MOMENT GRAPHS AND SCHUBERT CALCULUS

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Abstract. These are informal notes for the mini-course held during the Workshop "Forms, Flags, Graphs and Beyond", University of Ottawa, 9–12 May 2019.

1. Schubert's enumerative geometry

We start our introduction to Schubert calculus by discussing the original problems the German mathematician Hermann Hannibal Caesar Schubert (1848–1911) was interested in. In the next sections, we will then rephrase his questions in modern language, that is by mean of Grassmann varieties and their Schubert varieties. We took very much inspiration from [KL].

In 1879, Schubert published his book on "Calculus of enumerative geometry" [Schu], which is seen as the origin of what is nowadays called Schubert calculus. The central focus of his work, as the name of his book suggests, was the determination of the number of solutions to linear intersection problems, a topic which was of great interest to many mathematicians at that time (Grassmann, Giambelli, Pieri, Severi,...).

Let us look at three tipical enumerative geometric questions:

(QN1) How many lines do pass through 2 given points in the plane?
As we all know, the answer is
- **1** if the two points are distinct,
- ∞ if they coincide.

(QN2) How many points do belong to two given lines in the plane?
Again, we know the answer and we know that we have to distinguish three cases:
- **1** if the two lines are distinct and not parallel,
- **0** if the lines are parallel,
- ∞ if they coincide.

(QN3) How many lines in the space do intersect four given lines?
Finally, a less trivial question! The answer Schubert gives
- **2** if the four lines are in *general position*.

In his text, Schubert explains to us what are the conditions that the four lines L_1, L_2, L_3, L_4 have to satisfy to be in a general position: they can be subdivided into two pairs, say $\{L_1, L_2\}$ and $\{L_3, L_4\}$, such that each line intersects the other line of its pair in exactly one point and does not meet the other two, moreover the two planes generated by the two pairs $\pi_1 = \langle L_1, L_2\rangle$ and $\pi_2 = \langle L_3, L_4\rangle$ are not parallel. At this point, if we draw the (real) picture we see immediately that there are indeed two lines meeting all of L_1, L_2, L_3, L_4: the one passing through the two

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intersection points $L_1 \cap L_2$ and $L_3 \cap L_4$, and the one obtained by intersecting the planes, i.e. $\pi_1 \cap \pi_2$.

Afterwards, Schubert would conclude that the number of solutions to **(QN3)** would be always 2, when counted with multiplicities, for any configuration of the four lines which provide a finite number of solutions. This is the so-called *principle of special position* or *of conservation of numbers.*

The notion of *general position* is unluckily not rigorous and this caused several mistakes in solution counting, which is of course a problem if you are doing enumerative geometry! Making enumerative geometry rigorous was considered so interesting that it was among Hilbert problems (the 15th). This is the origin of Schubert Calculus.

2. Grassmannians and matrices

There is a vast literature on geometry and combinatorics of Grassmannians, their Schubert varieties and related topics. One standard reference is [Fu]

We want now to rephrase Schubert's questions in more modern language. Before doing so, we should notice that in the way we stated it, the principle of conservation of numbers seems to fail already for the second question. In fact, among the answers to **(QN2)** there are two which are finite: 0 and 1. Here the problem is that we automatically interpreted the words *plane*, resp. *space* as $\mathbb{C}^2$, resp. $\mathbb{C}^3$, while Schubert was dealing with projetive geometry! In fact, all lines meet in $\mathbb{P}^2$, and hence we can only have 1 or ∞ as solutions. Schubert's questions are hence

(QN1) How many lines pass through 2 given points in $\mathbb{P}^2$?
(QN2) How many points belong to two given lines in $\mathbb{P}^2$?
(QN3) How many lines in $\mathbb{P}^3$ intersect four given lines?

At this point, even if we got convinced that the (complex) projective world is the correct setting for enumerative geometric questions, we would still like to work with affine spaces. In order to get back to the affine world, it is enough to remember that k-spaces in $\mathbb{P}^n$ can be seen as $k+1$-dimensional subspaces of $\mathbb{C}^{n+1}$, being defined as the solution set of system of $n-k$ linear homogeneous equations in $n+1$ variables.

We will use column vectors between square brackets to denote points in $\mathbb{P}^n$, e.g.

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{P}^2.$$

In a few lines, we will rephrase our three questions in terms of linear subspaces of appropriate complex vector spaces. Let us hence introduce notation and denote

$$\mathrm{Gr}(k,n) = \{U \subseteq \mathbb{C}^n \mid U \ k\text{-dimensional subspace of } \mathbb{C}^n\}, \qquad n \geq k \geq 0$$

the Grassmannian (or Grassmann variety) of k-spaces in $\mathbb{C}^n$. We will soon consider the Grassmannian in more detail, but we first want to get back to **(QN1), (QN2), (QN3)**:

(QN1) How many 2-dimensional subspaces of $\mathbb{C}^3$ do pass through 2 given lines (in $\mathbb{C}^3$)?
The two lines are points of the Grassmannian, $L^1, L^2 \in \mathrm{Gr}(1,3)$, therefore if we define for $i = 1, 2$,

$$Y_i := \{U \in \mathrm{Gr}(2,3) \mid L^i \subseteq U\} = \{U \in \mathrm{Gr}(2,3) \mid \dim(U \cap L^i) \geq 1\},$$

the answer to our question is given by the cardinality of the set $Y_1 \cap Y_2$.

(QN2) How many lines in $\mathbb{C}^3$ belong to two given 2-dimensional subspaces of $\mathbb{C}^3$? We proceed as in the previous item and we fix $W^1, W^2 \in \mathrm{Gr}(2,3)$. Moreover, we define, for $i = 1, 2$,

$$Y_i = \{U \in \mathrm{Gr}(1,3) \mid U \subseteq W^i\} = \{U \in \mathrm{Gr}(1,3) \mid \dim(U \cap W^i) \geq 1\},$$

and see that the answer of **(QN2)** is given by the cardinality of the set $Y_1 \cap Y_2$.

(QN3) How many 2-dimensional subspaces in $\mathbb{C}^4$ intersect four given 2-dimensional subspaces of $\mathbb{C}^4$?
We proceed as in the previous items and we fix $W^1, W^2, W^3, W^4 \in \mathrm{Gr}(2,3)$. Moreover, we define, for $i = 1, 2, 3, 4$,

$$Y_i = \{U \in \mathrm{Gr}(1,3) \mid U \cap W^i \neq \{0\}\} = \{U \in \mathrm{Gr}(1,3) \mid \dim(U \cap W^i) \geq 1\},$$

and see that the answer of **(QN3)** is given by the cardinality of the set $Y_1 \cap Y_2 \cap Y_3 \cap Y_4$.

The varieties Y_i's appearing in the answers to our questions are examples of Schubert varieties, as we will see soon.

2.1. **Matrices.** We want now to briefly discuss a convienent way of representing points in the Grassmannian $\mathrm{Gr}(k,n)$ via $(n \times k)$-matrices.

Let $U \in \mathrm{Gr}(k,n)$, pick a basis $b_1, \dots, b_k$ of it and arrange this vectors in an $(n \times k)$-matrix M (as for the projective space, we are using column notation for the elements in $\mathbb{C}^n$).

By linear algebra we know that there exists a unique column reduced matrix M_U that we can get from M by performing elementary column operations (that is rescaling a column by a non-zero scalar, exchanging columns and adding columns). Here by column reduced we mean a matrix such that in each column the lowest (i.e. with maximal row index) non-zero entry is a 1 (called *pivot* or *leading 1*), which also has the property that all the matrix entries on its right are zero; moreover if column j has its leading 1 on row i_j, then the leading one of column $j+1$ lies on a row with index $i_{j+1} > i_j$.

We obtain in this way a well defined map $\mathrm{Gr}(k,n) \to \mathrm{Mat}_{n\times k}(\mathbb{C})$.

We remember at this point that GL_k is generated by elementary operations and we notice that we can only obtain maxmal rank matrice, so that we have really gotten a bijective map

$$\mathrm{Gr}(k,n) \to \{M \in \mathrm{Mat}_{n\times k} \mid \mathrm{rk}(M) = k\} / GL_k,$$

or, equivalentily,

$$\mathrm{Gr}(k,n) \to \{M \in \mathrm{Mat}_{n\times k} \mid \mathrm{rk}(M) = k\} / \sim,$$

where $M \sim M'$ if and only if you can obtain M' from M via a finite sequence of elementary column operations.

This identification allows us to equip $\mathrm{Gr}(k,n)$ with the structure of complex projective algebraic variety:

- first, denote by $I_{k,n}$ the set of strickly increasing sequences of k numbers between 1 and n and consider it as a totally ordered set with respect to the lexicografic order (and recall that $\#I_{k,n} = \binom{n}{k}$);
- then, consider the map

$$\widetilde{P_{k,n}} : \mathrm{Mat}_{n\times k} \to \mathbb{C}^{\binom{n}{k}}, \qquad M \mapsto (p_{\underline{i}}(M))_{\underline{i} \in I_{k,n}},$$

where $p_{\underline{i}}$ denotes the determinant of the $k \times k$-minor of M consisting of the rows having indices $i_1, \ldots, i_k$;

- finally, notice that if M has maximal rank its image will never be the zero vector and that if $M' = Mg$ for a matrix $g \in GL_k$, then

$$\widetilde{P_{k,m}}(M') = \det(g)\widetilde{P_{k,m}}(M'),$$

that is we have an induced map

$$\{M \in \mathrm{Mat}_{n\times k} \mid \mathrm{rk}(M) = k\} / GL_k \to \mathbb{P}^{\binom{n}{k}-1},$$

and hence a map

$$P_{k,n} : \mathrm{Gr}(k,n) \to \mathbb{P}^{\binom{n}{k}-1}, \qquad U \mapsto \widetilde{P_{k,n}}(M_U).$$

The above map is injective and called *Plücker embedding*. Its image is the zero locus of quadratic equations (the *Plücker relations*) which we will not give here.

Example 2.1. *In the case of $Gr(2,4)$, its image under the Plücker embedding is the zero locus of the following equation in the Plücker coordinates:*

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}.$$

3. Schubert varieties

Recall that for any $U \in \mathrm{Gr}(k,n)$ we defined the (reduced column echelon form) matrix $M_U \in \mathrm{Mat}_{n\times k}$.

Definition 3.1. *Let $U \in Gr(k,n)$. Its position is the sequence $\underline{i} \in I_{k,n}$ whose j-th component is the row index of the leading one of the j-th column of M_U. We denote such a sequence by $pos(U)$.*

Let $\mathcal{E} = (e_1, \ldots, e_n)$ be the standard basis of $\mathbb{C}^n$. From now on, we denote by

$$E_r = \mathrm{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_r\}, \qquad r \in \{1, \ldots, n\}.$$

We are hence able to give the (first) definition of Schubert cell.

Definition 3.2. *Let $\underline{i} \in I_{k,n}$. The Schubert cell $C_{\underline{i}}$ is*

$$\begin{aligned} C_{\underline{i}} &= \{U \in Gr(k,n) \mid pos(U) = \underline{i}\} \\ &= \{U \in Gr(k,n) \mid dim(U \cap E_r) = j \text{ for any } i_j \leq r < i_{j+1}\}, \end{aligned}$$

where, by convention, we put $i_0 = 0$ and $i_{k+1} = n+1$.

Example 3.1. *Let $U \in Gr(2,4)$ and assume that $\underline{i} = (1,3)$. Then, U had position $(1,3)$ if and only if its associate matrix M_U is of the form*

$$\begin{pmatrix} 1 & 0 \\ 0 & a \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

for some $a \in \mathbb{C}$, meaning that there is an $a \in \mathbb{C}$ such that $U = span_{\mathbb{C}}\{e_1, ae_2 + e_3\}$. This is the case if and only if

$$dim(U \cap E_1) = 1, \quad dim(U \cap E_2) = 1, \quad dim(U \cap E_3) = 2, dim(U \cap E_4) = 2,$$

which are exactly the conditions which appear in the second definition.

Notice that in our example $C_{\underline{i}} \simeq \mathbb{C}$. This is in fact a general true that $C_{\underline{i}}$ is an affine space, whose dimension is nothing but the entries that the required column echelon form leaves free, and can therefore be computed from $\underline{i}$:

$$C_{\underline{i}} \simeq \mathbb{C}^N, \quad \text{where } N = \sum_j (i_j - j).$$

The reason for this formula is that to get a matrix of prescribed echelon form you first right all the leading ones, then the zeroes unde and on the right to the leading ones, and then can put in all other entries whatever you like. Clearly the entries in column j which are free are $i_j - 1 - (j-1) = i_j - j$.

Definition 3.3. *The dimension of $C_{\underline{i}}$ is said* the length *of $\underline{i}$ and denote it by $\ell(\underline{i})$.*

The motivation for the previous

At this point we should notice that by definition the Grassmann variety is the disjoint union of the $C_{\underline{i}}$'s:

$$\mathrm{Gr}(k,n) = \bigsqcup_{\underline{i} \in I_{k,n}} C_{\underline{i}}.$$

In fact, the decomposition of the Grassmannian into affine spaces provides the Grassmannian with a structure of CW-complex. If you are not familiar with CW-complexes, you should think of them as a homotopy-theoretic generalization of the notion of a simplicial complex, that is a space built up in a hierarchical way starting with a collection of points, then attaching some disks along their boundaries, then attaching along the boundaries of this space 2-dimensional disks and so on.What you should anyway keep in mind is that this structure makes computing (co)homology easy.

Finally, we can give the (first) definition of Schubert variety.

Definition 3.4. *Let $\underline{i} \in I_{k,n}$. The Schubert variety $X_{\underline{i}}$ is the (projective) closure of $C_{\underline{i}}$ in $Gr(k,n)$, that is*

$$X_{\underline{i}} = \left\{ U \in Gr(k,n) \mid dim(U \cap E_{i_j}) \geq j, \text{ for any } j = 1 \ldots, k \right\}.$$

Example 3.2. *Let $k=2$ and $n=4$. Consider this time $\underline{i} = (2,4)$. Then*

$$\begin{aligned} X_{1,4} &= \{U \in Gr(2,4) \mid dim(U \cap E_2) \geq 1, \ dim(U \cap E_4) \geq 2\} \\ &= \{U \in Gr(2,4) \mid dim(U \cap E_2) \geq 1\} \end{aligned}$$

which coincides with the Y_i in the answer to Schubert's third question if we assume $L^i = E_2$.

Example 3.3. *It is an easy check to see that $X_{(1,2,3,\ldots,k)} = \{E_\bullet\}$. This is of course compatible with the dimension counting ($\sum_j (j-j) = 0$) and the fact that all Schubert varieties are irreducible, since obtained as closures of irreducible varieties.*

Example 3.4. *Observe that all $U \in Gr(k,n)$ satisfy the conditions*

$$dim(U \cap E_{n-k+j}) \geq j \quad j = 1, \ldots, k,$$

so that $Gr(k,n) = X_{(n-k+1,\ldots,n)}$.

In order to proper restate Schubert's questions we recall the notion of flag (which you have seen also in Lara Bossinger and Dave Anderson's talks). A (complete) flag $F_\bullet = (F_r)_{r=1,\ldots,n-1}$ in $\mathbb{C}^n$ is a collection of nested subspaces:

$$\{0\} \subset F_1 \subset F_2 \subset \ldots \subset F_{n-1} \subset \mathbb{C}^n.$$

Notice that since the inclusions are all proper it has to hold $F_r \in \mathrm{Gr}(r,n)$. The collection $E_\bullet = (E_r)$ is usually called the *standard flag*.

At this point we look back at the definition of Schubert varieties and we notice that we could have just replaced the standard flag by any other flag to obtain an isomorphic variety:

$$X_{\underline{i}}(F_\bullet) = \left\{ U \in \mathrm{Gr}(k,n) \mid \dim(U \cap F_{i_j}) \geq j, \text{ for any } j = 1\ldots,k \right\}.$$

Now Schubert's concept of *general position* corresponds to *transversality* of flags.

Definition 3.5. *Two flags $F_\bullet^{(1)}$ and $F_\bullet^{(2)}$ in $\mathbb{C}^n$ are said to be transverse if*

$$F_i^{(1)} \cap F_{n-i}^{(2)} = 0, \quad \textit{for any } i = 1,\ldots,n-1.$$

An example of a pair of transverse flags consists of the standard one together with the so-called opposite flag $E_\bullet^{\mathrm{opp}}$, where the constituing subspaces are

$$E_r^{\mathrm{opp}} := \langle e_n, e_{n-1},\ldots,e_{n-r+1}\rangle, \qquad r = 1,\ldots,n-1.$$

We are ready to rephrase Schubert's first two questions:

(QN1) How many 2-dimensional subspaces of $\mathbb{C}^3$ do pass through 2 given lines (in $\mathbb{C}^3$)?
Since we want the two given lines to be in general position, we can just choose them as $L^1 = E_1$ and $L^2 = E_1^{\mathrm{opp}}$, so that

$$Y_1 := \{U \in \mathrm{Gr}(2,3) \mid L^i \subseteq U\} = \{U \in \mathrm{Gr}(2,3) \mid \ \dim(U \cap L^i) \geq 1\} = X_{1,3}(E_\bullet),$$

$$Y_2 := \{U \in \mathrm{Gr}(2,3) \mid L^i \subseteq U\} = \{U \in \mathrm{Gr}(2,3) \mid \ \dim(U \cap L^i) \geq 1\} = X_{1,3}(E_\bullet^{\mathrm{opp}}),$$

the answer to our question is hence $\# X_{1,3}(E_\bullet) \cap X_{1,3}(E_\bullet^{\mathrm{opp}})$.

(QN2) How many lines in $\mathbb{C}^3$ do belong to two given 2-dimensional subspaces of $\mathbb{C}^3$?
As for the previous point, the answer is given by $\# X_{2,3}(E_\bullet) \cap X_{2,3}(E_\bullet^{\mathrm{opp}})$.

In the next section we will relate the cardinality of these intersections to the cohomology of the Grassmannian.

3.1. **The Bruhar order.**

Let us know assume that all our cells and vareties are given with respect to a fixed flag, and that this flag is the standard one, so that we save some notation.

Observe that every Schubert variety is a (disjoint) union of Schubert cells. This is a consequence of the CW-complex structure on the Grassmannian, which imples that the closure of each cell is a union of cells, or, if you are familiar with this language, it is just a consequence of the set theoretic description in terms of the relative position with respect to the standard flag. The orbit closure relation equips the set $I_{k,n}$ with a partial order:

$$\underline{i} \leq \underline{l} \ \Leftrightarrow \ C_{\underline{i}} \subset X_{\underline{l}}.$$

We see immediately that this is just the partial order given by

$$\underline{i} \leq \underline{l} \ \Leftrightarrow \ i_j \leq l_j \text{ for any } j = 1,\ldots,k.$$

Observe that there ia a unique maximal and a unique minimal element in the Bruhat order.

The Bruhat order will play a role tomorrow, while defining equivariant Schubert classes.

4. Schubert calculus and cohomology of the Grassmannian

We will briefly explain here why the cohomology of the Grassmannian, appeared in several other talks, is relevant to our questions.

It can be proven that the Grassmannian $\mathrm{Gr}(k,n)$ is a complex manifold of dimension $k(n-k)$, therefore algebraic topology tells us that

$$H^i(\mathrm{Gr}(nk,n),\mathbb{Z})=0, \qquad \text{for all } i\notin\{0,2,4,\dots,2k(n-k)\}.$$

Thanks to the CW-complex structure, we can assign a natural cohomology class to each (closed) subvariety of $\mathrm{Gr}(k,n)$. If two subvarieties are members of a continuous family of subvarieties, then they are assigned the same cohomology class.

Definition 4.1. *Let $\underline{i}\in I_{k,n}$ and let $F_\bullet$ be a flag in $\mathbb{C}^n$, then the class of the Schubert variety $X_{\underline{i}}(F_\bullet)$ is called Schubert cycle.*

If F and F' are two distinct flags in $\mathbb{C}^n$, then we can continuously go from $X_{\underline{i}}(F)$ to $X_{\underline{i}}(F')$ (by change of basis) and so they are assigned the same cohomology class, which, hence only depends on $\underline{i}$. This consideration allows us to simply denote by $\sigma_{\underline{i}}$ such a class.

The Grassmannian is a smooth variety and therefore Poincaré duality holds (as Ben Elias in his talk reminded us). The class $\sigma_{\underline{i}}$ is really the Poincaré dual of the cycle in homology corresponding to $X_{\underline{i}}$. Since $X_{\underline{i}}$ had complex dimension $\ell(\underline{i})=\sum(i_j-j)$, the corresponding cycle in homology is $[X_{\underline{i}}]\in H_{2\ell(\underline{i})}(\mathrm{Gr}(k,n),\mathbb{Z})$. We conclude that

$$\sigma_{\underline{i}}\in H^{2(k(n-k)-\ell(\underline{i}))}(\mathrm{Gr}(k,n),\mathbb{Z}).$$

Again, thanks to the CW-complex structure, we know that

$$H^{2(k(n-k)-r)}(\mathrm{Gr}(k,n),\mathbb{Z})=\mathbb{Z}-\mathit{span}\{\sigma_{\underline{i}}\mid \ell(\underline{i})=r\}.$$

We are now able to get back to enumerative geometry, since the ring structure of the cohomology of $\mathrm{Gr}(k,n)$ is indeed controlled by intersections of Schubert varieties, thas is

$$\sigma_{\underline{i}}\sigma_{\underline{j}}=[X_{\underline{i}}(F)\cap X_{\underline{l}}(F')]^\vee,$$

where F and F' are transverse.

Observe that $H^{2k(n-k)}(\mathrm{Gr}(k,n),\mathbb{Z})=\mathbb{Z}\sigma_{\mathrm{pt}}$, where $\mathrm{pt}=(1,2,3\dots,k)$, and we can consider the projection onto the top cohomology group, which induces a projection onto $\mathbb{Z}$:

$$\delta:\bigoplus_{i\in\mathbb{Z}}H^i(\mathrm{Gr}(k,n),\mathbb{Z})\to\mathbb{Z}, \quad \sum a_{\underline{i}}\sigma_{\underline{i}}\mapsto a_{\mathrm{pt}}.$$

Notice that this map already appeared in Ben Elias' talk in the more general setting of (generalised) flag varieties: it was the responsable of the Frobenius structure of the cohomology ring of these varieties.

A fundamental fact is now that if the intersection problem $X_{\underline{i}}(F)\cap X_{\underline{l}}(F')$ admits a finite number of solutions, then this number is given by $\delta(\sigma_{\underline{i}}\sigma_{\underline{l}})$.

5. Moment graphs and GKM theory

I hope that yesterday I convinced you that understanding the ring structure of $H^*(\mathrm{Gr}(k,n))$ is an interesting problem. Maybe you were already interested in it, but now you have some more motivation coming from enumerative combinatorics!

Let us notice that GL_n acts on $\mathrm{Gr}(k,n)$ by change of basis, hence also its maximal torus of diagonal matrices acts on the Grassmann variety. If you like to think of points of the Grassmannian in terms of matrices, this action is nothing but left multiplication. We want to exploit such an action to study the cohomology ring. We will hence abbandon the Grassmannian for a bit and focus on a class of nice algebraic varieties equipped with a torus action.

Assume we are given a complex algebraic variety X, and that an algebraic torus T acts on it with a finite number of fixed points and 1-dimensional orbits. Assume moreover that the torus action is *equivariantly formal*. In [GKM] you can find several, equivalent, characterisations of equivariant formality. For us, today, it is enough to know that all varieties whose odd (ordinary) cohomology groups vanish are equivariantly formal. These assumptions on the torus action are way to strong and the main result I will state soon holds is way greater generality, but for today we will make these assumptions, since they allows us to apply this theory anyway to Grassmannians (and generalised flag varieties and their Schubert varieties).

In the above hypotheses, it can be shown that the closure of each one-dimensional orbit of T in X is isomorphic to $\mathbb{P}^1$ and containes exactly two fixed points in its closure. This tells us that if we look at the 1-skeleton of the torus action we obtain a graph!

Example 5.1. *While developping the general theory, we will keep an eye on the Grassmannian case. As we have mentioned, the maximal torus of GL_n, consisting of diagonal invertible matrices acts on $Gr(k,n)$.*

First, let us consider the easiest case, that is $\mathbb{P}^1 \simeq Gr(1,2)$. The torus acts as follows

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \lambda_1 a \\ \lambda_2 b \end{bmatrix} \quad \textit{for any } \lambda_1, \lambda_2 \in \mathbb{C}^\times, \ \textit{for any } \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{P}^1$$

and we see immediately that we only have 2 fixed points, that is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. *In terms of 1-dimensional subspaces of $\mathbb{C}^2$, this means that the fixed points are $\mathbb{C}e_1, \ \mathbb{C}e_2 \subseteq \mathbb{C}^2$.*

More in general,

$$Gr(k,n)^T = \{\langle e_{i_1}, e_{i_2}, \dots, e_{i_k}\rangle \mid \underline{i} \in I_{k,n}\},$$

and we can index fixed points by elements in $I_{k,n}$. It will be useful for us to set

$$E_k^{\underline{i}} := \langle e_{i_1}, \dots, e_{i_k}\rangle.$$

Let us get back to the baby case, for which our two fixed points are E_1^1 and E_1^2. For $Gr(1,2)$, we only have one 1-dimensional orbit, that is

$$\left\{ \begin{bmatrix} a \\ b \end{bmatrix} \mid a, b \in \mathbb{C}^\times \right\}.$$

Taking the closure results in allowing a or b to be 0, and hence we get the two fixed points.

Observe that this is what happens in the more general case: assume that we have two tuples $\underline{i}$ and $\underline{l}$ which agree for all entries but one, then the corresponding k-dimensional spaces $E_k^{\underline{i}}$ and $E_k^{\underline{l}}$ intersect in a $k-1$-dimensional subspace, say W, *and there are indices $p \neq q$ such that*

$$E^{\underline{i}}_k = W \oplus \mathbb{C}e_p, \quad E^{\underline{j}}_k = W \oplus \mathbb{C}e_q$$

so that there is a one dimensional orbit whose closure contains them:

$$\{W \oplus \mathbb{C}(ae_p + be_q) \mid a, b \in \mathbb{C}^\times\}.$$

It can be shown that these are all 1-dimensional orbits.

We should notice right away that we have already obtained a combinatorial criterion: there is a 1-dimensional orbit between two fixed points $\underline{i}$ and $\underline{l}$ if and only if one can obtain a tuple from the other by exchanging two indices.

At this point we have assigned a graph to any variety equipped with nice torus action, but we want a little more: we want to color the edges of the graph via torus characters. This is easy: take a 1-dimensional orbit, then the torus has to act on it via a character. This character (well-defined up to a sign, as we are about to see) will be the label. We will refer to such a graph as the *moment graph* coming from the T action on X.

Example 5.2. *Since all the 1-dimensional orbits are isomorphic $\mathbb{P}^1$, it is enough to look at the baby case to understand how labels work: we want to determine the character by which $T \subseteq GL_2$ acts on*

$$\mathcal{O} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \mid a, b \in \mathbb{C}^\times \right\}.$$

Recall that the homomorphisms of algebraic groups between T and $\mathbb{C}^\times$ are all of the form

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mapsto \lambda_1^d \lambda_2^e, \text{ for } d, e \in \mathbb{Z},$$

so that $Hom(T, \mathbb{C}^\times) \simeq \mathbb{Z}^2$ and, as an abelian group, it is generated by

$$\epsilon_1 : \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mapsto \lambda_1, \quad \epsilon_2 : \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mapsto \lambda_2$$

We have to choose an isomorphism of the 1-dimensional orbit with $\mathbb{C}^\times$. Let us pick

$$\mathcal{O} \to \mathbb{C}^\times, \qquad \begin{bmatrix} a \\ b \end{bmatrix} \mapsto a/b$$

and study the torus action:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \lambda_1 a \\ \lambda_2 b \end{bmatrix} \mapsto (\lambda_1 a)/(\lambda_2 b) = (\lambda_1 \lambda_2^{-1})(a/b),$$

which corresponds to the character $\epsilon_1 - \epsilon_2$. Notice that if we had chosen the isomorphism

$$\mathcal{O} \to \mathbb{C}^\times, \quad = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \mid a, b \in \mathbb{C}^\times \right\} \mapsto b/a,$$

we would have gotten the character $\epsilon_2 - \epsilon_1$. This is in fact always the case: there are two possibile isomorphisms of the 1-dimensional orbit to $\mathbb{C}^\times$, depending on the chosen affine chart, and the resulting character is hence well-defined up to a sign.

Recall that in the general case of $Gr(k, n)$ all 1-dimensional orbits are of the form

$$\mathcal{O} = \{W \oplus \mathbb{C}(ae_p + be_q) \mid a, b \in \mathbb{C}^\times\}$$

for appropriate $p \neq q \in \{1, \dots, n\}$. As before we can show that T acts on $\mathcal{O}$ via $\pm(\epsilon_p - \epsilon_q)$.

We will denote by $\mathcal{V}$, resp. $\mathcal{E}$ the set of vertices, resp. edges a the moment graph $\mathcal{G}$. Given an edge x —— y of $\mathcal{G}$, we will denote by $\alpha(x$—$y)$ the corresponding torus character (this is defined, as we saw, up to a sign, but the sign will not play any role).

We want to use the data encoded in the moment graph to determine the T-equivariant cohomology of X. This is (following Borel's construction) just the singular cohomology of $(X \times ET)/T$, where $ET \to BT$ is a universal principal bundle, i.e. a principal bundle whose totale space is contractible. As in Alexander Merkurjev's talk, $BT \simeq ET/T$ denotes the classifying space of T.

Equivariant cohomology has a lot of nice properties such as functoriality and a ring structure, therefore the projection

$$X \hookrightarrow \{\mathrm{pt}\}$$

induces a structure of $S := H^\bullet_T(\mathrm{pt}, R)$-module on $H^\bullet_T(X, R)$, where R is some commutative ring: functoriality induces a map $H^\bullet_T(\mathrm{pt}, R) \to H^\bullet_T(X, R)$, while ring structure allows to multiply the classes lifted from $H^*_T(\mathrm{pt}, R)$ with classes in $H^\bullet_T(X, R)$.

We have already seen in Alexander Merkurjev's talk that $BT = \prod_{i=1}^r \mathbb{P}^\infty$ if $T \simeq (\mathbb{C}^\times)^r$, so that its cohomology is isomorphic to a polynomial ring in r-variables, which can be identified with the symmetric algebra (over R) of the character lattice of the torus, with $\mathbb{Z}$-grading given by declaring the degree of the variable equal to 2.

Let us consider now the inclusion of the fixed point locus

$$X^T \hookrightarrow X.$$

By functoriality we get a ring homomorphism ψ

$$H^\bullet_T(X, R) \to H^\bullet_T(X^T, R) = \bigoplus_{x \in X^T} H^\bullet_T(x, R) \simeq \bigoplus_{x \in X^T} H^\bullet_T(x, R)$$

From now on, we fix $R = \mathbb{Q}$, and we drop the coefficient ring in our notation. The fundamental result for us is the following:

Theorem 5.1. *The above ring homomorphism if injective and*

$$\psi(H^\bullet_T(X)) = \{(f_x) \in \bigoplus S \mid f_x - f_y \in \alpha(x \text{ --- } y)S \text{ for any } x \text{ --- } y \in \mathcal{E}\}$$

The study of $H^\bullet_T(X)$ via its isomorphic image $\psi(H^\bullet_T(X))$, translates a topological problem into a problem which is more combinatorial in nature, since all information you need for describing $\psi(H^\bullet_T(X))$ are encoded in the moment graph of the T-action on X.

This approach is usually referred to as *GLM-Theory*, due to the seminal paper by Goresky-Kottwitz-MacPherson. As Dave Anderson stressed out, though, the history of this theorem is very long, coming from an application of localisation techniques, which had been applied for about 35 years before [GKM]. The idea of restricting the attention to 1-dimensional orbits is certainly due to Chang and Skjelbred [CS]. For a detailed historical account, we refer to [GKM, S1.7].

Since we assumed the action to be equivariantly formal, it holds

$$H_T^\bullet(X) \simeq H^\bullet(X) \otimes_{\mathbb{Q}} S \text{ and } H(X) = \frac{H_T^\bullet(X)}{I.H_T^\bullet(X)},$$

where the first one is an isomorphism of ($\mathbb{Z}$-graded) S-modules in the second formula I is the augmentation ideal of S.

Example 5.3. *Let us look at the $\mathbb{P}^1$-case. Then $S = \mathbb{Q}[\epsilon_1, \epsilon_2]$ and Teorem 5 tells us that*

$$\begin{aligned}\psi(H_T(\mathbb{P}^1)) &= \{(f_1 - f_2) \in S \oplus S \mid f - g \in (\epsilon_1 - \epsilon_2)S\} \\ &= (1,1)S \oplus (0, \epsilon_1 - \epsilon_2)S \simeq S \oplus S[-2].\end{aligned}$$

We see hence that, as free S-module, we have a generator in degree 0 and one in degree 2.

This is compatible with

$$H^0(\mathbb{P}^1) = \begin{cases} \mathbb{Q} & \textit{if } i = 0, 2, \\ 0 & \textit{otherwise.} \end{cases}$$

5.1. **Equivariant Schubert classes.** We see here that we have an equivariant analogue of Schubert cycles which are sent to Schubert cycles under base change. We can therefore study multiplication of Schubert cycles in the equivariant setting and then get back to the non-equivariant.

Assume now $\mathcal{G}$ to be the moment graph associated with $\mathrm{Gr}(k, n)$. Then we recall that the set of vertices is indexed by $I_{k,n}$ and it is equipped with a partial order (the Bruhat order). We use this to define a nice basis $\{\xi^{\underline{i}}\}$ of $H_T^\bullet(\mathrm{Gr}(k, n))$.

Definition 5.1. *For any $\underline{i} \in I_{k,n}$, there exists a unique element $\xi^{\underline{i}} \in \psi(H_T^\bullet(Gr(k, n)))$ defined by the following conditions*

(SH1) $\xi^{\underline{i}}_{\underline{l}} = 0$ *unless* $\underline{i} \leq \underline{j}$*;*

(SH2) $\xi^{\underline{i}}_{\underline{i}} = \prod_{(p,q)\underline{i} \geq \underline{i}} (\epsilon_p - \epsilon_q)$.

For general (well behaved) moment graphs these classes were defined by Tymoczko [Ty], who named them *Knutson-Tao classes*, since Knutson and Tao introduced them in the Grassmannian case [KT].

Example 5.4. *Look at $Gr(2, 3)$. In this case (but this is very special!) we get a complete graph with three vertices indexed by 12, 13, 23 ordered in lexicographic order (which in this case coincides with the Bruhat order). Write a general element in $\psi(Gr(2, 3))$ as a triple (f_{12}, f_{13}, f_{23}). Then the Knutson-Tao basis is*

$$\xi^{12} = ((\epsilon_2 - \epsilon_3)(\epsilon_1 - \epsilon_3), 0, 0), \quad \xi^{13} = (\epsilon_1 - \epsilon_3, \epsilon_1 - \epsilon_2, 0), \quad \xi^{23} = (1, 1, 1).$$

At this point to answer (QN1) we have to look at the coefficient of ξ^{12} in the expansion of $(\xi^{13})^2$:

$$\begin{aligned}(\xi^{13})^2 &= ((\epsilon_1 - \epsilon_3)^2, (\epsilon_1 - \epsilon_2)^2, 0) \\ &= (\epsilon_1 - \epsilon_2)\xi^{13} + \xi^{12},\end{aligned}$$

and we conclude that given 2 points in the plane there is a unique line passing through them!

Another way of obtaining these classes would be by using the analogue of Demazure operators, but, unluckily, we do not have time to talk about this.

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