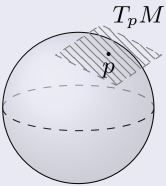
Holomorphic realization of positive energy representations of the Virasoro group

Hadi Salmasian
Department of Mathematics and Statistics
University of Ottawa

April 23, 2014

The *tangent bundle* of a manifold M will be denoted by TM. Note that

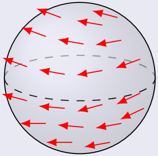
$$TM = \bigcup_{p \in M} T_pM.$$



Review of manifolds and Lie groups

A *vector field* on M is a smooth map

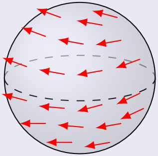
$$\mathbf{v} : M \to TM \ , \ p \mapsto \mathbf{v}(p) \in T_pM.$$



Review of manifolds and Lie groups

A *vector field* on M is a smooth map

$$\mathbf{v} : M \to TM \ , \ p \mapsto \mathbf{v}(p) \in T_pM.$$



The *Lie bracket* of $\mathbf{v}, \mathbf{w} \in \mathrm{Vect}(M)$ is defined by

$$[\mathbf{v}, \mathbf{w}](p) := \frac{d}{dt}\Big((\phi_t^{\mathbf{v}})^* \mathbf{w}(p)\Big)\Big|_{t=0} \text{ for every } p \in M,$$

where

$$(\phi_t^{\mathbf{v}})^* \mathbf{w}(p) := \mathsf{d}_{\mathbf{w}(\phi_t^{\mathbf{v}}(p))} \phi_{-t}^{\mathbf{v}} \big(\phi_t^{\mathbf{v}}(p)\big).$$

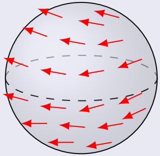
The Lie bracket $\mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ is bilinear and satisfies

$$[\mathbf{w}, \mathbf{v}] = -[\mathbf{v}, \mathbf{w}] \quad \text{and} \quad [\mathbf{u}, [\mathbf{v}, \mathbf{w}]] + [\mathbf{v}, [\mathbf{w}, \mathbf{u}]] + [\mathbf{w}, [\mathbf{u}, \mathbf{v}]] = 0.$$

Review of manifolds and Lie groups

A *vector field* on M is a smooth map

$$\mathbf{v} : M \to TM \ , \ p \mapsto \mathbf{v}(p) \in T_pM.$$



The *Lie bracket* of $\mathbf{v}, \mathbf{w} \in \mathrm{Vect}(M)$ is defined by

$$[\mathbf{v}, \mathbf{w}](p) := \frac{d}{dt}\Big((\phi_t^{\mathbf{v}})^* \mathbf{w}(p)\Big)\Big|_{t=0} \text{ for every } p \in M,$$

where

$$(\phi_t^{\mathbf{v}})^* \mathbf{w}(p) := \mathsf{d}_{\mathbf{w}(\phi_t^{\mathbf{v}}(p))} \phi_{-t}^{\mathbf{v}} \big(\phi_t^{\mathbf{v}}(p)\big).$$

The Lie bracket $\mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ is bilinear and satisfies

$$[\mathbf{w}, \mathbf{v}] = -[\mathbf{v}, \mathbf{w}] \quad \text{and} \quad [\mathbf{u}, [\mathbf{v}, \mathbf{w}]] + [\mathbf{v}, [\mathbf{w}, \mathbf{u}]] + [\mathbf{w}, [\mathbf{u}, \mathbf{v}]] = 0.$$

Review of manifolds and Lie groups

A *Lie group* is a smooth manifold G with a group structure such that the maps

$$G \times G \to G\ ,\ (g,h) \mapsto gh \quad \text{and} \quad G \mapsto G\ ,\ g \mapsto g^{-1}$$

are smooth.

- **Examples:** $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{C}^*$, $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{R})$, etc.

Let G be a Lie group. Set $\ell_g : G \to G$, $\ell_g(h) = gh$.

- Every $x \in T_{\mathbf{1}}G$ results in a *left-invariant vector field* $\mathbf{v}_x \in \mathrm{Vect}(G)$:

$$\mathbf{v}_x(g) := \mathrm{d}_x\ell_g(\mathbf{1}) = \frac{d}{dt}(g\exp(tx))\Big|_{t=0}.$$

- The *Lie algebra* of G is $\mathrm{Lie}(G) := T_{\mathbf{1}}G$, equipped with the Lie bracket

$$\mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(G)\ ,\ [x,y] := [\mathbf{v}_x, \mathbf{v}_y](\mathbf{1}).$$

- The map $[\cdot,\cdot] : \mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ is bilinear and satisfies

$$[y,x] = -[x,y] \quad \text{and} \quad [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

Review of manifolds and Lie groups

A *Lie group* is a smooth manifold G with a group structure such that the maps

$$G \times G \to G \ , \ (g,h) \mapsto gh \quad \text{and} \quad G \mapsto G \ , \ g \mapsto g^{-1}$$

are smooth.

- **Examples:** $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{C}^*$, $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{R})$, etc.

Let G be a Lie group. Set $\ell_g : G \to G$, $\ell_g(h) = gh$.

- Every $x \in T_1G$ results in a *left-invariant vector field* $\mathbf{v}_x \in \mathrm{Vect}(G)$:

$$\mathbf{v}_x(g) := \mathrm{d}_x\ell_g(\mathbf{1}) = \frac{d}{dt}\big(g\exp(tx)\big)\Big|_{t=0} .$$

- The *Lie algebra* of G is $\mathrm{Lie}(G) := T_1G$, equipped with the Lie bracket

$$\mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(G) \ , \ [x,y] := [\mathbf{v}_x, \mathbf{v}_y](\mathbf{1}).$$

- The map $[\cdot,\cdot] : \mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ is bilinear and satisfies

$$[y,x] = -[x,y] \quad \text{and} \quad [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

Review of manifolds and Lie groups

A *Lie group* is a smooth manifold G with a group structure such that the maps

$$G \times G \to G \ , \ (g,h) \mapsto gh \quad \text{and} \quad G \mapsto G \ , \ g \mapsto g^{-1}$$

are smooth.

- **Examples:** $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{C}^*$, $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{R})$, etc.

Let G be a Lie group. Set $\ell_g : G \to G$, $\ell_g(h) = gh$.

- Every $x \in T_{\mathbf{1}}G$ results in a *left-invariant vector field* $\mathbf{v}_x \in \mathrm{Vect}(G)$:

$$\mathbf{v}_x(g) := \mathsf{d}_x \ell_g(\mathbf{1}) = \frac{d}{dt}\big(g \exp(tx)\big)\Big|_{t=0} .$$

- The *Lie algebra* of G is $\mathrm{Lie}(G) := T_{\mathbf{1}}G$, equipped with the Lie bracket

$$\mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(G) \ , \ [x,y] := [\mathbf{v}_x, \mathbf{v}_y](\mathbf{1}).$$

- The map $[\cdot,\cdot] : \mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ is bilinear and satisfies

$$[y,x] = -[x,y] \quad \text{and} \quad [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

Review of manifolds and Lie groups

A *Lie group* is a smooth manifold G with a group structure such that the maps

$$G \times G \to G \ , \ (g,h) \mapsto gh \quad \text{and} \quad G \mapsto G \ , \ g \mapsto g^{-1}$$

are smooth.

- **Examples:** $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{C}^*$, $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{R})$, etc.

Let G be a Lie group. Set $\ell_g : G \to G$, $\ell_g(h) = gh$.

- Every $x \in T_{\mathbf{1}}G$ results in a *left-invariant vector field* $\mathbf{v}_x \in \mathrm{Vect}(G)$:

$$\mathbf{v}_x(g) := \mathsf{d}_x\ell_g(\mathbf{1}) = \frac{d}{dt}\big(g\exp(tx)\big)\Big|_{t=0} .$$

- The *Lie algebra* of G is $\mathrm{Lie}(G) := T_{\mathbf{1}}G$, equipped with the Lie bracket

$$\mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(G) \ , \ [x,y] := [\mathbf{v}_x, \mathbf{v}_y](\mathbf{1}).$$

- The map $[\cdot,\cdot] : \mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ is bilinear and satisfies

$$[y,x] = -[x,y] \quad \text{and} \quad [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

Review of manifolds and Lie groups

A *Lie group* is a smooth manifold G with a group structure such that the maps

$$G \times G \to G \ , \ (g,h) \mapsto gh \quad \text{and} \quad G \mapsto G \ , \ g \mapsto g^{-1}$$

are smooth.

- **Examples:** $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{C}^*$, $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{R})$, etc.

Let G be a Lie group. Set $\ell_g : G \to G$, $\ell_g(h) = gh$.

- Every $x \in T_{\mathbf{1}}G$ results in a *left-invariant vector field* $\mathbf{v}_x \in \mathrm{Vect}(G)$:

$$\mathbf{v}_x(g) := \mathsf{d}_x \ell_g(\mathbf{1}) = \frac{d}{dt}\big(g \exp(tx)\big)\Big|_{t=0}.$$

- The *Lie algebra* of G is $\mathrm{Lie}(G) := T_{\mathbf{1}}G$, equipped with the Lie bracket

$$\mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(G) \ , \ [x,y] := [\mathbf{v}_x, \mathbf{v}_y](\mathbf{1}).$$

- The map $[\cdot,\cdot] : \mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ is bilinear and satisfies

$$[y,x] = -[x,y] \quad \text{and} \quad [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

Example

- $\mathrm{Diff}_+(S^1)$: the group of orientation-preserving diffeomorphisms of S^1.
- (Leslie '67, Hamilton '82, Milnor '84) $\mathrm{Vect}(S^1) = \mathrm{Lie}(\mathrm{Diff}_+(S^1))$.

- Every $\mathbf{v} \in \mathrm{Vect}(S^1)$ can be written as

$$\mathbf{v}(e^{i\theta}) = f(e^{i\theta})\frac{d}{d\theta} \quad , \quad 0 \leq \theta \leq 2\pi$$

where $f : S^1 \to \mathbb{R}$ is smooth.

$\mathrm{Diff}_+(S^1)$

Example

- $\mathrm{Diff}_+(S^1)$: the group of orientation-preserving diffeomorphisms of S^1.
- (Leslie '67, Hamilton '82, Milnor '84) $\mathrm{Vect}(S^1) = \mathrm{Lie}(\mathrm{Diff}_+(S^1))$.

- Every $\mathbf{v} \in \mathrm{Vect}(S^1)$ can be written as

$$\mathbf{v}(e^{i\theta}) = f(e^{i\theta})\frac{d}{d\theta} \quad , \quad 0 \leq \theta \leq 2\pi$$

where $f : S^1 \to \mathbb{R}$ is smooth.

- Complexification:

$$\mathfrak{d}_\infty := \mathrm{Vect}(S^1) \otimes_{\mathbb{R}} \mathbb{C}$$

- Set $\mathbf{d}_n = ie^{in\theta}\frac{d}{d\theta}$ and

$$\mathfrak{d} := \mathrm{Span}_{\mathbb{C}}\{\mathbf{d}_n : n \in \mathbb{Z}\} \subset \mathfrak{d}_\infty.$$

$\mathrm{Diff}_+(S^1)$

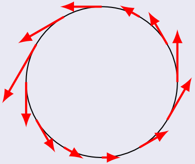
Example

- $\mathrm{Diff}_+(S^1)$: the group of orientation-preserving diffeomorphisms of S^1.
- (Leslie '67, Hamilton '82, Milnor '84) $\mathrm{Vect}(S^1) = \mathrm{Lie}(\mathrm{Diff}_+(S^1))$.

- Every $\mathbf{v} \in \mathrm{Vect}(S^1)$ can be written as

$$\mathbf{v}(e^{i\theta}) = f(e^{i\theta})\frac{d}{d\theta} \quad , \quad 0 \le \theta \le 2\pi$$

where $f : S^1 \to \mathbb{R}$ is smooth.



- Complexification:

$$\mathfrak{d}_\infty := \mathrm{Vect}(S^1) \otimes_{\mathbb{R}} \mathbb{C}$$

- Set $\mathbf{d}_n = ie^{in\theta}\dfrac{d}{d\theta}$ and

$$\mathfrak{d} := \mathrm{Span}_{\mathbb{C}}\{\mathbf{d}_n \ : \ n \in \mathbb{Z}\} \subset \mathfrak{d}_\infty .$$

$\mathrm{Diff}_+(S^1)$

Example

- $\mathrm{Diff}_+(S^1)$: the group of orientation-preserving diffeomorphisms of S^1.
- (Leslie '67, Hamilton '82, Milnor '84) $\mathrm{Vect}(S^1) = \mathrm{Lie}(\mathrm{Diff}_+(S^1))$.

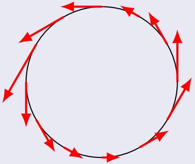
- Every $\mathbf{v} \in \mathrm{Vect}(S^1)$ can be written as

$$\mathbf{v}(e^{i\theta}) = f(e^{i\theta})\frac{d}{d\theta} \quad , \quad 0 \leq \theta \leq 2\pi$$

where $f : S^1 \to \mathbb{R}$ is smooth.

- Complexification:

$$\mathfrak{d}_\infty := \mathrm{Vect}(S^1) \otimes_{\mathbb{R}} \mathbb{C}$$



- Set $\mathbf{d}_n = ie^{in\theta}\frac{d}{d\theta}$ and

$$\mathfrak{d} := \mathrm{Span}_{\mathbb{C}}\{\mathbf{d}_n \ : \ n \in \mathbb{Z}\} \subset \mathfrak{d}_\infty .$$

Example

- $\mathrm{Diff}_+(S^1)$: the group of orientation-preserving diffeomorphisms of S^1.
- (Leslie '67, Hamilton '82, Milnor '84) $\mathrm{Vect}(S^1) = \mathrm{Lie}(\mathrm{Diff}_+(S^1))$.

- Every $\mathbf{v} \in \mathrm{Vect}(S^1)$ can be written as

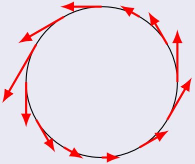
$$\mathbf{v}(e^{i\theta}) = f(e^{i\theta})\frac{d}{d\theta} \quad , \quad 0 \leq \theta \leq 2\pi$$

  where $f : S^1 \to \mathbb{R}$ is smooth.
- Complexification:

$$\mathfrak{d}_\infty := \mathrm{Vect}(S^1) \otimes_{\mathbb{R}} \mathbb{C}$$

- Set $\mathbf{d}_n = ie^{in\theta}\dfrac{d}{d\theta}$ and

$$\mathfrak{d} := \mathrm{Span}_{\mathbb{C}}\{\mathbf{d}_n \ : \ n \in \mathbb{Z}\} \subset \mathfrak{d}_\infty.$$



The Virasoro algebra

$$\mathbf{d}_n = ie^{in\theta}\frac{d}{d\theta} \quad \text{and} \quad \mathfrak{d} := \mathrm{Span}_{\mathbb{C}}\{\mathbf{d}_n \ : \ n \in \mathbb{Z}\} \subset \mathfrak{d}_\infty := \mathrm{Vect}(S^1) \otimes_{\mathbb{R}} \mathbb{C}.$$

Theorem (Gelfand–Fuks '68)

The Lie algebra $\mathfrak{d}$ has a unique central extension $\widehat{\mathfrak{d}} := \mathfrak{d} \oplus \mathbb{C}\kappa$.

The Lie bracket of $\widehat{\mathfrak{d}}$ is uniquely determined by

$$[\mathbf{d}_m, \mathbf{d}_n] = (n-m)\mathbf{d}_{m+n} + \delta_{m,-n}\frac{m(m^2-1)}{12}\kappa.$$

Triangular decomposition

We can write $\widehat{\mathfrak{d}} = \widehat{\mathfrak{d}}_- \oplus \widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+$, where

$$\widehat{\mathfrak{d}}_\circ := \mathbb{C}\mathbf{d}_0 \oplus \mathbb{C}\kappa \quad \text{and} \quad \widehat{\mathfrak{d}}_\pm = \mathrm{Span}_{\mathbb{C}}\{\mathbf{d}_{\pm n} \ : \ n = 1, 2, 3, \ldots\}.$$

$$[\widehat{\mathfrak{d}}_+, \widehat{\mathfrak{d}}_+] \subseteq \widehat{\mathfrak{d}}_+ \quad , \quad [\widehat{\mathfrak{d}}_-, \widehat{\mathfrak{d}}_-] \subseteq \widehat{\mathfrak{d}}_- \quad , \quad [\widehat{\mathfrak{d}}_\circ, \widehat{\mathfrak{d}}_\circ] \subseteq \widehat{\mathfrak{d}}_\circ \quad , \quad .$$

$$[\widehat{\mathfrak{d}}_\circ, \widehat{\mathfrak{d}}_+] \subseteq \widehat{\mathfrak{d}}_+ \quad , \quad [\widehat{\mathfrak{d}}_\circ, \widehat{\mathfrak{d}}_-] \subseteq \widehat{\mathfrak{d}}_-.$$

The Virasoro algebra

$\mathbf{d}_n = ie^{in\theta}\dfrac{d}{d\theta}$ and $\mathfrak{d} := \mathrm{Span}_{\mathbb{C}}\{\mathbf{d}_n \;:\; n \in \mathbb{Z}\} \subset \mathfrak{d}_\infty := \mathrm{Vect}(S^1) \otimes_{\mathbb{R}} \mathbb{C}.$

Theorem (Gelfand–Fuks '68)

The Lie algebra $\mathfrak{d}$ has a unique central extension $\widehat{\mathfrak{d}} := \mathfrak{d} \oplus \mathbb{C}\kappa$.

The Lie bracket of $\widehat{\mathfrak{d}}$ is uniquely determined by

$$[\mathbf{d}_m, \mathbf{d}_n] = (n-m)\mathbf{d}_{m+n} + \delta_{m,-n}\frac{m(m^2-1)}{12}\kappa.$$

Triangular decomposition

We can write $\widehat{\mathfrak{d}} = \widehat{\mathfrak{d}}_- \oplus \widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+$, where

$$\widehat{\mathfrak{d}}_\circ := \mathbb{C}\mathbf{d}_0 \oplus \mathbb{C}\kappa \quad \text{and} \quad \widehat{\mathfrak{d}}_\pm = \mathrm{Span}_{\mathbb{C}}\{\mathbf{d}_{\pm n} \;:\; n = 1, 2, 3, \ldots\}.$$

$$[\widehat{\mathfrak{d}}_+, \widehat{\mathfrak{d}}_+] \subseteq \widehat{\mathfrak{d}}_+ \quad , \quad [\widehat{\mathfrak{d}}_-, \widehat{\mathfrak{d}}_-] \subseteq \widehat{\mathfrak{d}}_- \quad , \quad [\widehat{\mathfrak{d}}_\circ, \widehat{\mathfrak{d}}_\circ] \subseteq \widehat{\mathfrak{d}}_\circ \quad , \quad .$$

$$[\widehat{\mathfrak{d}}_\circ, \widehat{\mathfrak{d}}_+] \subseteq \widehat{\mathfrak{d}}_+ \quad , \quad [\widehat{\mathfrak{d}}_\circ, \widehat{\mathfrak{d}}_-] \subseteq \widehat{\mathfrak{d}}_-.$$

The Virasoro algebra

$$\mathbf{d}_n = ie^{in\theta}\frac{d}{d\theta} \quad \text{and} \quad \mathfrak{d} := \mathrm{Span}_{\mathbb{C}}\{\mathbf{d}_n \ : \ n \in \mathbb{Z}\} \subset \mathfrak{d}_\infty := \mathrm{Vect}(S^1) \otimes_{\mathbb{R}} \mathbb{C}.$$

Theorem (Gelfand–Fuks '68)

The Lie algebra $\mathfrak{d}$ has a unique central extension $\widehat{\mathfrak{d}} := \mathfrak{d} \oplus \mathbb{C}\kappa$.

The Lie bracket of $\widehat{\mathfrak{d}}$ is uniquely determined by

$$[\mathbf{d}_m, \mathbf{d}_n] = (n-m)\mathbf{d}_{m+n} + \delta_{m,-n}\frac{m(m^2-1)}{12}\kappa.$$

Triangular decomposition

We can write $\widehat{\mathfrak{d}} = \widehat{\mathfrak{d}}_- \oplus \widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+$, where

$$\widehat{\mathfrak{d}}_\circ := \mathbb{C}\mathbf{d}_0 \oplus \mathbb{C}\kappa \quad \text{and} \quad \widehat{\mathfrak{d}}_\pm = \mathrm{Span}_{\mathbb{C}}\{\mathbf{d}_{\pm n} \ : \ n = 1, 2, 3, \ldots\}.$$

$$[\widehat{\mathfrak{d}}_+, \widehat{\mathfrak{d}}_+] \subseteq \widehat{\mathfrak{d}}_+ \quad , \quad [\widehat{\mathfrak{d}}_-, \widehat{\mathfrak{d}}_-] \subseteq \widehat{\mathfrak{d}}_- \quad , \quad [\widehat{\mathfrak{d}}_\circ, \widehat{\mathfrak{d}}_\circ] \subseteq \widehat{\mathfrak{d}}_\circ \quad , \quad .$$

$$[\widehat{\mathfrak{d}}_\circ, \widehat{\mathfrak{d}}_+] \subseteq \widehat{\mathfrak{d}}_+ \quad , \quad [\widehat{\mathfrak{d}}_\circ, \widehat{\mathfrak{d}}_-] \subseteq \widehat{\mathfrak{d}}_-.$$

Examples of $\widehat{\mathfrak{d}}$-modules

$$\widehat{\mathfrak{d}} \times V \to V \quad , \quad [x,y]\cdot v = x\cdot(y\cdot v) - y\cdot(x\cdot v) \quad \text{for all } x,y \in \widehat{\mathfrak{d}};\ v \in V.$$

Fock space representations (string theory)

- $V := \mathbb{C}[x_1, x_2, x_3, \ldots]$
- For $n \in \mathbb{Z}$, define $\mathbf{a}_n \in \mathrm{End}(V)$ by $\mathbf{a}_n(p) := \begin{cases} \dfrac{\partial p}{\partial x_n} & n > 0 \\ n x_{-n} p & n < 0 \\ p & n = 0 \end{cases}$
- For $k \in \mathbb{Z}$, set $L_k := -\dfrac{1}{2} \sum_{n \in \mathbb{Z}} :\mathbf{a}_{-j}\mathbf{a}_{j+k}:$

  where $:\mathbf{a}_i\mathbf{a}_j: = \begin{cases} \mathbf{a}_i\mathbf{a}_j & i \le j \\ \mathbf{a}_j\mathbf{a}_i & i > j \end{cases}$.

  Then
  $$[L_m, L_n] = (n-m)L_{m+n} + \delta_{m,-n}\frac{m(m^2-1)}{12}.$$

Examples of $\widehat{\mathfrak{d}}$-modules

$$\widehat{\mathfrak{d}} \times V \to V \quad , \quad [x,y]\cdot v = x\cdot(y\cdot v) - y\cdot(x\cdot v) \quad \text{for all } x,y \in \widehat{\mathfrak{d}};\ v \in V.$$

Fock space representations (string theory)

- $V := \mathbb{C}[x_1, x_2, x_3, \ldots]$
- For $n \in \mathbb{Z}$, define $\mathbf{a}_n \in \mathrm{End}(V)$ by $\mathbf{a}_n(p) := \begin{cases} \dfrac{\partial p}{\partial x_n} & n > 0 \\ n x_{-n} p & n < 0 \\ p & n = 0 \end{cases}$
- For $k \in \mathbb{Z}$, set $L_k := -\dfrac{1}{2} \sum_{n \in \mathbb{Z}} {:}\mathbf{a}_{-j}\mathbf{a}_{j+k}{:}$

  where ${:}\mathbf{a}_i\mathbf{a}_j{:} = \begin{cases} \mathbf{a}_i\mathbf{a}_j & i \leq j \\ \mathbf{a}_j\mathbf{a}_i & i > j \end{cases}.$

  Then

  $$[L_m, L_n] = (n-m)L_{m+n} + \delta_{m,-n}\frac{m(m^2-1)}{12}.$$

Examples of $\widehat{\mathfrak{d}}$-modules

$$\widehat{\mathfrak{d}} \times V \to V \quad , \quad [x,y]\cdot v = x\cdot(y\cdot v) - y\cdot(x\cdot v) \quad \text{for all } x,y \in \widehat{\mathfrak{d}};\ v \in V.$$

Fock space representations (string theory)

- $V := \mathbb{C}[x_1, x_2, x_3, \ldots]$
- For $n \in \mathbb{Z}$, define $\mathbf{a}_n \in \mathrm{End}(V)$ by $\mathbf{a}_n(p) := \begin{cases} \dfrac{\partial p}{\partial x_n} & n > 0 \\ n x_{-n} p & n < 0 \\ p & n = 0 \end{cases}$
- For $k \in \mathbb{Z}$, set $L_k := -\dfrac{1}{2} \sum_{n\in\mathbb{Z}} {:}\mathbf{a}_{-j}\mathbf{a}_{j+k}{:}$

  where ${:}\mathbf{a}_i\mathbf{a}_j{:} = \begin{cases} \mathbf{a}_i\mathbf{a}_j & i \le j \\ \mathbf{a}_j\mathbf{a}_i & i > j \end{cases}.$

  Then

$$[L_m, L_n] = (n-m)L_{m+n} + \delta_{m,-n}\frac{m(m^2-1)}{12}.$$

Examples of $\widehat{\mathfrak{d}}$-modules

$$\widehat{\mathfrak{d}} \times V \to V \quad , \quad [x,y]\cdot v = x\cdot(y\cdot v) - y\cdot(x\cdot v) \quad \text{for all } x,y \in \widehat{\mathfrak{d}};\ v \in V.$$

Fock space representations (string theory)

- $V := \mathbb{C}[x_1, x_2, x_3, \ldots]$
- For $n \in \mathbb{Z}$, define $\mathbf{a}_n \in \mathrm{End}(V)$ by $\mathbf{a}_n(p) := \begin{cases} \dfrac{\partial p}{\partial x_n} & n > 0 \\ n x_{-n} p & n < 0 \\ p & n = 0 \end{cases}$
- For $k \in \mathbb{Z}$, set $L_k := -\dfrac{1}{2} \displaystyle\sum_{n \in \mathbb{Z}} {:}\mathbf{a}_{-j}\mathbf{a}_{j+k}{:}$

  where ${:}\mathbf{a}_i\mathbf{a}_j{:} = \begin{cases} \mathbf{a}_i\mathbf{a}_j & i \leq j \\ \mathbf{a}_j\mathbf{a}_i & i > j \end{cases}.$

Then

$$[L_m, L_n] = (n-m)L_{m+n} + \delta_{m,-n}\frac{m(m^2-1)}{12}.$$

$$\widehat{\mathfrak{d}} \times V \to V \quad , \quad [x,y]\cdot v = x\cdot(y\cdot v) - y\cdot(x\cdot v) \quad \text{for all } x,y \in \widehat{\mathfrak{d}};\ v \in V.$$

Fock space representations (string theory)

- $V := \mathbb{C}[x_1, x_2, x_3, \ldots]$
- For $n \in \mathbb{Z}$, define $\mathbf{a}_n \in \mathrm{End}(V)$ by $\mathbf{a}_n(p) := \begin{cases} \dfrac{\partial p}{\partial x_n} & n > 0 \\ n x_{-n} p & n < 0 \\ p & n = 0 \end{cases}$
- For $k \in \mathbb{Z}$, set $L_k := -\dfrac{1}{2} \sum_{n\in\mathbb{Z}} {:}\mathbf{a}_{-j}\mathbf{a}_{j+k}{:}$

  where ${:}\mathbf{a}_i\mathbf{a}_j{:} = \begin{cases} \mathbf{a}_i\mathbf{a}_j & i \le j \\ \mathbf{a}_j\mathbf{a}_i & i > j \end{cases}.$

  Then

  $$[L_m, L_n] = (n-m)L_{m+n} + \delta_{m,-n}\frac{m(m^2-1)}{12}.$$

Highest weight modules

$$\widehat{\mathfrak{d}} = \widehat{\mathfrak{d}}_- \oplus \widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+$$

$$\widehat{\mathfrak{d}}_\circ := \mathbb{C}\mathbf{d}_0 \oplus \mathbb{C}\kappa \quad \text{and} \quad \widehat{\mathfrak{d}}_\pm = \mathrm{Span}_{\mathbb{C}}\{\mathbf{d}_{\pm n} \ : \ n = 1, 2, 3, \ldots\}.$$

The modules $V_{h,c}$

Fix $h, c \in \mathbb{C}$.

- $\mathbb{C}_{h,c}$: one-dimensional $\widehat{\mathfrak{d}}_\circ$-module.

$$\mathbf{d}_0 \cdot v := hv \quad \text{and} \quad \kappa \cdot v = cv \quad \text{for every } v \in \mathbb{C}_{h,c}.$$

- $\mathbb{C}_{h,c}$ extends (trivially on $\widehat{\mathfrak{d}}_+$) to a module for $\widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+$.
- Verma module:

$$\mathbf{U}(\widehat{\mathfrak{d}}) \otimes_{\mathbf{U}(\widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+)} \mathbb{C}_{h,c} \qquad\qquad \mathbf{U}(\widehat{\mathfrak{d}}) : \mathbf{T}(\widehat{\mathfrak{d}}) / \langle x \otimes y - y \otimes x - [x,y] \rangle$$

- Unique irreducible quotient:

$$V_{h,c} := \left(\mathbf{U}(\widehat{\mathfrak{d}}) \otimes_{\mathbf{U}(\widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+)} \mathbb{C}_{h,c} \right) \Big/ M_{h,c}.$$

Highest weight modules

$$\widehat{\mathfrak{d}} = \widehat{\mathfrak{d}}_- \oplus \widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+$$

$$\widehat{\mathfrak{d}}_\circ := \mathbb{C}\mathbf{d}_0 \oplus \mathbb{C}\kappa \quad \text{and} \quad \widehat{\mathfrak{d}}_\pm = \mathrm{Span}_{\mathbb{C}}\{\mathbf{d}_{\pm n} \ : \ n = 1, 2, 3, \ldots\}.$$

The modules $V_{h,c}$

Fix $h, c \in \mathbb{C}$.

- $\mathbb{C}_{h,c}$: one-dimensional $\widehat{\mathfrak{d}}_\circ$-module.

$$\mathbf{d}_0 \cdot v := hv \quad \text{and} \quad \kappa \cdot v = cv \quad \text{for every } v \in \mathbb{C}_{h,c}.$$

- $\mathbb{C}_{h,c}$ extends (trivially on $\widehat{\mathfrak{d}}_+$) to a module for $\widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+$.
- Verma module:

$$\mathbf{U}(\widehat{\mathfrak{d}}) \otimes_{\mathbf{U}(\widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+)} \mathbb{C}_{h,c} \qquad \mathbf{U}(\widehat{\mathfrak{d}}) : \mathbf{T}(\widehat{\mathfrak{d}}) / \langle x \otimes y - y \otimes x - [x, y] \rangle$$

- Unique irreducible quotient:

$$V_{h,c} := \left(\mathbf{U}(\widehat{\mathfrak{d}}) \otimes_{\mathbf{U}(\widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+)} \mathbb{C}_{h,c} \right) / M_{h,c}.$$

Highest weight modules

$$\widehat{\mathfrak{d}} = \widehat{\mathfrak{d}}_- \oplus \widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+$$

$$\widehat{\mathfrak{d}}_\circ := \mathbb{C}\mathbf{d}_0 \oplus \mathbb{C}\kappa \quad \text{and} \quad \widehat{\mathfrak{d}}_\pm = \mathrm{Span}_{\mathbb{C}}\{\mathbf{d}_{\pm n} \ : \ n = 1, 2, 3, \ldots\}.$$

The modules $V_{h,c}$

Fix $h, c \in \mathbb{C}$.

- $\mathbb{C}_{h,c}$: one-dimensional $\widehat{\mathfrak{d}}_\circ$-module.

$$\mathbf{d}_0 \cdot v := hv \quad \text{and} \quad \kappa \cdot v = cv \quad \text{for every } v \in \mathbb{C}_{h,c}.$$

- $\mathbb{C}_{h,c}$ extends (trivially on $\widehat{\mathfrak{d}}_+$) to a module for $\widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+$.
- Verma module:

$$\mathbf{U}(\widehat{\mathfrak{d}}) \otimes_{\mathbf{U}(\widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+)} \mathbb{C}_{h,c} \qquad \mathbf{U}(\widehat{\mathfrak{d}}) : \mathbf{T}(\widehat{\mathfrak{d}}) / \langle x \otimes y - y \otimes x - [x, y] \rangle$$

- Unique irreducible quotient:

$$V_{h,c} := \left(\mathbf{U}(\widehat{\mathfrak{d}}) \otimes_{\mathbf{U}(\widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+)} \mathbb{C}_{h,c} \right) \Big/ M_{h,c}.$$

Highest weight modules

$$\widehat{\mathfrak{d}} = \widehat{\mathfrak{d}}_- \oplus \widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+$$

$$\widehat{\mathfrak{d}}_\circ := \mathbb{C}\mathbf{d}_0 \oplus \mathbb{C}\kappa \quad \text{and} \quad \widehat{\mathfrak{d}}_\pm = \mathrm{Span}_{\mathbb{C}}\{\mathbf{d}_{\pm n} \ : \ n = 1, 2, 3, \ldots\}.$$

The modules $V_{h,c}$

Fix $h, c \in \mathbb{C}$.

- $\mathbb{C}_{h,c}$: one-dimensional $\widehat{\mathfrak{d}}_\circ$-module.

$$\mathbf{d}_0 \cdot v := hv \quad \text{and} \quad \kappa \cdot v = cv \qquad \text{for every } v \in \mathbb{C}_{h,c}.$$

- $\mathbb{C}_{h,c}$ extends (trivially on $\widehat{\mathfrak{d}}_+$) to a module for $\widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+$.
- Verma module:

$$\mathbf{U}\big(\widehat{\mathfrak{d}}\big) \otimes_{\mathbf{U}(\widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+)} \mathbb{C}_{h,c} \qquad\qquad \mathbf{U}\big(\widehat{\mathfrak{d}}\big) : \mathbf{T}\big(\widehat{\mathfrak{d}}\big)/\langle x \otimes y - y \otimes x - [x, y]\rangle$$

- Unique irreducible quotient:

$$V_{h,c} := \Big(\mathbf{U}\big(\widehat{\mathfrak{d}}\big) \otimes_{\mathbf{U}(\widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+)} \mathbb{C}_{h,c}\Big) \Big/ M_{h,c}.$$

Highest weight modules

$$\widehat{\mathfrak{d}} = \widehat{\mathfrak{d}}_- \oplus \widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+$$

$$\widehat{\mathfrak{d}}_\circ := \mathbb{C}\mathbf{d}_0 \oplus \mathbb{C}\kappa \quad \text{and} \quad \widehat{\mathfrak{d}}_\pm = \mathrm{Span}_\mathbb{C}\{\mathbf{d}_{\pm n} \ : \ n = 1, 2, 3, \ldots\}.$$

The modules $V_{h,c}$

Fix $h, c \in \mathbb{C}$.

- $\mathbb{C}_{h,c}$: one-dimensional $\widehat{\mathfrak{d}}_\circ$-module.

$$\mathbf{d}_0 \cdot v := hv \quad \text{and} \quad \kappa \cdot v = cv \quad \text{for every } v \in \mathbb{C}_{h,c}.$$

- $\mathbb{C}_{h,c}$ extends (trivially on $\widehat{\mathfrak{d}}_+$) to a module for $\widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+$.
- Verma module:

$$\mathbf{U}(\widehat{\mathfrak{d}}) \otimes_{\mathbf{U}(\widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+)} \mathbb{C}_{h,c} \qquad\qquad \mathbf{U}(\widehat{\mathfrak{d}}) : \mathbf{T}(\widehat{\mathfrak{d}}) / \langle x \otimes y - y \otimes x - [x, y] \rangle$$

- Unique irreducible quotient:

$$V_{h,c} := \left(\mathbf{U}(\widehat{\mathfrak{d}}) \otimes_{\mathbf{U}(\widehat{\mathfrak{d}}_\circ \oplus \widehat{\mathfrak{d}}_+)} \mathbb{C}_{h,c} \right) \Big/ M_{h,c}.$$

Unitarizable highest weight modules

The real form $\widehat{\mathfrak{d}}_{\mathbb{R}}$

- Complex conjugation $\widehat{\mathfrak{d}} \to \widehat{\mathfrak{d}}$, $x \mapsto \overline{x}$ defined by $\overline{\mathbf{d}_n} = -\mathbf{d}_{-n}$, $\overline{\kappa} = -\kappa$.

Unitarizable modules

A $\widehat{\mathfrak{d}}$-module V is called *unitarizable* if it is equipped with a positive definite Hermitian form $(\cdot,\cdot) : V \times V \to \mathbb{C}$ which satisfies

$$(x \cdot v, w) = -(v, \overline{x} \cdot w) \quad \text{for all } x \in \widehat{\mathfrak{d}};\ v, w \in V.$$

Theorem (FQS '85; GKO '86; Kac–Wakimoto '85; Langlands '86; . . .)

The highest weight module $V_{h,c}$ is unitarizable if and only if one of the following conditions hold:

$$\begin{cases} c \geq 1 \\ h \leq 0 \end{cases}$$

$$\begin{cases} c = 1 - \dfrac{6}{(m+2)(m+3)} \\ h = \dfrac{1 - ((m+3)r - (m+2)s)^2}{4(m+2)(m+3)} \end{cases}$$

for integers $1 \leq s \leq r \leq m+1$.

Unitarizable highest weight modules

The real form $\widehat{\mathfrak{d}}_{\mathbb{R}}$

- Complex conjugation $\widehat{\mathfrak{d}} \to \widehat{\mathfrak{d}}$, $x \mapsto \overline{x}$ defined by $\overline{\mathbf{d}_n} = -\mathbf{d}_{-n}$, $\overline{\kappa} = -\kappa$.

Unitarizable modules

A $\widehat{\mathfrak{d}}$-module V is called *unitarizable* if it is equipped with a positive definite Hermitian form $(\cdot,\cdot) : V \times V \to \mathbb{C}$ which satisfies

$$(x \cdot v, w) = -(v, \overline{x} \cdot w) \quad \text{for all } x \in \widehat{\mathfrak{d}};\ v, w \in V.$$

Theorem (FQS '85; GKO '86; Kac–Wakimoto '85; Langlands '86; ...)

The highest weight module $V_{h,c}$ is unitarizable if and only if one of the following conditions hold:

$$\begin{cases} c \geq 1 \\ h \leq 0 \end{cases}$$

$$\begin{cases} c = 1 - \dfrac{6}{(m+2)(m+3)} \\ h = \dfrac{1 - ((m+3)r - (m+2)s)^2}{4(m+2)(m+3)} \end{cases}$$

for integers $1 \leq s \leq r \leq m+1$.

Unitarizable highest weight modules

The real form $\widehat{\mathfrak{d}}_{\mathbb{R}}$

- Complex conjugation $\widehat{\mathfrak{d}} \to \widehat{\mathfrak{d}}$, $x \mapsto \overline{x}$ defined by $\overline{\mathbf{d}_n} = -\mathbf{d}_{-n}$, $\overline{\kappa} = -\kappa$.

Unitarizable modules

A $\widehat{\mathfrak{d}}$-module V is called *unitarizable* if it is equipped with a positive definite Hermitian form $(\cdot,\cdot) : V \times V \to \mathbb{C}$ which satisfies

$$(x \cdot v, w) = -(v, \overline{x} \cdot w) \quad \text{for all } x \in \widehat{\mathfrak{d}};\ v, w \in V.$$

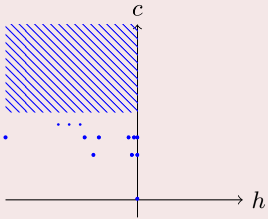
Theorem (FQS '85; GKO '86; Kac–Wakimoto '85; Langlands '86; . . .)

The highest weight module $V_{h,c}$ is unitarizable if and only if one of the following conditions hold:

$$\begin{cases} c \geq 1 \\ h \leq 0 \end{cases}$$

$$\begin{cases} c = 1 - \dfrac{6}{(m+2)(m+3)} \\ h = \dfrac{1 - ((m+3)r - (m+2)s)^2}{4(m+2)(m+3)} \end{cases}$$

for integers $1 \leq s \leq r \leq m+1$.



The group Vir

The Bott–Virasoro group

- $\mathrm{Vir} := \mathrm{Diff}_+(S^1) \times \mathbb{R}$ with the group operation

$$(\varphi_1, t_1)(\varphi_2, t_2) := \big(\varphi_1 \circ \varphi_2, t_1 + t_2 + \mathbf{B}(\varphi_1, \varphi_2)\big)$$

where $\mathbf{B}(\varphi_1, \varphi_2) := \dfrac{1}{2}\displaystyle\int_0^{2\pi} \log\big((\varphi_1 \circ \varphi_2)'\big)\, d\log(\varphi_2').$

- $\mathrm{Lie}(\mathrm{Vir}) \otimes_{\mathbb{R}} \mathbb{C} \cong \widehat{\mathfrak{d}}_\infty := \mathfrak{d}_\infty \oplus \mathbb{C}$ (central extention).

The Bott–Virasoro group

- $\mathrm{Vir} := \mathrm{Diff}_+(S^1) \times \mathbb{R}$ with the group operation
$$(\varphi_1, t_1)(\varphi_2, t_2) := \big(\varphi_1 \circ \varphi_2, t_1 + t_2 + \mathbf{B}(\varphi_1, \varphi_2)\big)$$
where $\mathbf{B}(\varphi_1, \varphi_2) := \dfrac{1}{2}\displaystyle\int_0^{2\pi} \log\big((\varphi_1 \circ \varphi_2)'\big)\, d\log(\varphi_2')$.
- $\mathrm{Lie}(\mathrm{Vir}) \otimes_{\mathbb{R}} \mathbb{C} \cong \widehat{\mathfrak{d}}_\infty := \mathfrak{d}_\infty \oplus \mathbb{C} \quad$ (central extention).

Unitary representations

Unitary representations

By a *unitary representation* of a Lie group G on a Hilbert space $\mathscr{H}$, we mean a group homomorphism

$$\pi : G \mapsto \mathrm{U}(\mathscr{H})$$

with continuous orbit maps $\pi^v : G \to \mathscr{H}$, $\pi^v(g) := \pi(g)v$ for every $v \in \mathscr{H}$.

Infinitesimal action

- Given a unitary representation $(\pi, \mathscr{H})$ of G, we can define

$$\mathrm{d}\pi(x)v := \lim_{t\to 0} \frac{1}{t}(\pi(\exp(tx))v - v) \quad x \in \mathrm{Lie}(G);\ v \in \mathscr{H}.$$

  if the limit exists.
- $\mathscr{H}^\infty := \{v \in \mathscr{H} \ : \ \pi^v : G \to \mathscr{H} \text{ is smooth } \}$ is a Lie(G)-module.

Unitary representations

Unitary representations

By a *unitary representation* of a Lie group G on a Hilbert space $\mathscr{H}$, we mean a group homomorphism

$$\pi : G \mapsto \mathrm{U}(\mathscr{H})$$

with continuous orbit maps $\pi^v : G \to \mathscr{H}$, $\pi^v(g) := \pi(g)v$ for every $v \in \mathscr{H}$.

Infinitesimal action

- Given a unitary representation $(\pi, \mathscr{H})$ of G, we can define

  $$\mathsf{d}\pi(x)v := \lim_{t\to 0} \frac{1}{t}\big(\pi(\exp(tx))v - v\big) \quad x \in \mathrm{Lie}(G);\ v \in \mathscr{H}.$$

  if the limit exists.
- $\mathscr{H}^\infty := \{v \in \mathscr{H} \ : \ \pi^v : G \to \mathscr{H} \text{ is smooth } \}$ is a Lie(G)-module.

Unitary representations

Unitary representations

By a *unitary representation* of a Lie group G on a Hilbert space $\mathscr{H}$, we mean a group homomorphism

$$\pi : G \mapsto \mathrm{U}(\mathscr{H})$$

with continuous orbit maps $\pi^v : G \to \mathscr{H}$, $\pi^v(g) := \pi(g)v$ for every $v \in \mathscr{H}$.

Infinitesimal action

- Given a unitary representation $(\pi, \mathscr{H})$ of G, we can define

  $$\mathsf{d}\pi(x)v := \lim_{t \to 0} \frac{1}{t}\big(\pi(\exp(tx))v - v\big) \quad x \in \mathrm{Lie}(G);\ v \in \mathscr{H}.$$

  if the limit exists.
- $\mathscr{H}^\infty := \{v \in \mathscr{H} \ : \ \pi^v : G \to \mathscr{H}$ is smooth $\}$ is a Lie(G)-module.

Integrating highest weight modules $V_{h,c}$

$$\mathbf{d}_n = ie^{in\theta}\frac{d}{d\theta} \quad , \quad \mathfrak{d} := \mathrm{Span}_{\mathbb{C}}\{\mathbf{d}_n \ : \ n \in \mathbb{Z}\} \quad , \quad \widehat{\mathfrak{d}} := \mathfrak{d} \oplus \mathbb{C}\kappa$$

Conjecture (Kac '82). Suppose that $V_{h,c}$ is unitarizable. Then there exists a unitary representation $\pi_{h,c}$ of Vir on the Hilbert completion of $V_{h,c}$ such that

$$\mathsf{d}\pi_{h,c}(x)v = x \cdot v \quad \text{for all} \quad x \in \widehat{\mathfrak{d}};\ v \in V_{h,c}.$$

- Proved by Goodman–Wallach ('85).
- (Neretin, Segal, Toledano-Laredo,...).

Integrating highest weight modules $V_{h,c}$

$$\mathbf{d}_n = ie^{in\theta}\frac{d}{d\theta} \quad , \quad \mathfrak{d} := \mathrm{Span}_{\mathbb{C}}\{\mathbf{d}_n \ : \ n \in \mathbb{Z}\} \quad , \quad \widehat{\mathfrak{d}} := \mathfrak{d} \oplus \mathbb{C}\kappa$$

Conjecture (Kac '82). Suppose that $V_{h,c}$ is unitarizable. Then there exists a unitary representation $\pi_{h,c}$ of Vir on the Hilbert completion of $V_{h,c}$ such that

$$d\pi_{h,c}(x)v = x \cdot v \quad \text{for all} \quad x \in \widehat{\mathfrak{d}};\ v \in V_{h,c}.$$

- Proved by Goodman–Wallach ('85).
- (Neretin, Segal, Toledano-Laredo,...).

Integrating highest weight modules $V_{h,c}$

$$\mathbf{d}_n = ie^{in\theta}\frac{d}{d\theta} \quad , \quad \mathfrak{d} := \mathrm{Span}_{\mathbb{C}}\{\mathbf{d}_n \ : \ n \in \mathbb{Z}\} \quad , \quad \widehat{\mathfrak{d}} := \mathfrak{d} \oplus \mathbb{C}\kappa$$

Conjecture (Kac '82). Suppose that $V_{h,c}$ is unitarizable. Then there exists a unitary representation $\pi_{h,c}$ of Vir on the Hilbert completion of $V_{h,c}$ such that

$$\mathrm{d}\pi_{h,c}(x)v = x \cdot v \quad \text{for all} \quad x \in \widehat{\mathfrak{d}};\ v \in V_{h,c}.$$

- Proved by Goodman–Wallach ('85).
- (Neretin, Segal, Toledano-Laredo,...).

Positive energy representations

$\pi_{h,c}$: the unitary representation obtained from the completion of $V_{h,c}$.

- The operator $\mathsf{d}\pi_{h,c}(-\mathbf{d}_0)$ is diagonalizable with **non-negative** eigenvalues.

Positive energy representation

A unitary representation of Vir is called a *positive energy representation* if

$$\mathrm{Spec}\big(\mathsf{d}\pi(-\mathbf{d}_0)\big) \subseteq [0, \infty).$$

Problem

Describe <u>all</u> the positive energy representations of Vir.

Example

$\mathbb{R}$ acting on $\mathscr{H} := \mathrm{L}^2(\mathbb{R})$ by $t \cdot \phi(x) := \phi(x+t)$. Then for every measurable set $\Omega \subseteq \mathbb{R}$, the subspace

$$\mathscr{H}_\Omega := \left\{ \phi \in \mathrm{L}^2(\mathbb{R}) \;:\; \widehat{\phi}\big|_\Omega \equiv 0 \right\} \subseteq \mathscr{H}$$

is closed and translation-invariant.

Positive energy representations

$\pi_{h,c}$: the unitary representation obtained from the completion of $V_{h,c}$.

- The operator $\mathsf{d}\pi_{h,c}(-\mathbf{d}_0)$ is diagonalizable with **non-negative** eigenvalues.

Positive energy representation

A unitary representation of Vir is called a *positive energy representation* if

$$\operatorname{Spec}\big(\mathsf{d}\pi(-\mathbf{d}_0)\big) \subseteq [0, \infty).$$

Problem

Describe all the positive energy representations of Vir.

Example

$\mathbb{R}$ acting on $\mathscr{H} := \mathrm{L}^2(\mathbb{R})$ by $t \cdot \phi(x) := \phi(x+t)$. Then for every measurable set $\Omega \subseteq \mathbb{R}$, the subspace

$$\mathscr{H}_\Omega := \left\{ \phi \in \mathrm{L}^2(\mathbb{R}) \ : \ \widehat{\phi}\big|_\Omega \equiv 0 \right\} \subseteq \mathscr{H}$$

is closed and translation-invariant.

Positive energy representations

$\pi_{h,c}$: the unitary representation obtained from the completion of $V_{h,c}$.

- The operator $\mathsf{d}\pi_{h,c}(-\mathbf{d}_0)$ is diagonalizable with **non-negative** eigenvalues.

Positive energy representation

A unitary representation of Vir is called a *positive energy representation* if

$$\operatorname{Spec}\big(\mathsf{d}\pi(-\mathbf{d}_0)\big) \subseteq [0, \infty).$$

Problem

Describe all the positive energy representations of Vir.

Example

$\mathbb{R}$ acting on $\mathscr{H} := \mathrm{L}^2(\mathbb{R})$ by $t \cdot \phi(x) := \phi(x+t)$. Then for every measurable set $\Omega \subseteq \mathbb{R}$, the subspace

$$\mathscr{H}_\Omega := \left\{ \phi \in \mathrm{L}^2(\mathbb{R}) \ : \ \widehat{\phi}\big|_\Omega \equiv 0 \right\} \subseteq \mathscr{H}$$

is closed and translation-invariant.

Positive energy representations

$\pi_{h,c}$: the unitary representation obtained from the completion of $V_{h,c}$.

- The operator $\mathsf{d}\pi_{h,c}(-\mathbf{d}_0)$ is diagonalizable with **non-negative** eigenvalues.

Positive energy representation

A unitary representation of Vir is called a *positive energy representation* if

$$\operatorname{Spec}\big(\mathsf{d}\pi(-\mathbf{d}_0)\big) \subseteq [0, \infty).$$

Problem

Describe all the positive energy representations of Vir.

Example

$\mathbb{R}$ acting on $\mathscr{H} := \mathrm{L}^2(\mathbb{R})$ by $t \cdot \phi(x) := \phi(x+t)$. Then for every measurable set $\Omega \subseteq \mathbb{R}$, the subspace

$$\mathscr{H}_\Omega := \left\{ \phi \in \mathrm{L}^2(\mathbb{R}) \ : \ \widehat{\phi}\big|_\Omega \equiv 0 \right\} \subseteq \mathscr{H}$$

is closed and translation-invariant.

Positive energy representations

$\pi_{h,c}$: the unitary representation obtained from the completion of $V_{h,c}$.

- The operator $\mathsf{d}\pi_{h,c}(-\mathbf{d}_0)$ is diagonalizable with **non-negative** eigenvalues.

Positive energy representation

A unitary representation of Vir is called a *positive energy representation* if

$$\operatorname{Spec}\big(\mathsf{d}\pi(-\mathbf{d}_0)\big) \subseteq [0, \infty).$$

Problem

Describe <u>all</u> the positive energy representations of Vir.

Example

$\mathbb{R}$ acting on $\mathscr{H} := \mathrm{L}^2(\mathbb{R})$ by $t \cdot \phi(x) := \phi(x+t)$. Then for every measurable set $\Omega \subseteq \mathbb{R}$, the subspace

$$\mathscr{H}_\Omega := \left\{ \phi \in \mathrm{L}^2(\mathbb{R}) \ : \ \widehat{\phi}\big|_\Omega \equiv 0 \right\} \subseteq \mathscr{H}$$

is closed and translation-invariant.

A direct integral representation

- $H := \exp(\widehat{\mathfrak{d}}_\circ \cap \widehat{\mathfrak{d}}_\mathbb{R}) \subset \mathrm{Vir}$, $H \cong \mathbb{R}^2$.
- $\widehat{H}_+$: the "blue" region

$(\pi_\mu, \mathscr{H}_\mu)$

- $\mu : \mathfrak{B}(\widehat{H}_+) \to [0, \infty)$: finite positive Borel measure.
- $\mathscr{H}_\mu$: the vector space of all maps $\mathbf{e} : \widehat{H}_+ \to \prod_{(h,c)\in\widehat{H}_+} \mathscr{H}_{h,c}$ such that:

  (a) $\mathbf{e}(h, c) \in \mathscr{H}_{h,c}$.

  (b) $\int_{\widehat{H}_+} \|\mathbf{e}(\cdot)\|^2 d\mu(\cdot) < \infty$.

- Set $\big(\pi_\mu(g)\mathbf{e}\big)(h, c) := \pi_{h,c}(g)\mathbf{e}(h, c)$ for $g \in \mathrm{Vir}$ and $\mathbf{e} \in \mathscr{H}_\mu$.

A direct integral representation

- $H := \exp(\widehat{\mathfrak{d}}_\circ \cap \widehat{\mathfrak{d}}_\mathbb{R}) \subset \mathrm{Vir} \quad , \quad H \cong \mathbb{R}^2.$
- $\widehat{H}_+$: the "blue" region

$(\pi_\mu, \mathscr{H}_\mu)$

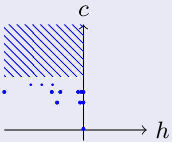
- $\mu : \mathfrak{B}(\widehat{H}_+) \to [0, \infty)$: finite positive Borel measure.
- $\mathscr{H}_\mu$: the vector space of all maps $\mathbf{e} : \widehat{H}_+ \to \prod_{(h,c)\in\widehat{H}_+} \mathscr{H}_{h,c}$ such that:

  (a) $\mathbf{e}(h,c) \in \mathscr{H}_{h,c}$.

  (b) $\int_{\widehat{H}_+} \|\mathbf{e}(\cdot)\|^2 d\mu(\cdot) < \infty$.
- Set $\big(\pi_\mu(g)\mathbf{e}\big)(h,c) := \pi_{h,c}(g)\mathbf{e}(h,c)$ for $g \in \mathrm{Vir}$ and $\mathbf{e} \in \mathscr{H}_\mu$.

A direct integral representation

- $H := \exp(\widehat{\mathfrak{d}}_\circ \cap \widehat{\mathfrak{d}}_{\mathbb{R}}) \subset \mathrm{Vir}$, $H \cong \mathbb{R}^2$.
- $\widehat{H}_+$: the "blue" region



$(\pi_\mu, \mathscr{H}_\mu)$

- $\mu : \mathfrak{B}(\widehat{H}_+) \to [0, \infty)$: finite positive Borel measure.
- $\mathscr{H}_\mu$: the vector space of all maps $\mathbf{e} : \widehat{H}_+ \to \prod_{(h,c)\in\widehat{H}_+} \mathscr{H}_{h,c}$ such that:

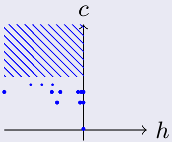
  (a) $\mathbf{e}(h,c) \in \mathscr{H}_{h,c}$.

  (b) $\int_{\widehat{H}_+} \|\mathbf{e}(\cdot)\|^2 d\mu(\cdot) < \infty$.

- Set $(\pi_\mu(g)\mathbf{e})(h,c) := \pi_{h,c}(g)\mathbf{e}(h,c)$ for $g \in \mathrm{Vir}$ and $\mathbf{e} \in \mathscr{H}_\mu$.

A direct integral representation

- $H := \exp(\widehat{\mathfrak{d}}_\circ \cap \widehat{\mathfrak{d}}_\mathbb{R}) \subset \mathrm{Vir}$, $H \cong \mathbb{R}^2$.
- $\widehat{H}_+$: the "blue" region

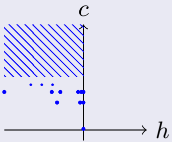


$(\pi_\mu, \mathscr{H}_\mu)$

- $\mu : \mathfrak{B}(\widehat{H}_+) \to [0, \infty)$: finite positive Borel measure.
- $\mathscr{H}_\mu$: the vector space of all maps $\mathbf{e} : \widehat{H}_+ \to \prod_{(h,c)\in\widehat{H}_+} \mathscr{H}_{h,c}$ such that:
  - (a) $\mathbf{e}(h, c) \in \mathscr{H}_{h,c}$.
  - (b) $\int_{\widehat{H}_+} \|\mathbf{e}(\cdot)\|^2 d\mu(\cdot) < \infty$.
- Set $(\pi_\mu(g)\mathbf{e})(h, c) := \pi_{h,c}(g)\mathbf{e}(h, c)$ for $g \in \mathrm{Vir}$ and $\mathbf{e} \in \mathscr{H}_\mu$.

A direct integral representation

- $H := \exp(\widehat{\mathfrak{d}}_\circ \cap \widehat{\mathfrak{d}}_\mathbb{R}) \subset \mathrm{Vir}$, $H \cong \mathbb{R}^2$.
- $\widehat{H}_+$: the "blue" region

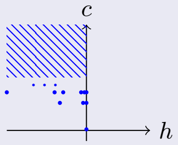


$(\pi_\mu, \mathscr{H}_\mu)$

- $\mu : \mathfrak{B}(\widehat{H}_+) \to [0, \infty)$: finite positive Borel measure.
- $\mathscr{H}_\mu$: the vector space of all maps $\mathbf{e} : \widehat{H}_+ \to \prod_{(h,c)\in\widehat{H}_+} \mathscr{H}_{h,c}$ such that:
  - (a) $\mathbf{e}(h,c) \in \mathscr{H}_{h,c}$.
  - (b) $\int_{\widehat{H}_+} \|\mathbf{e}(\cdot)\|^2 d\mu(\cdot) < \infty$.
- Set $(\pi_\mu(g)\mathbf{e})(h,c) := \pi_{h,c}(g)\mathbf{e}(h,c)$ for $g \in \mathrm{Vir}$ and $\mathbf{e} \in \mathscr{H}_\mu$.

A direct integral representation

- $H := \exp(\widehat{\mathfrak{d}}_\circ \cap \widehat{\mathfrak{d}}_\mathbb{R}) \subset \mathrm{Vir}$, $H \cong \mathbb{R}^2$.
- $\widehat{H}_+$: the "blue" region



$(\pi_\mu, \mathscr{H}_\mu)$

- $\mu : \mathfrak{B}(\widehat{H}_+) \to [0, \infty)$: finite positive Borel measure.
- $\mathscr{H}_\mu$: the vector space of all maps $\mathbf{e} : \widehat{H}_+ \to \prod_{(h,c)\in\widehat{H}_+} \mathscr{H}_{h,c}$ such that:

  (a) $\mathbf{e}(h, c) \in \mathscr{H}_{h,c}$.

  (b) $\int_{\widehat{H}_+} \|\mathbf{e}(\cdot)\|^2 d\mu(\cdot) < \infty$.

- Set $\big(\pi_\mu(g)\mathbf{e}\big)(h, c) := \pi_{h,c}(g)\mathbf{e}(h, c)$ for $g \in \mathrm{Vir}$ and $\mathbf{e} \in \mathscr{H}_\mu$.

The main theorem

Proposition

- $(\pi_\mu, \mathscr{H}_\mu)$ is a positive energy unitary representation of Vir.

Theorem (Neeb–S. '14)

Let $(\pi, \mathscr{H})$ be a positive energy unitary representation of Vir. Then there exist mutually singular finite positive Borel measures $\mu_1, \mu_2, \ldots, \mu_\infty$ on $\widehat{H}_+$ such that

$$\pi \cong \pi_{\mu_1} \oplus 2\pi_{\mu_2} \oplus \cdots \oplus \infty\pi_{\mu_\infty}.$$

Furthermore, the measures $\mu_1, \mu_2, \ldots, \mu_\infty$ are unique up to equivalence.

The main theorem

Proposition

- $(\pi_\mu, \mathscr{H}_\mu)$ is a positive energy unitary representation of Vir.

Theorem (Neeb–S. '14)

Let $(\pi, \mathscr{H})$ be a positive energy unitary representation of Vir. Then there exist mutually singular finite positive Borel measures $\mu_1, \mu_2, \ldots, \mu_\infty$ on $\widehat{H}_+$ such that

$$\pi \cong \pi_{\mu_1} \oplus 2\pi_{\mu_2} \oplus \cdots \oplus \infty\pi_{\mu_\infty}.$$

Furthermore, the measures $\mu_1, \mu_2, \ldots, \mu_\infty$ are unique up to equivalence.

Complex structures on G/H

- G Lie group, $H \subseteq G$ Lie subgroup, $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$.
- The quotient space G/H is a G-manifold.

$$G \times G/H \to G/H \ , \ (g, g'H) \mapsto gg'H.$$

- Set $M := G/H \quad , \quad \mathfrak{q} : G \to M$, $\mathfrak{q}(g) := gH \quad , \quad m_\circ = \mathfrak{q}(\mathbf{1})$.
- $T_{m_\circ}M \cong \mathfrak{g}/\mathfrak{h}$.
- Assume that M is a G-invariant complex manifold.

$$J : T_{m_\circ}M \to T_{m_\circ}M \ , \ J^2 = -I \qquad \text{“}v \mapsto \sqrt{-1}v\text{”}.$$

- $J : (T_{m_\circ}M) \otimes_{\mathbb{R}} \mathbb{C} \to (T_{m_\circ}M) \otimes_{\mathbb{R}} \mathbb{C}$ results in the decomposition

$$\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}} \cong T_{m_\circ}M \otimes_{\mathbb{R}} \mathbb{C} \cong T^{1,0}_{m_\circ}M \oplus T^{0,1}_{m_\circ}M.$$

Theorem (Frölicher ’55)

Every complex structure on $M := G/H$ corresponds to a unique Lie subalgebra $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$ satisfying

$$\mathrm{Ad}(H)\mathfrak{q} = \mathfrak{q} \quad , \quad \mathfrak{q} \cap \overline{\mathfrak{q}} = \mathfrak{h}_{\mathbb{C}} \quad , \quad \mathfrak{q} + \overline{\mathfrak{q}} = \mathfrak{g}_{\mathbb{C}}.$$

The converse holds when $\dim G < \infty$ (Newlander–Nirenberg Theorem).

Complex structures on G/H

- G Lie group, $H \subseteq G$ Lie subgroup, $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$.
- The quotient space G/H is a G-manifold.

$$G \times G/H \to G/H \ , \ (g, g'H) \mapsto gg'H.$$

- Set $M := G/H$, $\mathfrak{q} : G \to M$, $\mathfrak{q}(g) := gH$, $m_\circ = \mathfrak{q}(\mathbf{1})$.
- $T_{m_\circ}M \cong \mathfrak{g}/\mathfrak{h}$.
- Assume that M is a G-invariant complex manifold.

$$J : T_{m_\circ}M \to T_{m_\circ}M \ , \ J^2 = -I \qquad \text{“} v \mapsto \sqrt{-1}v \text{”}.$$

- $J : (T_{m_\circ}M) \otimes_{\mathbb{R}} \mathbb{C} \to (T_{m_\circ}M) \otimes_{\mathbb{R}} \mathbb{C}$ results in the decomposition

$$\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}} \cong T_{m_\circ}M \otimes_{\mathbb{R}} \mathbb{C} \cong T^{1,0}_{m_\circ}M \oplus T^{0,1}_{m_\circ}M.$$

Theorem (Frölicher ’55)

Every complex structure on $M := G/H$ corresponds to a unique Lie subalgebra $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$ satisfying

$$\mathrm{Ad}(H)\mathfrak{q} = \mathfrak{q} \quad , \quad \mathfrak{q} \cap \overline{\mathfrak{q}} = \mathfrak{h}_{\mathbb{C}} \quad , \quad \mathfrak{q} + \overline{\mathfrak{q}} = \mathfrak{g}_{\mathbb{C}}.$$

The converse holds when $\dim G < \infty$ (Newlander–Nirenberg Theorem).

Complex structures on G/H

- G Lie group, $H \subseteq G$ Lie subgroup, $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$.
- The quotient space G/H is a G-manifold.

$$G \times G/H \to G/H \ , \ (g, g'H) \mapsto gg'H.$$

- Set $M := G/H \quad , \quad \mathfrak{q} : G \to M, \mathfrak{q}(g) := gH \quad , \quad m_\circ = \mathfrak{q}(\mathbf{1})$.
- $T_{m_\circ}M \cong \mathfrak{g}/\mathfrak{h}$.
- Assume that M is a G-invariant complex manifold.

$$J : T_{m_\circ}M \to T_{m_\circ}M \ , \ J^2 = -I \qquad \text{“}v \mapsto \sqrt{-1}v\text{”}.$$

- $J : (T_{m_\circ}M) \otimes_{\mathbb{R}} \mathbb{C} \to (T_{m_\circ}M) \otimes_{\mathbb{R}} \mathbb{C}$ results in the decomposition

$$\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}} \cong T_{m_\circ}M \otimes_{\mathbb{R}} \mathbb{C} \cong T^{1,0}_{m_\circ}M \oplus T^{0,1}_{m_\circ}M.$$

Theorem (Frölicher '55)

Every complex structure on $M := G/H$ corresponds to a unique Lie subalgebra $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$ satisfying

$$\mathrm{Ad}(H)\mathfrak{q} = \mathfrak{q} \quad , \quad \mathfrak{q} \cap \overline{\mathfrak{q}} = \mathfrak{h}_{\mathbb{C}} \quad , \quad \mathfrak{q} + \overline{\mathfrak{q}} = \mathfrak{g}_{\mathbb{C}}.$$

The converse holds when $\dim G < \infty$ (Newlander–Nirenberg Theorem).

Complex structures on G/H

- G Lie group, $H \subseteq G$ Lie subgroup, $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$.
- The quotient space G/H is a G-manifold.

$$G \times G/H \to G/H \ , \ (g, g'H) \mapsto gg'H.$$

- Set $M := G/H \quad , \ \mathsf{q} : G \to M, \mathsf{q}(g) := gH \ , \ m_\circ = \mathsf{q}(\mathbf{1})$.
- $T_{m_\circ}M \cong \mathfrak{g}/\mathfrak{h}$.
- Assume that M is a G-invariant complex manifold.

$$J : T_{m_\circ}M \to T_{m_\circ}M \ , \ J^2 = -I \qquad \text{“}v \mapsto \sqrt{-1}v\text{”}.$$

- $J : (T_{m_\circ}M) \otimes_{\mathbb{R}} \mathbb{C} \to (T_{m_\circ}M) \otimes_{\mathbb{R}} \mathbb{C}$ results in the decomposition

$$\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}} \cong T_{m_\circ}M \otimes_{\mathbb{R}} \mathbb{C} \cong T^{1,0}_{m_\circ}M \oplus T^{0,1}_{m_\circ}M.$$

Theorem (Frölicher '55)

Every complex structure on $M := G/H$ corresponds to a unique Lie subalgebra $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$ satisfying

$$\mathrm{Ad}(H)\mathfrak{q} = \mathfrak{q} \quad , \quad \mathfrak{q} \cap \bar{\mathfrak{q}} = \mathfrak{h}_{\mathbb{C}} \quad , \quad \mathfrak{q} + \bar{\mathfrak{q}} = \mathfrak{g}_{\mathbb{C}}.$$

The converse holds when $\dim G < \infty$ (Newlander–Nirenberg Theorem).

Complex structures on G/H

- G Lie group, $H \subseteq G$ Lie subgroup, $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$.
- The quotient space G/H is a G-manifold.

$$G \times G/H \to G/H \ , \ (g, g'H) \mapsto gg'H.$$

- Set $M := G/H$, $\mathsf{q} : G \to M$, $\mathsf{q}(g) := gH$, $m_\circ = \mathsf{q}(\mathbf{1})$.
- $T_{m_\circ}M \cong \mathfrak{g}/\mathfrak{h}$.
- Assume that M is a G-invariant complex manifold.

$$J : T_{m_\circ}M \to T_{m_\circ}M \ , \ J^2 = -I \qquad \text{“} v \mapsto \sqrt{-1}v \text{”}.$$

- $J : (T_{m_\circ}M) \otimes_{\mathbb{R}} \mathbb{C} \to (T_{m_\circ}M) \otimes_{\mathbb{R}} \mathbb{C}$ results in the decomposition

$$\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}} \cong T_{m_\circ}M \otimes_{\mathbb{R}} \mathbb{C} \cong T^{1,0}_{m_\circ}M \oplus T^{0,1}_{m_\circ}M.$$

Theorem (Frölicher '55)

Every complex structure on $M := G/H$ corresponds to a unique Lie subalgebra $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$ satisfying

$$\mathrm{Ad}(H)\mathfrak{q} = \mathfrak{q} \quad , \quad \mathfrak{q} \cap \bar{\mathfrak{q}} = \mathfrak{h}_{\mathbb{C}} \quad , \quad \mathfrak{q} + \bar{\mathfrak{q}} = \mathfrak{g}_{\mathbb{C}}.$$

The converse holds when $\dim G < \infty$ (Newlander–Nirenberg Theorem).

Complex structures on G/H

- G Lie group, $H \subseteq G$ Lie subgroup, $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$.
- The quotient space G/H is a G-manifold.

$$G \times G/H \to G/H \ , \ (g, g'H) \mapsto gg'H.$$

- Set $M := G/H$, $\mathsf{q} : G \to M$, $\mathsf{q}(g) := gH$, $m_\circ = \mathsf{q}(\mathbf{1})$.
- $T_{m_\circ}M \cong \mathfrak{g}/\mathfrak{h}$.
- Assume that M is a G-invariant complex manifold.

$$J : T_{m_\circ}M \to T_{m_\circ}M \ , \ J^2 = -I \qquad \text{“}v \mapsto \sqrt{-1}v\text{”}.$$

- $J : (T_{m_\circ}M) \otimes_{\mathbb{R}} \mathbb{C} \to (T_{m_\circ}M) \otimes_{\mathbb{R}} \mathbb{C}$ results in the decomposition

$$\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}} \cong T_{m_\circ}M \otimes_{\mathbb{R}} \mathbb{C} \cong T^{1,0}_{m_\circ}M \oplus T^{0,1}_{m_\circ}M.$$

Theorem (Frölicher '55)

Every complex structure on $M := G/H$ corresponds to a unique Lie subalgebra $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$ satisfying

$$\mathrm{Ad}(H)\mathfrak{q} = \mathfrak{q} \quad , \quad \mathfrak{q} \cap \bar{\mathfrak{q}} = \mathfrak{h}_{\mathbb{C}} \quad , \quad \mathfrak{q} + \bar{\mathfrak{q}} = \mathfrak{g}_{\mathbb{C}}.$$

The converse holds when $\dim G < \infty$ (Newlander–Nirenberg Theorem).

Complex structures on G/H

- G Lie group, $H \subseteq G$ Lie subgroup, $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$.
- The quotient space G/H is a G-manifold.

$$G \times G/H \to G/H \ , \ (g, g'H) \mapsto gg'H.$$

- Set $M := G/H \quad , \ \mathsf{q} : G \to M, \mathsf{q}(g) := gH \ , \ m_\circ = \mathsf{q}(\mathbf{1})$.
- $T_{m_\circ}M \cong \mathfrak{g}/\mathfrak{h}$.
- Assume that M is a G-invariant complex manifold.

$$J : T_{m_\circ}M \to T_{m_\circ}M \ , \ J^2 = -I \qquad \text{“} v \mapsto \sqrt{-1}v \text{”}.$$

- $J : (T_{m_\circ}M) \otimes_{\mathbb{R}} \mathbb{C} \to (T_{m_\circ}M) \otimes_{\mathbb{R}} \mathbb{C}$ results in the decomposition

$$\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}} \cong T_{m_\circ}M \otimes_{\mathbb{R}} \mathbb{C} \cong T^{1,0}_{m_\circ}M \oplus T^{0,1}_{m_\circ}M.$$

Theorem (Frölicher '55)

Every complex structure on $M := G/H$ corresponds to a unique Lie subalgebra $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$ satisfying

$$\mathrm{Ad}(H)\mathfrak{q} = \mathfrak{q} \quad , \quad \mathfrak{q} \cap \overline{\mathfrak{q}} = \mathfrak{h}_{\mathbb{C}} \quad , \quad \mathfrak{q} + \overline{\mathfrak{q}} = \mathfrak{g}_{\mathbb{C}}.$$

The converse holds when $\dim G < \infty$ (Newlander–Nirenberg Theorem).

Complex structures on G/H

- G Lie group, $H \subseteq G$ Lie subgroup, $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$.
- The quotient space G/H is a G-manifold.

$$G \times G/H \to G/H \ , \ (g, g'H) \mapsto gg'H.$$

- Set $M := G/H \quad , \ \mathsf{q} : G \to M, \ \mathsf{q}(g) := gH \quad , \ m_\circ = \mathsf{q}(\mathbf{1})$.
- $T_{m_\circ}M \cong \mathfrak{g}/\mathfrak{h}$.
- Assume that M is a G-invariant complex manifold.

$$J : T_{m_\circ}M \to T_{m_\circ}M \ , \ J^2 = -I \qquad \text{“}v \mapsto \sqrt{-1}v\text{”}.$$

- $J : (T_{m_\circ}M) \otimes_{\mathbb{R}} \mathbb{C} \to (T_{m_\circ}M) \otimes_{\mathbb{R}} \mathbb{C}$ results in the decomposition

$$\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}} \cong T_{m_\circ}M \otimes_{\mathbb{R}} \mathbb{C} \cong T^{1,0}_{m_\circ}M \oplus T^{0,1}_{m_\circ}M.$$

Theorem (Frölicher '55)

Every complex structure on $M := G/H$ corresponds to a unique Lie subalgebra $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$ satisfying

$$\mathrm{Ad}(H)\mathfrak{q} = \mathfrak{q} \quad , \quad \mathfrak{q} \cap \overline{\mathfrak{q}} = \mathfrak{h}_{\mathbb{C}} \quad , \quad \mathfrak{q} + \overline{\mathfrak{q}} = \mathfrak{g}_{\mathbb{C}}.$$

The converse holds when $\dim G < \infty$ (Newlander–Nirenberg Theorem).

Complex structures on G/H

- G Lie group, $H \subseteq G$ Lie subgroup, $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$.
- The quotient space G/H is a G-manifold.

$$G \times G/H \to G/H \ , \ (g, g'H) \mapsto gg'H.$$

- Set $M := G/H$, $\mathsf{q} : G \to M$, $\mathsf{q}(g) := gH$, $m_\circ = \mathsf{q}(\mathbf{1})$.
- $T_{m_\circ}M \cong \mathfrak{g}/\mathfrak{h}$.
- Assume that M is a G-invariant complex manifold.

$$J : T_{m_\circ}M \to T_{m_\circ}M \ , \ J^2 = -I \qquad \text{“}v \mapsto \sqrt{-1}v\text{”}.$$

- $J : (T_{m_\circ}M) \otimes_{\mathbb{R}} \mathbb{C} \to (T_{m_\circ}M) \otimes_{\mathbb{R}} \mathbb{C}$ results in the decomposition

$$\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}} \cong T_{m_\circ}M \otimes_{\mathbb{R}} \mathbb{C} \cong T^{1,0}_{m_\circ}M \oplus T^{0,1}_{m_\circ}M.$$

Theorem (Frölicher ’55)

Every complex structure on $M := G/H$ corresponds to a unique Lie subalgebra $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$ satisfying

$$\mathrm{Ad}(H)\mathfrak{q} = \mathfrak{q} \quad , \quad \mathfrak{q} \cap \overline{\mathfrak{q}} = \mathfrak{h}_{\mathbb{C}} \quad , \quad \mathfrak{q} + \overline{\mathfrak{q}} = \mathfrak{g}_{\mathbb{C}}.$$

The converse holds when $\dim G < \infty$ (Newlander–Nirenberg Theorem).

Complex structures on G/H

- G Lie group, $H \subseteq G$ Lie subgroup, $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$.
- The quotient space G/H is a G-manifold.

$$G \times G/H \to G/H \ , \ (g, g'H) \mapsto gg'H.$$

- Set $M := G/H$, $\mathsf{q} : G \to M$, $\mathsf{q}(g) := gH$, $m_\circ = \mathsf{q}(\mathbf{1})$.
- $T_{m_\circ}M \cong \mathfrak{g}/\mathfrak{h}$.
- Assume that M is a G-invariant complex manifold.

$$J : T_{m_\circ}M \to T_{m_\circ}M \ , \ J^2 = -I \qquad \text{“}v \mapsto \sqrt{-1}v\text{”}.$$

- $J : (T_{m_\circ}M) \otimes_{\mathbb{R}} \mathbb{C} \to (T_{m_\circ}M) \otimes_{\mathbb{R}} \mathbb{C}$ results in the decomposition

$$\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}} \cong T_{m_\circ}M \otimes_{\mathbb{R}} \mathbb{C} \cong T^{1,0}_{m_\circ}M \oplus T^{0,1}_{m_\circ}M.$$

Theorem (Frölicher ’55)

Every complex structure on $M := G/H$ corresponds to a unique Lie subalgebra $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$ satisfying

$$\mathrm{Ad}(H)\mathfrak{q} = \mathfrak{q} \quad , \quad \mathfrak{q} \cap \overline{\mathfrak{q}} = \mathfrak{h}_{\mathbb{C}} \quad , \quad \mathfrak{q} + \overline{\mathfrak{q}} = \mathfrak{g}_{\mathbb{C}}.$$

The converse holds when $\dim G < \infty$ (Newlander–Nirenberg Theorem).

G-invariant complex structures on $\mathbb{V}_\rho := G \times_H V_\rho$

- $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$, $M := G/H$, $\mathfrak{q} : G \to G/H$.
- M complex manifold (characterized by $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$).
- V_ρ : a representation of H , $\rho : H \to \mathrm{GL}(V_\rho)$.
- $\mathbb{V}_\rho := G \times_H V_\rho = G \times V_\rho / H$ w.r.t. the H-action

$$(g, v) \cdot h := (gh, \rho(h^{-1})v).$$

Theorem (Tirao–Wolf '70)

The G-invariant complex structures of $\mathbb{V}_\rho$ are uniquely determined by Lie algebra homomorphisms $\beta : \overline{\mathfrak{q}} \to \mathrm{End}(V_\rho)$ satisfying

$$\beta\big|_{\mathfrak{h}} = \mathrm{d}\rho \quad \text{and} \quad \beta(\mathrm{Ad}(h)x) = \mathrm{Ad}(\rho(h))\beta(x).$$

The converse holds when $\dim G < \infty$.

G-invariant complex structures on $\mathbb{V}_\rho := G \times_H V_\rho$

- $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$, $M := G/H$, $\mathfrak{q} : G \to G/H$.
- M complex manifold (characterized by $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$).
- V_ρ : a representation of H , $\rho : H \to \mathrm{GL}(V_\rho)$.
- $\mathbb{V}_\rho := G \times_H V_\rho = G \times V_\rho / H$ w.r.t. the H-action

$$(g, v) \cdot h := (gh, \rho(h^{-1})v).$$

Theorem (Tirao–Wolf '70)

The G-invariant complex structures of $\mathbb{V}_\rho$ are uniquely determined by Lie algebra homomorphisms $\beta : \overline{\mathfrak{q}} \to \mathrm{End}(V_\rho)$ satisfying

$$\beta\big|_{\mathfrak{h}} = \mathrm{d}\rho \quad \text{and} \quad \beta(\mathrm{Ad}(h)x) = \mathrm{Ad}(\rho(h))\beta(x).$$

The converse holds when $\dim G < \infty$.

G-invariant complex structures on $\mathbb{V}_\rho := G \times_H V_\rho$

- $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$, $M := G/H$, $\mathfrak{q} : G \to G/H$.
- M complex manifold (characterized by $\mathfrak{q} \subseteq \mathfrak{g}_\mathbb{C}$).
- V_ρ : a representation of H , $\rho : H \to \mathrm{GL}(V_\rho)$.
- $\mathbb{V}_\rho := G \times_H V_\rho = G \times V_\rho / H$ w.r.t. the H-action

$$(g, v) \cdot h := (gh, \rho(h^{-1})v).$$

Theorem (Tirao–Wolf '70)

The G-invariant complex structures of $\mathbb{V}_\rho$ are uniquely determined by Lie algebra homomorphisms $\beta : \overline{\mathfrak{q}} \to \mathrm{End}(V_\rho)$ satisfying

$$\beta\big|_{\mathfrak{h}} = \mathrm{d}\rho \quad \text{and} \quad \beta(\mathrm{Ad}(h)x) = \mathrm{Ad}(\rho(h))\beta(x).$$

The converse holds when $\dim G < \infty$.

G-invariant complex structures on $\mathbb{V}_\rho := G \times_H V_\rho$

- $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$, $M := G/H$, $\mathsf{q} : G \to G/H$.
- M complex manifold (characterized by $\mathfrak{q} \subseteq \mathfrak{g}_\mathbb{C}$).
- V_ρ : a representation of H , $\rho : H \to \mathrm{GL}(V_\rho)$.
- $\mathbb{V}_\rho := G \times_H V_\rho = G \times V_\rho / H$ w.r.t. the H-action

$$(g, v) \cdot h := (gh, \rho(h^{-1})v).$$

Theorem (Tirao–Wolf '70)

The G-invariant complex structures of $\mathbb{V}_\rho$ are uniquely determined by Lie algebra homomorphisms $\beta : \overline{\mathfrak{q}} \to \mathrm{End}(V_\rho)$ satisfying

$$\beta\big|_{\mathfrak{h}} = \mathrm{d}\rho \quad \text{and} \quad \beta(\mathrm{Ad}(h)x) = \mathrm{Ad}(\rho(h))\beta(x).$$

The converse holds when $\dim G < \infty$.

G-invariant complex structures on $\mathbb{V}_\rho := G \times_H V_\rho$

- $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$, $M := G/H$, $\mathfrak{q} : G \to G/H$.
- M complex manifold (characterized by $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$).
- V_ρ : a representation of H , $\rho : H \to \mathrm{GL}(V_\rho)$.
- $\mathbb{V}_\rho := G \times_H V_\rho = G \times V_\rho / H$ w.r.t. the H-action

$$(g, v) \cdot h := (gh, \rho(h^{-1})v).$$

Theorem (Tirao–Wolf '70)

The G-invariant complex structures of $\mathbb{V}_\rho$ are uniquely determined by Lie algebra homomorphisms $\beta : \overline{\mathfrak{q}} \to \mathrm{End}(V_\rho)$ satisfying

$$\beta\big|_{\mathfrak{h}} = \mathsf{d}\rho \quad \text{and} \quad \beta(\mathrm{Ad}(h)x) = \mathrm{Ad}(\rho(h))\beta(x).$$

The converse holds when $\dim G < \infty$.

G-invariant complex structures on $\mathbb{V}_\rho := G \times_H V_\rho$

- $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$, $M := G/H$, $\mathsf{q} : G \to G/H$.
- M complex manifold (characterized by $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$).
- V_ρ : a representation of H , $\rho : H \to \mathrm{GL}(V_\rho)$.
- $\mathbb{V}_\rho := G \times_H V_\rho = G \times V_\rho / H$ w.r.t. the H-action

$$(g, v) \cdot h := (gh, \rho(h^{-1})v).$$

Theorem (Tirao–Wolf '70)

The G-invariant complex structures of $\mathbb{V}_\rho$ are uniquely determined by Lie algebra homomorphisms $\beta : \overline{\mathfrak{q}} \to \mathrm{End}(V_\rho)$ satisfying

$$\beta\big|_{\mathfrak{h}} = \mathsf{d}\rho \quad \text{and} \quad \beta(\mathrm{Ad}(h)x) = \mathrm{Ad}(\rho(h))\beta(x).$$

The converse holds when $\dim G < \infty$.

The complex principal bundle $\mathbb{S}$

- $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$, $M := G/H$, $\mathfrak{q} : G \to G/H$.
- M complex manifold (characterized by $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$).
- S : complex Lie group, $\mathfrak{s} := \mathrm{Lie}(S)$, $\gamma : H \to S$.
- $\mathbb{S} := G \times_H S = G \times S/H$ $\quad (g, s) \cdot h := (gh, \gamma(h)^{-1}s)$.

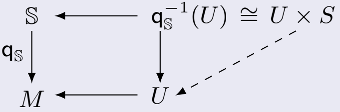
The complex principal bundle $\mathbb{S}$

- $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$, $M := G/H$, $\mathsf{q} : G \to G/H$.
- M complex manifold (characterized by $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$).
- S : complex Lie group, $\mathfrak{s} := \mathrm{Lie}(S)$, $\gamma : H \to S$.
- $\mathbb{S} := G \times_H S = G \times S\,/H \qquad (g, s) \cdot h := (gh, \gamma(h)^{-1} s)$.

$$\begin{array}{ccccc}
\mathbb{S} & \longleftarrow & \mathsf{q}_{\mathbb{S}}^{-1}(U) & \cong & U \times S \\
\big\downarrow \mathsf{q}_{\mathbb{S}} & & \big\downarrow & \swarrow & \\
M & \longleftarrow & U & &
\end{array}$$

The complex principal bundle $\mathbb{S}$

- $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$, $M := G/H$, $\mathsf{q} : G \to G/H$.
- M complex manifold (characterized by $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$).
- S : complex Lie group, $\mathfrak{s} := \mathrm{Lie}(S)$, $\gamma : H \to S$.
- $\mathbb{S} := G \times_H S = G \times S\,/H \qquad (g, s) \cdot h := (gh, \gamma(h)^{-1}s)$.



(a) The complex principal bundle structures on $\mathbb{S}$ are parametrized by Lie algebra homomorphisms $\beta : \overline{\mathfrak{q}} \to \mathfrak{s}$ satisfying

$$\beta\big|_{\mathfrak{h}} = \mathsf{d}\gamma \quad \text{and} \quad \beta(\mathrm{Ad}(h)x) = \mathrm{Ad}(\gamma(h))\beta(x).$$

(b) Given such a $\beta : \overline{\mathfrak{q}} \to \mathfrak{s}$, there exists a corresponding complex principal bundle structure on $\mathbb{S}$ if and only if the "$\overline{\partial}$-equation"

$$L_x f(g) = -\beta(x) \cdot f(g) \quad , \quad x \in \overline{\mathfrak{q}}$$

has a local solution $f : \mathsf{q}^{-1}(U) \to S$.

The complex principal bundle $\mathbb{S}$

- $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$, $M := G/H$, $\mathsf{q} : G \to G/H$.
- M complex manifold (characterized by $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$).
- S : complex Lie group, $\mathfrak{s} := \mathrm{Lie}(S)$, $\gamma : H \to S$.
- $\mathbb{S} := G \times_H S = G \times S \,/ H \qquad (g,s)\cdot h := (gh, \gamma(h)^{-1}s)$.

$$\begin{array}{ccccc} \mathbb{S} & \longleftarrow & \mathsf{q}_{\mathbb{S}}^{-1}(U) & \cong & U \times S \\ \downarrow \mathsf{q}_{\mathbb{S}} & & \downarrow & \swarrow & \\ M & \longleftarrow & U & & \end{array}$$

(a) The complex principal bundle structures on $\mathbb{S}$ are parametrized by Lie algebra homomorphisms $\beta : \overline{\mathfrak{q}} \to \mathfrak{s}$ satisfying

$$\beta\big|_{\mathfrak{h}} = \mathsf{d}\gamma \quad \text{and} \quad \beta(\mathrm{Ad}(h)x) = \mathrm{Ad}(\gamma(h))\beta(x).$$

(b) Given such a $\beta : \overline{\mathfrak{q}} \to \mathfrak{s}$, there exists a corresponding complex principal bundle structure on $\mathbb{S}$ if and only if the "$\overline{\partial}$-equation"

$$L_x f(g) = -\beta(x)\cdot f(g) \quad , \quad x \in \overline{\mathfrak{q}}$$

has a local solution $f : \mathsf{q}^{-1}(U) \to S$.

The complex principal bundle $\mathbb{S}$

- $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$, $M := G/H$, $\mathsf{q} : G \to G/H$.
- M complex manifold (characterized by $\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$).
- S : complex Lie group, $\mathfrak{s} := \mathrm{Lie}(S)$, $\gamma : H \to S$.
- $\mathbb{S} := G \times_H S = G \times S\,/H \qquad (g, s) \cdot h := (gh, \gamma(h)^{-1}s)$.

$$\begin{array}{ccc} \mathbb{S} & \longleftarrow & \mathsf{q}_{\mathbb{S}}^{-1}(U) \cong U \times S \\ {\scriptstyle \mathsf{q}_{\mathbb{S}}}\downarrow & & \downarrow \quad \swarrow \\ M & \longleftarrow & U \end{array}$$

(c) Let $\mathbb{V}_\rho$ be a representation of H. Suppose that ρ extends to a complex representation $\widetilde{\rho} : S \to \mathrm{GL}(V_\rho)$ (i.e., $\widetilde{\rho} \circ \gamma = \rho$). Then the canonical isomorphism

$$\mathbb{S} \times_{\widetilde{\rho}} V_\rho \cong \mathbb{V}_\rho \quad , \quad [[g, s], v] \mapsto [g, \widetilde{\rho}(s)v]$$

induces a complex structure on $\mathbb{V}_\rho$.

- $G := \mathrm{Vir}$, $H := \exp(\widehat{\mathfrak{d}}_\circ \cap \widehat{\mathfrak{d}}_{\mathbb{R}}) \quad \Rightarrow \quad G/H \cong \mathrm{Diff}_+(S^1)/S^1$
- $\mathbb{D} := \left\{ z \in \mathbb{C} \ : \ |z| \leq 1 \right\}$.
- $\mathcal{K} := \left\{ f : \mathbb{D} \to \mathbb{C} \ : \ f\big|_{\partial\mathbb{D}} \text{ is } C^\infty,\ f\big|_{\mathbb{D}^\circ} \text{ is bihol., } f(0) = 0,\ f'(0) = 1 \right\}$.
- Riemann Mapping Theorem $\Rightarrow$ bijection $\mathcal{K} \ \leftrightsquigarrow \left\{ f\big|_{\partial\mathbb{D}} \ : \ f \in \mathcal{K} \right\}$

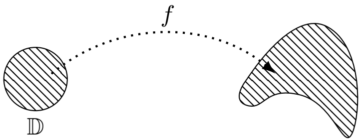
- $G := \mathrm{Vir}$, $H := \exp(\widehat{\mathfrak{d}}_\circ \cap \widehat{\mathfrak{d}}_\mathbb{R}) \quad \Rightarrow \quad G/H \cong \mathrm{Diff}_+(S^1)/S^1$
- $\mathbb{D} := \left\{ z \in \mathbb{C} \ : \ |z| \leq 1 \right\}$.
- $\mathcal{K} := \left\{ f : \mathbb{D} \to \mathbb{C} \ : \ f\big|_{\partial\mathbb{D}} \text{ is } C^\infty,\ f\big|_{\mathbb{D}^\circ} \text{ is bihol.},\ f(0) = 0,\ f'(0) = 1 \right\}$.
- Riemann Mapping Theorem $\Rightarrow$ bijection $\mathcal{K} \ \leftrightsquigarrow \left\{ f\big|_{\partial\mathbb{D}} \ : \ f \in \mathcal{K} \right\}$

- $G := \mathrm{Vir}$, $H := \exp(\widehat{\mathfrak{d}}_\circ \cap \widehat{\mathfrak{d}}_\mathbb{R}) \quad \Rightarrow \quad G/H \cong \mathrm{Diff}_+(S^1)/S^1$
- $\mathbb{D} := \{z \in \mathbb{C} \ : \ |z| \leq 1\}$.
- $\mathcal{K} := \left\{f : \mathbb{D} \to \mathbb{C} \ : \ f\big|_{\partial\mathbb{D}} \text{ is } C^\infty,\ f\big|_{\mathbb{D}^\circ} \text{ is bihol.},\ f(0) = 0,\ f'(0) = 1\right\}$.
- Riemann Mapping Theorem $\Rightarrow$ bijection $\mathcal{K} \leftrightsquigarrow \left\{f\big|_{\partial\mathbb{D}} \ : \ f \in \mathcal{K}\right\}$

- $G := \mathrm{Vir}$, $H := \exp(\widehat{\mathfrak{d}}_\circ \cap \widehat{\mathfrak{d}}_\mathbb{R})$ $\quad\Rightarrow\quad G/H \cong \mathrm{Diff}_+(S^1)/S^1$
- $\mathbb{D} := \{z \in \mathbb{C} \;:\; |z| \leq 1\}$.
- $\mathcal{K} := \Big\{ f : \mathbb{D} \to \mathbb{C} \;:\; f\big|_{\partial\mathbb{D}} \text{ is } C^\infty,\ f\big|_{\mathbb{D}^\circ} \text{ is bihol.},\ f(0) = 0,\ f'(0) = 1 \Big\}$.
- Riemann Mapping Theorem $\Rightarrow$ bijection $\mathcal{K} \leftrightsquigarrow \Big\{ f\big|_{\partial\mathbb{D}} \;:\; f \in \mathcal{K} \Big\}$

$\mathrm{Diff}_+(S^1)/S^1$ (Kirillov–Yuriev '88, Bowick–Rajeev '88, Witten '87,...)

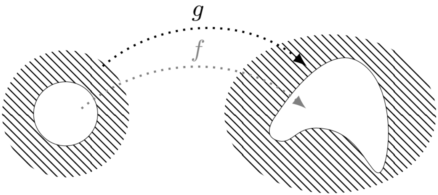
- $G := \mathrm{Vir}$, $H := \exp(\widehat{\mathfrak{d}}_\circ \cap \widehat{\mathfrak{d}}_\mathbb{R}) \quad \Rightarrow \quad G/H \cong \mathrm{Diff}_+(S^1)/S^1$
- $\mathbb{D} := \{ z \in \mathbb{C} \ : \ |z| \leq 1 \}$.
- $\mathcal{K} := \left\{ f : \mathbb{D} \to \mathbb{C} \ : \ f\big|_{\partial\mathbb{D}} \text{ is } C^\infty,\ f\big|_{\mathbb{D}^\circ} \text{ is bihol.},\ f(0) = 0,\ f'(0) = 1 \right\}$.
- Riemann Mapping Theorem $\Rightarrow$ bijection $\mathcal{K} \leftrightsquigarrow \left\{ f\big|_{\partial\mathbb{D}} \ : \ f \in \mathcal{K} \right\}$



$f \in \mathcal{K} \quad$ (unique)

$\mathrm{Diff}_+(S^1)/S^1$ (Kirillov–Yuriev '88, Bowick–Rajeev '88, Witten '87,...)

- $G := \mathrm{Vir}$, $H := \exp(\widehat{\mathfrak{d}}_\circ \cap \widehat{\mathfrak{d}}_\mathbb{R}) \quad \Rightarrow \quad G/H \cong \mathrm{Diff}_+(S^1)/S^1$
- $\mathbb{D} := \big\{ z \in \mathbb{C} \ : \ |z| \le 1 \big\}$.
- $\mathcal{K} := \Big\{ f : \mathbb{D} \to \mathbb{C} \ : \ f\big|_{\partial\mathbb{D}} \text{ is } C^\infty,\ f\big|_{\mathbb{D}^\circ} \text{ is bihol.},\ f(0) = 0,\ f'(0) = 1 \Big\}$.
- Riemann Mapping Theorem $\Rightarrow$ bijection $\mathcal{K} \ \rightsquigarrow \ \Big\{ f\big|_{\partial\mathbb{D}} \ : \ f \in \mathcal{K} \Big\}$

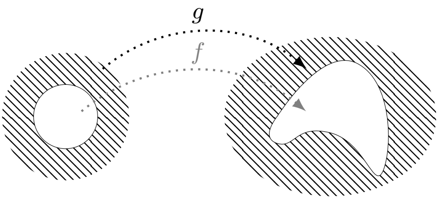


g bihol., $\ g(\infty) = \infty$

$\mathrm{Diff}_+(S^1)/S^1 \cong \mathcal{K}$ via the map

$$f \in \mathcal{K} \ \mapsto \ \gamma_f := f\big|_{\partial\mathbb{D}} \ \mapsto \ f^{-1} \circ g\big|_{\partial\mathbb{D}}$$

- $G := \mathrm{Vir}$, $H := \exp(\widehat{\mathfrak{d}}_\circ \cap \widehat{\mathfrak{d}}_\mathbb{R}) \quad \Rightarrow \quad G/H \cong \mathrm{Diff}_+(S^1)/S^1$
- $\mathbb{D} := \{z \in \mathbb{C} \ : \ |z| \leq 1\}$.
- $\mathcal{K} := \left\{ f : \mathbb{D} \to \mathbb{C} \ : \ f\big|_{\partial\mathbb{D}} \text{ is } C^\infty,\ f\big|_{\mathbb{D}^\circ} \text{ is bihol.},\ f(0) = 0,\ f'(0) = 1 \right\}$.
- Riemann Mapping Theorem $\Rightarrow$ bijection $\mathcal{K} \leftrightsquigarrow \left\{ f\big|_{\partial\mathbb{D}} \ : \ f \in \mathcal{K} \right\}$



g bihol., $g(\infty) = \infty$

$\mathrm{Diff}_+(S^1)/S^1 \cong \mathcal{K}$ via the map

$$f \in \mathcal{K} \ \mapsto \ \gamma_f := f\big|_{\partial\mathbb{D}} \ \mapsto \ f^{-1} \circ g\big|_{\partial\mathbb{D}}$$

- $G := \mathrm{Vir}$, $H := \exp(\widehat{\mathfrak{d}}_\circ \cap \widehat{\mathfrak{d}}_\mathbb{R}) \quad \Rightarrow \quad G/H \cong \mathrm{Diff}_+(S^1)/S^1$
- Recall: $\mathrm{Lie}(\mathrm{Diff}_+(S^1)) \otimes_\mathbb{R} \mathbb{C} = \mathfrak{d}_\infty := \mathrm{Vect}(S^1) \otimes_\mathbb{R} \mathbb{C}$.

$$\mathfrak{d}_\infty = \left\{ \sum_{n\in\mathbb{Z}} a_n \mathbf{d}_n \ : \ \sum_{n\in\mathbb{Z}} (|n|+1)^k |a_n| < \infty \right\}.$$

- $\mathfrak{d}_\infty = \mathfrak{q} + \overline{\mathfrak{q}}$ where $\mathfrak{q} := \left\{ x \in \mathfrak{d}_\infty \ : \ x = \sum_{n\geq 0} a_n \mathbf{d}_n \right\}$.
- $S := H_\mathbb{C}$, $\gamma : H \to H_\mathbb{C}$ canonical injection $\quad \Rightarrow \quad \mathbb{S} = G \times_H H_\mathbb{C}$.

Theorem (Lempert '95)

The group Vir has a **one-parameter** family of left-invariant complex structures (depending on $\tau \in \mathbb{C}$ such that $\tau \neq \overline{\tau}$) corresponding to

$$\widehat{\mathfrak{d}}_\infty = \mathfrak{q}_\tau \oplus \overline{\mathfrak{q}}_\tau = \widehat{\mathfrak{d}}^{1,0}_{\infty,\tau} \oplus \widehat{\mathfrak{d}}^{0,1}_{\infty,\tau}$$

where

$$\widehat{\mathfrak{d}}^{1,0}_{\infty,\tau} := \left\{ \tau a_0 \kappa + \sum_{n\geq 0} a_n \mathbf{d}_n \right\} \quad \text{and} \quad \widehat{\mathfrak{d}}^{0,1}_{\infty,\tau} := \left\{ \overline{\tau} a_0 \kappa + \sum_{n\leq 0} a_n \mathbf{d}_n \right\}.$$

- $G := \mathrm{Vir}$, $H := \exp(\widehat{\mathfrak{d}}_\circ \cap \widehat{\mathfrak{d}}_{\mathbb{R}}) \quad \Rightarrow \quad G/H \cong \mathrm{Diff}_+(S^1)/S^1$
- Recall: $\mathrm{Lie}(\mathrm{Diff}_+(S^1)) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{d}_\infty := \mathrm{Vect}(S^1) \otimes_{\mathbb{R}} \mathbb{C}$.

$$\mathfrak{d}_\infty = \left\{ \sum_{n\in\mathbb{Z}} a_n \mathbf{d}_n \;:\; \sum_{n\in\mathbb{Z}} (|n|+1)^k |a_n| < \infty \right\}.$$

- $\mathfrak{d}_\infty = \mathfrak{q} + \overline{\mathfrak{q}}$ where $\mathfrak{q} := \left\{ x \in \mathfrak{d}_\infty \;:\; x = \sum_{n\geq 0} a_n \mathbf{d}_n \right\}$.
- $S := H_{\mathbb{C}}$, $\gamma : H \to H_{\mathbb{C}}$ canonical injection $\quad \Rightarrow \quad \mathbb{S} = G \times_H H_{\mathbb{C}}$.

Theorem (Lempert '95)

The group Vir has a **one-parameter** family of left-invariant complex structures (depending on $\tau \in \mathbb{C}$ such that $\tau \neq \overline{\tau}$) corresponding to

$$\widehat{\mathfrak{d}}_\infty = \mathfrak{q}_\tau \oplus \overline{\mathfrak{q}}_\tau = \widehat{\mathfrak{d}}^{1,0}_{\infty,\tau} \oplus \widehat{\mathfrak{d}}^{0,1}_{\infty,\tau}$$

where

$$\widehat{\mathfrak{d}}^{1,0}_{\infty,\tau} := \left\{ \tau a_0 \kappa + \sum_{n\geq 0} a_n \mathbf{d}_n \right\} \quad \text{and} \quad \widehat{\mathfrak{d}}^{0,1}_{\infty,\tau} := \left\{ \overline{\tau} a_0 \kappa + \sum_{n\leq 0} a_n \mathbf{d}_n \right\}.$$

- $G := \mathrm{Vir}$, $H := \exp(\widehat{\mathfrak{d}}_\circ \cap \widehat{\mathfrak{d}}_\mathbb{R}) \quad \Rightarrow \quad G/H \cong \mathrm{Diff}_+(S^1)/S^1$
- Recall: $\mathrm{Lie}(\mathrm{Diff}_+(S^1)) \otimes_\mathbb{R} \mathbb{C} = \mathfrak{d}_\infty := \mathrm{Vect}(S^1) \otimes_\mathbb{R} \mathbb{C}$.

$$\mathfrak{d}_\infty = \left\{ \sum_{n\in\mathbb{Z}} a_n \mathbf{d}_n \ : \ \sum_{n\in\mathbb{Z}} (|n|+1)^k |a_n| < \infty \right\}.$$

- $\mathfrak{d}_\infty = \mathfrak{q} + \overline{\mathfrak{q}}$ where $\mathfrak{q} := \left\{ x \in \mathfrak{d}_\infty \ : \ x = \sum_{n\geq 0} a_n \mathbf{d}_n \right\}$.
- $S := H_\mathbb{C}$, $\gamma : H \to H_\mathbb{C}$ canonical injection $\quad \Rightarrow \quad \mathbb{S} = G \times_H H_\mathbb{C}$.

Theorem (Lempert '95)

The group Vir has a **one-parameter** family of left-invariant complex structures (depending on $\tau \in \mathbb{C}$ such that $\tau \neq \overline{\tau}$) corresponding to

$$\widehat{\mathfrak{d}}_\infty = \mathfrak{q}_\tau \oplus \overline{\mathfrak{q}}_\tau = \widehat{\mathfrak{d}}^{1,0}_{\infty,\tau} \oplus \widehat{\mathfrak{d}}^{0,1}_{\infty,\tau}$$

where

$$\widehat{\mathfrak{d}}^{1,0}_{\infty,\tau} := \left\{ \tau a_0 \kappa + \sum_{n\geq 0} a_n \mathbf{d}_n \right\} \quad \text{and} \quad \widehat{\mathfrak{d}}^{0,1}_{\infty,\tau} := \left\{ \overline{\tau} a_0 \kappa + \sum_{n\leq 0} a_n \mathbf{d}_n \right\}.$$

- $G := \mathrm{Vir}$, $H := \exp(\widehat{\mathfrak{d}}_\circ \cap \widehat{\mathfrak{d}}_\mathbb{R}) \quad \Rightarrow \quad G/H \cong \mathrm{Diff}_+(S^1)/S^1$
- Recall: $\mathrm{Lie}(\mathrm{Diff}_+(S^1)) \otimes_\mathbb{R} \mathbb{C} = \mathfrak{d}_\infty := \mathrm{Vect}(S^1) \otimes_\mathbb{R} \mathbb{C}$.

$$\mathfrak{d}_\infty = \left\{ \sum_{n\in\mathbb{Z}} a_n \mathbf{d}_n \; : \; \sum_{n\in\mathbb{Z}} (|n|+1)^k |a_n| < \infty \right\}.$$

- $\mathfrak{d}_\infty = \mathfrak{q} + \overline{\mathfrak{q}}$ where $\mathfrak{q} := \left\{ x \in \mathfrak{d}_\infty \; : \; x = \sum_{n\geq 0} a_n \mathbf{d}_n \right\}$.
- $S := H_\mathbb{C}$, $\gamma : H \to H_\mathbb{C}$ canonical injection $\quad \Rightarrow \quad \mathbb{S} = G \times_H H_\mathbb{C}$.

Theorem (Lempert '95)

The group Vir has a **one-parameter** family of left-invariant complex structures (depending on $\tau \in \mathbb{C}$ such that $\tau \neq \overline{\tau}$) corresponding to

$$\widehat{\mathfrak{d}}_\infty = \mathfrak{q}_\tau \oplus \overline{\mathfrak{q}}_\tau = \widehat{\mathfrak{d}}^{1,0}_{\infty,\tau} \oplus \widehat{\mathfrak{d}}^{0,1}_{\infty,\tau}$$

where

$$\widehat{\mathfrak{d}}^{1,0}_{\infty,\tau} := \left\{ \tau a_0 \kappa + \sum_{n\geq 0} a_n \mathbf{d}_n \right\} \text{ and } \widehat{\mathfrak{d}}^{0,1}_{\infty,\tau} := \left\{ \overline{\tau} a_0 \kappa + \sum_{n\leq 0} a_n \mathbf{d}_n \right\}.$$

Reproducing kernel Hilbert spaces

Theorem (Neeb–S. '14)

Let $(\pi, \mathscr{H})$ be a positive energy unitary representation of Vir. Then there exist mutually singular finite positive borel measures $\mu_1, \mu_2, \ldots, \mu_\infty$ such that

$$\pi \cong \pi_{\mu_1} \oplus 2\pi_{\mu_2} \oplus \cdots \oplus \infty\pi_{\mu_\infty}.$$

Furthermore, the measures $\mu_1, \mu_2, \ldots, \mu_\infty$ are unique up to equivalence.

Outline of proof

- Lempert's Theorem $\Rightarrow$ complex structure on $\mathbb{S}$
  $\Rightarrow$ complex structure on $\mathbb{V}_\rho$
- **Key idea:** Positive energy rep's $(\pi, \mathscr{H})$ of Vir can be realized as

$$(\pi, \mathscr{H}) \hookrightarrow \Gamma_{\mathrm{hol}}(\mathbb{V}_\rho) \cong \mathcal{O}(\mathbb{S}, V_\rho)$$

  where $\mathcal{O}(\mathbb{S}, V_\rho) := \Big\{ f : \mathbb{S} \to V_\rho \;:\; \forall_{p \in \mathbb{S}} \forall_{s \in S} \left(f(p \cdot s) = \widetilde{\rho}(s)^{-1} f(p) \right) \Big\}$.
- $\mathscr{H}$ is a *reproducing kernel hilbert space*:

$$K : \mathbb{S} \times \overline{\mathbb{S}} \to \mathrm{End}(V_\rho)\ ,\ K(s_1, s_2) = \mathrm{ev}_{s_1} \circ \mathrm{ev}_{s_2}^* \quad \text{is holomorphic.}$$

- $(\pi, \mathscr{H})$ is uniquely determined by $K : \mathbb{S} \times \overline{\mathbb{S}} \to \mathrm{End}(V_\rho)$.

Reproducing kernel Hilbert spaces

Theorem (Neeb–S. '14)

Let $(\pi, \mathscr{H})$ be a positive energy unitary representation of Vir. Then there exist mutually singular finite positive borel measures $\mu_1, \mu_2, \ldots, \mu_\infty$ such that

$$\pi \cong \pi_{\mu_1} \oplus 2\pi_{\mu_2} \oplus \cdots \oplus \infty\pi_{\mu_\infty}.$$

Furthermore, the measures $\mu_1, \mu_2, \ldots, \mu_\infty$ are unique up to equivalence.

Outline of proof

- Lempert's Theorem $\Rightarrow$ complex structure on $\mathbb{S}$
  $\Rightarrow$ complex structure on $\mathbb{V}_\rho$
- **Key idea:** Positive energy rep's $(\pi, \mathscr{H})$ of Vir can be realized as
  $$(\pi, \mathscr{H}) \hookrightarrow \Gamma_{\mathrm{hol}}(\mathbb{V}_\rho) \cong \mathcal{O}(\mathbb{S}, V_\rho)$$
  where $\mathcal{O}(\mathbb{S}, V_\rho) := \left\{ f : \mathbb{S} \to V_\rho \;:\; \forall_{p \in \mathbb{S}} \forall_{s \in S} \left(f(p \cdot s) = \widetilde{\rho}(s)^{-1} f(p) \right) \right\}$.
- $\mathscr{H}$ is a *reproducing kernel hilbert space:*
  $$K : \mathbb{S} \times \overline{\mathbb{S}} \to \mathrm{End}(V_\rho)\ ,\ K(s_1, s_2) = \mathrm{ev}_{s_1} \circ \mathrm{ev}_{s_2}^* \quad \text{is holomorphic.}$$
- $(\pi, \mathscr{H})$ is uniquely determined by $K : \mathbb{S} \times \overline{\mathbb{S}} \to \mathrm{End}(V_\rho)$.

Reproducing kernel Hilbert spaces

Theorem (Neeb–S. '14)

Let $(\pi, \mathscr{H})$ be a positive energy unitary representation of Vir. Then there exist mutually singular finite positive borel measures $\mu_1, \mu_2, \ldots, \mu_\infty$ such that

$$\pi \cong \pi_{\mu_1} \oplus 2\pi_{\mu_2} \oplus \cdots \oplus \infty\pi_{\mu_\infty}.$$

Furthermore, the measures $\mu_1, \mu_2, \ldots, \mu_\infty$ are unique up to equivalence.

Outline of proof

- Lempert's Theorem $\Rightarrow$ complex structure on $\mathbb{S}$
  $\Rightarrow$ complex structure on $\mathbb{V}_\rho$
- **Key idea:** Positive energy rep's $(\pi, \mathscr{H})$ of Vir can be realized as

$$(\pi, \mathscr{H}) \hookrightarrow \Gamma_{\text{hol}}(\mathbb{V}_\rho) \cong \mathcal{O}(\mathbb{S}, V_\rho)$$

where $\mathcal{O}(\mathbb{S}, V_\rho) := \left\{ f : \mathbb{S} \to V_\rho \;\; : \;\; \forall_{p \in \mathbb{S}} \forall_{s \in S} \left(f(p \cdot s) = \widetilde{\rho}(s)^{-1} f(p) \right) \right\}.$

- $\mathscr{H}$ is a *reproducing kernel hilbert space:*

$$K : \mathbb{S} \times \overline{\mathbb{S}} \to \text{End}(V_\rho)\ , \ K(s_1, s_2) = \text{ev}_{s_1} \circ \text{ev}_{s_2}^* \quad \text{is holomorphic.}$$

- $(\pi, \mathscr{H})$ is uniquely determined by $K : \mathbb{S} \times \overline{\mathbb{S}} \to \text{End}(V_\rho)$.

Reproducing kernel Hilbert spaces

Theorem (Neeb–S. '14)

Let $(\pi, \mathscr{H})$ be a positive energy unitary representation of Vir. Then there exist mutually singular finite positive borel measures $\mu_1, \mu_2, \ldots, \mu_\infty$ such that

$$\pi \cong \pi_{\mu_1} \oplus 2\pi_{\mu_2} \oplus \cdots \oplus \infty\pi_{\mu_\infty}.$$

Furthermore, the measures $\mu_1, \mu_2, \ldots, \mu_\infty$ are unique up to equivalence.

Outline of proof

- Lempert's Theorem $\Rightarrow$ complex structure on $\mathbb{S}$
  $\Rightarrow$ complex structure on $\mathbb{V}_\rho$
- **Key idea:** Positive energy rep's $(\pi, \mathscr{H})$ of Vir can be realized as
  $$(\pi, \mathscr{H}) \hookrightarrow \Gamma_{\mathrm{hol}}(\mathbb{V}_\rho) \cong \mathcal{O}(\mathbb{S}, V_\rho)$$
  where $\mathcal{O}(\mathbb{S}, V_\rho) := \left\{ f : \mathbb{S} \to V_\rho \;:\; \forall_{p \in \mathbb{S}} \forall_{s \in S} \left(f(p \cdot s) = \widetilde{\rho}(s)^{-1} f(p) \right) \right\}$.
- $\mathscr{H}$ is a *reproducing kernel hilbert space*:
  $$K : \mathbb{S} \times \overline{\mathbb{S}} \to \mathrm{End}(V_\rho)\ ,\ K(s_1, s_2) = \mathrm{ev}_{s_1} \circ \mathrm{ev}_{s_2}^* \quad \text{is holomorphic.}$$
- $(\pi, \mathscr{H})$ is uniquely determined by $K : \mathbb{S} \times \overline{\mathbb{S}} \to \mathrm{End}(V_\rho)$.

Reproducing kernel Hilbert spaces

Theorem (Neeb–S. '14)

Let $(\pi, \mathscr{H})$ be a positive energy unitary representation of Vir. Then there exist mutually singular finite positive borel measures $\mu_1, \mu_2, \ldots, \mu_\infty$ such that

$$\pi \cong \pi_{\mu_1} \oplus 2\pi_{\mu_2} \oplus \cdots \oplus \infty\pi_{\mu_\infty}.$$

Furthermore, the measures $\mu_1, \mu_2, \ldots, \mu_\infty$ are unique up to equivalence.

Outline of proof

- Lempert's Theorem $\Rightarrow$ complex structure on $\mathbb{S}$
  $\Rightarrow$ complex structure on $\mathbb{V}_\rho$
- **Key idea:** Positive energy rep's $(\pi, \mathscr{H})$ of Vir can be realized as

  $$(\pi, \mathscr{H}) \hookrightarrow \Gamma_{\mathrm{hol}}(\mathbb{V}_\rho) \cong \mathcal{O}(\mathbb{S}, V_\rho)$$

  where $\mathcal{O}(\mathbb{S}, V_\rho) := \Big\{ f : \mathbb{S} \to V_\rho \ : \ \forall_{p \in \mathbb{S}} \forall_{s \in S} \big(f(p \cdot s) = \widetilde{\rho}(s)^{-1} f(p) \big) \Big\}.$
- $\mathscr{H}$ is a *reproducing kernel hilbert space*:

  $$K : \mathbb{S} \times \overline{\mathbb{S}} \to \mathrm{End}(V_\rho)\ ,\ K(s_1, s_2) = \mathrm{ev}_{s_1} \circ \mathrm{ev}_{s_2}^* \quad \text{is holomorphic.}$$
- $(\pi, \mathscr{H})$ is uniquely determined by $K : \mathbb{S} \times \overline{\mathbb{S}} \to \mathrm{End}(V_\rho)$.

Thank you!