_i \times \beta_i$. Observe that by definition $\phi_\star \colon pt_{\mathcal{X}} \mapsto pt_{\mathcal{X}'}$. Similarly, interchanging $\mathcal{X}$ and $\mathcal{X}'$ we obtain a rational cycle $\phi' \in \mathrm{Ch}_{\dim \mathcal{X}'}(\mathcal{X}' \times \mathcal{X})$ such that $\phi'_\star \colon pt_{\mathcal{X}'} \mapsto pt_{\mathcal{X}}$. Restricting ϕ and ϕ' to the direct summands $\overline{\mathcal{R}_p(G)}$ and $\overline{\mathcal{R}_p(G')}$ of $M(\overline{\mathcal{X}})$ and $M(\overline{\mathcal{X}}')$ respectively we obtain the rational maps $\phi_R \colon \overline{\mathcal{R}_p(G)} \to \overline{\mathcal{R}_p(G')}$ and $\phi'_R \colon \overline{\mathcal{R}_p(G')} \to \overline{\mathcal{R}_p(G)}$.

Since the motive $\mathcal{R}_p(G)$ is indecomposable and $\operatorname{rk} \mathrm{Ch}^i(\overline{\mathcal{R}_p(G)}) \leq 1$ for all i, the ring of rational endomorphisms of $\overline{\mathcal{R}_p(G)}$ is generated by the identity endomorphism Δ. The same holds for the ring of rational endomorphisms of $\overline{\mathcal{R}_p(G')}$. Since $(\phi'_R)_\star \circ (\phi_R)_\star \colon pt_{\mathcal{X}} \mapsto pt_{\mathcal{X}}$, the composition $\phi'_R \circ \phi_R = \Delta$. Similarly we obtain $\phi_R \circ \phi'_R = \Delta'$. By the Rost nilpotence since ϕ_R and ϕ'_R are rational, the motives $\mathcal{R}_p(G)$ and $\mathcal{R}_p(G')$ are isomorphic. □

$\mathbb{Z}$-coefficients. Let $G = {}_\xi\hat{G}$ be a twisted form by means of a cocycle $\xi \in Z^1(F, \hat{G})$, where $\hat{G}$ is a group of type F_4 or E_6^{sc}. Assume that G is not split by a field extension of degree less than 4. Observe that such a group always splits by an extension of degree 6.

Let X be a generically split projective homogeneous G-variety. Then according to Proposition 5.4.6 the Chow motive of X with integer coefficients splits as a direct sum of twisted copies of an indecomposable motive $\mathcal{R}(G)$ such that

$$\mathcal{R}(G) \otimes \mathbb{Z}/2 = \bigoplus_{i=0,1,2,6,7,8} \mathcal{R}_2(G)(i), \qquad P(\overline{\mathcal{R}_2(G)}, t) = 1 + t^3,$$
$$\mathcal{R}(G) \otimes \mathbb{Z}/3 = \bigoplus_{i=0,1,2,3} \mathcal{R}_3(G)(i), \qquad P(\overline{\mathcal{R}_3(G)}, t) = 1 + t^4 + t^8,$$
$$P(\overline{\mathcal{R}(G)}, t) = 1 + t + t^2 + \ldots + t^{11}.$$

7.3.6 Remark. In particular, we provided a uniform proof of the main results of papers [Bo03] and [NSZ], where the cases of G_2- and F_4-varieties were considered.

The case $r > 1$. According to Table 5.2.5 this holds for split groups $\hat{G}$ of remaining classical types B_n, D_n for $p = 2$; exceptional types E_7, E_8 for $p = 2$ and E_6^{ad}, E_8 for $p = 3$.

Projective Quadric. Consider a generically split projective quadric X. By [EKM, 25.6, 28.2] a quadric X is generically split if and only if it is a Pfister quadric or its codimension one neighbor. Let ϕ be the k-fold Pfister form (or its codimension one neighbor) defining X. Then $X = X_\Theta$ is a projective homogeneous G-variety, where $G = \mathrm{O}(\phi)$ is the orthogonal group of ϕ, and Θ is the subset of the respective Dynkin diagram obtained by removing the first vertex.

Assume $J_2(G) \neq (0, \ldots, 0)$. In view of Corollary 7.2.6 this holds if and only if $n_2(G) \neq 1$. By Springer's Theorem the latter holds if and only if ϕ is not split. By Theorem 5.4.2 we obtain the decomposition

$$M(X;\mathbb{Z}/2) \simeq \bigoplus_{i\geq 0} \mathcal{R}_2(G)(i)^{\oplus a_i}$$

where the motive $\mathcal{R}_2(G)$ is indecomposable. Moreover, by Theorem 3.5.19 the same decomposition holds with $\mathbb{Z}$-coefficients.

We now compute $J_2(G)$. Observe that the group G splits over the function field $F(X)$ and X splits over $F(x)$ for any $x \in X$. It is known (see [EKM, §72]) that the Chow groups $\mathrm{Ch}^l(\overline{X})$ for $l < 2^{k-1}-1$ consist of rational cycles, i.e. the restriction maps $\mathrm{Ch}^l(X) \to \mathrm{Ch}^l(X_{F(x)})$ are surjective for $l < 2^{k-1}-1$. Then by Proposition 5.2.1 and Table 5.2.5 we obtain that $j_i = 0$ for $1 \leq i < 2^{k-2}$. Therefore,

$$J_2(G) = (0,\dots,0,1) \text{ and } P(\overline{\mathcal{R}_2(G)},t) = 1 + t^{2^{k-1}-1}. \tag{7.3}$$

Finally, by Corollary 5.3.14 the motive $\mathcal{R}_2(G)$ coincides with the motive introduced in [Ro98], called the *Rost motive*.

In this way we obtain the Rost decomposition of the motive of a Pfister quadric and its codimension one neighbor.

Maximal orthogonal Grassmannian. Let X be a connected component of a maximal orthogonal Grassmannian. Then $X = X_\Theta$ is a projective homogeneous G-variety, where $G = \mathrm{O}(q)$ is the orthogonal group of q, and Θ is the subset of the respective Dynkin diagram obtained by removing the last vertex. According to [Ti66, p.55-56] X is generically split.

By Theorem 5.4.2 we obtain the decomposition

$$M(X;\mathbb{Z}/2) \simeq \bigoplus_{i\geq 0} \mathcal{R}_2(G)(i)^{\oplus a_i},$$

where the motive $\mathcal{R}_2(G)$ is indecomposable. Comparing the Poincaré polynomials of $M(X;\mathbb{Z}/2)$ and $\mathcal{R}_2(G)$ we obtain the following two extreme cases:

- If the group G is given by a generic cocycle, i.e. q is generic, the motive $M(X;\mathbb{Z}/2)$ coincides with $\mathcal{R}_2(G)$ and, therefore, is indecomposable. This corresponds to the maximal value of the J-invariant.

- If $q = \phi$ is a Pfister form or its codimension one neighbor, then by the previous example $\mathcal{R}_2(G)$ coincides with the Rost motive. This corresponds to the minimal non-trivial value of the J-invariant (7.3).

7.4 Non-hyperbolicity of orthogonal involutions

In the present section we provide an application of the J-invariant to the problem of hyperbolicity of orthogonal involutions. Namely, we prove the following

7.4.1 Theorem. *(cf. [Ga07, Appendix]) Let (A,σ) be a central simple algebra with orthogonal involution over a field F of characteristic $\neq 2$. If $\deg A/\operatorname{ind} A$ is odd, then the involution σ is not hyperbolic over the function field of the Severi-Brauer variety of A.*

7.4.2. The index 2 case is [PSS, Prop. 3.3]. The case of a division algebra, i.e. $\operatorname{ind} A = \deg A$, is [Ka00, Thm. 5.3]. Note that our theorem follows also from [Ka08, Thm. 3.3].

Proof. We may assume that A has index at least 4 and hence degree is divisible by 4. Further, we may assume that (A,σ) is in I^3 as in Example 1.2(3), otherwise the conclusion is obvious.

Consider the groups $G = \mathrm{HSpin}(A,\sigma)$ and $G' = \mathrm{PGL}(A)$. Let X and X' be respective varieties of Borel subgroups.

$$\text{Assume that } \sigma \text{ is hyperbolic over the function field } F(\mathrm{SB}(A)). \qquad (*)$$

Then the group G is split over $F(\mathrm{SB}(A))$ and, hence, over $F(X')$. Since the group G is split over $F(X)$, the algebra A and the group G' are split over $F(X)$.

By Theorem 5.4.2 to any simple linear algebraic group of inner type over k and its torsion prime p one may associate an indecomposable Chow motive $\mathcal{R}_p$ such that over the algebraic closure of F the generating function of $\mathcal{R}_p$ is given by the product of r cyclotomic polynomials

$$\prod_{i=1}^{r} \frac{1-t^{d_i 2^{j_i}}}{1-t^{d_i}}\ , \text{ where } 0 \le j_i \le k_i \text{ and } d_i > 0$$

the explicit values of the parameters d_i and bounds k_i are provided in Table 5.2.5 and the r-tuple of integers $(j_1, j_2, \dots, j_r)$ is the J-invariant.

Let $\mathcal{R}_2(G)$ and $\mathcal{R}_2(G')$ be the respective motives for the groups G and G' and for $p = 2$. By Proposition 5.4.3.(iii) applied to G over $F(X')$ and G' over $F(X)$ we obtain the following motivic reformulation of the assumption $(*)$:

$$\mathcal{R}_2(G) \simeq \mathcal{R}_2(G'). \qquad (**)$$

Since the group G' is a twisted form of the group $\mathrm{PGL}_{\deg A}$, by the first line of Table 5.2.5 the J-invariant of G' has only one entry ($r = 1$), and the parameter d_1 is 1. As in the proof of Lemma 7.3.1 we obtain that the J-invariant is the list consisting of the single element s, where 2^s is the index of A. Hence the generating function of $\mathcal{R}_2(G')$ is $(1-t^{2^s})/(1-t)$.

Similarly, since the group G is a twisted form of the group $\mathrm{HSpin}_{\deg A}$, by Table 5.2.5 the J-invariant of G has $\frac{1}{4}\deg A$ entries with $d_i = 2i-1$ and the following inequality holds

$$j_1 \le k_1 = v_2(\tfrac{1}{2}\deg A) = v_2(2^{s-1}\cdot \tfrac{\deg A}{\operatorname{ind} A}) = s-1 < s,$$

where v_2 is the 2-adic valuation (here we essentially use that $\frac{\deg A}{\operatorname{ind} A}$ is odd).

The isomorphism $(**)$ implies the equality of the respective generating functions, namely

$$\frac{1-t^{2^s}}{1-t} = \prod_{i=1}^{r} \frac{1-t^{(2i-1)2^{j_i}}}{1-t^{2i-1}} \text{ , where } j_1 < s.$$

We claim that it never holds. Indeed, comparing the coefficients at t^2 and t^3 of the polynomials at the left and the right hand side, we conclude that $j_1 \geq 2$ and $j_2 = 0$. Then comparing them consequently at powers t^{2i} and t^{2i+1}, $i \geq 2$ we conclude that $2^{j_1} \geq 2i+1$ and $j_{i+1} = 0$. Therefore, $j_2 = j_3 = \ldots = j_r = 0$ and j_1 must coincide with s, which is not the case, since $j_1 < s$.

Hence, the assumption $(*)$ fails and the theorem is proven. $\square$

7.5 Classification of algebras with involutions of small degrees

To be filled

Chapter 8

Cobordism cycles

In paper [Vi07] A. Vishik using the techniques of symmetric operations in algebraic cobordism (see [Vi06]) proved that changing the base field by the function field of a smooth projective quadric doesn't change the property of being rational for cycles of small codimension. This fact which he calls the Main Tool Lemma plays the crucial role in his construction of fields with u-invariant 2^r+1.

In this chapter we prove the Main Tool Lemma for i) a class of varieties introduced by M. Rost in the proof of the Bloch-Kato conjecture, namely, for varieties which possess a *special correspondence* (see [Ro06, Definition 5.1]), and ii) for projective homogeneous F_4-varieties.

As in Vishik's proof the main technical tools are the algebraic cobordism of M. Levine and F. Morel, the generalised degree formula and the divisibility of Chow traces of certain Landweber-Novikov operations.

In the exposition we follow paper [Za09]. We assume that the base field F has characteristic 0.

8.1 Algebraic cobordism and mod-p operations

In the present section we recall several facts about Landweber-Novikov operations on algebraic cobordism and Rost characteristic numbers. The goal is to introduce certain operations

$$\phi_p^{q(t)}\colon \Omega(X) \to \mathrm{Ch}(X) \text{ parametrised by } q(t) \in \mathrm{Ch}(X)[[t]]$$

from the ring of algebraic cobordism $\Omega(X)$ of a smooth projective variety X over F to the Chow ring $\mathrm{Ch}(X)$ modulo p-torsion with $\mathbb{Z}/p$-coefficients of a smooth variety X, where p is a given prime.

8.1.1. The group $\Omega^m(X)$ of cobordism cycles is generated by classes of proper morphisms $[Z \to X]$ of pure codimension m with Z smooth (see [LM07]). There are cohomological operations on Ω parametrised by partitions called Landweber-Novikov operations and denoted by S_{LN}. These operations commute with pullbacks, satisfy projection and Cartan formulas. We will deal only with operations parametrized by partitions $(p-1, p-1, \dots, p-1)$. Such an operation will be denoted by S^i_{LN}, where i is the length of a partition.

There is a commutative diagram for any integer m

$$\begin{tikzcd}
\Omega^m(X) \arrow[r, "pr"] \arrow[d, "S^i_{LN}"'] & \mathrm{CH}^m(X)/p\text{-tors} \arrow[r] & \mathrm{Ch}^m(X) \arrow[d, "S^i"] \\
\Omega^{m+i(p-1)}(X) \arrow[r, "pr"] & \mathrm{CH}^{m+i(p-1)}(X)/p\text{-tors} \arrow[r] & \mathrm{Ch}^{m+i(p-1)}(X)
\end{tikzcd}$$

where S^i_{LN} is the Landweber-Novikov operation, $pr\colon \Omega(X) \to \mathrm{CH}(X)$ is the canonical morphism of oriented theories and S^i is the i-th reduced p-power operation.

By properties of reduced power operations $S^i = 0$ if $i > m$. By commutativity of the diagram it means that the composite $pr \circ S^i_{LN}$ is divisible by p in the Chow ring modulo p-torsion $\mathrm{CH}^{m+i(p-1)}(X)/p$-tors. Define (cf. [Vi06, 3.3])

$$\phi_p^{t^{(p-1)a}} = \tfrac{1}{p}(pr \circ S^i_{LN}) \mod p, \text{ where } a = i - m > 0. \tag{8.1}$$

If r is not divisible by $(p-1)$, then we set $\phi_p^{t^r} = 0$. Hence, we have constructed an operation $\phi_p^{t^r}$, $r > 0$, which maps $\Omega^m(X)$ to $\mathrm{Ch}^{r+pm}(X)$.

Finally, given a power series $q(t) \in \mathrm{Ch}(X)[[t]]$ define

$$\phi_p^{q(t)} = \sum_{r\geq 0} q_r \phi_p^{t^r}, \text{ where } q(t) = \sum_{r\geq 0} q_r t^r.$$

By the very definition operations $\phi_p^{q(t)}$ are additive and respect pull-backs.

8.1.2 Definition. Let $[U]$ be the class of a smooth projective variety U of dimension d in the Lazard ring $\mathbb{L}_d = \Omega^{-d}(pt)$. Assume that $(p-1)$ divides d. Then the integer $\phi^{t^{dp}}([U]) \in \mathrm{Ch}^0(pt) = \mathbb{Z}/p$ will be called the *Rost number* of U and will be denoted by $\eta_p(U)$.

Using the definition of S_{LN} the number $\eta_p(U)$ can be computed as follows. Let $\xi_1, \xi_2, \dots, \xi_d$ be the roots of the total Chern class of the tangent bundle of U. Define $c(T_U)^{(p)} = \prod_{j=1}^d (1 + \xi_j^{p-1})$. Then

$$\eta_p(U) = \tfrac{1}{p} \deg \left(c(T_U)^{(p)}\right)^{-1}. \tag{8.2}$$

Indeed, it coincides with the number $b_d(U)/p$ introduced in [Ro06, Sect. 9].

8.1.3 Lemma. *(cf. [Vi07, Prop.2.3]) Let U be a smooth projective variety of positive dimension d and $[U]$ be its class in the Lazard ring. Let $\beta \in \Omega^j(X)$. Then*

$$\phi_p^{t^r}([U]\cdot\beta) = \eta_p(U) \cdot pr(S^i_{LN}(\beta)), \textit{ where } r = (p-1)(i-j) + dp > 0.$$

Observe that $\phi_p^{t^r}([U]\cdot\beta) = 0$ if d is not divisible by $(p-1)$.

Proof. Let $q\colon X \to pt$ be the structure map. Then by Cartan formula

$$\tfrac{1}{p}(pr \circ S^i_{LN})([U]\cdot\beta) = \sum_{\alpha+\beta=i} \tfrac{1}{p} pr(S^{\alpha}_{LN}(q^*[U]))\cdot pr(S^{\beta}_{LN}(\beta)).$$

To finish the proof observe that if $\alpha \neq \frac{d}{p-1}$, then $pr(S^{\alpha}_{LN}(q^*[U])) = q^*(pr \circ S^{\alpha}_{LN}([U])) = 0$ in $\mathrm{CH}(X)/p$-tors by dimension reasons. □

8.1.4 Corollary. *(cf. [LM07, Lemma 4.4.20]) Let U and V be smooth projective varieties of positive dimensions. Then $\eta_p(U\cdot V) = 0$ in $\mathbb{Z}/p$.*

Proof. Apply Lemma 8.1.3 to $X = pt$ and $\beta = [V]$. Then $\eta_p(U\cdot V) = \eta_p(U)\cdot pr(S^i_{LN}([V]))$, where the last factor is divisible by p and, hence, becomes trivial modulo p. □

According to [LM07, Remark 4.5.6] the kernel of the canonical morphism $pr\colon \Omega(X)\to \mathrm{Ch}(X)$ is generated by classes of positive dimensions, i.e. $ker(pr) = \mathbb{L}_{>0}\cdot\Omega(X)$. Hence, any $\gamma \in ker(pr)$ can be written as

$$\gamma = \sum_{u_Z\in\mathbb{L}_{>0}} u_Z\cdot[Z\to X]. \tag{8.3}$$

Let $\gamma_{pt}\in\mathbb{L}$ denote the class u_Z corresponding to the point $Z = pt$.

8.1.5 Lemma. *Let $\pi\in\Omega(X\times X)$ be an idempotent and $\gamma\in\Omega(X)$ be such that $pr(\pi_\star(\gamma)-\gamma) = 0$, where $\pi_\star$ is the realization. Then $\eta_p(\pi_\star(\gamma)_{pt}) = \eta_p(\gamma_{pt})$.*

Proof. In presentation (8.3) let $\pi_\star(\gamma)-\gamma = \sum_{u_Z\in\mathbb{L}_{>0}} u_Z\cdot[Z\to X]$. Since π is an idempotent and $\pi_\star$, q_* are $\mathbb{L}$-module homomorphisms, we obtain

$$0 = q_*\pi_\star(\pi_\star(\gamma)-\gamma) = \sum_{u_Z\in\mathbb{L}_{>0}} u_Z\cdot q_*\pi_\star([Z\to X]).$$

Apply η_p to the both sides of the equality. By Corollary 8.1.4 all summands with $\dim Z > 0$ become trivial (modulo p). Hence, $\eta_p(u_{pt}) = 0 \mod p$. □

8.1.6 Corollary. *Let π and γ be as above. Then $\eta_p(q_*(\pi_\star(\gamma)-\gamma)) = 0$, where $q\colon X\to pt$ is the structure map.*

Proof. Observe that $\eta_p(q_*(\pi_\star(\gamma)-\gamma)) = \sum_{u_Z\in\mathbb{L}_{>0}}\eta_p(u_Z\cdot[Z])$, where all summands with $\dim Z > 0$ are trivial by Corollary 8.1.4 and the summand with $Z = pt$ is trivial by Lemma 8.1.5. □

The next important lemma is a direct consequence of the result by M. Rost [Ro06, Lemma 9.3].

8.1.7 Lemma. *Let X be a variety which possesses a special correspondence. Then for any $\gamma\in\Omega_{>0}(X)$ the $\deg pr(S^i_{LN}(\gamma))$ is divisible by p in $\mathrm{CH}(X)/p$-tors.*

Proof. We have $S^\bullet = S_\bullet \cdot c_\Omega(-T_{X_{F_a}})$, where ($S^\bullet$) $S_\bullet$ are the (co-)homological operations. Hence, $\deg pr(S^i_{LN}(\gamma)) = \deg\big(pr(S_\bullet(\gamma))\cdot pr(c_\Omega(-T_{X_{F_a}}))\big)$. Since all Chern classes of the tangent bundle of X_{F_a} are defined over F and X possesses a special correspondence, according to [Ro06, Lemma 9.3] we obtain that

$$\deg\big(pr(S_\bullet(\gamma))\cdot pr(c_\Omega(-T_{X_{F_a}}))\big) = \deg pr(S_i(\gamma)) \mod p.$$

Since $S_\bullet$ respect push-forwards and γ has positive dimension, $\deg pr(S_i(\gamma))$ is divisible by p as well. □

8.2 The main tool lemma

8.2.1. Let L/F be a field extension and X be a variety over F. We say a cycle $z \in \mathrm{CH}(X_L)$ is defined over F if z belongs to the image of the restriction map $res_{L/F}$. Let $\mathrm{Ch}(X)$ denote the Chow group of X modulo p-torsion with $\mathbb{Z}/p$-coefficients, where p is a prime. Let F_a denote the algebraic closure of F.

8.2.2. Let X be a smooth proper irreducible variety over a field F of dimension n, p be a prime and d be an integer $0 \le d \le n$. Assume that X has no zero-cycles of degree coprime to p. We say X is a *d-splitting variety mod p* if for any smooth quasi-projective variety Y over F, for any $m < d$ and for any cycle $y \in \mathrm{Ch}^m(Y_{F_a})$ the following condition holds

$$y \text{ is defined over } k \Longleftrightarrow y_{F_a(X)} \text{ is defined over } F(X). \tag{8.4}$$

8.2.3 Example. Let Q be an anisotropic projective quadric over F of dimension $n > 2$ and $p = 2$. Then according to A. Vishik

(a) Q is a $[\frac{n+1}{2}]$-splitting variety [Vi07, Cor. 3.5.(1)];

(b) Q is a n-splitting variety if and only if Q possesses a Rost projector (the proof is unpublished).

In the present section we prove the following generalisation of 8.2.3.(b)

8.2.4 Theorem. *Let X be a smooth proper irreducible variety of dimension n over a field F of characteristic 0. Assume that X has no zero-cycles of degree coprime to p. If X possesses a special correspondence in the sense of Rost, then X is a $\frac{n}{p-1}$-splitting variety and the value $\frac{n}{p-1}$ is optimal.*

Proof. The proof consists of several steps. First, following Vishik's arguments for a given $y \in \mathrm{Ch}^m(X_{F_a})$ we construct a cycle $\bar\omega$ defined over F in the cobordism ring of the product $\Omega(X_{F_a} \times Y_{F_a})$. To do this we essentially use the surjectivity of the canonical map $pr\colon \Omega \to \mathrm{CH}$. The motivic decomposition of X provides an idempotent cycle π. Applying the realization of π to $\bar\omega$ we obtain a cycle ρ defined over F which can be written in the form (8.5). To finish the proof we apply two operations $\phi_p^{t^r} \circ p_{Y*}$ and $p_{Y*} \circ \phi_p^{t^{r'}}$ to the cycle ρ. The direct computations which are based on the generalised degree formula and [Ro06, Lemma 9.3] show that the difference $(\phi_p^{t^r} \circ p_{Y*} - p_{Y*} \circ \phi_p^{t^{r'}})(\rho)$ is defined over F and provides the cycle y.

I. We start as in the proof of [Vi07, Thm. 3.1]. Let Y be a smooth quasi-projective variety over F. Let $y \in \mathrm{Ch}^m(Y_{F_a})$ be such that $y_{F_a(X)}$ is defined over $F(X)$. We want to show that y is defined over F for all $m < d$.

Consider the commutative diagram

$$\begin{tikzcd}
\omega \arrow[d, mapsto] & \Omega^m(X \times Y) \arrow[r, two heads, "pr"] \arrow[d, "res"'] & \mathrm{Ch}^m(X \times Y) \arrow[r, two heads, "pr_1^*"] \arrow[d, "res"] & \mathrm{Ch}^m(Y_{F(X)}) \arrow[d, "res"] & u \arrow[d, mapsto] \\
\bar{\omega} & \Omega^m(X_{F_a} \times Y_{F_a}) \arrow[r, two heads, "pr"] & \mathrm{Ch}^m(X_{F_a} \times Y_{F_a}) \arrow[r, two heads, "\bar{pr}_1^*"] & \mathrm{Ch}^m(Y_{F_a(X)}) & y_{F_a(X)}
\end{tikzcd}$$

where the pull-back pr_1^* is surjective by the localisation sequence and pr is surjective due to [LM07, Thm.4.5.1]. By the hypothesis there exists a preimage u of $y_{F_a(X)}$ by means of res. By the surjectivity of pr and pr_1^*, there exists a preimage ω of u. Set $\bar{\omega} = res(\omega)$.

II. Let X be a variety which possesses a special correspondence and has no zero-cycles of degree coprime to p. By the results of M. Rost it follows that

(a) $\eta_p(X_{F_a}) \neq 0 \mod p$ (see [Ro06, Thm. 9.9]),

(b) $n = p^s - 1$ (see [Ro06, Cor. 9.12]),

(c) the Chow motive of X contains an indecomposable summand M which over F_a splits as a direct sum of Tate motives twisted by the multiples of $d = \frac{n}{p-1}$ (see [Ro06, Prop. 7.14])

$$M_{F_a} \simeq \bigoplus_{i=0}^{p-1} \mathbb{Z}/p\{di\}.$$

Let π be an idempotent defining the respective Ω-motive M. Then the realization $\rho = \pi_\star(\bar{\omega})$ is defined over F and can be written as (cf. [Vi07, p.368])

$$\rho = x_n \times y_n + \sum_{i=1}^{p-2} x_{di} \times y_{di} + x_0 \times y_0 \in \Omega^m(X_{F_a} \times Y_{F_a}), \tag{8.5}$$

where $x_j \in \Omega_j(X_{F_a})$, $y_j \in \Omega^{m-n+j}(Y_{F_a})$, $x_0 = [pt \hookrightarrow X_{F_a}]$, $x_n = \pi_\star(1)$ and $pr(y_n) = y$ (cf. [Vi07, Lemma 3.2]).

III. Let p_X^* and p_Y^* denote the pull-backs induced by projections $X \times Y \to X, Y$. Since $m < d$, $r = (dp - m)(p - 1) > 0$. Consider the cycle $\phi_p^{t^r}(p_{Y*}(\rho))$. It is defined over k and has codimension m.

8.2.5 Lemma. $\phi_p^{t^r}(p_{Y*}(\rho)) = \eta_p(X_{F_a}) \cdot y + \phi_p^{t^r}(y_0)$ *in* $\mathrm{Ch}^m(Y_{F_a})$.

Proof. By the projection formula

$$\phi_p^{t^r}(p_{Y*}(x_j \times y_j)) = \phi_p^{t^r}(p_{Y*}(p_X^*(x_j) \cdot p_Y^*(y_j))) = \phi_p^{t^r}(q_*(x_j) \cdot y_j),$$

where $x_j \times y_j$ is a summand of (8.5) and $q\colon X \to pt$ is the structure map.

Assume $0 < j < n$. By Lemma 8.1.3 we obtain

$$\phi_p^{t^r}(p_{Y*}(x_j \times y_j)) = \eta_p(q_*(x_j)) \cdot pr(S_{LN}^l(y_j)), \text{ where } l = \tfrac{n-j}{p-1}.$$

Since $codim(y_j) = m - n + j \leq m - n + (p-2)d = m - d < 0$, we have $pr(S_{LN}^l(y_i)) = S^l(pr(y_j)) = 0$ in $\mathrm{Ch}^m(X_{F_a})$. Therefore, only the very right and the left summands of (8.5) remain non-trivial after applying $\phi_p^{t^r} \circ p_{Y*}$.

Now by Lemma 8.1.6 the first summand is equal to

$$\phi_p^{t^r}(p_{Y*}(x_n \times y_n)) = \eta_p(q_*(x_n)) \cdot pr(y_n) = \eta_p(q_*(\pi_*(1))) \cdot y = \eta_p(X_{F_a}) \cdot y.$$

and the last summand $\phi_p^{t^r}(p_{Y*}(x_0 \times y_0)) = \phi^{t^r}(q_*(x_0) \cdot y_0) = \phi_p^{t^r}(y_0)$. □

Since $m < d$, $r' = (d-m)(p-1) > 0$. Consider the cycle $p_{Y*}(\phi_p^{t^{r'}}(\rho))$. It is defined over k and has codimension m.

8.2.6 Lemma. *$p_{Y*}(\phi_p^{t^{r'}}(\rho)) = \phi_p^{t^r}(y_0)$ in $\mathrm{Ch}^m(Y_{F_a})$.*

Proof. By the very definition $p_{Y*}\big(\phi_p^{t^{r'}}(x_j \times y_j)\big) = \frac{1}{p} p_{Y*} pr(S_{LN}^d(x_j \times y_j))$. By the projection and Cartan formulas the latter can be written as

$$\tfrac{1}{p} \deg\big(pr(S_{LN}^a(x_j))\big) \cdot pr(S_{LN}^{d-a}(y_j)), \text{ where } a = \tfrac{j}{p-1}.$$

Since $m < d$, the cycles y_j have negative codimensions for all $j < n$ and, therefore, $pr(S_{LN}^{d-a}(y_j))$ is divisible by p for all $j < n$. On the other hand, by Lemma 8.1.7 the $\deg\big(pr(S_{LN}^a(x_j))\big)$ is divisible by p for all $j > 0$. Hence, $p_{Y*}\big(\phi_p^{t^{r'}}(x_j \times y_j)\big) = 0$ for all $0 < j < n$.

Consider the case $j = n$. Recall that $x_n = \pi_\star(1)$. Then we have

$$\tfrac{1}{p} \deg\big(pr(S_{LN}^d(x_n))\big) = \sum_{u_Z \in \mathbb{L}_{>0}} \eta_p(u_Z) \cdot \deg pr(S_{LN}^{d-a}([Z \to X])),$$

where $a = \frac{\dim Z}{p-1}$ and $x_n - 1 = \sum u_Z[Z \to X]$ in presentation (8.3). Observe that it is trivial mod p, since for all $\dim Z > 0$ the degree of the cycle $pr(S_{LN}^{d-a}([Z \to X]))$ is trivial by Lemma 8.1.7 and for $\dim Z = 0$ the Rost number $\eta_p(u_{pt})$ is trivial by Lemma 8.1.5.

Hence, only the very last summand, i.e. $x_0 \times y_0$, remains non-trivial after applying $p_{Y*} \circ \phi^{t^{r'}}$ which gives $p_{Y*}(\phi^{t^{r'}}(x_0 \times y_0)) = \phi^{t^r}(y_0)$. □

IV. By Lemmas 8.2.5 and 8.2.6 the following cycle is defined over F

$$\phi_p^{t^r}(p_{Y*}(\rho)) - p_{Y_*}(\phi_p^{t^{r'}}(\rho)) = \eta_p(X_{F_a}) \cdot y.$$

Since $\eta_p(X_{F_a}) \neq 0 \mod p$, the cycle y is defined over F, therefore, X is a d-splitting variety.

To see that $d = \frac{n}{p-1}$ is an optimal value take $Y = X$ and consider the cycle $y \in \mathrm{Ch}^d(X_{F_a})$ which generates the Chow group of the Tate motive $\mathbb{Z}/p\{n-d\}$ in the decomposition of M_{F_a} over F_a. Observe that y coincides with the cycle H introduced in [Ro06, Sect. 5]. Since M splits over $F(X)$, $y_{F_a(X)}$ is defined over $F(X)$. By condition (8.4) we obtain that y is defined over F, i.e. M splits over F which contradicts to the indecomposability of M. The theorem is proven. □

8.3 F_4-varieties

In the present section we apply the techniques developed in the proof of 8.2.4 to describe all d-splitting varieties of type F_4. Namely, we prove the following

8.3.1 Corollary. *Let X be a projective homogeneous variety of type F_4 and p be one of its torsion primes (2 or 3). Assume that X has no zero-cycles of degree coprime to p. Then depending on p we have*

$p=2$: *If X is of type F_4/P_4, then X is a $(\dim X)$-splitting variety. For all other types X is a 3-splitting variety and this value is optimal.*

$p=3$: *X is always a 4-splitting variety and this value is optimal.*

Proof. We follow the steps of the proof of Theorem 8.2.4

I. Let X be a smooth geometrically cellular variety over F of dimension n. As in the beginning of the previous section given a cycle $y \in \mathrm{Ch}^m(Y_{F_a})$, where Y is smooth quasi-projective, we construct a cobordism cycle $\bar{\omega} \in \Omega^m(X_{F_a} \times Y_{F_a})$ defined over k.

II. Assume that the motive of X contains a motive $M = (X, \pi)$ such that

$$\pi_{F_a} = \gamma \times \gamma^\vee + \sum_{i=0}^{p-2} x_{di} \times x_{di}^\vee, \text{ where } d > 1,$$

the cycle $\gamma \in \Omega_{(p-1)d}(X_{F_a})$ is defined over F, $\gamma^\vee$ denotes its Poincare dual, i.e. $pr(\gamma \cdot \gamma^\vee) = pt$, and $x_j \in \Omega_j(X_{F_a})$. Then the realization $\rho = \pi_\star(p_X^*(\gamma) \cdot \bar{\omega}) \in \Omega^{m+n-g}(X_{F_a} \times Y_{F_a})$ is defined over F and can be written as (cf. (8.5))

$$\rho = x_g \times y_g + \sum_{i=1}^{p-2} x_{di} \times y_{di} + x_0 \times y_0,$$

where $g = (p-1)d$, $x_g = \pi_\star(\gamma)$ and $pr(y_g) = y$.

The transposed cycle π^t defines an opposite direct summand $M^t = (X, \pi^t)$ of the motive of X (the one which contains the generic point of X over F_a). The realization $\rho' = \pi^t_\star(\bar{\omega})$ is defined over F and can be written as

$$\rho' = x^{(0)} \times y^{(0)} + \sum_{i=1}^{p-2} x^{(di)} \times y^{(di)} + x^{(g)} \times y^{(g)} \in \Omega^m(X_{F_a} \times Y_{F_a}),$$

where $x^{(j)} \in \Omega^j(X_{F_a})$ and $y^{(g)} = y_0$.

III. Since the Chow group of the cellular variety X_{F_a} is torsion-free, we may use Mod-p operations over F_a. We now apply the operations $\phi^{t^r} \circ p_{Y*}$ and $p_{Y*} \circ (p_X^*(\gamma) \cdot \phi^{t^{r'}})$ to the cycles ρ and ρ' respectively.

Applying the first operation and repeating the arguments of the proof of Lemma 8.2.5 we obtain that for any $m < d$, $r = (dp-m)(p-1)$, the following cycle is defined over F

$$\phi_p^{t^r}(p_{Y*}(\rho)) = \eta_p(\gamma) \cdot y + \phi_p^{t^r}(y_0) \text{ in } \mathrm{Ch}^m(Y_{F_a}).$$

To apply the second operation for any $m < d$ consider the cycle $\delta = p_{Y*}(p_X^*(\gamma) \cdot \phi_p^{t^{r'}}(\rho'))$, where $r' = (d-m)(p-1)$. It is defined over k and has codimension m. Since we don't have the version of [Ro06, Lemma 9.3] for an arbitrary variety, to compute δ we have to treat each torsion prime case separately.

For $p = 2$ we obtain $\delta = \phi_p^{t^r}(y_0)$ by dimension reasons. Indeed, in this case the cycle ρ' consists only of two summands

$$\rho' = x^{(0)} \times y^{(0)} + \gamma^\vee \times y^{(g)},$$

where the first summand vanishes, since $S^\alpha_{LN}(x^{(0)}) = 0$ if $|\alpha| > 0$ and the second summand gives the required cycle $\phi_p^{t^r}(y_0)$.

For $p = 3$ the cycle ρ' consists of three terms

$$\rho' = x^{(0)} \times y^{(0)} + x^{(d)} \times y^{(d)} + \gamma^\vee \times y^{(g)},$$

where again the first summand vanishes, the last gives $\phi^{t^r}(y_0)$ and the middle gives

$$\deg\big(pr(\gamma) \cdot pr(S^\alpha_{LN}(x^{(d)}))\big) \cdot \tfrac{1}{p} pr(S^\beta_{LN}(y^{(d)})), \text{ where } \alpha = \beta = d/2. \tag{8.6}$$

Hence, following the part IV of the previous section to prove that X is a d-splitting variety for $p = 2$ or 3 it is enough to assume that $\eta_p(\gamma) \neq 0 \mod p$ and that (8.6) vanishes for $p = 3$.

IV. We use the following notation. Let G be a simple linear algebraic group over F. Recall that a projective homogeneous G-variety X is of type $\mathcal{D}$, if the group G has a root system of type $\mathcal{D}$. Moreover, if X_{F_a} is the variety of parabolic subgroups of G defined by the subset of simple roots S of $\mathcal{D}$, then we say that X is of type $\mathcal{D}/P_S$.

Given an F_4-variety X we provide a cycle γ satisfying $\eta_p(\gamma) \neq 0 \mod p$ as follows

$p = 2$**:** If X is generically split over the 2-primary closure of F, then we may assume that X is of type F_4/P_1. In this case X has dimension 15 and by Theorem 5.4.2 the Chow motive of X with $\mathbb{Z}/p$-coefficients splits as a direct sum of twisted copies of a certain motive $M = (X, \pi)$ with the generating function $P(M_{F_a}, t) = 1 + t^3$. Since the Chow group $\mathrm{Ch}_r(X_{F_a})$ has rank 1 for $r = 0 \dots 3$, the idempotent π_{F_a} can be written as $\pi_{F_a} = \gamma \times \gamma^\vee + pt \times 1$, where γ is represented by a 3-dimensional subquadric $Q_3 \hookrightarrow X_{F_a}$ which is an additive generator of $\mathrm{Ch}_3(X_{F_a})$ defined over k. Since $\eta_2(Q_3) = 1$, X is a d-splitting variety with $d = 3$.

If X is not generically split, i.e. X is of the type F_4/P_4, then X is a splitting variety of the symbol given by the cohomological invariant f_5. By the result of Rost [Ro06, Rem. 2.3 and § 8] X is a variety which possesses a special correspondence. Hence, by Thm. 8.2.4 X is a 15-splitting variety.

$p = 3$**:** In this case all F_4-varieties are generically split over the 3-primary closure of F and we may assume that X is of type F_4/P_4. Similar to the previous case using the motivic decomposition of Theorem 5.4.2 we obtain an idempotent

$$\pi_{F_a} = \gamma \times \gamma^\vee + x_4 \times x_4^\vee + pt \times 1,$$

where $\gamma \in \Omega_8(X_{F_a})$ is defined over F and $x_4 \in \Omega_4(X_{F_a})$ is not. By the explicit formulae from [NSZ, 5.5] we may identify $pr(\gamma)$ with the 7-th power of the generator H of the Picard group of X_{F_a}. which is the only cycle in $\mathrm{Ch}_8(X_{F_a})$ defined over F. Since H is very ample, H^7 can be represented by a smooth projective subvariety Z of X_{F_a}. Hence, we may identify γ with the class $[Z \hookrightarrow X_{F_a}]$. Then the direct computations using the adjunction formula [Fu98, Example 3.2.12] show that $\eta_3(Z) \neq 0 \mod 3$.

To prove the vanishing of the cycle (8.6) it is enough to prove the vanishing of the cycle

$$\deg\big(pr(\gamma) \cdot S^2(pr(x^{(4)}))\big) \cdot \phi_3^{t^{12-2m}}(y^{(4)}).$$

Direct computations show that $S^2(\mathrm{Ch}^4(X_{F_a}))$ is trivial, hence, $\delta = \phi_p^{t^r}(y_0)$ and X is a d-splitting variety with $d = 4$. The Corollary 8.3.1 is proven. □

8.3.2 Remark. Let X be a d-splitting geometrically cellular variety. As an immediate consequence of [KM06, Cor. 4.11] we obtain the following bound for a canonical p-dimension of X

$$\mathrm{cd}_p(X) \geq d.$$

In the case of a variety of type F_4/P_4 it gives $\mathrm{cd}_2(X) = \dim X = 15$.

Chapter 9

Cohomological invariants versus motivic invariants

9.1 Cohomological invariants

9.2 Basic correspondence of a splitting variety (after M. Rost)

9.3 Symbols and Rost motives

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