E8- and new class of commutative non-associative algebras with a continuous Pierce Spectrum

COMMENTS ABOUT EXCEPTIONAL LIE GROUPS

G2, F4, E6 and E7 share with classical Lie groups the possibility of being described as Automorphisms of a multilinear form(s):

- G2= Automorphisms of a trilinear alternating form in $C^7$
- F4= Automorphisms of a trilinear symmetric form and a quadratic form in $C^{26}$
- E6= Automorphisms of a trilinear symmetric form in $C^{27}$
- E7= Connected component of automorphims of a symmetric form degree 4 in $C^{56}$

In all the above cases, as is the case for classical groups, the representation has the smallest possible dimension.

In the cases G2 and F4, it can be shown that the trilinear form is accompanied by a non-degenerate bilinear symmetric form, so the group can alternatively be described as the automorphims group of a non-associative algebra:

- G2= Automorphims of a skew vector product in $C^7$
- F4= Automorphims of a commutative algebra in $C^{26}$

It is also known that

- E6= Connected component of automorphisms of the 56 dimensional algebra of 2x2 matrices with Albert algebra entries and scalar diagonals
THE SINGULAR POSITION OF E8

The smallest dimensional representation of E8 is the adjoint representation.

- E8 = Automorphims of the Lie Algebra e8.  (The dog chasing its tail!)
- T.A. Springer had noticed that E8 acts on the second smallest representation V(3875) with a unique invariant symmetric cubic form and a unique bilinear non-degenerate symmetric form. Hence, E8 acts as automorphims of a unique commutative algebra structure on V(3875)
- S. Garibaldi and R. Guralnik have shown that E8 is the full automorphism group of the algebra
- The representation V(3875) is a sub-representation, called \( Y_2^* \) of \( S^2(e8) \) and
  \[
  S^2(e8) = C + Y_2^* + L(2\alpha), \text{ where } \alpha = \text{the highest root for e8}
  \]

The E8 invariant algebra structure on \( Y_2^* \) is unique.

- The same type splitting occurs for the other exceptional groups as put into evidence by P.Deligne. **
  \[
  S^2(g) = C + Y_2^* + L(2\alpha)
  \]

For G2, F4, E6 and E7, the space \( Y_2^* \) has at least two distinct G-invariant algebra structures

<table>
<thead>
<tr>
<th>Lie group Type</th>
<th>G2</th>
<th>F4</th>
<th>E6</th>
<th>E7</th>
<th>E8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dimension of ( Y_2^* )</td>
<td>27</td>
<td>324</td>
<td>650</td>
<td>1539</td>
<td>3875</td>
</tr>
<tr>
<td>Dual Coxeter number ( h^* )</td>
<td>4</td>
<td>9</td>
<td>12</td>
<td>18</td>
<td>30</td>
</tr>
</tbody>
</table>

Table 1

CONSTRUCTION OF THE ALGEBRA $A(e8)$

**First Question:** How do we single out $Y_2^*$ or $C + Y_2^*$? (If we want an algebra with a unit)

There are 3 natural E8 intertwining maps $S^2(e8) \rightarrow H(e8) \subset \text{End}(e8)$, which are **linearly independent**.

Let us designate the Killing form on $e8$ by $K(.,.)$, and consider the following maps:

1. A map $P: S^2(e8) \rightarrow H(e8) = K$-symmetric endomorphism of $e8$ as:
   \[ P(X, Y) = \frac{1}{2} (X \otimes K(Y, \cdot) + Y \otimes K(X, \cdot)) \]
   This map is an isomorphism of $e8$-modules

2. A map $R: S^2(e8) \rightarrow H(e8)$ given as:
   \[ R(X, Y) = \text{ad}(X) \cdot \text{ad}(Y) \]
   (the Jordan Product = $\frac{1}{2}$ the symmetrised product)

3. There is also a natural intertwining map “the scalar map” $I: S^2(e8) \rightarrow H(e8)$ given as:
   \[ I(X, Y) = K(X, Y) \cdot e \]
   where $e$ is the identity map on $e8$

**Answer to the first question:** Some linear combination of $P$ and $R$ has to exclude $L(2\alpha)$.

**Lemma:** For any simple Lie algebra $g$, the mapping $S: S^2(g) \rightarrow H(g)$ defined as:

\[ S(X, Y) = h^\vee \text{ad}(X) \cdot \text{ad}(Y) + P(X, Y) \]

has its image outside of $L(2\alpha)$

Where $h^\vee$ = dual Coxeter number = (squared norm of the highest root) $^{-1}$

**Definition:** $A(g) = \text{Image of } S$

**Lemma:** $A(e8) = C e + Y_2^*$, in particular $S^+(X, Y) = h^\vee \text{ad}(X) \cdot \text{ad}(Y) + P(X, Y) \cdot \frac{(h^\vee + 1)K(X, Y)}{\dim(E8)} \cdot e$

belongs to $Y_2^*$
**Second Question:** What does the product rule on $Y_2^*$ look like?

In other words, how can express the product between two elements $S^+(X, X) \otimes S^+(Y, Y)$ in $Y_2^*$?

Recall: for e8, there is only one algebra structure on $Y_2^*$.

How can we express this invariant product?

A. Clearly, the Jordan product $S^+(X, X) \ast S^+(Y, Y)$, which takes values in $H(e8)$, has an E8 covariant projection onto $Y_2^*$.

Any non-zero multiple of $Proj_{Y_2^*} (S^+(X, X) \ast S^+(Y, Y))$ qualifies as $S^+(X, X) \otimes S^+(Y, Y)$

B. There is an E8 intertwining operator (a sandwich Casimir) $\Omega : H(e8) \to H(e8)$ given as

$$\Omega(T) = \sum_1^d \text{ad}(Y_j) \text{ad}(Y_j),$$

where $\{Y_j\}_{1}^{d}$ is a K-orthonormal basis for e8

C. For the exceptional Lie algebras, the ring of invariant polynomials $I(g)$ on the Lie algebra $g$, has generators of degrees: $2 < l_1 < l_2 \ldots < l_r$ with $l_r > 5$, from which it follows that:

C.1 Trace $(\text{ad}(X)^4) = \alpha (K(X,X))^2$ (see formulas of Okubo**)

In addition, we also have, **only for E8:**

C.2 Trace $(\text{ad}(X)^6) = \gamma (K(X,X))^3$

For e8 the parameters are $\gamma = \frac{1}{120}$ and $\alpha = \frac{1}{10}$

D. As a consequence of C.1 one may compute the action of the “sandwich Casimir”

On the images of the 3 linearly independent intertwining maps $P, R, I : S^2(g) \to H(g)$:

$$\Omega(P(XX)) = \text{ad}(X)^2$$ and $$\Omega(ad(X)^2) = \frac{1}{6} \text{ad}(X)^2 + \frac{4}{6} P(X,X) + \frac{2}{6} K(X,X)e.$$

This leads to a complete description of the action of $\Omega$ on $H(g)$.

The eigenvalues of $\Omega$ are:

$$\lambda = 1$$ corresponds to the submodule $C$ of $H(g)$

$$\lambda_+ = \frac{1}{\hbar_+}$$ corresponds to the submodule $L(2\alpha)$ of $H(g)$

$$\lambda_- = \frac{1}{\hbar_-}$$ corresponds to the submodule $Y_2^*$

Therefore, the product $S^+(X, X) \cdot S^+(Y, Y)$ in $Y_2^*$ can be obtained theoretically, by applying the projection operator:

$$\text{Proj}_{Y_2^*} = \text{arbitrary choice of non-zero constant (} \Omega^{-1} \text{) } (\Omega + \frac{1}{\hbar})$$

to the Jordan product $S^+(X, X) \cdot S^+(Y, Y)$.

In practice this means being able to express the action of $\Omega$ on $\text{ad}(X)^2 \cdot \text{ad}(Y)^2$.

This is information is not extractable from C.1 but it is available in the e8 case because of C.2: In fact, we have:

$$\Omega(\text{ad}(X)^4) = -\frac{1}{\hbar} \ast \text{ad}(X)^4 + \frac{a}{3} \ast K(X, X) \ast \text{ad}(X)^2 + \frac{4y}{5} \ast K(X, X) \ast P(X, X) + \frac{y}{5} \ast K(X, X)^2 \ast I$$

This can be linearized twice to yield an expression for $\Omega(\text{ad}(X)^2 \cdot \text{ad}(Y)^2)$

**Lemma:** For the case e8, the product on $Y_2^*$ is given by (constant chosen to make the formula “look nice”):

$$S^*(X^2) \cdot S^*(Y^2) = h \ast S^*(\text{ad}(X)^2(Y), Y) + h \ast S^*(\text{ad}(Y)^2(X), X) + h \ast S^*([X, Y]^2) + K(X, Y)S^*(X, Y)$$

$$- \frac{(h^* + 1)}{\dim(e8)} K(X, X)S^*(Y^2) - \frac{(h^* + 1)}{\dim(e8)} K(Y, Y)S^*(X^2)$$
**Third Question:** How do we turn A(e8) into an algebra with unit element? And if possible can we have the identity map of on e8 as the unit?

- Given a G-invariant algebra \((V, \square)\) with a non-degenerate G-invariant bilinear form \(\langle.,.\rangle\), there is a known process for constructing an algebra \(A= C e + V\), with unit \(e\) via:

\[(ae+x) \circ (be+y) = (x \square y + ay + bx) + (ab + \mu \langle x,y \rangle) e, \text{ for some constant } \mu\]

- The \(Y_2^+\) algebra for E8, has an E8 invariant non degenerate bilinear form:

\[\langle S^+(X, X), S^+(Y, Y) \rangle = \text{Trace} (S^+(X, X) \cdot S^+(Y, Y))\]. In fact, Okubo’s formula C.1 for all exceptional Lie algebras gives:

\[\langle S^+(X, X), S^+(Y, Y) \rangle = \frac{(h^+ +12)}{6} \left\{ -h^+ K([X, Y], [X, Y]) + K(X, Y)^2 - \frac{(h^+ +1)}{\dim(g)} K(X, X) K(Y, Y) \right\}\]

A suitable choice of \(\mu\) (again, for esthetic reasons) gives the following expression for the product on the algebra with unit A(e8):

\[S(X^2) \circ S(Y^2) = h^+ S(ad(X)^2(Y), Y) + h^+ S(ad(Y)^2(X), X) + h^+ S([X, Y]^2) + K(X,Y)S(X,Y)\]

A. The identity operator \(e\) of End(e8) acts as a unit in this algebra

B. The normalized trace map \(\tau(a) = \frac{1}{\dim(e8)} \text{Trace}(a)\) on \(A(e8)\) is associative and non-degenerate meaning:

\[\tau ((a \circ b) \circ c) = \tau (a \circ (b \circ c))\] and \(\tau (x \circ y) = \tau (x \circ y)\) is a non-degenerate bilinear symmetric (E8 invariant) form. In fact \(\tau (x, y)\) it is positive definite for the compact real form of e8.

C. The algebra A(e8) is a simple algebra.

There is nothing to prevent extending this expression, as a definition of the product in the algebra A(g), for all other exceptional Lie algebras. In fact for g exceptional, all of the above statements A, B, C remain true.
THE CORRESPONDENCE: \( g \) (simple) \( \rightarrow A(g) \)

**Fourth Question:** Does this work for any simple Lie algebra in general? Specifically, is it true that:

A. \( H(g) = L(2\alpha) + A(g) \) ?
B. \( \tau(x,y) = \tau(x \circ y) \) is non degenerate on \( A(g) \) and positive definite if \( g \) is a compact real form?
C. Is \( A(g) \) a simple algebra?

The answer is **yes** to all of the above:

A. For the classical Lie algebras, \( H(g) = Ce + J_1 + J_2 + L(2\alpha) \) except for the so(8) case where \( H(g) = Ce + J_1 + J_2 + J_3 + L(2\alpha) \) and the sl(3) case where \( H(g) = Ce + J_1 + L(2\alpha) \).

   We show in the article that all dominant weight vectors of \( H(g) \) modulo \( L(2\alpha) \), lie inside \( A(g) \).

   In fact, besides the Cartan component \( L(2\alpha) \) of \( H(g) \) with dominant weight \( 2\alpha \),
   It can be shown that the other irreducible modules \( J_k \), have a dominant weight vector with non-zero image \( S(X_\alpha, X_\beta) (\alpha \) the highest root and \( \beta \) a root strongly orthogonal to \( \alpha \), or in cases \( An \) and \( Cn \), a dominant weight vector in \( A(g) \) corresponding to the highest short root.

B. The associativity of the trace form \( \tau((a \circ b) \circ c) = \tau(a \circ (b \circ c)) \) is just an algebraic consequence of the product formula. The non –degeneracy of the bilinear form \( \tau(x \circ y) \) is shown on a case by case basis as well as the fact that \( \tau(x \circ y) \) is positive definite for a compact real form \( g \).

C. The simplicity of the algebra \( A(g) \) results directly from the non-degeneracy of \( \tau(x \circ y) \) on each of the irreducible components of \( A(g) \).
SOME CONCLUDING REMARKS

- Among the Algebras $A(g)$, the cases $g=\text{sl}(2)$ and $g=\text{sl}(3)$ are special because the product on the “vector” part of $A(g)$ is trivial. This turns $A(\text{sl}(3))$ into a well known Jordan algebra and $A(\text{sl}(2))$ is reduced to the scalars.

- Excluding the case $A(\text{sl}(3))$. For any $h \in g$ with $K(h,h) \neq 0$, the element $u = S(h^2)/K(h,h)$ is an idempotent in $A(g)$. If $h$ belongs to a Cartan sub-algebra of $g$, then the action of $u$ on a dominant weight vector $S(X_\alpha, X_\beta)$ takes all values at least in $[0,1/2]$ as $h$ varies over the Cartan sub-algebra. Consequently, the Pierce Spectrum contains the full unit interval by duality. As a consequence the algebras $A(g)$ are not Jordan (with the exception of $A(\text{sl}(3))$).

- With the exception of $A(\text{sl}(3))$, one can show that $\text{AUT}(A(g))$ lies in the connected component of $G$.

- We have searched for “classical” identities for $A(g2)$. That is, homogeneous identities among non-associative monomials of a given degree, for orders $\leq 7$ and have found none. However the search for homogeneous weighted identities has been more successful. There are no weighted identities of degree $\leq 6$ and there is a unique weighted identity of degree 7 with polynomial ($G2$ invariant) coefficients. There is strong indication that a similar identity holds for $A(e8)$.

- As it turns out, for the cases $G2$, $F4$, $E6$ and $E7$, there is a naturally defined embedding of $A(g)$ into $\text{End}(V)$ where $V$ is the smallest dimensional representation of $G$ (dimensions 7, 26, 27 and 56 respectively). This provides an alternative way for a computer based model of these algebras.

\[
\sigma(S(X^2)) = 6h^\vee \pi(X)^2 - \frac{1}{2} K(X,X)I \quad X \in g
\]

- For the cases $G2$, $F4$, $E6$ and $E7$ we know that $Y_2^*$ has at least two distinct $G$-invariant algebra structures. At this point there is no indication of how these other algebra structures differ from $A(g)$. 

\[
\sigma(S(X^2)) = 6h^\vee \pi(X)^2 - \frac{1}{2} K(X,X)I \quad X \in g
\]