CLASSIFYING SPACES

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1. Cohomology theories on smooth varieties

1.1. **Chow and Grothendieck rings.** Let X be a smooth integral variety over a field F. We use the following notation:

$\mathrm{CH}(X) = \coprod_{i\geq 0} \mathrm{CH}^i(X)$ is the graded Chow ring of classes of algebraic cycles on X.

$K(X)$ is the Grothendieck ring of any of the following three categories: vector bundles over X, locally free $\mathcal{O}_X$-modules of finite rank, coherent $\mathcal{O}_X$-modules.

$I(X)$ is the kernel of the rank homomorphism $K(X) \to \mathbb{Z}$, $I(X)$ is called the *fundamental ideal* of $K(X)$.

In practice, it is easier to compute $K(X)$ than $\mathrm{CH}(X)$.

1.2. **Chow filtration.** For every $i \geq 1$, let $K(X)^{(i)}$ be the subgroup of $K(X)$ generated by the classes of coherent $\mathcal{O}_X$-modules with codimension of support at least i, or equivalently, by the classes $[\mathcal{O}_Z]$, where $Z \subset X$ is a closed irreducible subset of codimension at least i. In particular, $K(X)^{(1)} = I(X)$.

We have the following finite *Chow filtration* (or *topological filtration*) on $K(X)$:

$$K(X) = K(X)^{(0)} \supset K(X)^{(1)} \supset K(X)^{(2)} \supset \dots$$

We write

$$\mathrm{Chow}^i K(X) := K(X)^{(i)}/K(X)^{(i+1)}.$$

$\mathrm{Chow}\, K(X) = \coprod_{i\geq 0} \mathrm{Chow}^i K(X)$ is a graded ring.

There is a surjective graded ring homomorphism

$$\boxed{\varphi_X : \mathrm{CH}(X) \twoheadrightarrow \mathrm{Chow}\, K(X)}$$

taking the class $[Z]$ of a closed irreducible subset $Z \subset X$ to the class of $\mathcal{O}_Z$.

1.3. **Chern classes.** There are *Chern class* maps

$$c_i : K(X) \to \mathrm{CH}^i(X), \quad i \geq 0,$$

functorial in X, satisfying the following properties:

(1) $c_0(a) = 1$ for all $a \in K(X)$,
(2) For a line bundle $L \to X$, the class $c_1(L)$ in $\mathrm{CH}^1(X)$ is the image of the class of L under the isomorphism $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{CH}^1(X)$,
(3) $c_i(E) = 0$ for a vector bundle $E \to X$ and $i > \mathrm{rank}(E)$,
(4) $c_n(a + b) = \sum_{i+j=n} c_i(a)c_j(b)$, i.e., the *total Chern class* $c_t(a) := \sum_{i\geq 0} c_i(a)t^i$ is additive-multiplicative.

A *splitting principle* asserts that for every element $a \in K(X)$ there is a morphism $Y \to X$ of smooth varieties such that the homomorphism $\mathrm{CH}(X) \to \mathrm{CH}(Y)$ is injective and the image of a in $K(Y)$ is a linear combination of the classes of line bundles with integer coefficients. It follows that the properties as above determine the Chern classes uniquely.

The *Chern subring* of $\mathrm{CH}(X)$ is the graded subring generated by all Chern classes. In general, the Chern subring of $\mathrm{CH}(X)$ is not equal to $\mathrm{CH}(X)$.

The restriction of the Chern class c_i on $K(X)^{(i)}$ is a homomorphism trivial on $K(X)^{(i+1)}$, hence defining a homomorphism

$$\psi^i_X : \mathrm{Chow}^i K(X) \to \mathrm{CH}^i(X).$$

Both compositions of ψ^i_X with φ^i_X are multiplications by $(-1)^{i-1}(i-1)!$. In particular, ψ^i_X with φ^i_X are isomorphisms for $i \leq 2$ and $\mathrm{Ker}(\varphi^i_X)$ is a torsion group killed by $(i-1)!$. It follows that ψ^i_X is isomorphism after tensoring with $\mathbb{Q}$.

There are also K-theoretic *Chern class* maps

$$c^K_i : K(X) \to K(X), \quad i \geq 0,$$

satisfying similar properties, where property (2) should be replaced by

(2′): For a line bundle $L \to X$, the class $c^K_1(L)$ is equal to $1 - [L]^{-1}$.

In fact,

$$c^K_i(a) := \gamma^i(\mathrm{rank}(a) - a^\vee),$$

where γ^i is the gamma-operation defined by $\gamma^t = \lambda_{t/(1-t)}$ with λ the *lambda* operation given by the exterior powers of vector bundles.

Note that $c^K_1(L) \in K(X)^{(1)}$. Moreover, $\mathrm{Im}(c^K_i) \subset K(X)^{(i)}$ for all i. In particular, there are Chern classes

$$\bar{c}^K_i : K(X) \to \mathrm{Chow}^i K(X), \quad i \geq 0.$$

Lemma 1.1. *The following diagram is commutative:*

$$\begin{array}{ccc} K(X) & & \\ {\scriptstyle c_i}\downarrow & \searrow^{\bar{c}^K_i} & \\ \mathrm{CH}^i(X) & \xrightarrow{\varphi^i_X} & \mathrm{Chow}^i K(X) \end{array}$$

Proof. Let $j : D \hookrightarrow X$ be an irreducible divisor. Then for the locally free sheaf $L = \mathcal{L}(D)$, we have $c_1(L) = [D]$. It follows from the exact sequence

$$0 \to \mathcal{L}(-D) \to \mathcal{O}_X \to j_*\mathcal{O}_D \to 0$$

that

$$\varphi(c_1(L)) = \varphi([D]) = j_*[\mathcal{O}_D] = [\mathcal{O}_X] - [\mathcal{L}(-D)] = 1 - L^{-1} = \bar{c}_1(L). \qquad \square$$

1.4. **Chern filtration.** There is another filtration on $K(X)$ that is easier to compute than $K(X)^{(i)}$. Let $K(X)^{[i]}$ be generated by the products $c_{i_1}^K(a_1)\cdots c_{i_n}^K(a_n)$ with $a_j \in K(X)$ and $i_1 + \cdots + i_n \geq i$. This is the smallest ring filtration satisfying $c_i^K(a) \in K(X)^{[i]}$ for all i and $a \in K(X)$. We have

$$K(X) = K(X)^{[0]} \supset K(X)^{[1]} \supset K(X)^{[2]} \supset \dots$$

The formula $a = -c_1^K(a^\vee)$ for all $a \in I(X)$ shows that $K(X)^{[1]} = I(X)$, hence

$$\boxed{I(X)^i \subset K(X)^{[i]} \subset K(X)^{(i)}}$$

We write

$$\operatorname{Chern}^i K(X) := K(X)^{[i]}/K(X)^{[i+1]}.$$

Then $\operatorname{Chern} K(X) = \coprod_{i\geq 0} \operatorname{Chern}^i K(X)$ is the graded ring. We have a diagram of graded ring homomorphisms

$$\begin{array}{ccc} & & \operatorname{Chern} K(X) \\ & & \big\downarrow{\scriptstyle \rho_X} \\ \operatorname{CH}(X) & \xrightarrow{\varphi_X} & \operatorname{Chow} K(X) \end{array}$$

In general, ρ_X is neither injective nor surjective. It is known that

$$I(X) = K(X)^{[1]} = K(X)^{(1)}, \quad K(X)^{[2]} = K(X)^{(2)}, \quad K(X)^{[i]}_{\mathbb{Q}} = K(X)^{(i)}_{\mathbb{Q}}.$$

In particular, $\operatorname{Ker}(\rho_X)$ and $\operatorname{Coker}(\rho_X)$ are torsion groups. We have

$$\operatorname{Chern}^i K(X) = \operatorname{Chow}^i K(X) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \operatorname{CH}^1(X) = \operatorname{Pic}(X), & \text{if } i = 1. \end{cases}$$

Example 1.2. We have $\operatorname{CH}(\mathbb{P}^n_F) = \mathbb{Z}[h]/(h^{n+1})$, where $h \in \operatorname{CH}^1(\mathbb{P}^n_F)$ is the class of a hyperplane section, $K(\mathbb{P}^n_F) = \mathbb{Z}[l]/(l-1)^{n+1}$, where $l = [\mathcal{O}(1)]$. Also, $c_1(l) = h$ and $c_1^K(l) = 1 - l^{-1}$,

$$I(\mathbb{P}^n_F)^i = K(\mathbb{P}^n_F)^{[i]} = K(\mathbb{P}^n_F)^{(i)} = (l-1)^i K(\mathbb{P}^n_F).$$

We have the following properties of the three filtrations on $K(X)$:

$\operatorname{CH}(X)$ is generated by Chern classes	$\Leftarrow$	$\operatorname{CH}(X)$ is generated by $\operatorname{CH}^1(X)$	$\Rightarrow$	$K(X)$ is generated by classes of line bundles
$\Downarrow$		$\Downarrow$		$\Downarrow$
$K(X)^{[i]} = K(X)^{(i)}$		$I(X)^i = K(X)^{[i]} = K(X)^{(i)}$		$I(X)^i = K(X)^{[i]}$
$\Uparrow$				
$\operatorname{Chern} K(X)$ is torsion-free				

For example, suppose $\operatorname{CH}(X)$ is generated by Chern classes. By Lemma 1.1, the ring $\operatorname{Chow} K(X)$ is generated by Chern classes, hence ρ_X is surjective. By descending induction on i, we see that $K(X)^{[i]} = K(X)^{(i)}$.

2. Classifying spaces

2.1. **Torsors.** Let G be a linear algebraic group over F. We just assume that G is of finite type (and don't assume smoothness or connectedness). Suppose G acts on a variety Y (on the right), acts trivially on a variety X and $f : Y \to X$ is a G-equivariant morphism. Consider the morphism

$$\theta : Y \times G \to Y \times_X Y, \quad (y, g) \mapsto (yg, y).$$

We say that f is a *G-torsor* is θ is an isomorphism and f is faithfully flat. The first condition means that for every commutative F-algebra R and every point $x \in X(R)$, either the fiber of $Y(R) \to X(R)$ over x is empty or $G(R)$ acts simply transitively on the fiber. We think of X as a variety of G-orbits in Y and often write $X = Y/G$.

A G-torsor $E \to \operatorname{Spec} F$ is called a *principal homogeneous space* of G.

Example 2.1. GL_n-torsors over X are essentially vector bundles over X of rank n. Precisely, if $E \to X$ is a vector bundle of rank n, then the variety $\operatorname{Iso}_X(\mathbb{1}_X^n, E)$ of isomorphisms between E and the trivial vector bundle $\mathbb{1}_X^n$ is a GL_n-torsor, and every torsor is of this form for a unique vector bundle $E \to X$ up to canonical isomorphism.

2.2. **Descent.** Let $Y \to X$ be a G-torsor and $W \to Y$ be a G-vector bundle (i.e., G acts linearly on W and the morphism is G-equivariant). Let $p_i : Y \times_X Y \to Y$, $i = 1, 2$, be the two projections. We have the two isomorphisms

$$p_i^*(W) := (Y \times_X Y) \times_{Y,p_i} W \simeq W \times G.$$

The automorphism $W \times G \to W \times G$ taking (w, g) to (wg, g) yields a descent data on W in the fppf (flat) topology (an isomorphism $p_1^*(W) \xrightarrow{\sim} p_2^*(W)$ satisfying the cocycle condition). It is known that descent holds for vector bundles (locally free sheaves). Hence the G-vector bundle $W \to Y$ descents to a vector bundle $W/G \to X$. Moreover, is $W' \subset W$ is an open G-invariant subset, then W' descents to an open subset $W'/G \subset W/G$.

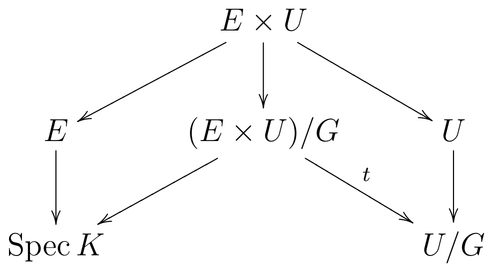
Example 2.2. Let $Y \to X$ be a G-torsor and V a linear G-representation. Then the G-vector bundle $V \times Y \to Y$ descends to a vector bundle $(V \times Y)/G \to X$. We call the Chern classes of this bundle by the Chern classes of the representation (if the torsor is clear).

2.3. **Versal torsors.** A G-torsor $f : Y \to X$ is called *weakly versal* if for every p.h.s. $E \to \operatorname{Spec} K$ with K a field extension of F with K infinite there is a point $x \in X(K)$ such that $E \to \operatorname{Spec} K$ is isomorphic to the pull-back of f with respect to x.

$$\begin{array}{ccc} E & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ \operatorname{Spec} K & \xrightarrow{x} & X \end{array}$$

We say that $f : Y \to X$ is *versal* if for every nonempty subset $U \subset X$, the G-torsor $f^{-1}(U) \to U$ is weakly versal.

Let V be a G-representation over F and let $U \subset V$ be a nonempty G-invariant open subset such that there is a G-torsor $f : U \to U/G$. Then f is a versal G-torsor. Indeed, let $E \to \operatorname{Spec} K$ be a G-torsor, where K is a field extension of F with K infinite. Consider the diagram



with two fiber squares. As $(E\times U)/G$ is an open subset in the vector bundle (vector space) $(E \times V)/G$ over K and K is infinite, there is a rational point $s : \operatorname{Spec} K \to (E \times U)/G$ over K. Then $E \to \operatorname{Spec} K$ is the pull-back of f with respect to the composition $t \circ s$.

A morphism of varieties $f : Y \to X$ over a field F is called *weakly split* if there is a rational morphism $g : X \dashrightarrow Y$ such that $f \circ g$ is the identity of X. We say that f is *split* if for every nonempty open subset $U \subset Y$ there is a rational morphism $g : X \dashrightarrow Y$ such that $\operatorname{Im}(g) \cap U \neq \emptyset$ and $f \circ g = \mathrm{id}_X$.

A variety X over F is *weakly retract rational* (respectively, *retract rational*) if there is a nonempty open subvariety $Y \subset \mathbb{A}^n_F$ for some n and a weakly split (respectively, split) morphism $f : Y \to X$ over F.

Every stably rational variety is retract rational and hence weakly retract rational.

We say that the G-torsors over field extensions of F are *rationally parameterized* if there is a versal G-torsor $Y \to X$ with X a rational variety.

Proposition 2.3. *The G-torsors over field extensions of F are rationally parameterized if and only if the classifying space* BG *is retract rational over F.*

Example 2.4. Let n be a positive integer that is neither divisible by 8 nor by p^2, where p is an odd prime. Then BPGL(n) is retract rational. I conjecture that otherwise, BPGL(n) is not retract rational.

Example 2.5. The space BSpin(n) is retract rational if $n \leq 14$ This follows from the classification of quadratic forms with trivial discriminant and Clifford invariant of dimension at most 14. I conjecture that BSpin(n) is not retract rational if $n \geq 15$.

2.4. **Approximations of** BG**.** In topology the classifying space BG of a topological group G is defined as EG/G, where EG is a contractible space with a free G-action.

Let G be an algebraic group over F. We don't define BG (it makes sense as an algebraic stack but not an algebraic variety), but we define "approximations" of BG as algebraic varieties.

Let V be a G-representation over F and $U \subset V$ be a nonempty G-invariant open subset such that there is a G-torsor $f : U \to U/G$. We say that U/G is an *n-approximation of* BG if $\operatorname{codim}_V(V \setminus U) \geq n$. Every group G admits n-approximations for every n.

Example 2.6. Embed $G \hookrightarrow \mathrm{GL}(m)$ with $m > 0$ and choose an integer $N \geq 0$. Let U_N be the open subset of all injective linear maps $F^m \to F^{m+N}$ in the vector space $V = \operatorname{Hom}(F^m, F^{m+N})$. We have $\operatorname{codim}_V(V \setminus U_N) = N + 1$. The group $\mathrm{GL}(m + N)$ acts linearly (by composition) on V (on the left) and acts transitively on U_N with the stabilizer

$\begin{pmatrix} 1_m & * \\ 0 & \mathrm{GL}(N) \end{pmatrix}$ of the canonical inclusion $F^m \hookrightarrow F^m \oplus F^N = F^{m+N}$. The group G acts on U_N via the right action of $\mathrm{GL}(m)$ on U_N by composition and if we define

$$U_N/G := \mathrm{GL}(m+N)/\begin{pmatrix} G & * \\ 0 & \mathrm{GL}(N) \end{pmatrix}$$

we have a G-torsor $U_N \to U_N/G$ and U_N/G is an $(N+1)$-approximation of $\mathrm{B}G$. Note that $U_N/\mathrm{GL}(m)$ is naturally isomorphic to the Grassmannian variety $\mathrm{Gr}(m, m+N)$.

2.5. **Definition of $\mathrm{CH}(\mathrm{B}G)$ by Totaro.** Fix $i \geq 0$. Let U/G and U'/G be two n-approximations of $\mathrm{B}G$ for $n > i$. The "roof" diagram

$$\begin{array}{ccccc}
(U \times V')/G & \xleftarrow{\ open\ } & (U \times U')/G & \xrightarrow{\ open\ } & (V \times U')/G \\
{\scriptstyle vector}\big\downarrow{\scriptstyle bundle} & \swarrow & & \searrow & {\scriptstyle vector}\big\downarrow{\scriptstyle bundle} \\
U/G & & & & U'/G
\end{array}$$

yields two pull-back isomorphisms

$$\mathrm{CH}^i(U/G) \xleftarrow{\sim} \mathrm{CH}^i((U \times U')/G) \xrightarrow{\sim} \mathrm{CH}^i(U'/G)$$

since $i < n$. In other words, the group $\mathrm{CH}^i(U/G)$ does not depend up to canonical isomorphism on the choice of an n-approximations U/G of $\mathrm{B}G$ with $n > i$. We set

$$\mathrm{CH}^i(\mathrm{B}G) := \mathrm{CH}^i(U/G).$$

Example 2.7. Let $G = \mathbb{G}_m$ over F. Consider the standard action of $\mathbb{G}_m$ on $\mathbb{A}^n_F$. Taking $U_n := \mathbb{A}^n_F \setminus \{0\}$ we get an n-approximation $\mathbb{P}^{n-1}_F = U_n/\mathbb{G}_m$ of $\mathrm{B}\mathbb{G}_m$. It follows that

$$\mathrm{CH}(\mathrm{B}\mathbb{G}_m) = \mathbb{Z}[h],$$

where h is the class of a hyperplane section. Let T be a split torus and $x \in \widehat{T} = \mathrm{Hom}(T, \mathbb{G}_m)$ a character of T. Choose an approximation U/T of $\mathrm{B}T$. Let L_x be the line bundle $(\mathbb{A}^1 \times U)/T \to U/T$, where T acts on $\mathbb{A}^1$ via the character x. (For example, if $T = \mathbb{G}_m$, the line bundle L_x for the tautological character x is the canonical line bundle on $\mathbb{P}^{n-1}$ (having the nickname $\mathcal{O}(1)$) with $c_1(L_x) = h$.) The map $\widehat{T} \to \mathrm{CH}^1(\mathrm{B}T)$ taking a character x to $c_1(L_x)$ extends to an isomorphism

$$\mathrm{CH}(\mathrm{B}T) \simeq \mathrm{Sym}(\widehat{T}),$$

where Sym is the symmetric ring.

3. Invariants of G-torsors

3.1. **Definition.** Let Q be a contravariant functor from *Sm*(F) to *AbGroups*. Let G be a group over F. An *invariant* of G with values in Q is an assignment to every G-torsor $E \to X$ over a smooth variety X over F an element in $Q(X)$. We assume that this assignment is functorial in X. All invariants of G with values in Q form an abelian group $\mathrm{Inv}(G, Q)$.

We consider invariants of G with values in CH^i for a given i. We get a graded ring

$$\mathrm{Inv}(G, \mathrm{CH}) := \coprod_{i \geq 0} \mathrm{Inv}(G, \mathrm{CH}^i).$$

Example 3.1. In view of Example 2.1, a polynomial of classical Chern classes c_k with integer coefficients yields an invariant of the group GL_n.

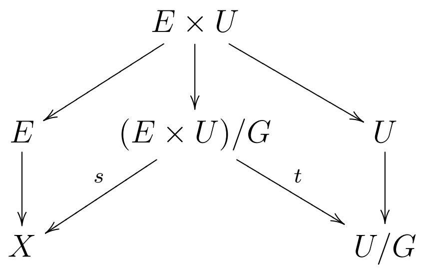
If $a \in \mathrm{Inv}(G, \mathrm{CH}^i)$ we define an element $\alpha(a)$ in $\mathrm{CH}^i(\mathrm{B}G)$ as follows. Choose an n-approximation U/G of $\mathrm{B}G$ where $U \subset V$ and $n > i$ and set

$$\alpha(a) = a(U \to U/G) \in \mathrm{CH}^i(U/G) = \mathrm{CH}^i(\mathrm{B}G).$$

Thus, we get a homomorphism

$$\alpha : \mathrm{Inv}(G, \mathrm{CH}^i) \to \mathrm{CH}^i(\mathrm{B}G).$$

We can define a homomorphism in the other direction. Choose an n-approximation U/G of $\mathrm{B}G$ for $n > i$ as above. Let $E \to X$ be a G-torsor with smooth X. Consider the "roof" diagram



By assumption, the pull back homomorphism $s^* : \mathrm{CH}^i(X) \to \mathrm{CH}^i((E \times U)/G)$ is an isomorphism since it is equal to the composition of two isomorphisms

$$\mathrm{CH}^i(X) \xrightarrow{\sim} \mathrm{CH}^i((E \times V)/G) \xrightarrow{\sim} \mathrm{CH}^i((E \times U)/G).$$

The first map is homotopy invariance isomorphism, the second map is an isomorphism since $n > i$.

Define

$$\beta : \mathrm{CH}^i(\mathrm{B}G) \to \mathrm{Inv}(G, \mathrm{CH}^i)$$

by

$$\beta(c)(E \to X) = (s^*)^{-1}(t^*(c))$$

for any $c \in \mathrm{CH}^i(U/G) = \mathrm{CH}^i(\mathrm{B}G)$.

Theorem 3.2. *(Totaro) The maps α and β are isomorphisms inverse to each other,* $\mathrm{Inv}(G, \mathrm{CH}) \simeq \mathrm{CH}(\mathrm{B}G)$.

3.2. **Torsors trivial in Zariski topology.** We say that a G-torsor $E \to X$ is *Zariski trivial* if there is a Zariski cover $X = \cup U_i$ such that the restrictions $E|_{U_i} \to U_i$ are trivial torsors for all i.

A *Zariski invariant* of G with values in a functor Q is a functorial assignment to every Zariski trivial G-torsor $E \to X$ over a smooth variety X an element in $Q(X)$. All Zariski invariants with values in Q form an abelian group $\mathrm{Inv}_{Zar}(G, Q)$.

We study the graded ring

$$\mathrm{Inv}_{Zar}(G,\mathrm{CH}) := \coprod_{i\geq 0} \mathrm{Inv}_{Zar}(G,\mathrm{CH}^i).$$

We have the restriction graded ring homomorphism

$$\mathrm{Inv}(G,\mathrm{CH}) \to \mathrm{Inv}_{Zar}(G,\mathrm{CH}).$$

Now assume that G is a split reductive group with maximal split torus T. If a G-torsor is split at the generic point, then by a theorem of Colliot-Thélène and Ojanguren, the torsor is Zariski trivial. In particular, if G is a special group, every G-torsor is Zariski trivial, hence $\mathrm{Inv}(G,\mathrm{CH}) = \mathrm{Inv}_{Zar}(G,\mathrm{CH})$. In particular, by Example 2.7 and Theorem 3.2,

$$\mathrm{Inv}_{Zar}(T,\mathrm{CH}) = \mathrm{Inv}(T,\mathrm{CH}) = \mathrm{CH}(\mathrm{B}T) = \mathrm{Sym}(\widehat{T}).$$

Let N be the normalizer of T in G and $W = N/T$ the Weyl group of G. Note that the natural action of N by conjugation on the G-torsors is trivial and the N-action on the T-torsors factors through a W-action. Hence we have a restriction homomorphism

$$\mathrm{Inv}_{Zar}(G,\mathrm{CH}^i) \to \mathrm{Inv}_{Zar}(T,\mathrm{CH}^i)^W = \mathrm{Sym}^i(\widehat{T})^W.$$

Theorem 3.3. *(Edidin-Graham) Let G be a split reductive group with maximal split torus T and the Weyl group W. Then the map $\mathrm{Inv}_{Zar}(G,\mathrm{CH}) \to \mathrm{Sym}(\widehat{T})^W$ is an isomorphism.*

Proof. (Injectivity) Let $a \in \mathrm{Inv}_{Zar}(G,\mathrm{CH}^i)$ be such that $a|_T = 0$ and $p : E \to X$ a Zariski trivial G-torsor. We show that $a(E \to X) = 0$. The push-forward of the torsor $E \to E/T$ with respect to the inclusion of T to G is isomorphic to the pull-back of $E \to X$ under $q : E/T \to X$. Therefore,

$$q^*(a(E \to X)) = (\mathrm{res}\, a)(E \to E/T) = 0.$$

It suffices to show that $q^* : \mathrm{CH}(X) \to \mathrm{CH}(E/T)$ is injective. If B is a Borel subgroup containing T, all fibers of the projection $E/T \to E/B$ are affine spaces, the pull-back map $\mathrm{CH}(E/B) \to \mathrm{CH}(E/T)$ is an isomorphism, so we can replace T by B.

As $p : E \to X$ is Zariski trivial, there is a rational section of p, hence there is a rational section of $q : E/B \to X$. Let $Z \subset E/B$ be the closure of the image of this section. We have $q_*([Z]) = 1$ in $\mathrm{CH}(X)$. By the Projection Formula,

$$c = c \cdot q_*([Z]) = q_*(q^*(c) \cdot [Z])$$

for every $c \in \mathrm{CH}(X)$, hence q^* is split injective. □

Corollary 3.4. *If G is a special split reductive group, then the natural homomorphism, the restriction homomorphism $\mathrm{CH}(\mathrm{B}G) \to \mathrm{Sym}(\widehat{T})^W$ is an isomorphism.*

Remark 3.5. If G is a split reductive group (not necessarily special), the homomorphism $\mathrm{CH}(\mathrm{B}G) \to \mathrm{Sym}(\widehat{T})^W$ is an isomorphism after tensoring with $\mathbb{Q}$. It follows that the kernel this homomorphism coincides with $\mathrm{CH}(\mathrm{B}G)_{tors}$.

Question 3.6. *How to compute $\mathrm{Inv}_{Zar}(G,\mathrm{CH}^i)$ for an arbitrary reductive group G?*

Example 3.7. Let $G = \mathrm{GL}(n)$ (this is a special group), T the torus of diagonal matrices and $W = S_n$. Hence

$$\mathrm{Inv}(\mathrm{GL}(n), \mathrm{CH}) = \mathrm{CH}(\mathrm{BGL}(n)) = \mathrm{Sym}(\widehat{T})^W = \mathbb{Z}[x_1, x_2, \dots, x_n]^{S_n} = \mathbb{Z}[c_1, c_2, \dots, c_n],$$

where c_i are classical Chern classes (symmetric functions on the x_i). Thus, every invariant on vector bundles of rank n with values in CH is a polynomial in the Chern classes $c_1, c_2, \dots, c_n$. Similarly,

$$\mathrm{CH}(\mathrm{BSL}(n)) = \mathbb{Z}[c_2, \dots, c_n].$$

3.3. **Some computations of** $\mathrm{CH}(\mathrm{B}G)$**.** Embed a group G into $\mathrm{GL}(m)$ for some m. We can choose n-approximations $U/\mathrm{GL}(m)$ and U/G of $\mathrm{BGL}(m)$ and $\mathrm{B}G$, respectively. The morphism $U/G \to U/\mathrm{GL}(m)$ yields a graded ring homomorphism

$$\mathbb{Z}[c_1, c_2, \dots, c_m] = \mathrm{CH}(\mathrm{BGL}(m)) \to \mathrm{CH}(\mathrm{B}G).$$

We also denote by c_i their images in $\mathrm{CH}(\mathrm{B}G)$.

Lemma 3.8. *Suppose* $\mathrm{CH}^i(\mathrm{GL}(m)/G) = 0$ *for all* $i > 0$*. Then the homomorphism* $\mathrm{CH}(\mathrm{BGL}(m)) \to \mathrm{CH}(\mathrm{B}G)$ *is surjective, i.e.,* $\mathrm{CH}(\mathrm{B}G)$ *is generated by* $c_1, c_2, \dots, c_m$*.*

Proof. As the group $\mathrm{GL}(m)$ is special, all fibers of the morphism $\mathrm{B}G \to \mathrm{BGL}(m)$ are isomorphic to $\mathrm{GL}(m)/G$. The result follows from Rost's spectral sequence for this morphism. □

Remark 3.9. The generators $c_1, c_2, \dots, c_m$ of $\mathrm{CH}(\mathrm{B}G)$ are the Chern classes of the representation $G \hookrightarrow \mathrm{GL}(m)$.

Example 3.10. Let $G = \mathrm{Sp}(2n)$ (this is a special group) and consider the natural embedding of G into $\mathrm{GL}(2n)$. Then $\mathrm{GL}(2n)/G$ is isomorphic to the variety of nondegenerate symplectic forms of dimension $2n$ that is an open subset of an affine space. By Lemma 3.8, $\mathrm{CH}(\mathrm{Sp}(2n))$ is generated by $c_1, c_2, \dots, c_{2n}$. Note that for every vector bundle $E \to X$ with a nondegenerate symplectic form, the dual bundle $E^\vee$ is isomorphic to E, we have $2c_i(E) = 0$ if i is odd. By Corollary 3.4, $\mathrm{CH}(\mathrm{Sp}(2n))$ is torsion free, hence $c_i(E) = 0$ if i is odd. Restricting to a maximal split torus, we see that every nonzero polynomial in c_i with i even yields a nontrivial element in $\mathrm{CH}(\mathrm{Sp}(2n))$. Hence,

$$\mathrm{CH}(\mathrm{Sp}(2n)) = \mathbb{Z}[c_2, c_4, \dots, c_{2n}].$$

Example 3.11. Consider the natural embedding of the split orthogonal group $G = \mathrm{O}(n)$ into $\mathrm{GL}(n)$. Then $\mathrm{GL}(n)/G$ is isomorphic to the variety of nondegenerate quadratic forms of dimension n that is an open subset of an affine space. By the same argument as in Example 3.10, $\mathrm{CH}(\mathrm{O}(n))$ is generated by $c_1, c_2, \dots, c_n$ and $2c_i(E) = 0$ if i is odd. In fact, we have

$$\mathrm{CH}(\mathrm{O}(n)) = \mathbb{Z}[c_1, c_2, \dots, c_n]/(2c_i = 0,\ i \text{ odd}),$$

if $\mathrm{char}(F) \neq 2$.

Example 3.12. Since $\mathrm{O}(2m+1) = \mu_2 \times \mathrm{O}^+(2m+1)$, we have

$$\mathrm{CH}(\mathrm{O}^+(2m+1)) = \mathbb{Z}[c_2, c_3, \dots, c_{2m+1}]/(2c_i = 0,\ i \text{ odd}),$$

if $\mathrm{char}(F) \neq 2$.

Example 3.13. The ring $\mathrm{CH}(\mathrm{BO}^+(2m))$ is not generated by Chern classes if $m \geq 3$.

Example 3.14. Let T be a quasi-trivial torus, so T is a special group. Write $T = \mathrm{GL}_A(1)$, where A is an étale F-algebra. Similar to the split case, the variety $X_n = R_{A/F}(\mathbb{P}^n_A)$ is an approximation of $\mathrm{B}T$. The Chow motive of X_n is a direct sum of twists of 0-dimensional motives of the form $\operatorname{Spec} L$, where L/F is a separable finite field extension. For such motives the Chow groups satisfy Galois descent. It follows that

$$\mathrm{CH}^i(\mathrm{B}T) = \mathrm{CH}^i(X_n) = \mathrm{CH}^i(X_{n,\,\mathrm{sep}})^\Gamma = \mathrm{CH}^i(\mathrm{B}T_{\mathrm{sep}})^\Gamma = \mathrm{Sym}^i(\widehat{T}_{\mathrm{sep}})^\Gamma$$

for $i \leq n$ (here $\Gamma = \mathrm{Gal}(F_{\mathrm{sep}}/F)$). Thus,

$$\mathrm{CH}(\mathrm{B}T) \simeq \mathrm{Sym}(\widehat{T}_{\mathrm{sep}})^\Gamma.$$

Example 3.15. Let K/F be a cyclic cubic field extension and let $T = \mathrm{GL}_K(1)$. The character lattice $\widehat{T}_{\mathrm{sep}}$ has a basis $\{a, b, c\}$ cyclically permuted by the Galois group Γ. Therefore, $a^2b + b^2c + c^2a \in \mathbb{Z}[a, b, c]^\Gamma = \mathrm{CH}(\mathrm{B}T)$. A computation shows that every element in the Chern subring of $\mathrm{CH}(\mathrm{B}T)$ is stable modulo 2 under the action of the symmetric group S_3. Thus $\mathrm{CH}(\mathrm{B}T)$ is not generated by Chern classes, although T is a special group.

4. Representation ring $R(G)$

This is joint work with N. Karpenko.

Let G be an algebraic group over F. Write $R(G)$ for the representation ring of G. As an abelian group, $R(G)$ is free with basis the classes of irreducible representations. We think of $R(G)$ as an analog of the Grothendieck group $K(\mathrm{B}G)$.

Example 4.1. If T is a split torus, every irreducible representation of T is 1-dimensional, thus given by a character of T. Therefore,

$$R(T) = \mathbb{Z}[\widehat{T}].$$

It $x \in \widehat{T}$, we write e^x for the corresponding element in $R(T)$, so $e^{x+y} = e^x e^y$. If G is a split reductive group with a split maximal torus T, then the restriction homomorphism

$$R(G) \to R(T)^W = \mathbb{Z}[\widehat{T}]^W$$

is an isomorphism. (Note that similar homomorphism $\mathrm{CH}(\mathrm{B}G) \to \mathrm{Sym}(\widehat{T})^W$ is an isomorphism for special groups G but not isomorphism in general.)

Let $E \to X$ be a G-torsor, where X is a smooth variety. We have a canonical ring homomorphism

$$\alpha_E : R(G) \to K(X),$$

taking the class of a G-representation W to the class of the vector bundle

$$(W \times E)/G \to X.$$

We apply this to the G-torsor $U \to U/G$, where U/G is an n-approximation of $\mathrm{B}G$ for some n and $U \subset V$. In fact the resulting homomorphism

$$\alpha_U : R(G) \to K(U/G)$$

is surjective since it is equal to the composition

$$R(G) = K^G(pt) \xrightarrow{\sim} K^G(V) \twoheadrightarrow K^G(U) \xrightarrow{\sim} K(U/G).$$

Composing α_U with (classical) Chern classes on U/G yields Chern classes

$$c_i : R(G) \to \mathrm{CH}^i(\mathrm{B}G)$$

for $i < n$.

Example 4.2. The generators c_i of the ring $\mathrm{CH}^i(\mathrm{BGL}(n))$ are the Chern classes of the tautological representation of $\mathrm{GL}(n)$.

The augmentation ideal $I(G) \subset R(G)$ is the kernel of the ring homomorphism $R(G) \to \mathbb{Z}$ given by dimension of G-representations. The augmentation filtration on $R(G)$ is given by the powers $I(G)^i$, $i \geq 0$, of the augmentation ideal.

We simply write c_i^R for the Chern classes defined by

$$c_i^R(a) := \gamma^i(\mathrm{rank}(a) - a^\vee).$$

If a is a character of G (a 1-dimensional representation of G), then $c_1^R(a) = 1 - a^{-1}$.

We have the smallest ring *Chern filtration*

$$R(G) = R(G)^{[0]} \supset R(G)^{[1]} \supset \dots$$

with the property that $c_i^R(x) \in R(G)^{[i]}$ for all $x \in R(G)$ and any $i \geq 0$. We write $\mathrm{Chern}\, R(G)$ for the associated graded ring.

4.1. **Chow filtration on $R(G)$.** Our next goal is to define the Chow filtration on $R(G)$. Let U/G be an n-approximation of $\mathrm{B}G$ and $i \leq n$. We set

$$R(G)^{(i)} := (\alpha_U)^{-1}(K(U/G)^{(i)}).$$

This does not depend on the choice of the approximation U/G. We get the *Chow filtration*

$$R(G) = R(G)^{(0)} \supset R(G)^{(1)} \supset \dots$$

on $R(G)$.

We have

$$I(G)^i \subset R(G)^{[i]} \subset R(G)^{(i)}$$

for all i. (However none of the filtrations is finite in general.) The second inclusion induces a ring homomorphism $\mathrm{Chern}\, R(G) \to \mathrm{Chow}\, R(G)$ which is neither injective nor surjective in general.

Let U/G be an n-approximation of $\mathrm{B}G$ and $i < n$. The composition

$$\mathrm{CH}^i(\mathrm{B}G) = \mathrm{CH}^i(U/G) \xrightarrow{\varphi^i} \mathrm{Chow}^i K(U/G) = \mathrm{Chow}^i R(G)$$

yields a surjective graded ring homomorphism

$$\varphi_G : \mathrm{CH}(\mathrm{B}G) \twoheadrightarrow \mathrm{Chow}\, R(G).$$

We have a diagram

$$\begin{array}{ccc} & & \mathrm{Chern}\, R(G) \\ & & \downarrow{\scriptstyle \rho_G} \\ \mathrm{CH}(\mathrm{B}G) & \xrightarrow{\varphi_G} & \mathrm{Chow}\, R(G). \end{array}$$

The kernel of $\varphi^i : \mathrm{CH}^i(\mathrm{B}G) \to \mathrm{Chow}^i R(G)$ is killed by multiplication by $(i-1)!$. In particular, the maps φ^i are isomorphisms for $i \leq 2$. Also, $\varphi(c_i(a)) = c_i^R(a)$ modulo $R(G)^{(i+1)}$ for every $a \in R(G)$.

We have $I(G) = R(G)^{[1]} = R(G)^{(1)}$ and $R(G)^{[2]} = R(G)^{(2)}$. The map $\mathrm{Chern}\, R(G) \to \mathrm{Chow}\, R(G)$ becomes an isomorphism after tensoring with $\mathbb{Q}$ and

$$\mathrm{Chern}^i R(X) = \mathrm{Chow}^i R(X) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \widehat{G} = \mathrm{Pic}(\mathrm{B}G), & \text{if } i = 1. \end{cases}$$

$\mathrm{CH}(\mathrm{B}G)$ is generated by Chern classes	$\Leftarrow$	$\mathrm{CH}(\mathrm{B}G)$ is generated by $\mathrm{CH}^1(\mathrm{B}G)$	$\Rightarrow$	$R(G)$ is generated by classes of line bundles
$\Downarrow$		$\Downarrow$		$\Downarrow$
$R(G)^{[i]} = R(G)^{(i)}$		$I(G)^i = R(G)^{[i]} = R(G)^{(i)}$		$I(G)^i = R(G)^{[i]}$
$\Uparrow$				
$\mathrm{Chern}\, R(G)$ is torsion-free				

Example 4.3. For $G := O^+(2n)$ with any $n \geq 3$ over the field of complex numbers, the Chern filtration on $R(G)$ differs from the Chow filtration. Indeed, the Chow ring $\mathrm{CH}(\mathrm{B}G)$ is not generated by Chern classes. By calculation, the Chern subring of $\mathrm{CH}(\mathrm{B}G)$ (i.e., the subring of $\mathrm{CH}(\mathrm{B}G)$ generated by Chern classes) contains every element of finite order of the group $\mathrm{CH}(\mathrm{B}G)$. Since the kernel of the surjective ring homomorphism $\mathrm{CH}(\mathrm{B}G) \to \mathrm{Chow}\, R(G)$ consists of elements of finite order, the two above statements together imply that the ring $\mathrm{Chow}\, R(G)$ is not generated by Chern classes. Since the ring $\mathrm{Chern}\, R(G)$ is generated by Chern classes (for any G), the two filtrations (for $G = O^+(2n)$) are not the same.

Example 4.4. Consider the symplectic group $H = \mathrm{Sp}(2n)$. The group $\mathrm{CH}(\mathrm{B}H)$ is torsion-free. Since the kernel of the surjective ring homomorphism $\varphi : \mathrm{CH}(\mathrm{B}H) \to \mathrm{Chow}\, R(H)$ consists of torsion elements only, it follows that this map is an isomorphism. In particular, the group $\mathrm{Chow}\, R(H)$ is also torsion-free.

Since the ring $\mathrm{CH}(\mathrm{B}H)$ is generated by Chern classes, we conclude that the Chow filtration on $R(H)$ coincides with the Chern filtration. It follows that the group $\mathrm{Chern}\, R(H)$ is torsion-free.

The Weyl groups and character groups of maximal tori (as modules over the Weyl groups) of $\mathrm{Sp}(2n)$ and $G := O^+(2n+1)$ are isomorphic. Therefore, there is an isomorphism of the rings $R(H) \simeq R(G)$. It induces an isomorphism $\mathrm{Chern}\, R(H) \simeq \mathrm{Chern}\, R(G)$. In particular, the group $\mathrm{Chern}\, R(G)$ turns out to be torsion-free. This implies that the Chern filtration on $R(G)$ coincides with the Chow filtration. We conclude that the group $\mathrm{Chow}\, R(G)$ is torsion-free. But $\mathrm{CH}^3(\mathrm{B}G)$ contain torsion element c_3, hence

$$0 \neq c_3 \in \mathrm{Ker}\left(\mathrm{CH}^3(\mathrm{B}G) \xrightarrow{\varphi} \mathrm{Chow}^3 R(G)\right).$$

Replacing $\mathrm{B}G$ by a 4-approximation X, we get an example of a variety X such that $\mathrm{CH}(X)$ is generated by Chern classes, but φ_X is not injective (in degree 3).

4.2. **Equivalence of the three filtrations.**

Theorem 4.5. *For any group G and any n, we have $I(G)^n \supset R(G)^{(N)}$ for some N. In particular, the three filtrations define the same topology on $R(G)$.*

Proof. The statement is implied by the following two facts:

(1) For any n, there exists an approximation U/G of $\mathrm{B}G$ such that the kernel of the homomorphism $\alpha_U : R(G) \to K(U/G)$ is contained in $I(G)^n$.

(2) For any approximation U/G of $\mathrm{B}G$, the kernel of α_U contains $R(G)^{(N)}$ for some N.

The second fact is easier. Let $N = \dim(U/G) + 1$. Since α_U respects Chow filtration, we have

$$\alpha_U(R(G)^{(N)}) \subset K(U/G)^{(N)} = 0.$$

Now we prove (1). Let us fix an embedding $G \hookrightarrow \mathrm{GL}(m)$ for some m. For any N, consider the $(N+1)$-approximation $U/G = \mathrm{GL}(m+N)/H$ of $\mathrm{B}G$ as in Example 3.2, where $H = \begin{pmatrix} G & * \\ 0 & \mathrm{GL}(N) \end{pmatrix}$. Note that $R(H) = R(G \times \mathrm{GL}(N))$ since the unipotent radical of H acts trivially on all simple representations of H. Moreover, $R(G \times \mathrm{GL}(N)) = R(G) \otimes R(\mathrm{GL}(N))$.

By a theorem in equivariant K-theory,

$$K(U/G) = K(\mathrm{GL}(m+N)/H) = R(H)/IR(H) = [R(G) \otimes R(\mathrm{GL}(N))]/I[R(G) \otimes R(\mathrm{GL}(N))],$$

where $I = I(\mathrm{GL}(m+N))$.

Under this identification, the homomorphism $\alpha_U : R(G) \to K(U/G)$ (which we denote below by α^G) coincides with the natural (surjective) homomorphism

$$R(G) \to [R(G) \otimes R(\mathrm{GL}(N))]/I[R(G) \otimes R(\mathrm{GL}(N))].$$

It follows that

$$\alpha^G = \alpha^{\mathrm{GL}(m)} \otimes_{R(\mathrm{GL}(m))} R(G)$$

and therefore, the natural homomorphism

$$\mathrm{Ker}(\alpha^{\mathrm{GL}(m)}) \otimes_{R(\mathrm{GL}(m))} R(G) \to \mathrm{Ker}(\alpha^G)$$

is surjective.

We have $U/\mathrm{GL}(m) = \mathrm{Gr}(m, m+N)$. In fact, the kernel of $\alpha^{\mathrm{GL}(m)}$ is generated by some polynomials of degree at least $N+1$ in the Chern classes $c_1^R, \ldots, c_m^R \in R(\mathrm{GL}(m))$ of the standard representation of $\mathrm{GL}(m)$, where c_i^R is of degree i. Therefore, $\mathrm{Ker}(\alpha^G)$ is generated by polynomials in the images of $c_1^R, \ldots, c_m^R$ (these images are the Chern classes of the G-representation given by the fixed embedding $G \hookrightarrow \mathrm{GL}(m)$) of degree $> N$ and will indeed be contained in $I(G)^n$ for sufficiently large N. □

4.3. **Invariants with values in K.**

Let G be a group over F. We consider the group of invariants $\mathrm{Inv}(G, K)$ with values in K-theory. Consider the approximations U_N/G defined in the Example 2.6. Note that there is a natural G-equivariant closed embedding $U_N \hookrightarrow U_{N+1}$ inducing an embedding $U_N/G \hookrightarrow U_{N+1}/G$. By the proof of Theorem 4.5, the kernels J_N of the natural surjections $R(G) \to K(U_N/G)$ form a filtration that is equivalent to any of the three filtrations considered above. Therefore,

$$\lim_N K(U_N/G) \simeq \widehat{R(G)}.$$

Here $\widehat{R(G)}$ is the completion of $R(G)$ with respect to the powers of the fundamental ideal, i.e.,

$$\widehat{R(G)} = \lim_N (R(G)/I(G)^N).$$

An invariant in $\operatorname{Inv}(G,K)$ evaluated on the G-torsors $U_N \to U_N/G$ for all N yields an element in $\lim_N K(U_N/G) = \widehat{R(G)}$. Thus we get a map

$$\operatorname{Inv}(G,K) \to \widehat{R(G)}.$$

Theorem 4.6. *The map* $\operatorname{Inv}(G,K) \to \widehat{R(G)}$ *is an isomorphism.*

Note that if G is a split reductive group with a split maximal torus, then $\widehat{R(G)} = \widehat{R(T)}^W$. It follows that the group of Zariski invariants $\operatorname{Inv}_{\mathrm{Zar}}(G,K)$ is naturally isomorphic to $\operatorname{Inv}(G,K)$.

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