Hyperbolic Geometry and Modular Forms

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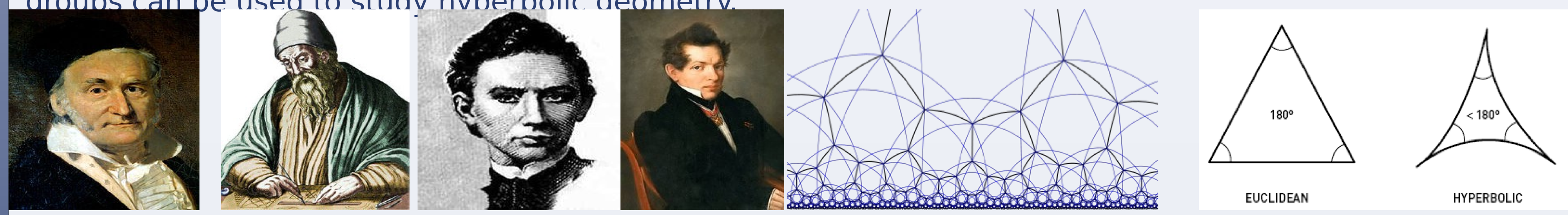
ABSTRACT

In his masterpiece, *The Elements*, Euclid laid down the foundations of the geometry of space by starting from a handful of self-evident axioms and postulates. One of these postulates, now known as the parallel postulate, states that for a give line *l* and a point *p* not lying on *l*, there is precisely one line through *p* that is parallel to *l*. While the parallel postulate appears to be intuitively true, mathematicians after Euclid were uncomfortable with the fact that the parallel postulate was not as self-evident as the other postulates. Indeed, there were countless failed attempts to derive the parallel postulate as a consequence of the other axioms. It was only in the 19th century that European mathematicians began to entertain the idea of a geometry that did not satisfy the parallel postulate. Such investigations eventually led to the invention of hyperbolic geometry- a geometry that fails to satisfy the parallel postulate and yet is internally consistent. Recent developments have produced many remarkable and deep connections between hyperbolic geometry and the theory of abstract mathematical objects known as groups. In particular, there are fascinating ways in which certain algebraic gadgets known as modular groups can be used to study hyperbolic geometry.



From left to right: Johann Carl Friedrich Gauss (1777-1855), Euclid of Alexandria (fl. 300 BC), Janos Bolyai (1802-1860), Nikolai Ivanovich Lobachevsky (1792-1856), Stellated regular heptagonal tiling of the upper half-plane model of hyperbolic geometry, Comparison of an Euclidean triangle with its hyperbolic cousin

INTRODUCTION

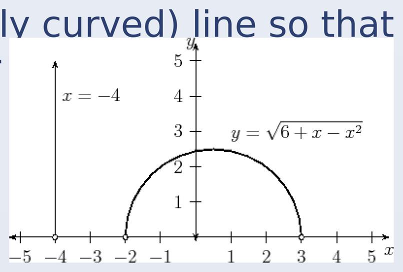
Just as Euclidean geometry is studied against the backdrop of ordinary two- or three-dimensional space, hyperbolic geometry and the objects relating to it are analysed in the context of the upper half-plane *H* which is defined as follows

$$\mathbb{H} = \{x + iy \mid y > 0; x, y \in \mathbb{R}\}.$$

Pictorially, we may think of *H* as being one-half of the familiar *xy*-plane of analytic geometry. We define hyperbolic lines to be the vertical half-lines and the semicircles centred on the horizontal axis of the plane. More precisely, we endow the half-plane with the metric

$$(ds)^2 = \frac{dx^2 + dy^2}{y^2}$$

where *s* measures the length of a (possibly curved) line so that the lines of shortest distance visually manifest as either semicircles centred at the origin or



Using this definition of a line, we can then construct various hyperbolic figures in the plane, including such interesting things as hyperbolic polygons. Once we are able to construct geometric figures in the hyperbolic plane, it becomes possible to study the transformations of these figures. (Note that, in mathematics, a transformation refers to a function which may alter certain geometrical features of the domain on which the function acts.) We focus on a special class of transformations known as linear fractional transformations and which are defined as

$$f(z) = \frac{az + b}{cz + d}$$

where the coefficients *a*, *b*, *c*, *d* are complex numbers satisfying *ad – bc=1*. The set of such transformations forms what is known as the modular group and we think of each element of this group acting on *H*. The modular group in turn contains certain subgroups called congruence subgroups and such subgroups are interesting because each of them gives rise to a region known as a fundamental domain in the upper half-plane. The fundamental domain associated to each group can be used to tessellate the plane as shown in the picture below

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \qquad \sigma = \Re(s) > 1.$$

Recent observations have suggested that the areas of the fundamental domains are connected in some way to the summations of certain types of infinite series. These summations are actually values of what is known as the Riemann zeta function which is defined below

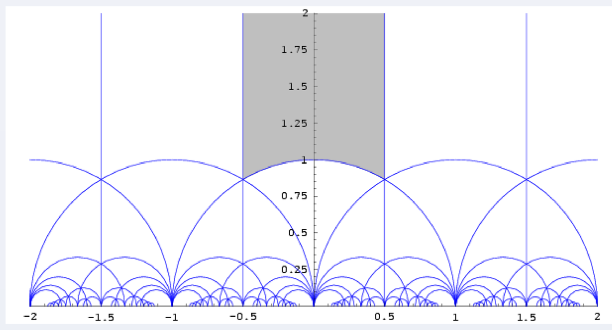
where *s* is a complex number with real part greater than 1and *n* ranges over the natural numbers starting at 1. Essentially, for each argument in the domain, the zeta function produces a convergent series and the value of the function at *s* is simply the limit to which the corresponding series converges. Named after the eminent German mathematician Bernhard Riemann, the Riemann zeta function is one of the most famous and most studied objects in mathematical history. This function lies at the heart of some of the deepest and most challenging questions in modern mathematics and it is at the core of what is known as the Riemann hypothesis.

OBSERVATIONS

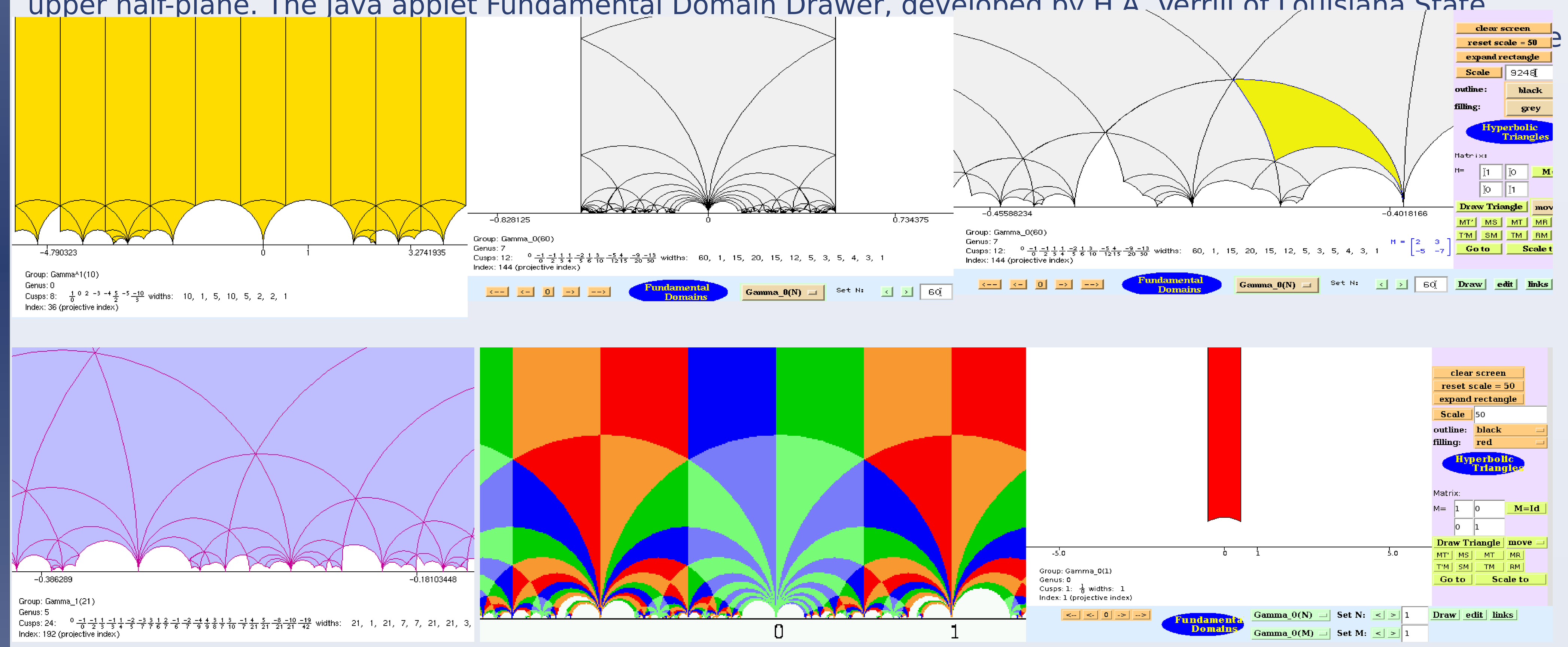
We begin the fundamental domain for the action of the modular group on the upper half-plane. This fundamental domain can be defined as

$$U = \left\{ z \in H : |z| > 1, |\operatorname{Re}(z)| < \frac{1}{2} \right\}.$$

The set *U* is geometrically represented as the shaded region in the following diagram



Using the familiar technique of integration by parts, we can compute this area. This might seem a bit surprising at first considering the fact that the shaded area appears to be unbounded and intuition tells us that the area should thus be infinite. However, just as we have a different notion of line and length in hyperbolic geometry, so too do we have a definition of area that varies from the ordinary Euclidean conception. Using the same method, we can actually compute areas of fundamental domains of different congruence subgroups of the modular group. The goal of these computations was to detect patterns that may underlie the areas of the congruence subgroups. To carry out the computations, it was first necessary to visualise the fundamental domains in the upper half-plane. The Java applet Fundamental Domain Drawer, developed by H.A. Verrill of Louisiana State



Clockwise from top left: Fundamental domain of refined congruence subgroup of level 10; Fundamental domain of full modular group of level 60; Fundamental domain of full modular group of level 60 with detail and with a hyperbolic triangle highlighted; Fundamental domain of full modular group of level 1; In this picture, a fundamental domain of the full modular group of level 2 can be constructed by any choice of 6 hyperbolic triangles of different colors; Fundamental domain of refined congruence subgroup of level 21.

Although the computations of the fundamental domains were not difficult, it was hard to detect patterns in their areas. The initial hypothesis that there are connections between the areas of fundamental domains of congruence subgroups and the Riemann zeta function is supported in the paper by Cazacu and Ghisa, where the authors use much broader techniques and theorems to establish some remarkable properties concerning fundamental domains. However, for the present, there are not many results apart from the calculations concerning the domains.

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