Direct limits of Schubert varieties and global sections of line bundles

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A B S T R A C T

We study the space of global sections $\Gamma(X, \mathcal{L})$ of a line bundle $\mathcal{L}$ on a $B$-stable ind-subvariety $X$ of $G/B$, where $G$ is a classical simple ind-group and $B$ is an arbitrary Borel subgroup of $G$. We give a necessary and sufficient condition for projectivity of $X$, and use it to prove that if $X$ is projective and $\mathcal{L}$ is globally generated, then $\Gamma(X, \mathcal{L})$ is the dual of a $B$-module $V_X(\mathcal{L})$ with finite-dimensional weight multiplicities. Finally, we describe the structure of these multiplicities in terms of Demazure modules.

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1. Introduction

A marvelous example of interplay between representation theory and algebraic geometry is the celebrated Borel–Weil–Bott theorem, which has been generalized vastly beyond reductive algebraic groups, e.g., to ind-groups and loop groups [DPW,Ku]. In connection with the Borel–Weil theorem, it is natural to study global sections (and higher cohomologies) of globally generated line bundles on $B$-stable subvarieties of a flag variety $G/B$, which are called Schubert varieties. Standard monomial theory and Demazure’s character formula are among the highlights of this subject.

A systematic study of representations of direct limit algebraic groups and their geometric realizations was initiated by Dimitrov, Penkov and Wolf [DP1,DPW]. In particular, in [DP2] the authors show that the infinite-dimensional homogeneous spaces $G/B$ for classical simple ind-groups $G$ are realizable as spaces of generalized flags. Somewhat surprisingly, they show that the ind-variety $G/B$ is rarely projective.

In this work our main purpose is to study $B$-stable subvarieties of $G/B$ (where $G$ is a classical simple ind-group and $B$ is a Borel subalgebra of $G$) which we call ind-Schubert varieties (see Section 2.4 for a precise definition). We are interested in the following two natural questions:

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**Question 1.** When is an ind-Schubert variety projective?
**Question 2.** For a globally generated line bundle \( \mathcal{L} \) on a projective ind-Schubert variety \( X \), what is the algebraic structure of the \( B \)-module \( \Gamma(X, \mathcal{L}) \)?

Our first result is a necessary and sufficient condition for projectivity of an ind-Schubert variety. Our result shows that although there is a rich class of infinite-dimensional projective ind-Schubert varieties, they are essentially subvarieties of products of projective \( G/B \)’s.

As for the second question, from general facts about ind-varieties it follows that the \( B \)-module \( \Gamma(X, \mathcal{L}) \) is canonically isomorphic to the dual of a \( B \)-module \( V_X(\mathcal{L}) \) (see Section 2.5). Our second result relates the geometry of \( X \) to the algebraic structure of \( V_X(\mathcal{L}) \). As an application of our projectivity criterion, we prove that when the ind-Schubert variety \( X \) is projective, for any globally generated line bundle \( \mathcal{L} \) on \( X \) the weight spaces of \( V_X(\mathcal{L}) \) are finite-dimensional. It will be seen that there exists a weight \( \lambda \) such that \( V_X(\mathcal{L})_\mu \neq \{0\} \) only if \( \lambda - \mu \) is a linear combination of positive roots. We conclude the paper by using Littelmann’s results on standard monomial theory [Li] to prove a theorem which provides a method of describing weight multiplicities of \( V_X(\mathcal{L}) \) in terms of weight multiplicities of certain Demazure modules (see Theorem 2).

This manuscript is organized as follows. In Section 2, we explain some preliminary results about ind-groups and ind-Schubert varieties. In Section 3 we study projectivity of ind-Schubert varieties. Section 4 is devoted to proving our results on weight multiplicities.

### 2. Preliminaries

#### 2.1. Classical simple ind-groups

Let \( G \) be either \( GL(\infty) \) or a classical simple ind-group of type \( B(\infty) \), \( C(\infty) \) or \( D(\infty) \). (For our purposes, there is no difference between the groups \( GL(\infty) \) and \( A(\infty) \). It is more convenient to work with \( GL(\infty) \), and we will do so.) For the reader’s convenience and future reference, we would like to describe explicit realizations of these ind-groups in more detail. We have \( GL(\infty) = \lim_{n \to \infty} GL_n(\mathbb{C}) \) where the monomorphism \( f_n : GL_n(\mathbb{C}) \to GL_{n+1}(\mathbb{C}) \) is given by

\[
X \mapsto \begin{bmatrix} X & 0 \\ 0 & 1 \end{bmatrix}.
\]

The group \( SO_{2n}(\mathbb{C}) \) (respectively \( SO_{2n+1}(\mathbb{C}) \) and \( Sp_{2n}(\mathbb{C}) \)) is realized as the subgroup of \( GL_{2n}(\mathbb{C}) \) (respectively \( GL_{2n+1}(\mathbb{C}) \) and \( GL_{2n}(\mathbb{C}) \)) of elements of determinant one which preserve a nondegenerate bilinear form \( \langle u, v \rangle \) on \( \mathbb{C}^{2n} \) (respectively on \( \mathbb{C}^{2n+1} \) and \( \mathbb{C}^{2n} \)) where

\[
\langle u, v \rangle = \begin{cases} 
\sum_{i=1}^{n} u_{2i-1}v_{2i} + u_{2i}v_{2i-1} & \text{for } SO_{2n}(\mathbb{C}), \\
 u_1v_1 + \sum_{i=1}^{n} u_{2i+1}v_{2i} + u_{2i}v_{2i+1} & \text{for } SO_{2n+1}(\mathbb{C}), \\
 \sum_{i=1}^{n} u_{2i-1}v_{2i} - u_{2i}v_{2i-1} & \text{for } Sp_{2n}(\mathbb{C}).
\end{cases}
\]

The group \( B(\infty) \) (respectively \( C(\infty) \) and \( D(\infty) \)) is defined as \( \lim G_n \) where \( G_n = SO_{2n+1}(\mathbb{C}) \) (respectively \( G_n = Sp_{2n}(\mathbb{C}) \) and \( G_n = SO_{2n}(\mathbb{C}) \)) and the monomorphism \( G_n \to G_{n+1} \) is given by \( f_{k+1} \circ f_k \) for \( k = 2n + 1 \) (respectively \( k = 2n \) and \( k = 2n \)). Note that \( B(\infty) \) and \( D(\infty) \) are isomorphic groups, but the isomorphism does not identify the root systems.

#### 2.2. Root systems and special subgroups

From now on we assume \( G = \lim G_n \) and \( i_n : G_n \to G_{n+1} \) are as in the standard realizations given above. As a root reductive ind-group [DPW], \( G \) has a standard Cartan subgroup \( H = \lim H_n \), where \( H_n \) is the subgroup of diagonal matrices in \( G_n \). Associated to \( H \) there is a root system \( \Delta = \lim \Delta_n \) of \( G \), which is a subset of the group of rational multiplicative characters of the ind-group \( H \). We
will use the explicit description of $\Delta$ given in Table 1 (note that $-\varepsilon_i(t) = \varepsilon_i(t^{-1})$). Every root $\alpha \in \Delta$ determines a reflection $s_{\alpha}$ of $\Delta$ in a standard way.

A Borel subgroup of $G$ is an ind-subgroup $B$ such that for every $n$, $B_n = B \cap G_n$ is a Borel subgroup of $G_n$. It can be seen that there exists a $\sigma \in \text{Aut}(G)$ such that $H \subset \sigma(B)$. From now on we assume that $H \subset B$. In the following proposition, which follows immediately from [DP1, Proposition 3], a $\mathbb{Z}_2$-linear order on $\{\pm \varepsilon_1, \pm \varepsilon_2, \ldots\}$ makes a linear order such that $x > y$ implies $-y > -x$.

**Proposition 1.** There is a one-to-one correspondence between Borel subgroups $B$ of $G$ containing $H$ and certain linearly ordered sets $\Omega$ as follows.

a. $G = GL(\infty)$: linear orders on $[\varepsilon_1, \varepsilon_2, \ldots]$.  
b. $G = B(\infty)$ or $G = C(\infty)$: $\mathbb{Z}_2$-linear orders on $\{\pm \varepsilon_1, \pm \varepsilon_2, \ldots\} \cup \{0\}$.  
c. $G = D(\infty)$: $\mathbb{Z}_2$-linear orders on $\{\pm \varepsilon_1, \pm \varepsilon_2, \ldots\} \cup \{0\}$ such that a minimal positive element, if exists, belongs to $[\varepsilon_1, \varepsilon_2, \ldots]$.

Let $B$ be a Borel subgroup of $G$ corresponding to an ordered set $\Omega$. Let $\Omega^+ = \Omega$ when $G = GL(\infty)$ and $\Omega^+ = \{x \mid x \in \Omega$ and $x > 0\}$ otherwise. Let $\Omega_n = \Omega^+ \cap \{\pm \varepsilon_1, \ldots, \pm \varepsilon_n\}$. The ordered set $\Omega_n$ is obtained from $\Omega_n$ by inserting either $\varepsilon_{n+1}$ or $-\varepsilon_{n+1}$ between the $k$th and $(k+1)$th largest elements of $\Omega_n$, where $k \in \{0, 1, \ldots, n\}$. (Obviously, $k = 0$ (respectively $k = n$) means that $\varepsilon_{n+1}$ becomes the largest (respectively smallest) element of $\Omega_{n+1}$.) The integer $k$ will be called the insertion location of $\Omega_n$.

For any subset $S$ of $\Omega^+$, let $H(S) = \bigcap_{t \in S} \ker \varepsilon_t$ and for any rational character $\nu$ of $H$ written formally as $\nu = \sum_{r \in \mathbb{Z}_2^+} v_r \varepsilon_r$, let $\nu(S) = \nu|_{H(S)} = \sum_{r \in S} v_r \varepsilon_r$. Clearly when $S = \emptyset$ we have $H(S) = \{1\}$.

### 2.3. Weyl groups and Schubert varieties

Suppose we have

$$\Omega_n = \{c_{\sigma(1)}\varepsilon_{\sigma(1)} > \cdots > c_{\sigma(n)}\varepsilon_{\sigma(n)}\}$$

where for any $l$, $c_l \in \{\pm 1\}$. Let $W_n$ represent the Weyl group of $G_n$. When $G = GL(\infty)$, we think of $W_n$ as $S_n$, the group of permutations of $\{\varepsilon_1, \ldots, \varepsilon_n\}$. When $G = B(\infty)$ or $G = C(\infty)$, we think of $W_n$ as $S_n \times \mathbb{Z}_2^n$ (the group acting by permutations and sign changes on $\{\pm c_{\sigma(1)}\varepsilon_{\sigma(1)}, \ldots, \pm c_{\sigma(n)}\varepsilon_{\sigma(n)}\}$) and when $G = D(\infty)$, we think of $W_n$ as the subgroup of $S_n \times \mathbb{Z}_2^n$ acting on $\{\pm c_{\sigma(1)}\varepsilon_{\sigma(1)}, \ldots, \pm c_{\sigma(n)}\varepsilon_{\sigma(n)}\}$ by permutations and sign changes where the number of negative signs is even (for more details, see [Hu, §12]).

The homomorphism $i_n$ induces a closed immersion $i_n^\ast : G_n/B_n \to G_{n+1}/B_{n+1}$ and one can consider the ind-variety $G/B = \lim_{\longrightarrow} G_n/B_n$. For any $\omega \in W_n$, the Schubert variety $B_n\omega_0 B_n \subset G_n/B_n$ will be denoted by $X_\omega$. For any point $p \in G_n/B_n$, $i_n(p)$ lies inside a Schubert cell in $G_{n+1}/B_{n+1}$. Consequently, for any $\omega \in W_n$, there exists a unique $\omega' \in W_{n+1}$ such that $i_n(X_\omega) \subseteq X_{\omega'}$ and moreover if $\omega'' \in W_{n+1}$ and $i_n(X_\omega) \subseteq X_{\omega''}$ then $\omega' \preceq \omega''$. (Here as well as in the rest of this paper, $\preceq$ represents the Bruhat order on the Weyl group with respect to the chosen Borel subgroup.) In fact the map $\Omega_n \to \Omega_{n+1}$ gives rise to a monomorphism $W_n \to W_{n+1}$, and $\omega'$ is the image of $\omega$ under the latter map.
2.4. Ind-varieties and ind-Schubert varieties

A locally irreducible ind-variety $X$ is a topological space with a filtration $X_1 \subseteq X_2 \subseteq \cdots$ such that each $X_n$ is an (irreducible) variety and the maps $X_n \to X_{n+1}$ are closed immersions. (Note that the definition is not independent of the filtration.) The topology of $X$ is the usual direct limit topology and its sheaf of regular functions is $\mathcal{O}_X = \varprojlim \mathcal{O}_{X_n}$. Our main example of a locally irreducible ind-variety is $G/B = \varprojlim G_n/B_n$.

A locally irreducible closed subvariety of $X$ is a closed subset $Y$ of $X$ such that for every $n$, $Y \cap X_n$ is an (irreducible) variety. A line bundle on $X$ is a sheaf of $\mathcal{O}_X$-modules locally isomorphic to $\mathcal{O}_X$. A line bundle $L$ on $X$ is called globally generated if $\Gamma(L)$ is globally generated for every $n$. It is called very ample if for every $n$, $L|_{X_n}$ is very ample and the map $\Gamma(X_n, L|_{X_n}) \to \Gamma(X_n, L|_{X_n})$ is surjective. An ind-variety $X$ is called projective whenever it has a very ample line bundle.

For any character $\lambda$ of $B_n$, let $\mathcal{L}_\lambda \in \text{Pic}(G_n/B_n)$ be the sheaf of sections of

$$G_n \times B_n \subseteq \lambda = \{(g, t) \mid g \in G, t \in \mathbb{C}, \text{ and } \forall b \in B_n: (gb, t) = (g, \lambda(b^{-1})t)\}$$

considered as a (geometric) line bundle over $G_n/B_n$. Any $L \in \text{Pic}(G_n/B_n)$ is of the form $\mathcal{L}_\lambda$ for some $\lambda$. Since $G/B = \varprojlim G_n/B_n$, we have $\text{Pic}(G/B) = \varprojlim \text{Pic}(G_n/B_n)$ and it follows that any element of $\text{Pic}(G/B)$ is of the form $\mathcal{L}_\lambda$ for a character $\lambda$ of $B$.

It is well known [Br] that a line bundle $\mathcal{L}_\lambda \in \text{Pic}(G_n/B_n)$ is globally generated (respectively, very ample) if and only if $\lambda$ is dominant (respectively, regular dominant). By Borel–Weil theorem, $\Gamma(G_n/B_n, \mathcal{L}_\lambda) \neq \{0\}$ if and only if $\mathcal{L}_\lambda$ is globally generated, in which case $\Gamma(G_n/B_n, \mathcal{L}_\lambda)$ is isomorphic to an irreducible $G_n$-module with lowest weight $-\lambda$. Its vector space dual $\Gamma(G_n/B_n, \mathcal{L}_\lambda)^*$ is an irreducible $G_n$-module with highest weight $\lambda$. The highest weight vector is given by the evaluation functional at $eB_n$ where $e \in G_n$ is the identity element.

By an ind-Schubert variety in $G/B = \varprojlim G_n/B_n$ we mean a $B$-stable locally irreducible closed ind-subvariety $X$ of $G/B$. It follows that $X = \varinjlim X_n$ where each $X_n \subseteq G_n/B_n$ is a Schubert variety.

From now on we assume that $X = \varinjlim X_n$ is an ind-Schubert variety where $X_n = X_{\omega_n} = B_n/\omega_n B_n$ for some $\omega_n \in \mathcal{W}_n$. It follows from Section 2.3 that $\omega_n \preceq \omega_{n+1}$.

2.5. The module $V_X(L)$

For any $L \in \text{Pic}(G_n/B_n)$, the space $\Gamma(X, L|_{X_n})$ is a $B_n$-module. If $L \in \text{Pic}(X)$, then the natural restriction map $\Gamma(X_n, L|_{X_n}) \to \Gamma(X_n, L|_{X_{n-1}})$ is a $B_n$-module homomorphism. As in [DPW, Proposition 10.3], the Mittag–Leffler property of the sequence

$$\cdots \to \Gamma(X_n, L|_{X_n}) \to \Gamma(X_{n-1}, L|_{X_{n-1}}) \to \cdots \to \Gamma(X_1, L|_{X_1})$$

implies that $\Gamma(X, L) = \varprojlim \Gamma(X_n, L|_{X_n})$. Therefore $\Gamma(X, L)$ has a natural $B$-module structure.

**Lemma 1.** Let $L \in \text{Pic}(X)$. If $L|_{X_n}$ is globally generated for every $n$, then the restriction map $\Gamma(X_n, L|_{X_n}) \to \Gamma(X_{n-1}, L|_{X_{n-1}})$ is surjective. In particular, if $L|_{X_n}$ is very ample for every $n$, then $L$ is a very ample line bundle of $X$ as well.

**Proof.** For every $n$, there exists an $L'_n \in \text{Pic}(G_n/B_n)$ such that $L'_n$ is globally generated and $L'_n|_{X_n} = L|_{X_n}$ (see [Ma] or [Br]). In the commutative diagram

$$\begin{array}{ccc}
\Gamma(X_{n-1}, L|_{X_{n-1}}) & \longrightarrow & \Gamma(X_n, L|_{X_n})^* \\
\downarrow & & \downarrow \\
\Gamma(G_{n-1}/B_{n-1}, L'_n|_{X_{n-1}}) & \longrightarrow & \Gamma(G_n/B_n, L'_n)^*
\end{array}$$

implies $\Gamma(X_n, L|_{X_n})^* \cong \Gamma(G_n/B_n, L'_n)^*$, which is surjective.
the lower horizontal map is nonzero and by Borel–Weil theorem, it is one-to-one. By Demazure’s theorem, the vertical maps are one-to-one too. Therefore the upper horizontal map has to be one-to-one. □

Let \( \mathcal{L} \in \text{Pic}(X) \) be globally generated. From Lemma 1 it follows that \( \Gamma(X, \mathcal{L}) \) is naturally dual to the \( B \)-module \( V_X(\mathcal{L}) = \lim_{\rightarrow} \Gamma(X_n, \mathcal{L}|_{X_n})^* \).

**Example.** Any Schubert variety in \( G_n/B_n \) has an open cell isomorphic to an affine space. Unlike Schubert varieties, an ind-Schubert variety may not contain an “open affine cell.” Let \( \omega_{4n} \) be the permutation of \( \{1, \ldots, 4n\} \) such that for any \( k \in \{0, \ldots, n-1\} \) and \( s \in \{1, 2, 3, 4\} \) we have \( \omega_{4n}(4k + s) = 4k + \sigma(s) \) where \( \sigma(1) = 3, \sigma(2) = 4, \sigma(3) = 1 \) and \( \sigma(4) = 2 \). Consider \( X_{\omega_{4n}} \subset GL_{4n}(\mathbb{C})/B_{4n} \) where \( B_{4n} \) is a subvariety of \( X_{\omega_{4n}} \). However, it is impossible to find an open subset \( U \) of \( X \) such that \( U \cap X_{\omega_{4n}} \) is an affine space for every \( k \). In fact by the criterion of Lakshmibai and Seshadr [LS] it follows that \( X_{\omega_{4n}} \) is inside the singular locus of \( X_{\omega_{4n+1}} \). Since \( U \cap X_{\omega_{4n}} \) is an affine space, it is smooth and therefore it does not intersect this singular locus; hence \( U \cap X_{\omega_{4n}} = \emptyset \), a contradiction.

### 3. Line bundles and projectivity of \( X \)

#### 3.1. A characterization of line bundles on \( X \)

The map \( i_n : X_n \to X_{n+1} \) induces a group homomorphism \( i_n^* : \text{Pic}(X_{n+1}) \to \text{Pic}(X_n) \) between the Picard groups of these varieties in a standard way, and we have \( \text{Pic}(X) = \lim_{\rightarrow} \text{Pic}(X_n) \).

**Proposition 2.** Any line bundle on \( X \) is the restriction of a line bundle on \( G/B \).

**Proof.** By the definition of inverse limit, the maps \( \text{Pic}(G/B) \to \text{Pic}(G_n/B_n) \to \text{Pic}(X_n) \) yield a canonical map \( \text{Pic}(G/B) \to \text{Pic}(X) \) and we need to show that it is surjective.

For simplicity we assume \( G = GL(\infty) \). The proof for other cases is quite similar. Let \( \Omega_n = \{ \varepsilon_{\sigma(1)} > \cdots > \varepsilon_{\sigma(n)} \} \). It is well known that \( \text{Pic}(G_n/B_n) = \mathbb{Z}/(1, \ldots, 1) \) where \( (1, \ldots, 1) \) is the coset of any element \( (a_1, \ldots, a_n) \in \mathbb{Z}^n \) corresponds to the line bundle on \( G_n/B_n \) associated to the character \( a_1 \varepsilon_{\sigma(1)} + \cdots + a_n \varepsilon_{\sigma(n)} \) of \( B_n \). If the insertion location of \( \Omega_n \) is \( k \), then the map

\[
i_n^* : \text{Pic}(G_{n+1}/B_{n+1}) \to \text{Pic}(G_n/B_n)
\]

is given by \( i_n^*(a_1, \ldots, a_{n+1}) = (a_1, \ldots, a_k, \widetilde{a}_{k+1}, \ldots, a_{n+1}) \). Let \( j_n : X_n \to G_n/B_n \) be the usual embedding. It is known (see [Ma] or [Br]) that the natural map

\[
j_n^* : \text{Pic}(G_n/B_n) \to \text{Pic}(X_n)
\]

is surjective and

\[
\ker j_n^* = \{(a_1, \ldots, a_{n+1}) : a_i = a_{i+1} \text{ whenever } s_{\varepsilon_{\sigma(i)} - \varepsilon_{\sigma(i+1)}} \leq \omega_n \}.
\]

For any positive integers \( p > q \), let \( i_{p-q}^* \) denote the map

\[
i_{p-q}^* : \text{Pic}(G_{p}/B_p) \to \text{Pic}(G_{q}/B_q)
\]

and let \( K_{p-q} = i_{p-q}^*(\ker j_{q}^*) \). Obviously \( K_{q+1-q} \supseteq K_{q+2-q} \supseteq K_{q+3-q} \supseteq \cdots \). Since \( K_{q+1} \) is given by equalities of coordinates, the same is true for any \( K_{p-q} \). This implies that the set \( \{K_{q+j-q} : j \in \mathbb{Z}^+\} \) is finite. Set \( K_{\infty} = \bigcap_{j \in \mathbb{Z}^+} K_{q+j-q} \).
To complete the proof it suffices to show that for any infinite sequence
\[ \cdots \mapsto m_n \mapsto m_{n-1} \mapsto \cdots \mapsto m_2 \mapsto m_1 \]
(where \( m_n \in \text{Pic}(X_n) \)) there exists a sequence
\[ \cdots \mapsto m'_n \mapsto m'_{n-1} \mapsto \cdots \mapsto m'_2 \mapsto m'_1 \]
where \( m'_n \in \text{Pic}(G_n/B_n) \) and \( j^*_n(m'_n) = m_n \). Fix \( n_o \in \mathbb{Z}^+ \), and for any \( n > n_o \) choose an element \( m''_n \in (j^*_n)^{-1}(m_n) \). The set of elements of \( \text{Pic}(G_n/B_n) \) which map to \( m_n \) is equal to the coset \( m''_n + \ker j^*_n \). For \( n \gg n_o \) we have \( i^*_n(m''_n + \ker j^*_n) = i^*_n(m''_n) + K_{\infty \mapsto n_o} \). Any two cosets of \( K_{\infty \mapsto n_o} \) are either identical or disjoint. Surjectivity of \( j^*_n+1 \) implies that the cosets \( i^*_n(m''_n) + K_{\infty \mapsto n_o} \) and \( i_{n+1 \mapsto n_o}^*(m''_n) + K_{\infty \mapsto n_o} \) are not disjoint. Therefore there exists an element \( \bar{m}_{n_o} \in \text{Pic}(G_{n_o}/B_{n_o}) \) such that for every \( n \gg n_o \),
\[ i^*_n(m''_n) + K_{\infty \mapsto n_o} = \bar{m}_{n_o} + K_{\infty \mapsto n_o}. \]

It is easily checked that \( j^*_{n_o}(\bar{m}_{n_o} + K_{\infty \mapsto n_o}) = m_{n_o} \). Moreover,
\[ i_{n_o+1 \mapsto n_o}^*(\bar{m}_{n_o+1} + K_{\infty \mapsto n_o+1}) = \bar{m}_{n_o} + K_{\infty \mapsto n_o}. \]

It is now clear how to obtain the required sequence of \( m'_n \)'s. \( \square \)

3.2. A combinatorial criterion for projectivity of \( X \)

The main purpose of this section is to give a combinatorial criterion for the projectivity of any ind-Schubert variety \( X \). This criterion will be used in the proof of Theorem 1 below. To state our criterion, we need to introduce a combinatorial invariant of \( X \), which we call the connectivity diagram, that is essentially an equivalence relation \( \sim \) on \( \Omega^+ \). To make our discussion more readable, we first define the connectivity diagram for an ind-Schubert variety of \( \text{GL}(\infty)/B \), state its main properties, and then show how its definition can be generalized to the remaining cases.

Suppose \( G = \text{GL}(\infty) \). Fix \( n \in \mathbb{Z}^+ \) and let \( \Omega_n = \{ \varepsilon_{\sigma(1)} > \cdots > \varepsilon_{\sigma(n)} \} \). We can define the following connectivity diagram for \( \Omega_n \). Imagine that the elements of \( \Omega_n \) are nodes, and connect any two consecutive nodes \( \varepsilon_{\sigma(r)} > \varepsilon_{\sigma(r+1)} \) by an edge if and only if \( s_{\sigma} \leq \omega_n \) where \( s_{\sigma} \) is the reflection corresponding to the root \( \alpha = \varepsilon_{\sigma(r)} - \varepsilon_{\sigma(r+1)} \). This graph is called the connectivity diagram of \( \Omega_n \).

The purpose of the connectivity diagram is to store information about the kernel of the map \( j^*_n : \text{Pic}(G_n/B_n) \rightarrow \text{Pic}(X_n) \) given in (3.1.1). In fact \( \varepsilon_{\sigma(n)} \) and \( \varepsilon_{\sigma(n+1)} \) are connected by an edge if and only if for any \( a = (a_1, \ldots, a_n) \in \ker j^*_n \), we have \( a_m = a_{m+1} \).

Using the commutative diagram
\[ \begin{array}{ccc}
\text{Pic}(G_{n+1}/B_{n+1}) & \xrightarrow{i^*_n} & \text{Pic}(G_n/B_n) \\
\downarrow{j^*_{n+1}} & & \downarrow{j^*_n} \\
\text{Pic}(X_{n+1}) & \xrightarrow{i^*_n} & \text{Pic}(X_n)
\end{array} \]

it can be seen that the connectivity diagrams of \( \Omega_n \) and \( \Omega_{n+1} \) are compatible. In other words, if \( \varepsilon_i \) and \( \varepsilon_j \) belong to the same connected component for the connectivity diagram of \( \Omega_n \), then they also belong to the same connected component of the connectivity diagram of \( \Omega_m \) for any \( m > n \). Therefore we have a well-defined connectivity diagram for \( \Omega^+ \). It is merely the equivalence relation \( \sim \) on \( \Omega^+ \) where \( \varepsilon_p \sim \varepsilon_q \) if and only if there exists an \( n \) such that \( \varepsilon_p \) and \( \varepsilon_q \) are in the same connected component of the connectivity diagram of \( \Omega_n \).
When $G$ is $B(\infty)$, $C(\infty)$ or $D(\infty)$, we use the same idea: Suppose

$$\Omega_n = \{c_{\sigma(1)}e_{\sigma(1)} > \cdots > c_{\sigma(n)} e_{\sigma(n)}\}$$

where the $c_{r}$’s are in $\{\pm 1\}$. We connect two consecutive nodes $c_{\sigma(m)}e_{\sigma(m)} > c_{\sigma(m+1)}e_{\sigma(m+1)}$ by an edge whenever $s_{\alpha} \preceq \omega_{n}$, where $\alpha \in \Delta_{n}$ is a simple positive root of the form $\alpha = c_{\sigma(m)}e_{\sigma(m)} + c_{\sigma(m+1)}e_{\sigma(m+1)}$. (Note that $\alpha = c_{\sigma(m)}e_{\sigma(m)} + c_{\sigma(m+1)}e_{\sigma(m+1)}$ happens only when $G = D(\infty)$.)

**Definition 1.** Let $n \in \mathbb{Z}^{+}$. When $G = B(\infty)$ (respectively $C(\infty)$ and $D(\infty)$), we call the positive simple root $\alpha \in \Delta_{n}$ special if $\alpha \in \sigma(c_{\sigma(n)} e_{\sigma(n)}$ (respectively $\alpha = 2c_{\sigma(n)} e_{\sigma(n)}$ and $c_{\sigma(n-1)}e_{\sigma(n-1)} + c_{\sigma(n)}e_{\sigma(n)}$).

We call a connected component $C$ of the connectivity diagram of $\Omega_{n}$ special if for the special simple root $\alpha \in \Delta_{n}$, we have $s_{\alpha} \preceq \omega_{n}$ and moreover $C$ contains $c_{\sigma(n)}e_{\sigma(n)}$.

Obviously, the connectivity diagram of $\Omega_{n}$ has at most one special connected component. Again one can see that connected components (and special connected components, if they exist) of $\Omega_{n}$’s are compatible. Therefore we obtain an equivalence relation $\sim$ on $\Omega^{+}$ (and there may be at most one special equivalence class).

**Proposition 3.** $X$ is projective if and only if every equivalence class of $\sim$ is isomorphic (as an ordered set) to a subset of $\mathbb{Z}$, and the special equivalence class, if exists, is isomorphic to a subset of $\mathbb{Z}^{+}$.

**Proof.** We give the details of the proof for the case $G = GL(\infty)$. For other cases the proof is quite similar and is left to the reader. By Proposition 2, we can start with an arbitrary line bundle $L = L_{1}$ (associated to a rational character $\lambda = \sum_{m \in \mathbb{Z}^{+}} \lambda_{m}e_{m}$ of $B$) on $G/B$ to see when $L_{|X}$ is very ample. (Note that the $\lambda_{m}$’s are integers.) Suppose $L_{n} = L_{|X_{n}}$ is very ample. It is known (see [Ma] or [Br]) that any very ample line bundle on $X_{n}$ is the restriction of a very ample line bundle on $G_{n}/B_{n}$. In other words, there exists an $L_{n}' \in \text{Pic}(G_{n}/B_{n})$ corresponding to $\{a_{1}, \ldots, a_{n}\} \in \mathbb{Z}^{n}/\{(1, \ldots, 1)\}$ such that $a_{1} > \cdots > a_{n}$ and $j_{n}^{*}(L_{n}') = L_{n}$. From the description of ker $j_{n}$ given in (3.1.2) we can conclude the following statement: for any line bundle $L_{n}' \in \text{Pic}(G_{n}/B_{n})$, corresponding to $\{b_{1}, \ldots, b_{n}\} \in \mathbb{Z}^{n}/\{(1, \ldots, 1)\}$, if $j_{n}^{*}(L_{n}') = L_{n}$ then the $b_{r}$’s corresponding to any connected component of the connectivity diagram of $\Omega_{n}$ form a strictly decreasing sequence. Since this statement holds for every $n$, it follows that the $\lambda_{m}$’s corresponding to every equivalence class of $\sim$ in $\Omega^{+}$ form a strictly decreasing sequence. It follows that the equivalence class is isomorphic to a subset of $\mathbb{Z}$.

Conversely, if every equivalence class of $\sim$ in $\Omega^{+}$ is isomorphic to a subset of $\mathbb{Z}$, then one can find integers $\lambda_{m}$ such that the $\lambda_{m}$’s corresponding to the $e_{m}$’s which belong to any equivalence class of $\sim$ form a strictly decreasing sequence. The line bundle on $G/B$ associated to $\sum_{m=1}^{\infty} \lambda_{m}e_{m}$ restricts to a very ample line bundle on $X$. \(\square\)

4. Main results

4.1. Finiteness of $H$-multiplicities of $V_{X}(\mathcal{L})$

Before we state and prove our theorems, we need to introduce some notation. Let $I$ be an index set and $\{C_{i}\}_{i \in I}$ be the family of equivalence classes of the connectivity diagrams of $\Omega^{+}$. Fix $n \in \mathbb{Z}^{+}$ and set $C_{n} = \Omega_{n} \cap C_{i}$. The partition of $\Omega_{n}$ by $C_{n,i}$’s corresponds to a unique standard parabolic subgroup $P_{n}$ of $G_{n}$ as follows: Set

$$\Sigma_{n,i} = \{\alpha \in \Delta_{n} \mid \alpha \text{ is a simple positive root, } \alpha \in \text{Span}_{\mathbb{R}}(C_{n,i}) \text{ and } s_{\alpha} \preceq \omega_{n}\}$$

and suppose $(\Sigma_{n,i})$ denotes the additive semigroup generated by $\Sigma_{n,i}$; Suppose $U_{n,i}$ is the unipotent subgroup of $G_{n}$ which corresponds to $(\Sigma_{n,i}) \cap \Delta_{n}$ in the standard way, and set $B_{n,i} = H_{n,i} \ltimes U_{n,i}$ where $H_{n,i} = H(C_{n,i})$. The group generated by $B_{n,i}$ and the reflections $s_{\alpha}$ for $\alpha \in \Sigma_{n,i}$ is a reductive subgroup of $G_{n}$ which we will denote by $G_{n,i}$. The parabolic $P_{n}$ is the group generated by $B_{n}$ and
the reflections $s_\alpha$ for $\alpha \in \bigcup_{i \in I} \Sigma_{n,i}$. The standard Levi subgroup of $P_n$ is isomorphic to the direct product $\prod_{i \in I} G_{n,i}$. We think of $G_{n,i}/B_{n,i}$ as well as $\prod_{i \in I} G_{n,i}/B_{n,i}$ as a Schubert variety embedded inside $G_n/B_n$ in the obvious way. Note that for all but finitely many $i \in I$, $C_{n,i} = \emptyset$ and consequently $G_{n,i}/B_{n,i}$ is just a point. Let $W_{n,i}$ denote the Weyl group of $G_{n,i}$, i.e., the group generated by $s_\alpha$ where $\alpha \in \Sigma_{n,i}$.

The compatibility of connectivity diagrams for $\Omega_n$'s implies that $i_0(P_n) \subset P_{n+1}$, and therefore $P = \lim_{n \to \infty} P_n$ is a standard parabolic subgroup of $G$. In fact $P$ is the unique standard parabolic subgroup of $G$ corresponding to the partition of $\Omega^+$ by $C_i$'s. The standard Levi factor of $P$ is isomorphic to $\prod_{i \in I} G_i$ where for every $i \in I$, $G^i = \lim_{n \to \infty} G_{n,i}$. Set $B^i = B \cap G^i$ and $H^i = H \cap G^i$.

Given any globally generated line bundle $L$ on $X$, by Proposition 2 we can assume $L = L_{C_i|X}$ where $L_{C_i}$ is the line bundle on $G/B$ associated to the character $\lambda = \sum_{m \in \mathbb{Z}_+} \lambda_m e_m$ of $H$. Note that for any $n$ and $i$ the weight $\lambda(C_{n,i})$ is dominant. Set $\lambda_{n,i} = \lambda(C_{n,i})$ and let $V(\lambda_{n,i})$ be the irreducible $B_{n,i}$-highest weight module for $G_{n,i}$ with highest weight $\lambda_{n,i}$. (If $C_{n,i} = \emptyset$ we take $V(\lambda_{n,i}) = \mathbb{C}$.) For any $i \in I$, let $V(\lambda^i)$ be the irreducible $B^i$-highest weight module of $G^i$ with highest weight $\lambda^i = \lambda(C_i)$.

The tensor products which appear below are all restricted in the following sense: Fix an ordering $I = \{i_1, i_2, \ldots \}$ of $I$. Suppose $\{M_i\}_{i \in I}$ is a family of complex vector spaces, each of which possessing a distinguished nonzero vector $m_i \in M_i$. Then $\bigotimes_{i \in I} M_i$ is defined as $\lim_{n \to \infty} \bigotimes_{i \in I} M_{n,i}$, where the map $\bigotimes_{i \in \{i_1, \ldots, i_k\}} M_i \to \bigotimes_{i \in \{i_1, \ldots, i_{k+1}\}} M_i$ is given by $w \mapsto w \otimes m_{i_{k+1}}$. In the modules which will appear below, the distinguished vectors are determined (up to a scalar) as follows: either $M_i = \mathbb{C}$, where we can take any nonzero vector, or $M_i$ is a weight module (for an appropriate subgroup of $H$) with a unique (up to scaling) highest weight vector. Formally, $\bigotimes_{i \in I} M_i$ is spanned by $\bigotimes_{i \in I} v_i$ where for all but finitely many $i$, $v_i = m_i$.

**Theorem 1.** Let $X = \lim X_n$ be a projective ind-Schubert variety in $G/B$. If $L = L_{C_i|X}$ is a globally generated line bundle on $X$, then the $H$-weight spaces of $V_X(L)$ are finite-dimensional.

**Proof.** For simplicity, assume $G = \text{GL}(\infty)$. The argument is easily adapted to other cases. Since $L$ is globally generated, it follows that for any integers $m, m' \in \{1, \ldots, n\}$, if $e_m \sim e_{m'}$ and in $\Omega_n$ we have $e_m \geq e_{m'}$. With a slight abuse of notation, let $L_n$ denote both $L_{|X_n}$ and $L_{\prod_{i \in I} G_{n,i}/B_{n,i}}$.

Recall that $X_n = X_{\omega_n}$ where $\omega_n \in W_n$. It is easily seen that $\omega_n$ is expressible as $\omega_n = \prod_{i \in I} \omega_{n,i}$ where $\omega_{n,i} \in W_{n,i}$. It follows that $X_n$ is a subvariety of $\prod_{i \in I} G_{n,i}/B_{n,i}$ of the form $\prod_{i \in I} X_{\omega_{n,i}}$ where each $X_{\omega_{n,i}}$ is a Schubert variety inside $G_{n,i}/B_{n,i}$ corresponding to $\omega_{n,i}$. From Demazure’s theorem it follows that the natural map

$$\Gamma\left(\prod_{i \in I} G_{n,i}/B_{n,i}, L_n\right) \to \Gamma(X_n, L_n)$$

is surjective, and therefore its dual map is a monomorphism of $B_n$-modules. But Borel–Weil theorem implies that as $\prod_{i \in I} G_{n,i}$-modules

$$\Gamma\left(\prod_{i \in I} G_{n,i}/B_{n,i}, L_n\right)^* \cong \bigotimes_{i \in I} V(\lambda_{n,i}). \quad (4.1.1)$$

We also have a commutative diagram

$$\begin{array}{ccc}
\Gamma(X_n, L_n)^* & \to & \Gamma(\prod_{i \in I} G_{n,i}/B_{n,i}, L_n)^* \\
\downarrow & & \downarrow \\
\Gamma(X_{n+1}, L_{n+1})^* & \to & \Gamma(\prod_{i \in I} G_{n+1,i}/B_{n+1,i}, L_{n+1})^*
\end{array}$$
A natural question is to find the dimension of the weight spaces \( V_X \). From \([DP1, \text{Theorem 5}]\) it follows that there exists a sequence \( S \) in which all of the maps are one-to-one. Therefore the induced map

\[
\lim \Gamma(X_n, \mathcal{L}_n)^* \rightarrow \lim \Gamma \left( \prod_{i \in I} G_{n,i}/B_{n,i}, \mathcal{L}_n \right)^*
\]  

is one-to-one.

Choose any \( H_n \)-weight vector \( v_\mu \) in \( \Gamma(\prod_{i \in I} G_{n,i}/B_{n,i}, \mathcal{L}_n)^* \) of weight

\[
\mu = \sum_{r=1}^n \mu_r \epsilon_r.
\]

Since the \( \prod_{i \in I} B_{n,i} \)-highest weight vector of \( \Gamma(\prod_{i \in I} G_{n,i}/B_{n,i}, \mathcal{L}_n)^* \) is the functional which evaluates global sections at the base point \( \prod_{i \in I} e_n B_{n,i} \) (where \( e_n \) is the identity element of \( G_{n,i} \)), a simple argument shows that the image of \( v_\mu \) under the map

\[
\Gamma \left( \prod_{i \in I} G_{n,i}/B_{n,i}, \mathcal{L}_n \right)^* \rightarrow \Gamma \left( \prod_{i \in I} G_{n+1,i}/B_{n+1,i}, \mathcal{L}_{n+1} \right)^*
\]

has \( H_{n+1} \)-weight \( \mu + \lambda_{n+1} \epsilon_{n+1} \). From this statement, \((4.1.1)\), and injectivity of \((4.1.2)\) it follows that \( \lim \Gamma(\prod_{i \in I} X_{n,i}, \mathcal{L}_n)^* \) is a \( B \)-submodule of \( \bigotimes_{i \in I} V(\lambda_i) \). Since \( X \) is projective, Proposition 3 implies that for every \( i \in I \), the Borel subgroup \( B^i \) has a basis (for the definition of a basis, see \([DP1, \text{Section 4}]\)). From \([DP1, \text{Theorem 5}]\) it follows that \( H^i \)-weight spaces of \( V(\lambda_i) \) are finite-dimensional. Therefore the \( H \)-weight spaces of \( V_X(\mathcal{L}) \) are finite-dimensional as well. \( \square \)

**Remark.** From the proof of Theorem 1 it follows easily that a weight space \( V_X(\mathcal{L})_\mu \) is nonzero only if \( \lambda - \mu \) is a sum of positive roots in \( \Delta \).

### 4.2. Demazure modules and \( H \)-multiplicities

We start with some notation. Let \( S \) be a finite subset of \( C^i \). Choose \( n \) such that \( S \subset \Omega_n \), \( S \) lies in a connected component of the connectivity diagram of \( \Omega_n \), and if \( C^i \) is special then the connected component of \( \Omega_n \) containing \( S \) is special as well. Let \( U(S) \) be the unipotent subgroup of \( G_n \), which corresponds to \( \langle \sum_n, \Delta_n \cap \text{Span}_\mathbb{Z}(S) \rangle \) in the standard way, and set \( B(\alpha) = H(S) \ltimes U(S) \). Let \( W(S) \) be the group generated by all reflections \( s_\alpha \) where \( \alpha \) is a simple positive root of \( \Delta_n \) such that \( \alpha \in \langle \sum_n, \Delta_n \cap \text{Span}_\mathbb{Z}(S) \rangle \), and let \( G(S) \) be the subgroup of \( G_n \) generated by \( B(\alpha) \) and \( W(S) \). It is easy to see that if \( n \) is sufficiently large then the notation is independent of \( n \).

Let \( \mathcal{L} = \mathcal{L}_{\lambda, |X|} \) be globally generated. Suppose \( E \) is the infinite-dimensional lattice \( \text{Span}_\mathbb{Z}(\{\epsilon_r\}_{r=1}^\infty) \). A natural question is to find the dimension of the weight spaces \( V_X(\mathcal{L})_\mu \) for any \( \mu \in \lambda + E \). (As mentioned in the above remark, the weights \( \mu \) for which \( V_X(\mathcal{L})_\mu \neq \{0\} \) lie in a cone in \( \lambda + E \) that is based at \( \lambda \).) It is reasonable to expect a connection between the dimensions of \( V_X(\mathcal{L})_\mu \), when \( \mu \) varies within finite-dimensional sublattices \( \lambda + E \) of \( \lambda + E \), and characters of finite-dimensional Demazure modules. The main goal of this section is to prove Theorem 2, which establishes such a connection.

**Theorem 2.** Let \( X = \lim X_n \) be projective and \( \mathcal{L} = \mathcal{L}_{\lambda, |X|} \) be a globally generated line bundle on \( X \). Then there exists a sequence \( S_1 \subset S_2 \subset \cdots \) of finite subsets of \( \Omega^+ \) such that \( \Omega^+ = \bigcup_{m \in \mathbb{Z}^+} S_m \) and

a. If \( V(m) \) represents the \( \prod_{i \in I} G(S_m \cap C^i) \)-highest weight module

\[
\bigotimes_{i \in I} V(\lambda(S_m \cap C^i))
\]
then $V_X(L)$ is isomorphic to a $B$-submodule of $\varinjlim V(m)$. Here the map $V(m) \to V(m+1)$ is $\bigotimes_{i \in I} \phi_{m,i}$, where

$$\phi_{m,i} : V(\lambda(S_m \cap C_i)) \to V(\lambda(S_{m+1} \cap C_i))$$

identifies a $B(S_m \cap C_i)$-highest weight vector inside $V(\lambda(S_m \cap C_i))$ with a $B(S_{m+1} \cap C_i)$-highest weight vector of $V(\lambda(S_{m+1} \cap C_i))$.

b. If we identify $V_X(L)$ with its image in $\varinjlim V(m)$, then for any $m \in \mathbb{Z}^+$ there exist elements $\sigma_{m,i} \in \mathcal{W}(S_m \cap C_i)$ such that

$$V_X(L) \cap V(m) = \bigotimes_{i \in I} V_{\sigma_{m,i}}(\lambda(S_m \cap C_i)).$$

Here $V_{\sigma_{m,i}}(\lambda(S_m \cap C_i))$ denotes the $B(S_m \cap C_i)$-module generated by the extremal weight vector of $V(\lambda(S_m \cap C_i))$ with weight $\sigma_{m,i} \cdot \lambda(S_m \cap C_i)$.

**Remark.** From the proof of Theorem 2 it follows that

$$V_X(L) \cap V(m) = \bigoplus_{\mu \in \lambda + \mathcal{E}_m} V_X(L)_\mu$$

where $\mathcal{E}_m$ is the finite-dimensional sublattice of $\mathcal{E}$ generated by $\{\varepsilon_r \mid r \in S_m\}$.

**Proof.** For simplicity we give the proof for $G = \text{GL}(\infty)$ and leave the necessary modifications for other cases to the reader. From the proof of Theorem 1 it follows that when $X = \varinjlim X_n$ is projective, we have $X_n = \prod_{i \in I} X_{n,i}$ where each $X_{n,i}$ is a Schubert variety inside $G_{n,i}/B_{n,i}$. Therefore as $\prod_{i \in I} B_{n,i}$-modules

$$\Gamma(X_n, L_n)^* \cong \bigotimes_{i \in I} V_{\omega_{n,i}}(\lambda_{n,i})$$

where $V_{\omega_{n,i}}(\lambda_{n,i})$ denotes the $B_{n,i}$-submodule of $V(\lambda_{n,i})$ generated by the extremal weight vector with weight $\omega_{n,i} \cdot \lambda_{n,i}$. Since $X$ is projective, we have $\Omega = \bigcup_{i \in I} C_i$ where for any $i \in I$ there exist $a_i, b_i$ satisfying $a_i < b_i$, $a_i \in \mathbb{Z} \cup \{-\infty\}$, and $b_i \in \mathbb{Z} \cup \{+\infty\}$ such that $C_i = \{\varepsilon_{r_m,i} \mid a_i < m < b_i\}$ and for any $a_i < m < m' < b_i$ we have $\varepsilon_{r_{m'},i} > \varepsilon_{r_m,i}$.

For any $p \in \mathbb{Z}^+$, set

$$C_{p,i} = \{\varepsilon_r \mid \varepsilon_r \in C_i \text{ and } r = r_{m,i} \text{ where } \max(-p, a_i) < m < \min(b_i, p)\}$$

and set $G_{p,i} = G(C_{p,i})$, $\lambda_{p,i} = \lambda(C_{p,i})$. Let $V(\lambda_{p,i})$ be the $B \cap G_{p,i}$-highest weight module of $G_{p,i}$ with highest weight $\lambda_{p,i}$.

Fix $q \in \mathbb{Z}^+$, choose any $i_1, \ldots, i_q \in I$, and let $p \in \mathbb{Z}^+$ be large enough such that for any $j \in \{1, \ldots, q\}$, $C_{p,i_j} \neq \emptyset$. Set $W = \bigotimes_{i=1}^q V(\lambda_{p,i_j})$. Choose $n_\varepsilon \in \mathbb{Z}^+$ large enough such that $\Omega_{n_\varepsilon} \supset C_{p,i_1} \cup \cdots \cup C_{p,i_q}$ and for any $i \in \{i_1, \ldots, i_q\}$, elements of $C_{p,i}$ lie in a connected component of the connectivity diagram of $\Omega_{n_\varepsilon}$.

By mapping the highest weight vector of $W$ (as a $G_{p,i_1} \times \cdots \times G_{p,i_q}$-module) to the highest weight vector of $\bigotimes_{i \in I} V(\lambda_{n_\varepsilon,i})$, $W$ can be identified with a $G_{p,i_1} \times \cdots \times G_{p,i_q}$-submodule of $\bigotimes_{i \in I} V(\lambda_{n_\varepsilon,i})$. (The image of $W$ in $\bigotimes_{i \in I} V(\lambda_{n_\varepsilon,i})$ is spanned by a set of weight vectors of weights of the form $\mu = \sum_{r \in \Omega_{n_\varepsilon}} \mu_r \varepsilon_r$ where $\mu_r = \lambda_r$ whenever $\varepsilon_r \notin C_{p,i_1} \cup \cdots \cup C_{p,i_q}$.) At the same time, from (4.2.1) it
follows that $\Gamma(X_{n_i}, L_{n_i})^*$ can be considered as a subspace of $\bigotimes_{i \in I} V(\lambda_{n_i}, i)$ in the obvious way. It is not difficult to see that $\Gamma(X_{n_i}, L_{n_i})^* \cap W = \bigotimes_{i \in I} W_i$ where

$$W_i = \begin{cases} V(\lambda, p, i) \cap V(\omega_{n_i}, \lambda_{n_i}, i) & \text{if } i \in \{i_1, \ldots, i_q\}, \\ V_{e}(\lambda_{n_i}, i) & \text{otherwise} \end{cases}$$

(here $e$ represents the trivial element of the corresponding Weyl group). Next we prove that for any $1 \leq m \leq q$, there exists an element $\sigma_{n_i, p, i_m}$ in the Weyl group $W(C^{p, i_m})$ of $G^{p, i_m}$ such that

$$V(\lambda, p, i_m) \cap V(\omega_{n_i}, \lambda_{n_i, i_m}) = V(\sigma_{n_i, p, i_m}, \lambda_{p, i_m}). \quad (4.2.2)$$

Before we prove (4.2.2), we show how the proof of Theorem 2 can be completed. The sequence $\sigma_{n_i, p, i_m}$ satisfies $\sigma_{n_i, p, i_m} \leq \sigma_{n_i, +1, p, i_m} \leq \cdots$ and therefore for $n_i$ sufficiently large it will be constant. Let us denote the stable value of the sequence by $\sigma_{p, i_m}$. It follows that $V_X(L) \cap W = \bigotimes_{i \in I} V(\sigma_{p, i_m}, \lambda(p, i))$.

To complete the proof of Theorem 2, we assume $I = \{i_1, i_2, \ldots\}$ and for any $q \in \mathbb{Z}^+$ we take $S_q = C_{N_q, i_1} \cup \cdots \cup C_{N_q, i_q}$ where $N_q$ is large enough such that for any $1 \leq m \leq q$, $C_{N_q, i_m} \neq \emptyset$.

The proof of (4.2.2) follows from Propositions 4 and 5 below, both of which are of independent interest. □

**Proposition 4.** Let $N$ be a positive integer and $X^1, X^2 \subseteq G_N/B_N$ be arbitrary Schubert varieties such that $X^1 \cap X^2$ is irreducible. Suppose $\tilde{L}$ is a globally generated line bundle on $G_N/B_N$. Then the intersections of the images of the canonical maps $\Gamma(X^1, \tilde{L})^* \to \Gamma(G_N/B_N, \tilde{L})^*$ and $\Gamma(X^2, \tilde{L})^* \to \Gamma(G_N/B_N, \tilde{L})^*$ is equal to the image of the canonical map $\Gamma(X^1 \cap X^2, \tilde{L})^* \to \Gamma(G_N/B_N, \tilde{L})^*$.

**Proof.** Since we are not aware of a reference, we would like to sketch a proof. Let $\eta_1, \eta_2 \in \mathcal{W}_N$ be the elements corresponding to $X^1, X^2$. The statement of the proposition is equivalent to showing that in the commutative diagram

$$\begin{CD}
\Gamma(G_N/B_N, \tilde{L}) @>\phi_1>> \Gamma(X^1, \tilde{L}) \\
@V\phi_2 VV @V\phi_3 VV \\
\Gamma(X^2, \tilde{L}) @>\phi_4>> \Gamma(X^1 \cap X^2, \tilde{L})
\end{CD}$$

we have $\ker \phi_4 \subseteq \phi_2(\ker \phi_1)$. From now on we use the notation of [Li] without defining it. By [Li, §5, Corollary 2], a basis for the kernel of $\phi_4$ is given by $p_\pi$’s such that $i(\pi) \leq \eta_2$ but $i(\pi) \neq \eta_1$, and [Li, §5, Corollary 2] immediately implies $\phi_4(p_\pi) = 0$. □

**Proposition 5.** Let $n \in \mathbb{Z}^+$, $X_{\omega_n} \subseteq G_n/B_n$ be a Schubert variety, and $Q_n$ be an arbitrary parabolic subgroup of $G_n$ containing $B_n$. Then $X_{\omega_n} \cap Q_n/B_n$ is an irreducible Schubert variety.

**Proof.** It is well known that there exists a one-parameter subgroup $\varphi : \mathbb{C}^\times \to H_n$ such that $Q_n = \{x \in G_n \mid \lim_{t \to 0} \varphi(t)x\varphi(t^{-1}) \text{ exists}\}$. Let $\bar{B}_n$ be the opposite Borel and $Y = \bar{B}_n/B_n$ be the open cell. Then $Y$ is $H_n$-equivariantly isomorphic to the tangent space $T_x(G_n/B_n) = \text{Lie}(G_n)/\text{Lie}(B_n)$ (where $e$ is the identity element of $G_n$), and therefore it has a natural $H_n$-module structure. Consequently, we have $Y = Y' \times Y''$ where $Y' = \{x \in Y \mid \forall t \in \mathbb{C}^\times : \varphi(t)x\varphi(t^{-1}) = x\}$ and $Y'' = \{x \in Y \mid \lim_{t \to \infty} \varphi(t)x\varphi(t^{-1}) = e\}$. Moreover, the map $\pi(x) = \lim_{t \to \infty} \varphi(t)x\varphi(t^{-1})$ is simply the projection morphism $Y \to Y'$.

Obviously $X_{\omega_n} \cap Y$ is irreducible, and therefore $\pi(X_{\omega_n} \cap Y)$ is irreducible. However, $\pi(X_{\omega_n} \cap Y) = X_{\omega_n} \cap Y \cap Q_n/B_n$. Every irreducible component of $X_{\omega_n} \cap Q_n/B_n$ contains $eB_n/B_n$ and therefore intersects $Y$ nontrivially. Therefore $X_{\omega_n} \cap Q_n/B_n$ must be irreducible as well. □
Remark. At least when \( G = \text{GL}(\infty) \), for any \( i \in I \) we have

\[
C_{n,i} = \{e_{u_1} > \cdots > e_{ux} > e_{ux+1} > \cdots > e_{uy} > e_{uy+1} > \cdots > e_{u2}\}
\]

and \( C^{p,i} = \{e_{ux+1} > \cdots > e_{uy}\} \). The proof of (4.2.2) follows from Proposition 4 when \( \eta_1 = \omega_{n,i} \) and

\[
\eta_2(t) = \begin{cases} 
  t & \text{if } j \leq x \text{ or } j > y, \\
  y + x + 1 - t & \text{otherwise}.
\end{cases}
\]

The irreducibility condition of Proposition 4 follows from Proposition 5.

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**References**


