A Tale of Two Matrices

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Fields Institute

August 29, 2015
What Can Two Matrices Tell Us About...
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Introducing Our Matrices

\[
P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
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Introducing Our Matrices

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Introducing Our Matrices

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**Natural Next Step: Multiply**

\[ PX = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = Z \]
What Next?

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Why Not Take Linear Combinations?

\[ pP + xX + zZ = \begin{pmatrix} 0 & p & z \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} := m(p, x, z) \]

where \( p, x, z \in \mathbb{R} \)
Result: A Lie Algebra

\[ \left\{ \begin{pmatrix} 0 & p & z \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}, \quad p, x, z \in \mathbb{R} \right\} = \{ m(p, x, z) : p, x, z \in \mathbb{R} \} \]

is a Lie Algebra.
What Is A Lie Algebra, Anyway?

A vector space with a binary operation called the bracket, denoted $[·,·]$, which satisfies:

1. $[x,x] = 0$
2. $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$
3. $[·,·]$ is linear in both arguments
What Is A Lie Algebra, Anyway?

- Kind of like a group!
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\left\{ \begin{pmatrix} 0 & p & z \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}, \quad p, x, z \in \mathbb{R} \right\} = \left\{ m(p, x, z) : p, x, z \in \mathbb{R} \right\}
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Examples of Lie Algebras

- The set $M_n$ of all $n \times n$ matrices, with bracket:

$$[A, B] = AB - BA.$$
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- The set $M_n$ of all $n \times n$ matrices, with bracket:
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- The set of linear maps over a vector space, with bracket:
  \[ [f, g] = f \circ g - g \circ f \]
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- The set $M_n$ of all $n \times n$ matrices, with bracket:
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- The set of linear maps over a vector space, with bracket:
  \[ [f, g] = f \circ g - g \circ f \]

Heisenberg Algebra $\mathfrak{h}$

- The set of all matrices of the form
  \[
  \begin{pmatrix}
  0 & p & z \\
  0 & 0 & x \\
  0 & 0 & 0
  \end{pmatrix}
  = m(p, x, z)
  \]
  with bracket
  \[
  [A, B] = AB - BA.
  \]
  \[
  [(p, x, z), (p', q', z')] = (0, 0, px' - xp')
  \]
Commutation Relations of Heisenberg Algebra $\mathfrak{h}$

Basis Matrices

\[
\begin{align*}
P &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, &
X &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, &
Z &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{align*}
\]

Computing The Bracket

\[
\begin{align*}
\{P, X\} &= PX - XP = Z \\
\{P, Z\} &= PZ - ZP = 0 \\
\{X, Z\} &= XZ - ZX = 0
\end{align*}
\]

These bracket relationships are called the canonical commutation relations.
Commutation Relations of Heisenberg Algebra $\mathfrak{h}$

### Basis Matrices

$$ P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} $$

### Computing The Bracket

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These bracket relationships are called the **canonical commutation relations**.

In Summary...
### Basis Matrices

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In Summary...

\[
[P, X] = Z
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[P, Z] = 0
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[X, Z] = 0
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The Heisenberg algebra has an elegant structure.
Algebras vs. Groups

- The Heisenberg algebra has an elegant structure
- But groups are more interesting algebraic objects
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- We can create a group out of a Lie algebra by using the exponential map
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- The Heisenberg algebra has an elegant structure
- But groups are more interesting algebraic objects
- We can create a group out of a Lie algebra by using the exponential map

Definition: The Exponential Map

Given a matrix Lie algebra defined by \( \{A\} \), we obtain a Lie group by the matrix exponential:

\[
\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \ldots.
\]
$\mathfrak{h} = \{pP + xX + zZ\}$
Let’s Exponentiate!

\[ \mathfrak{h} = \{ pP + xX + zZ \} \]

\[ \exp(P) = I + P + P^2 + \ldots \]

\[ P_{\mathcal{H}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
Let’s Exponentiate!

$$h = \{pP + xX + zZ\}$$

$$\exp(P) = I + P + P^2 + \ldots$$
$$P_{\mathcal{H}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\exp(X) = I + X + X^2 + \ldots$$
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Let’s Exponentiate!

\[ \mathfrak{h} = \{ pP + xX + zZ \} \]

\[
\begin{align*}
\exp(P) &= I + P + P^2 + \ldots \\
\exp(X) &= I + X + X^2 + \ldots \\
\exp(Z) &= I + Z + Z^2 + \ldots
\end{align*}
\]

\[
P_{\mathcal{H}} = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
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\]

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Z_{\mathcal{H}} = \begin{pmatrix}
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\exp(Z) = I + Z + Z^2 + \ldots \\
Z_{\mathcal{H}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\]

Take Linear Combinations

\[ \mathcal{H} = pP_{\mathcal{H}} + xX_{\mathcal{H}} + zZ_{\mathcal{H}}. \]
Let’s Exponentiate!

Result: A Lie Group, $\mathcal{H}$

$$\mathcal{H} = \{ \exp(\mathfrak{h}) \} = \left\{ \begin{pmatrix} 1 & p & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad p, x, z \in \mathbb{R} \quad := \{ M(p, x, z) \}$$
Let’s Exponentiate!

Result: A Lie Group, $\mathcal{H}$

$$\mathcal{H} = \{\exp(\hbar)\} = \left\{ \begin{pmatrix} 1 & p & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}, \quad p, x, z \in \mathbb{R} \right\} := \{M(p, x, z)\}$$

What Is A Lie Group, Anyway?

A Group is a set closed under a binary operation that satisfies:

1. Associative ✓
2. Every element has an inverse ✓
3. There exists an identity ✓

A Lie Group is a group with an additional manifold structure on which the group operation and inversion are smooth maps.

TL;DR A Lie Group has algebraic and differential structure.
Result: A Lie Group, $\mathcal{H}$

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- A Group is a set closed under a binary operation that satisfies:
  - 1. Associative ✓
  - 2. Every element has an inverse $M(p, x, z)^{-1} = M(-p, -x, px - z)$ ✓
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$C \cdot \text{Celebi, Hendricks, Jordan}$

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Let’s Exponentiate!

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What Is A Lie Group, Anyway?

- A **Group** is a set closed under a **binary operation** that satisfies:
  1. Associative \( \checkmark \)
  2. Every element has an inverse \( M(p, x, z)^{-1} = M(-p, -x, px - z) \) \( \checkmark \)
  3. There exists an identity \( M(0, 0, 0) \in \mathcal{H} \) \( \checkmark \)
Let’s Exponentiate!

Result: A Lie Group, $\mathcal{H}$

$$\mathcal{H} = \{ \exp(h) \} = \left\{ \begin{pmatrix} 1 & p & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} , \quad p, x, z \in \mathbb{R} \right\} := \{ M(p, x, z) \}$$

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- A **Lie Group** is a group with an additional manifold structure on which the group operation and inversion are smooth maps
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- TL;DR A Lie group has algebraic and differential structure
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\]
Commutation Relations Of Heisenberg Group $H$

**Basis Matrices**

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P_H = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X_H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad Z_H = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

**Computing The Bracket**

$$[P_H, X_H] = P_H X_H P_H^{-1} X_H^{-1} = Z_H$$

What luck! Yet again, we have the canonical commutation relations.

In Summary...

In both $h$ and $H$:

$$[P_H, X_H] = Z_H$$
$$[P_H, Z_H] = 0$$
$$[X_H, Z_H] = 0$$
Commutation Relations Of Heisenberg Group $\mathcal{H}$

### Basis Matrices

$$P_\mathcal{H} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X_\mathcal{H} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad Z_\mathcal{H} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

### Computing The Bracket

$$[P_\mathcal{H}, X_\mathcal{H}] = P_\mathcal{H}X_\mathcal{H}P_\mathcal{H}^{-1}X_\mathcal{H}^{-1} = Z_\mathcal{H}$$

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**Basis Matrices**

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\begin{align*}
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### Computing The Bracket

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What luck! Yet again, we have the canonical commutation relations.

### In Summary...

In both $\mathfrak{h}$ and $\mathcal{H}$:

- $[P_{\mathcal{H}}, X_{\mathcal{H}}] = Z_{\mathcal{H}}$
- $[P_{\mathcal{H}}, Z_{\mathcal{H}}] = 0$
- $[X_{\mathcal{H}}, Z_{\mathcal{H}}] = 0$
A representation $\pi$ of a group $G$ is a homomorphism from $G$ to the group $GL(V)$, where $GL(V)$ is the set of all invertible linear maps (operators) $T : V \rightarrow V$. That is:

$$\pi : G \rightarrow GL(V)$$

$$\pi(g)(v) = w, \text{ for some } v, w \in V, g \in G$$

$$\pi(g_1 g_2) = \pi(g_1) \pi(g_2)$$
What is a representation?

A representation $\pi$ of a group $G$ is a homomorphism from $G$ to the group $GL(V)$, where $GL(V)$ is the set of all invertible linear maps (operators) $T : V \rightarrow V$. That is:

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$$

$$
\pi(g_1g_2) = \pi(g_1)\pi(g_2)
$$

Linear invertible operator on $V$
The representation \( \pi \) is called \emph{unitary} if for every \( g \in G \) the operator \( \pi(g) \) is unitary on \( V \), i.e.,

\[
\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle \quad \text{for all } v, w \in V, g \in G
\]
Unitary Representation

The representation $\pi$ is called *unitary* if for every $g \in G$ the operator $\pi(g)$ is unitary on $V$, i.e.,

$$\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle \quad \text{for all } v, w \in V, g \in G$$

Invariant Subspace

A closed subspace $W \subset V$ is called *invariant* for $\pi$ if $\pi(g)W \subset W$ for every $g \in G$. 

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Unitary Representation

The representation \( \pi \) is called *unitary* if for every \( g \in G \) the operator \( \pi(g) \) is unitary on \( V \), i.e.,

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Invariant Subspace

A closed subspace \( W \subset V \) is called *invariant* for \( \pi \) if \( \pi(g)W \subset W \) for every \( g \in G \).

Irreducible Representation

The representation \( \pi \) is called *irreducible* if there is no proper (closed) invariant subspace, i.e., the only (closed) invariant subspaces are 0 and \( V \) itself.
Let’s go back to the Heisenberg group:

\[ \mathcal{H} = \left\{ \begin{pmatrix} 1 & p & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \middle| p, x, z \in \mathbb{R} \right\} \]

The center of the Heisenberg group is

\[ Z(\mathcal{H}) := \{(0, 0, z) \mid z \in \mathbb{R}\} \]
We will now focus on the representations of the Heisenberg group. As our representation space “V”, we choose the infinite-dimensional function space $L^2(\mathbb{R})$ where the functions satisfy the following property:

$$\|f\|_2^2 := \int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty$$

$L^2(\mathbb{R})$ is an inner product space. That is, it is a vector space provided with an inner product which is given for two functions $f, g \in L^2(\mathbb{R})$ as follows:

$$\langle f, g \rangle := \int_{\mathbb{R}} f(\xi) \overline{g(\xi)} \, d\xi$$
Some Basic Operators On $L^2(\mathbb{R})$

**Rotation Operator**

The rotation operator $R(x)$ for a function $f(\xi) \in L^2(\mathbb{R})$ is defined as follows:

$$R(x)f(\xi) := e^{ix\xi}f(\xi)$$
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**Remark**

The rotation and translation operators are unitary. In addition, they form one-parameter unitary groups as

$$\{R(x)\}_{x \in \mathbb{R}}, \{T(p)\}_{p \in \mathbb{R}}$$
For \((p, x, z) \in \mathcal{H}\) we define the **unitary representation** \(\pi_k(p, x, z)\) on \(L^2(\mathbb{R})\) by

\[
\pi_k(p, x, z)f(\xi) := e^{i(x\xi + z)k} f(\xi + p).
\]

And now, hold on to your seats... Here is the celebrated theorem!
For \((p, x, z) \in \mathcal{H}\) we define the unitary representation \(\pi_k(p, x, z)\) on \(L^2(\mathbb{R})\) by

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\]

Define the central character \(\chi_k\) of \(\pi_k\) as

\[
\chi_k(0, 0, z) := e^{izk}
\]
The Schrödinger Representation

Schrödinger Representation

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For \(k = 1\) we get

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\]
Notice that
\[
\pi(p, x, z) f(\xi) = e^{i(x\xi + z)} f(\xi + p)
= e^{ix\xi} \cdot e^{iz} \cdot f(\xi + p)
= R(x)\chi(z)T(p)
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\]
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= e^{ix\xi} \cdot e^{iz} \cdot f(\xi + p)
\]
\[
= R(x) \chi(z) T(p)
\]

And now, hold on to your seats... Here is the celebrated theorem!
Theorem: Stone-von Neumann

For $k \neq 0$ the unitary representation $\pi_k$ is irreducible. Every irreducible unitary representation of $\mathcal{H}$ with central character $\chi_k$ is isomorphic to $\pi_k$. 
A Quick Review

What Have We Done So Far?

Introduced $h$, and shown that $\exp(h) = H$.

Shown that both $h$ and $H$ have the canonical commutation relation.

Discovered that we can uniquely represent elements of $H$ as the product of three operators on $L^2(\mathbb{R})$.

What Now?

Show that the Schrödinger representations contains the foundations of quantum mechanics.

Bask in our knowledge of the one-ness of the universe.

Çelebi, Hendricks, Jordan
A Tale of Two Matrices
August 29, 2015
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A Closer Look at $R$ and $T$

### Schrödinger Representation

\[ \pi(p, x, z) = R(x)\mathcal{X}(z)T(p). \]
Schrödinger Representation

\[ \pi(p, x, z) = R(x)X(z)T(p). \]
A Closer Look at $R$ and $T$

Schrödinger Representation

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A Closer Look at $R$ and $T$

First, setting the scene:

$$\pi(p, 0, 0) = \pi(pP_H) = T(p) \quad \pi(0, x, 0) = \pi(xX_H) = R(x)$$

$$pP_H = p \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad xX_H = x \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

We associate the matrices $P_H$ and $X_H$ with the operators $T(p)$ and $R(x)$, respectively. Important: $P_H \leftrightarrow T(p)$ and $X_H \leftrightarrow R(x)$.
A Closer Look at $R$ and $T$

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- **Important:** $P_H \leftrightarrow T(p)$ and $X_H \leftrightarrow R(x)$. 
Stone’s Theorem (Simplified)

Every group of single-parameter unitary operators \( \{U(s)\}_{s \in \mathbb{R}} \) can be uniquely associated with a self-adjoint operator \( A \), such that

\[
U(s) = e^{isA} = \sum_{k=0}^{\infty} \frac{(isA)^k}{k!}
\]
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What Does That Even Mean?

It means that \( R(x) \) and \( T(p) \) can be viewed as the exponential of a self-adjoint operator \( A \).
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How, Though?

Easy to see for the rotation operator \( R(x) \), since

\[
R(x)f(\xi) = e^{ix\xi}f(\xi),
\]

So, by Stone’s Theorem, \( A = \hat{X} \), where \( \hat{X} \) is the operator that multiplies a function by its argument, i.e. \( \hat{X}f(\xi) = \xi f(\xi) \).
Stone’s Theorem

A Revelation (In Pictures)

\[ X_H \leftrightarrow \text{By Sch. rep} \rightarrow R(x) \]

Def’n of group \[ \exp(X) \leftrightarrow \text{Revelation!} \rightarrow \exp(ix\hat{X}) \]

Stone’s thm.
Stone’s Theorem

A Revelation (In Words)

Exponentiating the operator $\hat{X}$ gives the rotation operator. We associated the rotation operator with the matrix $X_H$. But $X_H \in \mathcal{H}$ was obtained by exponentiating $X \in \mathfrak{h}$. Logically, then, we should associate $X \in \mathfrak{h}$ with the $\hat{X}$ operator on $L^2(\mathbb{R})$.
Exponentiating the operator $\hat{X}$ gives the rotation operator.

Stone’s Theorem

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A Revelation (In Words)

- Exponentiating the operator $\hat{X}$ gives the rotation operator.
- We associated the rotation operator with the matrix $X_H$. 

$X_H \xleftarrow{\text{Def'n of group}} \xrightarrow{\text{By Sch. rep}} R(x) \xleftarrow{\text{Stone's thm.}} \xrightarrow{\text{exp}} \exp(iX) \xrightarrow{\text{Revelation!}} \exp(ix\hat{X})$
Stone’s Theorem

A Revelation (In Pictures)

\[ X_H \leftrightarrow \text{Def'n of group} \]
\[ \exp(X) \leftrightarrow \text{Revelation!} \]
\[ R(x) \rightarrow \text{By Sch. rep} \]
\[ \exp(ix\hat{X}) \rightarrow \text{Stone's thm.} \]

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- Exponentiating the operator \( \hat{X} \) gives the rotation operator.
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(1)
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(1)

Repeating For \( P \)

It turns out that:

\[
T(p) f(\xi) = \exp \left( ip \left( -i \frac{d}{d\xi} \right) \right) f(\xi) = \exp \left( \frac{d}{d\xi} \right) f(\xi)
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It turns out that:

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T(p)f(\xi) = \exp \left( ip \left( \frac{d}{-i d\xi} \right) \right) f(\xi) = \exp \left( \frac{d}{d\xi} \right) f(\xi)
\]

If You Learn Nothing Else Today…

\[
\exp \left( p \frac{d}{d\xi} \right) f(\xi) = \sum_{k=0}^{\infty} \frac{p^k f^k(\xi)}{k!} = f(\xi + p)
\]
We defined $\hat{P} = -i \frac{d}{d\xi}$.

\[ \begin{align*}
&\text{By Sch. rep} & \quad & \text{Def’n of group} \\
&P_{\mathcal{H}} & \quad & \text{Stone’s thm.} \\
&\exp(P) & \quad & \exp(ip\hat{P})
\end{align*} \]

Revelation!
We defined $\hat{P} = -i \frac{d}{d\xi}$.

By Sch. rep

$P_{\mathcal{H}} \leftrightarrow T(p)$

Def’n of group

Stone’s thm.

exp($P$) \begin{array}{c} \Leftarrow \text{Revelation!} \end{array} \text{exp} \left( ip\hat{P} \right)$
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By Sch. rep

$P_H \leftrightarrow T(p)$

Def’n of group

$\exp(P) \leftrightarrow \exp[ip\hat{P}]$

Revelation!

Stone’s thm.

Revelation (In Words)

- Exponentiating the operator $\hat{P}$ gives the translation operator.
Stone’s Theorem

**Another Revelation**

We defined $\hat{P} = -i \frac{d}{d\xi}$.

By Sch. rep

$P_{\mathcal{H}} \overset{\text{Def’n of group}}{\longrightarrow} \text{exp}(P) \overset{\text{Revelation!}}{\longrightarrow} \exp(ip\hat{P}) \overset{T(p)}{\longrightarrow} T(p)$

Stone’s thm.

We defined $\hat{P} = -i \frac{d}{d\xi}$.

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---

**Revelation (In Words)**

- Exponentiating the operator $\hat{P}$ gives the translation operator.
- We associated the translation operator with the matrix $P_{\mathcal{H}}$. 

---

Çelebi, Hendricks, Jordan

A Tale of Two Matrices

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Putting It All Together

**Schrödinger Representation**

\[
\pi(p, x, z) = R(x)\chi(z)T(p).
\]
Schrödinger Representation

\[ \pi(p, x, z) = R(x)\chi(z)T(p). \]

Putting It All Together

We can associate the matrices \( P \) and \( X \) with operators:

\[ P \leftrightarrow \hat{P} = -i \frac{\partial}{\partial \xi} \]
\[ X \leftrightarrow \hat{X} = \xi \]

We often call \( \hat{P} \) the momentum operator and \( \hat{X} \) the position operator. We'll find out why shortly!
Schrödinger Representation

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We often call \( \hat{P} \) the momentum operator and \( \hat{X} \) the position operator. We’ll find out why shortly!
What about the third basis matrix, $Z$? Remember our old friend, the canonical commutation relation? Let's try that on $\hat{P}$ and $\hat{X}$:

$$[\hat{P}, \hat{X}] f(\xi) = \hat{P}\hat{X}f(\xi) - \hat{X}\hat{P}f(\xi) = -i \frac{d}{d\xi}(\xi f(\xi)) - \xi \left(-i \frac{d}{d\xi} f(\xi)\right) = -if(\xi) = \hat{Z}f$$
Last Step

- What about the third basis matrix, $Z$?
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\[
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\]

\[
= -i \frac{d}{d\xi} (\xi f(\xi)) - \xi \left( -i \frac{d}{d\xi} f(\xi) \right)
\]
Finishing Up

Last Step

- What about the third basis matrix, \( Z \)?
- Remember our old friend, the canonical commutation relation? Let’s try that on \( \hat{P} \) and \( \hat{X} \):

\[
\begin{align*}
\left[ \hat{P}, \hat{X} \right] f(\xi) &= \hat{P} \hat{X} f(\xi) - \hat{X} \hat{P} f(\xi) \\
&= -i \frac{d}{d\xi} (\xi f(\xi)) - \xi \left( -i \frac{d}{d\xi} f(\xi) \right) \\
&= -if(\xi)
\end{align*}
\]
Last Step

- What about the third basis matrix, $Z$?
- Remember our old friend, the canonical commutation relation? Let’s try that on $\hat{P}$ and $\hat{X}$:

$$
\begin{align*}
\left[\hat{P}, \hat{X}\right] f(\xi) &= \hat{P}\hat{X} f(\xi) - \hat{X}\hat{P} f(\xi) \\
&= -i \frac{d}{d\xi} (\xi f(\xi)) - \xi \left(-i \frac{d}{d\xi} f(\xi)\right) \\
&= -i f(\xi) \\
&= \hat{Z} f
\end{align*}
$$
We have now successfully represented our magical matrices as operators on $L^2(\mathbb{R})$. The Stone-von Neumann theorem tells us that this is essentially the only way to construct them.
Success!

We have now successfully represented our magical matrices as operators on $L^2(\mathbb{R})$. The Stone-von Neumann theorem tells us that this is essentially the only way to construct them.

Matrices And Their Operators

\[
P \longleftrightarrow \hat{P} = -i \frac{d}{d\xi} \\
X \longleftrightarrow \hat{X} = \xi \\
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We have now successfully represented our magical matrices as operators on $L^2(\mathbb{R})$. The Stone-von Neumann theorem tells us that this is essentially the only way to construct them.

Matrices And Their Operators

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P \leftrightarrow \hat{P} = -i \frac{d}{d\xi}
\]
\[
X \leftrightarrow \hat{X} = \xi
\]
\[
Z \leftrightarrow \hat{Z} = -i
\]

In Summary…

\[
\left[ \hat{P}, \hat{X} \right] = \hat{Z}
\]
\[
\left[ \hat{P}, \hat{Z} \right] = 0
\]
\[
\left[ \hat{X}, \hat{Z} \right] = 0
\]

The canonical commutation relation lives!
Another Way of Getting the Momentum and Position Operators

Partial Derivative by $p$

\[
\left. \frac{\partial}{\partial p} \pi(p, 0, 0) \right|_{p=0} = \left. \frac{\partial}{\partial p} f(\xi + p) \right|_{p=0} = \lim_{p \to 0} \frac{f(\xi + p) - f(\xi)}{p} = \frac{d}{d\xi} f(\xi)
\]
Another Way of Getting the Momentum and Position Operators

**Partial Derivative by \( p \)**

\[
\left. \frac{\partial}{\partial p} \pi(p, 0, 0) \right|_{p=0} = \left. \frac{\partial}{\partial p} f(\xi + p) \right|_{p=0} = \lim_{\substack{p \to 0}} \left. \frac{f(\xi + p) - f(\xi)}{p} \right|_{p=0} = \frac{d}{d\xi} f(\xi)
\]

**Partial Derivative by \( x \)**

\[
\left. \frac{\partial}{\partial x} \pi(0, x, 0) \right|_{x=0} = \left. \frac{\partial}{\partial x} e^{ix\xi} f(\xi) \right|_{x=0} = \lim_{x \to 0} \frac{e^{ix\xi} f(\xi) - f(\xi)}{x}
\]

\[
\begin{align*}
&= \lim_{x \to 0} \frac{i \sin x\xi}{x} + \frac{\cos x\xi}{x} - \frac{1}{x} \\
&= i\xi f(\xi)
\end{align*}
\]
Given our position and momentum operators:

\[
\hat{X}(f(x)) = xf(x)
\]

\[
\hat{P}(f(x)) = -i \frac{d}{dx} f(x)
\]

We call this Laplacian the Hermite Operator.
Given our position and momentum operators:

\[ \hat{X}(f(x)) = xf(x) \]
\[ \hat{P}(f(x)) = -i \frac{d}{dx} f(x) \]

The Laplacian is given in Euclidean Two-Space by \( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \)
Hermite Operator

Given our position and momentum operators:

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\]

The Laplacian is given in Euclidean Two-Space by

\[
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\]

or

\[
\hat{X}(\hat{X}(f)) + \hat{P}(\hat{P}(f))
\]

or

\[
\left[ x^2 - \frac{d^2}{dx^2} \right] f(x)
\]

We call this Laplacian the **Hermite Operator**.
Self-Adjoint Operators

An operator $\mathcal{L}$ is self-adjoint over an inner product iff $\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}v \rangle$. Since the Hermite operator is self-adjoint over the $L^2(\mathbb{R})$ inner product, its eigenfunctions define an orthogonal basis for $L^2(\mathbb{R})$. Rescaling the Hermite operator like so,

$$H := -\frac{1}{4}\pi^2 \frac{d^2}{dx^2} + x^2$$

Gives us the orthonormal basis

$$h_k(x) = \frac{1}{\sqrt{k!}} \left(-\frac{1}{\sqrt{2}\pi}\right)^k e^{\pi x^2} dk e^{-\frac{2}{\pi}x^2} Hh_k = \frac{2k + 1}{2\pi} h_k.$$
Hermite Polynomials

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Given by

\[ \mathcal{F}(f) = \langle f(x), e^{2\pi i \xi x} \rangle = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} \, dx \]

So this will give me the strength of the function in each linear frequency, \( k \).
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\[ \mathcal{F}h_k = (-i)^k h_k \]
Properties of the Fourier Transform

Plancherel Theorem

\[ \langle f, f \rangle = \langle \mathcal{F}f, \mathcal{F}f \rangle \]
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**Plancherel Theorem**

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**Derivative as Multiplication**

\[ \mathcal{F} \frac{df(x)}{dx} = 2\pi i \xi \{ \mathcal{F} f(x) \}(\xi) \]

This means that the position and momentum operators are called Fourier Conjugates of each other.
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This means that the position and momentum operators are called **Fourier Conjugates** of each other.
Where We Are Going: Heisenberg’s Uncertainty Principle and Quantum Mechanics

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- Classical mechanics $\leftrightarrow$ finite dimensional spaces
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The Heisenberg Uncertainty Principle

\[ \| \hat{P} f \|_2^2 + \| \hat{X} f \|_2^2 \geq \frac{1}{2\pi} \| f \|_2^2 \]

The Derivation

\[ \langle H f, f \rangle \]
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\[ = \sum_{k=0}^{\infty} \frac{2k + 1}{2\pi} |\langle f, h_k \rangle|^2 \]

\[ \geq \frac{1}{2\pi} \sum_{k=0}^{\infty} |\langle f, h_k \rangle|^2 = \frac{1}{2\pi} \|f\|_2^2 \]

Çelebi, Hendricks, Jordan
A Tale of Two Matrices
August 29, 2015 34 / 36
Thank you!
The Cowling-Price Inequality

We attempted to find a shorter proof for the expression

\[ \| |x|^{\alpha} f(x) \|_p + \| |y|^{\beta} \hat{f}(y) \|_q \geq K \| f \|_2 \]

which is a known generalization of the Heisenberg Uncertainty Principle.