A NOTION OF RANK FOR UNITARY REPRESENTATIONS OF REDUCTIVE GROUPS BASED ON KIRILLOV'S ORBIT METHOD

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Abstract

We introduce a new notion of rank for unitary representations of semisimple groups over a local field of characteristic zero. The theory is based on Kirillov's method of orbits for nilpotent groups over local fields. When the semisimple group is a classical group, we prove that the new theory is essentially equivalent to Howe's theory of N-rank (see [Ho4], [L2], [Sc]). Therefore our results provide a systematic generalization of the notion of a small representation (in the sense of Howe) to exceptional groups. However, unlike previous works that used ad hoc methods to study different types of classical groups (and some exceptional ones; see [We], [LS]), our definition is simultaneously applicable to both classical and exceptional groups. The most important result of this article is a general "purity" result for unitary representations which demonstrates how similar partial results in these authors' works should be formulated and proved for an arbitrary semisimple group in the language of Kirillov's theory. The purity result is a crucial step toward studying small representations of exceptional groups. New results concerning small unitary representations of exceptional groups will be published in a forthcoming paper [S].

Contents

1. Introduction

1.1. Background

The theory of N-rank (see [Ho4], [L2], [Sc]) provides an effective tool for studying "small" unitary representations of classical semisimple groups. It is based on analyzing

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the restriction of a unitary representation to large commutative subgroups of such groups. The study of the notion of N-rank began in [Ho4], where the theory was developed for the metaplectic group. Roughly speaking, one can attach equivalence classes of bilinear symmetric forms to unitary representations, and these equivalence classes canonically correspond to orbits of the adjoint action of the Levi component of the Siegel parabolic on its unipotent radical. Later on, the theory was extended by J.-S. Li [L2] and R. Scaramuzzi [Sc] to classical groups, and an intrinsic connection with the theta correspondence was also found in [L1]. Representations that correspond to degenerate classes are called *singular* representations. They form a large class of nontempered representations, and therefore their study is essentially complementary to that of the tempered representations. It turns out that this class of representations is quite well behaved. In fact, Howe gave a construction of a large class of such representations (see [Ho5]), and they are applied in construction of automorphic forms (see [Ho3], [L3]).

The lack of a theory of N-rank which applies to the exceptional groups is, in a sense, a defect of this theory due to substantial interest in the small representations of these groups. Aside from some generalizations in the works of Loke and Savin [LS] and Dvorsky and Sahi [DS2], most recently Weissman [We] has provided an analogue for p-adic split simply laced groups excluding E_8 using the representation-theoretic analogue of a Fourier-Jacobi functor.

Here we develop a theory of rank which can also be applied to (almost) all exceptional groups. The word "almost" is explained more precisely in the remark after Proposition 3.1.1. Our theory relies on the structure of unipotent radicals of certain parabolic subgroups which are naturally expressible as a sequence of extensions by Heisenberg groups. In this article these unipotent radicals are called *H-tower* groups (see Definition 3.2.5).

The main point of this article is to show that although the abelian nilradicals in terms of which rank was defined for classical groups are usually not available in exceptional cases, the maximal unipotent subgroups of all semisimple groups have a common feature that enables us to define a similar notion of rank, which is essentially equivalent to the older one for classical groups. This result, apart from being interesting itself, provides the main tool needed to study small representations of exceptional groups. One interesting application of this result is that it enables us to obtain sharp bounds on the behavior of matrix coefficients of unitary representations of exceptional groups, a problem studied in [LZ], [O], and [LS]. Results along this line of research will be published elsewhere in [S].

It is also possible to generalize [Ho4, Theorem 4.2], which is the next step toward classification of small representations of exceptional groups. However, there does not seem to be any strong connection between the exceptional theta correspondences, which result from our work and the ones that already exist in the literature (see, e.g.,

[DS1], [MS], or [GRS]). This makes the classification problem both harder and more interesting at the same time. The work is still in its preliminary stages.

This article is organized as follows. Section 1.2 is devoted to the notation used throughout the article. In Section 2 we recall some basics about the Heisenberg groups and the Weil representation. In Section 3 we define a specific unipotent subgroup of a simple group with regard to which our rank is defined. The *rankable* representations of this unipotent group are defined and studied briefly in Section 4. Section 5 is devoted to the statement and proof of the main theorems, which essentially assert that rank is a nontrivial invariant of a representation. Section 6 shows why our new theory generalizes the older one.

1.2. Notation

In this section we introduce some notation that is used throughout this article. Throughout this article, we work with a local field $\mathbb{F}$ that is either $\mathbb{R}$, $\mathbb{C}$, or a finite extension of $\mathbb{Q}_p$, $p \neq 2$. Let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$, let $\mathbf{G}$ be the $\overline{\mathbb{F}}$-points of an absolutely simple, simply connected linear algebraic group defined over $\mathbb{F}$, and let $\mathbf{G}_{\mathbb{F}}$ be the group of $\mathbb{F}$-rational points of $\mathbf{G}$. We assume that $\mathbf{G}$ is $\mathbb{F}$-isotropic. By the Kneser-Tits conjecture, which holds in the case of local fields (see [PR, Section 7.2, Proposition 7.6, Theorem 7.6]), $\mathbf{G}_{\mathbb{F}}$ is equal to the group generated by its unipotent elements. (In the Archimedean case, $\mathbf{G}_{\mathbb{F}}$ is a connected Lie group.) Let G be a finite topological central extension of $\mathbf{G}_{\mathbb{F}}$. (For the definition of a topological central extension, the reader is referred to [Mo1, Chapter 1].)

Take a maximally split Cartan subgroup $\mathbf{H}$ of $\mathbf{G}$ (which is also defined over $\mathbb{F}$), and let $\mathbf{A}$ be the maximal split torus inside $\mathbf{H}$. Let A be the inverse image of $\mathbf{A}_{\mathbb{F}}$ in G.

We denote the Lie algebras of the groups $\mathbf{G}$, $\mathbf{H}$, $\mathbf{A}$, G, A by $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{a}$, $\mathfrak{g}_{\mathbb{F}}$, $\mathfrak{a}_{\mathbb{F}}$, respectively. More precisely, $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{a}$ are $\overline{\mathbb{F}}$-Lie algebras, and $\mathfrak{g}_{\mathbb{F}}$ and $\mathfrak{a}_{\mathbb{F}}$ are $\mathbb{F}$-forms of the corresponding algebras.

Let Δ be an absolute root system associated to $\mathfrak{h}$, and let Σ be the restricted (or relative, as some people call it) root system associated to $\mathfrak{a}$. The root space corresponding to any $\alpha \in \Delta$ is denoted by $\mathfrak{g}_\alpha$, and the coroot for α is denoted by H_α.

We fix a $\mathbf{G}$-invariant symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$, normalized in such a way that if for any two roots β, γ we define (β, γ) to be (H_β, H_γ), then $(\beta, \beta) = 2$ for any long root β. All roots of simply laced root systems are considered as long ones. For $\mathfrak{h}_{\mathbb{F}} = \mathfrak{h} \cap \mathfrak{g}_{\mathbb{F}}$, we have

$$\mathfrak{h}_{\mathbb{F}} = \mathfrak{a}_{\mathbb{F}} \oplus \mathfrak{t}_{\mathbb{F}},$$

where $\mathfrak{t}_{\mathbb{F}}$ is the orthogonal complement of $\mathfrak{a}_{\mathbb{F}}$ in $\mathfrak{h}_{\mathbb{F}}$ with respect to the invariant symmetric bilinear form $(\cdot, \cdot)$ introduced above.

As usual, Δ^+ and Σ^+ denote the set of positive roots. We assume that Δ^+, when restricted to $\mathfrak{a}$, contains Σ^+. Let Δ_B and Σ_B denote the bases for positive roots in the corresponding root systems such that Δ_B, when restricted to $\mathfrak{a}$, contains Σ_B.

The choice of a positive system determines a fixed Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ (or a minimal parabolic $\mathfrak{p}_{\min}$ of $\mathfrak{g}_{\mathbb{F}}$). A parabolic subalgebra of $\mathfrak{g}$ (resp., of $\mathfrak{g}_{\mathbb{F}}$) is called *standard* if it contains $\mathfrak{b}$ (resp., $\mathfrak{p}_{\min}$). We identify the standard parabolic subgroups and subalgebras of $\mathfrak{g}$ (resp., of $\mathfrak{g}_{\mathbb{F}}$) by the subsets of Δ_B (resp., of Σ_B) to which they naturally correspond. More precisely, for any subset $\Phi \subseteq \Delta_B$, by $\mathfrak{p}_\Phi$ we mean the standard parabolic subalgebra of $\mathfrak{g}$ which contains all root spaces $\mathfrak{g}_\alpha$ for any $\alpha \in \Delta^+$, as well as any $\mathfrak{g}_{-\alpha'}$, such that $\alpha' \in \Delta^+$ is in the semigroup generated by Φ. The definition is similar for the parabolic subalgebra (this time of $\mathfrak{g}_{\mathbb{F}}$) corresponding to any $\Phi \subseteq \Sigma_B$. Later on, when there is no risk of confusion between the parabolics of $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{F}}$, we use the more concise notation $\mathfrak{p}_\Phi$ for both situations. Also, again for simplicity and when there is no risk of ambiguity, Lie subalgebras of both $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{F}}$ may be denoted by Gothic letters in a similar way (i.e., without a subscript $\mathbb{F}$ in the case of a Lie subalgebra of $\mathfrak{g}_{\mathbb{F}}$).

The standard parabolic subgroups of $\mathbf{G}$ and G are denoted by $\mathbf{P}_\Phi$ and P_Φ, respectively. Note that a parabolic subgroup of G means the inverse image of a parabolic subgroup of $\mathbf{G}_{\mathbb{F}}$ in G. In other words, if the short exact sequence

$$1 \longrightarrow F \xrightarrow{\ i\ } G \xrightarrow{\ p\ } \mathbf{G}_{\mathbb{F}} \longrightarrow 1 \tag{1.1}$$

(where F is a finite subgroup of the center of G) represents our topological central extension, then a parabolic P of G is of the form $P = p^{-1}(\mathbf{P}_{\mathbb{F}})$ for an $\mathbb{F}$-parabolic $\mathbf{P}$ of $\mathbf{G}$. It follows from [Du, Chapitre II, Lemme 11] (or from an explicit construction of the universal central extension by Deodhar [D, Section 1.9]) that topological central extensions of semisimple groups over local fields are split over the unipotent subgroups,* and therefore one can have an analogous Levi decomposition in G as well; that is, if $P = p^{-1}(\mathbf{P}_{\mathbb{F}})$ and $\mathbf{P} = \mathbf{LN}$ is a Levi decomposition of $\mathbf{P}$ (with $\mathbf{L}$ being the reductive factor and $\mathbf{N}$ being the unipotent radical), then one can express P as $P = LN$, where

(1) $L = p^{-1}(\mathbf{L}_{\mathbb{F}})$;

(2) $N \subseteq p^{-1}(\mathbf{N}_{\mathbb{F}})$ is a normal subgroup of P, and the map

$$p : N \mapsto \mathbf{N}_{\mathbb{F}}$$

is an isomorphism of topological groups; existence of N is what follows from the results of Duflo or Deodhar cited above.

*In fact, an abstract (and hence topological) universal central extension of $\mathbf{G}_{\mathbb{F}}$ splits over the maximal unipotent subgroup. We are indebted to Gopal Prasad for pointing us to reference [D].

We may as well drop the subscript Φ of the standard parabolics P_Φ and $\mathbf{P}_\Phi$ when there is no risk of confusion about the identifying subset Φ of simple positive roots. Moreover, whenever a parabolic subgroup or subalgebra is standard, we assume that the Levi decomposition considered above is also standard.

Let K be an arbitrary locally compact group. Then the center of K is denoted by $\mathscr{Z}(K)$. Unitary representations of K are defined in the usual way (see [M2]) and denoted by Greek letters. The unitary dual of K (which is the set of equivalence classes of the irreducible unitary representations of K endowed with the Fell topology) is denoted by $\hat{K}$. Now suppose that K' is a closed subgroup of K, and let σ, σ' be unitary representations of K, K', respectively. Then, as usual, $\mathrm{Res}^K_{K'}\sigma$ and $\mathrm{Ind}^K_{K'}\sigma'$ mean restriction and unitary induction. If there is no ambiguity about K, we use the simpler notation $\sigma_{|K'}$ instead of $\mathrm{Res}^K_{K'}\sigma$.

Suppose that $\mathfrak{l}$ is an arbitrary Lie algebra. Then the center of $\mathfrak{l}$ is denoted by $\mathfrak{z}(\mathfrak{l})$.

2. Heisenberg group and Weil representation

2.1. Heisenberg groups and algebras

Let n be a positive integer. An $\mathbb{F}$-Heisenberg group (or simply a Heisenberg group) H_n is a two-step nilpotent $(2n+1)$-dimensional $\mathbb{F}$-group that is isomorphic to the subgroup of $\mathrm{GL}_{n+2}(\mathbb{F})$ of unipotent, upper-triangular matrices that do not have nonzero elements outside the diagonal, the first row, and the last column. This group has a one-dimensional center. We denote the Lie algebra of H_n by $\mathfrak{h}_n$. Since $H_n \subset \mathrm{GL}_{n+2}(\mathbb{F})$, $\mathfrak{h}_n$ sits in the usual way inside $\mathfrak{gl}_{n+2}(\mathbb{F})$.

The Lie algebra $\mathfrak{h}_n$ of H_n can also be described more abstractly as

$$\mathfrak{h}_n = W_n \oplus \mathfrak{z}$$

with the Lie bracket introduced below, where W_n is a $2n$-dimensional space endowed with a nondegenerate symplectic form $\langle\cdot,\cdot\rangle$ and $\mathfrak{z}$ is the center of $\mathfrak{h}_n$, a one-dimensional Lie subalgebra. We fix a Lie algebra isomorphism between $\mathfrak{z}$ and $\mathbb{F}$, and we denote the element in $\mathfrak{z}$ corresponding to $1 \in \mathbb{F}$ by Z. The Lie bracket on W_n is defined as follows:

$$\text{for every } X, Y \in W_n, \quad [X, Y] = \langle X, Y\rangle Z.$$

Viewing $\mathfrak{h}_n$ as a subalgebra of $\mathfrak{gl}_{n+2}(\mathbb{F})$, the exponential map of $\mathrm{GL}_{n+2}(\mathbb{F})$ gives a bijection between $\mathfrak{h}_n$ and H_n, and $\mathscr{Z}(H_n)$ is equal to $\exp(\mathfrak{z})$. This bijection is the exponential map of H_n.

2.2. Stone–von Neumann theorem

We briefly mention the structure of irreducible, infinite-dimensional unitary representations of a Heisenberg group. For a more detailed discussion, the reader can study, for example, [Ho2], [CG], or [T, Chapter 1].

The Stone–von Neumann theorem states that for any nontrivial unitary character of $\mathscr{Z}(H_n)$, up to unitary equivalence there is a unique infinite-dimensional irreducible unitary representation of H_n having this central character. We describe an explicit realization (the so-called Schrödinger model) for this representation below.

Consider an arbitrary polarization of W_n, that is, a decomposition of W_n as a direct sum

$$W_n = \mathfrak{x}_n \oplus \mathfrak{y}_n \tag{2.1}$$

of maximal isotropic subspaces. It is possible to choose bases $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_n\}$ of $\mathfrak{x}_n$ and $\mathfrak{y}_n$, respectively, such that

$$\langle X_i, Y_j \rangle = \delta_{i,j}. \tag{2.2}$$

There is an isomorphism

$$e : \mathfrak{x}_n \approx \mathfrak{y}_n^* \tag{2.3}$$

obtained via the symplectic form. We have $\mathscr{Z}(H_n) \approx \exp(\mathbb{F})$, and any nontrivial multiplicative (unitary) character of $\mathscr{Z}(H_n)$ is equal to $\chi \circ \exp^{-1}$ for some χ, where χ can be any nontrivial additive (unitary) character of $\mathfrak{z}(\mathfrak{h}_n) \approx \mathbb{F}$. Such characters χ are in one-to-one correspondence with elements of $\mathbb{F} - \{0\}$. For example, if $\mathbb{F} = \mathbb{R}$, then χ is of the form $\chi(x) = e^{\mathrm{i}xy}$ for some $y \in \mathbb{R} - \{0\}$ (here $\mathrm{i} = \sqrt{-1}$). This follows from the well-known fact that the unitary dual of a local field is identifiable to itself (see [W2]).

The (irreducible) representation ρ of H_n with central character χ may be realized on the Hilbert space

$$\mathscr{H}_\rho = L^2(\mathfrak{y}_n)$$

as follows. Let $y = \sum_{i=1}^n y_i Y_i \in \mathfrak{y}_n$. Then H_n acts on $\mathscr{H}_\rho$ as

$$\begin{aligned} \big(\rho(e^{tX_i})f\big)(y) &= \chi(-ty_i)f(y), \\ \big(\rho(e^{tY_i})f\big)(y) &= f(y + tY_i), \\ \big(\rho(e^{tZ})f\big)(y) &= \chi(t)f(y). \end{aligned} \tag{2.4}$$

This representation is irreducible and unitary with respect to the usual inner product of $\mathscr{H}_\rho$.

Notation. Henceforth, whenever a locally compact group N is isomorphic to a Heisenberg group, we denote the subset of $\hat{N}$ consisting of the family of the infinite-dimenional representations constructed above by $\hat{N}_\circ$.

2.3. The Weil representation

Let $\mathrm{Sp}(W_n)$ be the symplectic group associated to W_n. When $\mathbb{F} = \mathbb{C}$, take $\mathrm{Mp}(W_n) = \mathrm{Sp}(W_n)$; otherwise, let $\mathrm{Mp}(W_n)$ be the (metaplectic) double covering of $\mathrm{Sp}(W_n)$ (see [W1]). $\mathrm{Mp}(W_n)$ acts through $\mathrm{Sp}(W_n)$ on $H_n \approx \exp(\mathfrak{h}_n)$ (or, equivalently, on $\mathfrak{h}_n$) as follows. For any $w \oplus z \in W_n \oplus \mathfrak{z}$,

$$g : w \oplus z \to g \cdot w \oplus z.$$

One can consider the semidirect product $\mathrm{Mp}(W_n) \ltimes H_n$. We have the following (see [W1]).

PROPOSITION 2.3.1

Let χ be any nontrivial additive character of $\mathbb{F}$. Then the irreducible unitary representation ρ of H_n with central character χ can be extended to a unitary representation (still denoted by ρ) of $\mathrm{Mp}(W_n) \ltimes H_n$*; that is, for any $g \in$* $\mathrm{Mp}(W_n)$ *and any $h \in H_n$,*

$$\rho(g)\rho(h)\rho(g^{-1}) = \rho(ghg^{-1}).$$

3. Construction of H-tower groups

3.1. The Heisenberg parabolic

We use the notation of Section 1.2. Let $\tilde{\beta}$ be the highest root in Δ with respect to the positive system chosen in Section 1.2, and let $H_{\tilde{\beta}}$ be the coroot for $\tilde{\beta}$. Let $(\cdot, \cdot)$ denote the **G**-invariant symmetric bilinear form introduced in that section (i.e., $(\tilde{\beta}, \tilde{\beta}) = 2$). One obtains a grading

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j,$$

where $\mathfrak{g}_j$ is the j-eigenspace of $\mathrm{ad}(H_{\tilde{\beta}})$; that is,

$$\mathfrak{g}_j = \left\{ X \in \mathfrak{g} : [H_{\tilde{\beta}}, X] = jX \right\}.$$

Note that, in fact, $\mathfrak{g}_k = \{0\}$ for $|k| > 2$. The Jacobi identity implies that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$. Note that $\mathfrak{g}_1 = \mathfrak{g}_{-1} = \{0\}$ if and only if $\mathfrak{g} = \mathfrak{sl}_2$.

In the next proposition and at some points of this article, we consider $\mathfrak{g}$ as above such that

$$\mathfrak{g} \neq \mathfrak{sl}_2 \qquad \text{and} \qquad \tilde{\beta} \text{ is defined over } \mathbb{F}. \tag{3.1}$$

Condition (3.1) is not a crucial assumption. In fact, we believe that it can be removed (see the discussion in the remark after Proposition 3.1.1).

PROPOSITION 3.1.1
Let $\mathfrak{g}$ satisfy condition (3.1). Then we have the following.

(1) *The direct sum $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a parabolic subalgebra, $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is its nilradical, and*

$$\big[[\mathfrak{g}_0, \mathfrak{g}_0], \mathfrak{g}_2\big] = \{0\}.$$

The nilradical is a Heisenberg Lie algebra with center $\mathfrak{g}_2$ and symplectic space $\mathfrak{g}_1$. One can describe the symplectic form $\langle\cdot,\cdot\rangle_1$ on $\mathfrak{g}_1$ using the Lie bracket as follows. Fix $X_2 \in \mathfrak{g}_2 - \{0\}$, and let $\langle\cdot,\cdot\rangle_1$ be such that

$$\text{for every } X, Y \in \mathfrak{g}_1, \quad [X, Y] = \langle X, Y\rangle_1 \cdot X_2.$$

Then $\langle\cdot,\cdot\rangle_1$ is nondegenerate, and the (adjoint) representation of $[\mathfrak{g}_0, \mathfrak{g}_0]$ on $\mathfrak{g}_1$ is a symplectic representation with respect to $\langle\cdot,\cdot\rangle_1$.

(2) *When $\mathfrak{g}$ is not of type $\mathbf{A}_l$ $(l > 1)$, the parabolic of part (1) is characterized by the subset $\Delta_B - \{\beta\}$ of Δ_B for the unique simple root $\beta \in \Delta_B$ which satisfies $(\beta, \tilde{\beta}) \neq 0$. For type $\mathbf{A}_l$ $(l > 1)$, there are two such simple roots β', β'', and the parabolic corresponds to $\Delta_B - \{\beta', \beta''\}$.*

(3) *When $\mathfrak{g}$ is not of type $\mathbf{A}_l$ $(l > 1)$, β corresponds to a simple restricted root (still denoted by β). For $\mathfrak{g}$ of type $\mathbf{A}_l$ $(l > 2)$ and $\mathfrak{g}_{\mathbb{F}}$ nonsplit, the pair $\{\beta', \beta''\}$ corresponds to a simple restricted root β in the restricted root system.*

Let the sets $S \subset \Delta_B$ and $T \subset \Sigma_B$ be defined as follows. Set $S = \{\beta\}$ when $\mathfrak{g}$ is not of type $\mathbf{A}_l$ $(l > 1)$, and set $S = \{\beta', \beta''\}$ otherwise. Also, set $T = \{\beta', \beta''\}$ when $\mathfrak{g}_{\mathbb{F}}$ is split and $\mathfrak{g}$ is of type $\mathbf{A}_l$ $(l > 1)$, and set $T = \{\beta\}$ otherwise.

(4) *The parabolic $\mathfrak{p}_{\Delta_B - S}$ is defined over $\mathbb{F}$, and $(\mathfrak{p}_{\Delta_B - S})_{\mathbb{F}} = \mathfrak{p}_{\Sigma_B - T}$. Its nilradical is a Heisenberg algebra. The Lie algebra $\mathfrak{p}_{\Delta_B - S}$ is a maximal parabolic when $\mathfrak{g}$ is not of type $\mathbf{A}_l$ $(l > 1)$.*

Proof
See [GW, Section 2] and [To, Section 10]. □

Remark. The assumption that $\mathfrak{g}$ satisfies condition (3.1) holds for all but a very small class of Lie algebras $\mathfrak{g}$. Fox example, when $\mathbb{F} = \mathbb{R}$, the cases where $\tilde{\beta}$ is not defined over $\mathbb{R}$ are $\mathfrak{f}_{4(-20)}$, $\mathfrak{e}_{6(-26)}$, $\mathfrak{sp}(p, q)$, and $\mathfrak{sl}(n, \mathbb{H})$. The notion of rank in this article is based on the existence of a unipotent subgroup of G which is expressible as a tower of extensions by Heisenberg groups (see Definition 3.2.5), and therefore it is not applicable to the groups mentioned above. However, in all of these groups there

Table 1

$\mathfrak{g}$	$[\mathfrak{g}_0, \mathfrak{g}_0]$	$\mathfrak{g}_1$
$\mathbf{A_l}$ ($\mathbf{l} \geq 3$)	$\mathbf{A_{l-2}}$	$V_{\varpi_1} \oplus V^*_{\varpi_1}$
$\mathbf{B_2}$	$\mathbf{A_1}$	V_{ϖ_1}
$\mathbf{B_3}$	$\mathbf{A_1} \times \mathbf{A_1}$	$V_{\varpi_1} \hat{\otimes} V_{\varpi_1}$
$\mathbf{B_l}$ ($\mathbf{l} \geq 4$)	$\mathbf{A_1} \times \mathbf{B_{l-2}}$	$V_{\varpi_1} \hat{\otimes} V_{\varpi_1}$
$\mathbf{C_2}$	$\mathbf{A_1}$	V_{ϖ_1}
$\mathbf{C_l}$ ($\mathbf{l} \geq 3$)	$\mathbf{C_{l-1}}$	V_{ϖ_1}
$\mathbf{D_4}$	$\mathbf{A_1} \times \mathbf{A_1} \times \mathbf{A_1}$	$V_{\varpi_1} \hat{\otimes} V_{\varpi_1} \hat{\otimes} V_{\varpi_1}$
$\mathbf{D_5}$	$\mathbf{A_1} \times \mathbf{A_3}$	$V_{\varpi_1} \hat{\otimes} V_{\varpi_2}$
$\mathbf{D_l}$ ($\mathbf{l} \geq 6$)	$\mathbf{A_1} \times \mathbf{D_{l-2}}$	$V_{\varpi_1} \hat{\otimes} V_{\varpi_1}$
$\mathbf{E_6}$	$\mathbf{A_5}$	V_{ϖ_3}
$\mathbf{E_7}$	$\mathbf{D_6}$	V_{ϖ_6}
$\mathbf{E_8}$	$\mathbf{E_7}$	V_{ϖ_7}
$\mathbf{F_4}$	$\mathbf{C_3}$	V_{ϖ_3}
$\mathbf{G_2}$	$\mathbf{A_1}$	$V_{3\varpi_1}$

does exist a similar structure that is called an OKP subgroup in [HRW]. Therefore, in principle, the main results of this article (especially Theorem 5.3.2) should generalize to groups associated with these real forms. However, addressing the technical problems that arise with including those cases in this article makes it much more technical. To keep our presentation as simple and uniform as possible, we do not include those special cases in this article. We intend to deal with those cases elsewhere.

Definition 3.1.2
The parabolic subalgebras $\mathfrak{p}_{\Delta_B - S}$ of $\mathfrak{g}$ and $\mathfrak{p}_{\Sigma_B - T}$ of $\mathfrak{g}_{\mathbb{F}}$ or their corresponding parabolic subgroups $\mathbf{P}_{\Delta_B - S}$ of $\mathbf{G}$ and $P_{\Sigma_B - T} = (\mathbf{P}_{\Delta_B - S})_{\mathbb{F}}$ of $\mathbf{G}_{\mathbb{F}}$ (or the parabolic of G which corresponds to $P_{\Sigma_B - T}$) are called the *Heisenberg* parabolics.

We tend to drop their identifying subscripts for simplicity when there is no risk of confusion.

Table 1 demonstrates the structure of $\mathfrak{g}_1$ as a representation of $[\mathfrak{g}_0, \mathfrak{g}_0]$. Here V_ϖ denotes the representation with highest weight ϖ, and ϖ_i denotes the ith fundamental weight of the corresponding Lie algebra. We use the notation of [Bo, Planche I] for numbering fundamental weights (see also [Ti] for more explicit information).

3.2. The H-tower subgroup N_Γ of G
Let $\mathfrak{g}$ be as in condition (3.1). In this section we describe a nilpotent subalgebra of $\mathfrak{g}$ (and another one of $\mathfrak{g}_{\mathbb{F}}$) which is fundamental to our definition of rank. The construction of these nilpotent subalgebras is based on what is usually referred to as Kostant's cascade. We show that this nilpotent Lie subalgebra is actually the nilradical of a parabolic subalgebra (see Propositions 3.2.3, 3.2.4). The unipotent subgroup of

G which corresponds to this nilpotent subalgebra plays an important role in the rest of this article.

First, assume that $\mathfrak{g}$ splits over $\mathbb{F}$ (i.e., all roots of $\mathfrak{g}$ are defined over $\mathbb{F}$). Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ be the Heisenberg parabolic of $\mathfrak{g}$, obtained by Proposition 3.1.1, with the usual Levi decomposition (i.e., $\mathfrak{l}$ is the Levi factor and $\mathfrak{n}$ is the nilradical of $\mathfrak{p}$). In the case of orthogonal algebras, the commutator $[\mathfrak{l}, \mathfrak{l}]$ is not a simple Lie algebra. In fact, when $\mathfrak{g}$ is of types $\mathbf{B}_l$ ($l \geq 3$) or $\mathbf{D}_l$ ($l > 4$), the commutator $[\mathfrak{l}, \mathfrak{l}]$ is a direct sum of the form

$$[\mathfrak{l}, \mathfrak{l}] = \mathfrak{sl}_2 \oplus \mathfrak{s}, \tag{3.2}$$

where $\mathfrak{s}$ is simple. When $\mathfrak{g}$ is of type $\mathbf{D}_4$, we have

$$[\mathfrak{l}, \mathfrak{l}] = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2.$$

Definition 3.2.1
Let $\mathfrak{m}$ be defined as follows.

(1) If $\mathfrak{g}$ is of type $\mathbf{D}_l$ ($l > 4$) or $\mathbf{B}_l$ ($l > 3$), then $\mathfrak{m}$ is equal to the summand $\mathfrak{s}$ given in (3.2).
(2) If $\mathfrak{g}$ is of type $\mathbf{A}_2, \mathbf{A}_3, \mathbf{B}_2 = \mathbf{C}_2, \mathbf{B}_3, \mathbf{D}_4$, or $\mathbf{G}_2$, then $\mathfrak{m} = \{0\}$.
(3) Otherwise, $\mathfrak{m} = [\mathfrak{l}, \mathfrak{l}]$.

One can repeatedly apply Proposition 3.1.1 as follows. First, we apply it to $\mathfrak{m}$. If $\mathfrak{m}$ is nonzero, then $\mathfrak{m} \neq \mathfrak{sl}_2$ and Proposition 3.1.1 guarantees the existence of a Heisenberg parabolic in $\mathfrak{m}$. Let $\mathfrak{p}'$ be this parabolic of $\mathfrak{m}$, and let $\mathfrak{l}'$ be the Levi factor of $\mathfrak{p}'$. Let $\mathfrak{m}'$ be the subalgebra of $[\mathfrak{l}', \mathfrak{l}']$ which is defined in the same way that $\mathfrak{m}$ was defined as a subalgebra of $[\mathfrak{l}, \mathfrak{l}]$ in Definition 3.2.1. Then we apply Proposition 3.1.1 to $\mathfrak{m}'$, and so on. This process can be repeated as long as Proposition 3.1.1 can be applied. As a result, we obtain a sequence $S_1, \ldots, S_r$ of subsets of Δ_B, where each S_i, defined as in Proposition 3.1.1(2), contains either a simple root or a pair of simple roots. Each S_i corresponds to the highest root $\tilde{\beta}_i$ obtained in the ith step. We also denote the sequence of Heisenberg parabolics by $\mathfrak{p}^1, \ldots, \mathfrak{p}^r$ with the Levi decomposition

$$\mathfrak{p}^j = \mathfrak{l}^j \oplus \mathfrak{n}^j. \tag{3.3}$$

Therefore $\mathfrak{p}^1$ is the Heisenberg parabolic of $\mathfrak{g}$, $\mathfrak{p}^2$ is the Heisenberg parabolic of $\mathfrak{m}$, and so on. Each $\mathfrak{n}^j$ is normalized by any $\mathfrak{l}^{j'}$ for $j' \geq j$ and hence by $\mathfrak{n}^{j'}$. For a similar reason, $\mathfrak{n}^{j'}$ acts trivially on the center of $\mathfrak{n}^j$. Each $\mathfrak{n}^j$ is isomorphic to a Heisenberg algebra $\mathfrak{h}_{d_i}$ for some d_i. Therefore the Lie algebra $\mathfrak{n}^1 \oplus \cdots \oplus \mathfrak{n}^r$ is a tower of successive extensions by Heisenberg $\overline{\mathbb{F}}$-algebras. The following proposition is obvious.

PROPOSITION 3.2.2
The Lie algebra $\mathfrak{n}^1 \oplus \cdots \oplus \mathfrak{n}^r$ *is equal to the nilradical of the parabolic subalgebra* $\mathfrak{p}_\Gamma$ *of* $\mathfrak{g}$*, where*

$$\Gamma = \Delta_B - (S_1 \cup \cdots \cup S_r).$$

Now assume that $\mathfrak{g}$ satisfies condition (3.1) but is not necessarily $\mathbb{F}$-split. Recall that the positive system for Σ is compatible with the one for Δ. We can again apply Proposition 3.1.1 repeatedly. Note that the number of possible iterations for a nonsplit $\mathbb{F}$-form of $\mathfrak{g}$ is often smaller than the number of possible iterations in the split case. (In certain groups with small rank, they may be equal.) This is because the successive Levi factors may fail to satisfy condition (3.1).

Proposition 3.1.1(3) yields a sequence $T_1, \ldots, T_s$ of subsets of Σ_B (for some $s \leq r$), where each T_j contains a simple restricted root or a pair of them. Again, we obtain a nested sequence

$$\mathfrak{p}^1 \supset \cdots \supset \mathfrak{p}^s$$

of Heisenberg parabolics (this time inside $\mathfrak{g}_\mathbb{F}$), and the nilradical of any $\mathfrak{p}^i$ is normalized by the nilradical of any $\mathfrak{p}^{i'}$ when $i < i'$. Therefore $\mathfrak{p}^1$ is the Heisenberg parabolic subalgebra of $\mathfrak{g}_\mathbb{F}$. We warn the reader that the new $\mathfrak{p}^i$'s are different from those that appear immediately before Proposition 3.2.2; in fact, the older $\mathfrak{p}^i$'s are obtained from the newer $\mathfrak{p}^i$'s by an extension of scalars.

Tables 2 and 3 explain how Proposition 3.1.1 is applied iteratively to $\mathfrak{g}_\mathbb{F}$. For simplicity, the real and the p-adic cases are separated: Table 2 is for the p-adic case, and Table 3 is for the real case.

The column $\mathfrak{g}_\mathbb{F}$ in Table 2 shows the Tits index of $\mathfrak{g}_\mathbb{F}$, and the column $\mathfrak{m}_\mathbb{F}$ shows the Tits index of $\mathfrak{m}_\mathbb{F}$. For any $\mathfrak{g}_\mathbb{F}$, the number s is given too. Real exceptional Lie algebras in Table 3 are identified by the symmetric pairs $(\mathfrak{g}_\mathbb{R}, \mathfrak{k}_\mathbb{R})$. The only $\mathbb{F}$-forms of $\mathfrak{g}$ which appear in the columns $\mathfrak{g}_\mathbb{F}$ of both of the tables are those for which $\mathfrak{g}$ satisfies condition (3.1). If an entry in the column corresponding to $\mathfrak{m}_\mathbb{F}$ is equal to –, this means either that the Lie algebra $\mathfrak{m}$ is equal to $\{0\}$ or that the Lie algebra $\mathfrak{m}$ is semisimple but does not satisfy condition (3.1) (i.e., the highest root of $\mathfrak{m}$ is not defined over $\mathbb{F}$). Therefore one can use Tables 2 and 3 to see how many times Proposition 3.1.1 can be applied to a particular $\mathbb{F}$-form of $\mathfrak{g}$.

We denote the nilradical of the parabolic $\mathfrak{p}^j$ of $\mathfrak{g}_\mathbb{F}$ by $\mathfrak{h}^j$. It is an $\mathbb{F}$-Heisenberg algebra.

Table 2

$\mathfrak{g}_{\mathbb{F}}$	$\mathfrak{m}_{\mathbb{F}}$	s	$\mathfrak{g}_{\mathbb{F}}$	$\mathfrak{m}_{\mathbb{F}}$	s
$A_{2,2}$	–	1	$A_{3,3}$	–	1
$A_{r,r}$ $(r \geq 4)$	$A_{r-2,r-2}$	$\lfloor \frac{r}{2} \rfloor$	${}^2A^{(1)}_{3,2}$	–	1
${}^2A^{(1)}_{2r-1,r}$ $(r \geq 3)$	${}^2A^{(1)}_{2r-3,r-1}$	$r-1$	${}^2A^{(1)}_{2,1}$	–	1
${}^2A^{(1)}_{2r,r}$ $(r \geq 2)$	${}^2A^{(1)}_{2r-2,r-1}$	r	${}^2A^{(1)}_{3,1}$	–	1
${}^2A^{(1)}_{2r+1,r}$ $(r \geq 2)$	${}^2A^{(1)}_{2r-1,r-1}$	r	$B_{3,3}$	–	1
$B_{4,4}$	$C_{2,2}$	2	$B_{r,r}$ $(r \geq 4)$	$B_{r-2,r-2}$	$\lfloor \frac{r}{2} \rfloor$
$B_{3,2}$	–	1	$B_{4,3}$	–	1
$B_{r,r-1}$ $(r \geq 5)$	$B_{r-2,r-3}$	$\lfloor \frac{r-1}{2} \rfloor$	$C_{2,2}$	–	1
$C_{r,r}$ $(r \geq 3)$	$C_{r-1,r-1}$	$r-1$	${}^1D^{(1)}_{4,4}$	–	1
${}^1D^{(1)}_{5,5}$	$A_{3,3}$	2	${}^1D^{(1)}_{r,r}$ $(r \geq 6)$	${}^1D^{(1)}_{r-2,r-2}$	$\lfloor \frac{r-1}{2} \rfloor$
${}^1D^{(1)}_{4,2}$	–	1	${}^1D^{(1)}_{5,3}$	–	1
${}^1D^{(1)}_{r+2,r}$ $(r \geq 4)$	${}^1D^{(1)}_{r,r-2}$	$\lfloor \frac{r}{2} \rfloor$	${}^2D^{(1)}_{4,3}$	–	1
${}^2D^{(1)}_{5,4}$	${}^2A^{(1)}_{3,2}$	2	${}^2D^{(1)}_{r+1,r}$ $(r \geq 5)$	${}^2D^{(1)}_{r-1,r-2}$	$\lfloor \frac{r-1}{2} \rfloor$
${}^1D^{(2)}_{4,2}$	–	1	${}^1D^{(2)}_{2r,r}$ $(r \geq 3)$	${}^1D^{(2)}_{2r-2,r-1}$	$r-1$
${}^1D^{(2)}_{5,1}$	–	1	${}^1D^{(2)}_{2r+3,r}$ $(r \geq 2)$	${}^1D^{(2)}_{2r+1,r-1}$	r
${}^2D^{(2)}_{5,2}$	${}^2A^{(1)}_{3,1}$	2	${}^2D^{(2)}_{2r+1,r}$ $(r \geq 3)$	${}^2D^{(2)}_{2r-1,r-1}$	r
${}^2D^{(2)}_{4,1}$	–	1	${}^2D^{(2)}_{2r+2,r}$ $(r \geq 3)$	${}^2D^{(2)}_{2r,r-1}$	r
${}^3D^{2}_{4,2}$	–	1	${}^6D^{2}_{4,2}$	–	1
${}^1E^{16}_{6,2}$	–	1	${}^1E^{0}_{6,6}$	$A_{5,5}$	3
${}^2E^{2}_{6,4}$	${}^2A^{(1)}_{5,3}$	3	$E^{9}_{7,4}$	${}^1D^{(2)}_{6,3}$	3
$E^{0}_{7,7}$	${}^1D^{(1)}_{6,6}$	3	$E^{0}_{8,8}$	$E^{0}_{7,7}$	4
$F^{0}_{4,4}$	$C_{3,3}$	3	$G^{0}_{2,2}$	–	1

PROPOSITION 3.2.3
The nilpotent Lie subalgebra $\mathfrak{h}^1 \oplus \cdots \oplus \mathfrak{h}^s$ of $\mathfrak{g}_{\mathbb{F}}$ is equal to the nilradical of the parabolic subalgebra $\mathfrak{p}_\Gamma$ of $\mathfrak{g}_{\mathbb{F}}$, where

$$\Gamma = \Sigma_B - (T_1 \cup \cdots \cup T_s).$$

Moreover, we have the following proposition, which describes the relationship between the $\mathfrak{n}^i$'s and $\mathfrak{h}^i$'s.

Table 3

$\mathfrak{g}_{\mathbb{F}}$	$\mathfrak{m}_{\mathbb{F}}$	s	$\mathfrak{g}_{\mathbb{F}}$	$\mathfrak{m}_{\mathbb{F}}$	s
$\mathfrak{sl}_3(\mathbb{R})$	–	1	$\mathfrak{sl}_4(\mathbb{R})$	–	1
$\mathfrak{sl}_n(\mathbb{R})$ $(n \geq 5)$	$\mathfrak{sl}_{n-2}(\mathbb{R})$	$\lfloor \frac{n-1}{2} \rfloor$	$\mathfrak{su}(1,q)$ $(q > 1)$	–	1
$\mathfrak{su}(2,2)$	–	1	$\mathfrak{su}(r,q)$ $(2 \leq r < q)$	$\mathfrak{su}(r-1,q-1)$	r
$\mathfrak{su}(q,q)$ $(3 \leq q)$	$\mathfrak{su}(q-1,q-1)$	$q-1$	$\mathfrak{so}(1,q)$ $(q \geq 4)$	–	1
$\mathfrak{so}(2,q)$ $(q \geq 3)$	–	1	$\mathfrak{so}(3,q)$ $(q \geq 3)$	–	1
$\mathfrak{so}(4,4)$	–	1	$\mathfrak{so}(4,q)$ $(q \geq 5)$	$\mathfrak{so}(2,q-2)$	2
$\mathfrak{so}(r,q)$ $(5 \leq r < q)$	$\mathfrak{so}(r-2,q-2)$	$\lfloor \frac{r}{2} \rfloor$	$\mathfrak{so}(q,q)$ $(5 \leq q)$	$\mathfrak{so}(q-2,q-2)$	$\lfloor \frac{q-1}{2} \rfloor$
$\mathfrak{sp}_4(\mathbb{R})$	–	1	$\mathfrak{sp}_{2n}(\mathbb{R})$ $(n \geq 3)$	$\mathfrak{sp}_{2n-2}(\mathbb{R})$	$n-1$
$\mathfrak{so}^*(6)$	–	1	$\mathfrak{so}^*(8)$	–	1
$\mathfrak{so}^*(2r)$ $(r \geq 5)$	$\mathfrak{so}^*(2r-4)$	$\lfloor \frac{r-1}{2} \rfloor$	$(\mathfrak{e}_6, \mathfrak{sp}_4)$	$\mathfrak{sl}_6(\mathbb{R})$	3
$(\mathfrak{e}_6, \mathfrak{su}_6 \times \mathfrak{su}_2)$	$\mathfrak{su}(3,3)$	3	$(\mathfrak{e}_6, \mathfrak{so}(10) \times \mathfrak{u}(1))$	$\mathfrak{su}(1,5)$	2
$(\mathfrak{e}_7, \mathfrak{su}_8)$	$\mathfrak{so}(6,6)$	3	$(\mathfrak{e}_7, \mathfrak{so}(12) \times \mathfrak{su}_2)$	$\mathfrak{so}^*(12)$	3
$(\mathfrak{e}_7, \mathfrak{e}_6 \times \mathfrak{u}(1))$	$\mathfrak{so}(2,10)$	2	$(\mathfrak{e}_8, \mathfrak{so}(12))$	$(\mathfrak{e}_7, \mathfrak{su}_8)$	4
$(\mathfrak{e}_8, \mathfrak{e}_7 \times \mathfrak{su}_2)$	$(\mathfrak{e}_7, \mathfrak{e}_6 \times \mathfrak{u}(1))$	3	$(\mathfrak{f}_4, \mathfrak{sp}_3 \times \mathfrak{su}_2)$	$\mathfrak{sp}_6(\mathbb{R})$	3
$(\mathfrak{g}_2, \mathfrak{su}_2 \times \mathfrak{su}_2)$	–	1			

PROPOSITION 3.2.4
Let $\mathfrak{p}_\Gamma$ *be the parabolic subalgebra of* $\mathfrak{g}_{\mathbb{F}}$ *defined in Proposition 3.2.3 with Levi decomposition* $\mathfrak{p}_\Gamma = \mathfrak{l}_\Gamma \oplus \mathfrak{n}_\Gamma$. *Then*

$$\mathfrak{n}_\Gamma \otimes \overline{\mathbb{F}} = \mathfrak{n}^1 \oplus \cdots \oplus \mathfrak{n}^s$$

(where the $\mathfrak{n}^j$*'s are the same as those that appear in Proposition 3.2.2), and thus* $\mathfrak{n}_\Gamma \otimes \overline{\mathbb{F}}$ *is equal to the nilradical of the parabolic* $\mathfrak{p}_{\Gamma'}$ *of* $\mathfrak{g}$, *where*

$$\Gamma' = \Delta_B - (S_1 \cup \cdots \cup S_s).$$

Therefore $\mathbf{P}_{\Gamma'}$ is an $\mathbb{F}$-parabolic of $\mathbf{G}$ and $(\mathbf{P}_{\Gamma'})_{\mathbb{F}} = P_\Gamma$, where P_Γ is the parabolic of G which is associated to the set $\Gamma \subseteq \Sigma_B$ defined in Proposition 3.2.3. The group P_Γ plays a significant role in the rest of the article. Its Levi decomposition can be written as

$$P_\Gamma = L_\Gamma N_\Gamma. \tag{3.4}$$

Definition 3.2.5
Let U be the group of $\mathbb{F}$-points of a unipotent linear algebraic $\mathbb{F}$-group $\mathbf{U}$. U is said to be an *H-tower* group if and only if it satisfies one of the following two properties.
(1) $U = \{1\}$ (i.e., U is trivial).

(2) U is isomorphic to a semidirect product $U' \ltimes U''$, where
  (a) U'' is an $\mathbb{F}$-Heisenberg group and is the group of $\mathbb{F}$-points of an algebraic $\mathbb{F}$-subgroup $\mathbf{U}''$ of $\mathbf{U}$;
  (b) U' is the group of $\mathbb{F}$-rational points of an algebraic subgroup $\mathbf{U}'$ of $\mathbf{U}$, and the action of U' on U'' in the semidirect product $U' \ltimes U''$ comes from an algebraic action (defined over $\mathbb{F}$) of $\mathbf{U}'$ on $\mathbf{U}''$ by group automorphisms of $\mathbf{U}''$ (which, a fortiori, leave elements of $\mathscr{Z}(\mathbf{U}'')$ invariant and act on the symplectic space $\mathbf{U}''/\mathscr{Z}(\mathbf{U}'')$ via symplectic operators);*
  (c) U' is an H-tower group.

Therefore an H-tower group U can be expressed as a tower of successive extensions:

$$U = H^1 \cdot H^2 \cdots H^t = H^1 \rtimes \big(\cdots \rtimes (H^{t-1} \rtimes H^t) \cdots \big), \tag{3.5}$$

where each H^j is a Heisenberg group; that is, $H^j = H_{d_j}$ for some d_j (see Section 2.1).

Remarks

(1) The name H-tower is chosen so that it reminds the reader that the group is a tower of extensions by Heisenberg groups.
(2) Real H-tower groups form a subclass of OKP groups defined in [HRW].
(3) The number t in (3.5) is referred to as the *height* of the tower of extensions. It is denoted by $\mathrm{ht}(U)$.
(4) Consider an H-tower group U that is expressed as in (3.5). For any $j \in \{1, \ldots, \mathrm{ht}(U)\}$, we denote the quotient group $H^j \cdots H^{\mathrm{ht}(U)} \approx U/H^1 \cdots H^{j-1}$ by U_j. U_j is also an H-tower group.

In (3.5), if we take $U = N_\Gamma$ and $H^j = \exp \mathfrak{h}^j$, then by Proposition 3.2.3 we have the following.

PROPOSITION 3.2.6
N_Γ is an H-tower group, and $\mathrm{ht}(N_\Gamma) = s$, *where s is as in Proposition 3.2.3.*

4. Rankable representations of H-tower groups

4.1. Oscillator extension and rankable representations

Fix an H-tower group U, expressed as a tower of successive extensions as in (3.5). Let χ_1 be a nontrivial additive character of $\mathbb{F}$, and consider the unitary representation ρ_1 of H^1 with central character χ_1 (see Section 2.2). When restricted to the inverse image (in the metaplectic group) of the maximal unipotent subgroup of the symplectic group,

*This means that there is an $\mathbb{F}$-homomorphism of algebraic groups $\Phi : \mathbf{U}' \mapsto \mathbf{Sp}(\mathbf{U}''/\mathscr{Z}(\mathbf{U}''))$, and consequently, $\Phi(U') \subseteq \mathrm{Sp}(U''/\mathscr{Z}(U''))$.

the Weil representation factors through a representation of the maximal unipotent subgroup of the symplectic group. (In fact, the two-fold central extension of the symplectic group splits over the maximal unipotent subgroup, and, more importantly, this splitting is unique. This holds because any two splittings would differ by a finite-order character, whereas the maximal unipotent subgroup is a divisible group and therefore it does not have such characters.) Therefore any representation ρ of any Heisenberg group H_n with central character χ (see Section 2.2) is extendable to the unipotent radical of any Borel subgroup of the symplectic group $\mathrm{Sp}(W_n)$. This implies that the representation ρ_1 of H^1 is extendable (in at least one way) to U. We still denote this extension by ρ_1. Note that the extension of ρ_1 is not necessarily unique as one can, for example, twist it by a one-dimensional representation of U which is trivial on H^1. However, since the extension of ρ to the metaplectic group is unique, we can distinguish the extension of ρ_1 obtained by the restriction of the Weil representation without ambiguity. Let ρ_1 mean this specific extension.

In a similar fashion, for any $j > 1$ one can construct a representation of

$$U_j = U/H^1 \cdots H^{j-1} \approx H^j \cdots H^{\mathrm{ht}(U)}$$

as follows. We take an arbitrary nontrivial additive character χ_j of $\mathbb{F}$ and take the representation ρ_j of H^j with central character χ_j. As before, we use the Weil representation to extend ρ_j to U_j. Since

$$U_j \approx U/H^1 \cdots H^{j-1},$$

this representation can be extended to U, trivially on $H^1 \cdots H^{j-1}$. Thus, using ρ_j, we have constructed a representation of U. We still keep the notation ρ_j for this representation.

Definition 4.1.1
Let U be an H-tower group described as in (3.5). A representation of U is called *rankable* if and only if it is unitarily equivalent to a tensor product of the form

$$\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_k$$

for some $k \leq \mathrm{ht}(U)$, where each ρ_j is the representation of U which is obtained (using the Weil representation) by extending the irreducible representation of H^j with central character χ_j. The rank of a k-fold tensor product (including the case $k = 0$, i.e., the trivial representation) is defined as k. If σ is a rankable representation of rank k, we write $\mathrm{rank}(\sigma) = k$.

Remark. It follows from Corollary 4.2.3 that $\mathrm{rank}(\sigma)$ is well defined (i.e., it is an invariant of the unitary equivalence class of σ).

4.2. Kirillov theory for rankable representations

We denote by $\mathcal{O}_\sigma^*$ the coadjoint orbit attached to an irreducible unitary representation σ of a nilpotent group (see [Kr1], [Kr3, Chapter 3], [CG], or [Kr2, Section 2.2] for elaborated treatments of Kirillov's orbital theory). In his seminal article [Kr1], Kirillov developed his method of orbits for simply connected nilpotent real Lie groups. Later, however, in his 1966 International Congress of Mathematicians lecture in Moscow, he explained that essentially the same theory can be applied to algebraic unipotent groups over p-adic fields (see [GK]; see also [Mo2, Section 4] for more details).

LEMMA 4.2.1

Let N be the group of $\mathbb{F}$-rational points of a unipotent linear algebraic $\mathbb{F}$-group $\mathbf{N}$. Let σ be an irreducible unitary representation of N, and let $\mathcal{O}_\sigma^*$ be the coadjoint orbit attached to it. Then $\mathcal{O}_\sigma^*$ is an analytic manifold in the sense of [Se, Chapter 3, Section 2].

Proof

The lemma follows immediately from [PR, Section 3.1, Corollary 2]. □

Lemma 4.2.1 implies that one can speak of the dimension of a coadjoint orbit. (The transitivity under the group action implies that the dimension is the same around every point.) For any coadjoint orbit $\mathcal{O}^*$, let $\dim \mathcal{O}^*$ denote its dimension.

LEMMA 4.2.2

Let $\mathbf{N}$ be a unipotent algebraic group such that $\mathbf{N} = \mathbf{N}' \ltimes \mathbf{N}''$ as algebraic $\mathbb{F}$-groups (and the action of $\mathbf{N}'$ on $\mathbf{N}''$ is defined over $\mathbb{F}$), and let $N = N' \ltimes N''$ be the group of $\mathbb{F}$-rational points of $\mathbf{N}$ (where N', N'' are the $\mathbb{F}$-points of $\mathbf{N}'$, $\mathbf{N}''$). Let the Lie algebras of N', N'' be $\mathfrak{n}'$, $\mathfrak{n}''$, respectively. The Lie algebra of the semidirect product $N = N' \ltimes N''$ is $\mathfrak{n} = \mathfrak{n}' \oplus \mathfrak{n}''$ as a vector space, and we have a canonical isomorphism of dual spaces $\mathfrak{n}^* = \mathfrak{n}'^* \oplus \mathfrak{n}''^*$. Let σ', σ'' be, respectively, irreducible unitary representations of N', N'' such that

(1) *σ' is an irreducible unitary representation of N', extended trivially on N'' to a representation $\tilde{\sigma}'$ of N;*

(2) *σ'' is an irreducible unitary representation of N'' which extends to a unitary representation $\tilde{\sigma}''$ of N. (In other words, $\mathrm{Res}_{N''}^{N} \tilde{\sigma}'' = \sigma''$.)*

Then $\tilde{\sigma}' \otimes \tilde{\sigma}''$ is an irreducible representation of N, and

$$\dim \mathcal{O}_{\tilde{\sigma}' \otimes \tilde{\sigma}''}^* = \dim \mathcal{O}_{\sigma'}^* + \dim \mathcal{O}_{\sigma''}^*.$$

Proof
Consider the map $\mathbf{j}$ defined as the composition

$$N \mapsto N \times N \mapsto N/N'' \times N \approx N' \times N,$$

where the leftmost map is the diagonal embedding and the middle map (i.e., the map $N \times N \mapsto N/N'' \times N$) is the projection onto the first factor (i.e., it is given by $(n_1, n_2) \mapsto (n_1 N'', n_2)$). Observe that

$$\tilde{\sigma}' \otimes \tilde{\sigma}'' = \mathrm{Res}^{N' \times N}_{\mathbf{j}(N)} \sigma' \hat{\otimes} \tilde{\sigma}''. \tag{4.1}$$

It is easily seen by Mackey theory (see [M2, Theorem 3.12]) that $\tilde{\sigma}' \otimes \tilde{\sigma}''$ is irreducible. Now take a maximal chain of analytic subgroups

$$N' \times N = N^0 \supset N^1 \supset N^2 \supset \cdots \supset \mathbf{j}(N)$$

such that each N^j has codimension 1 in N^{j-1}. It follows from (4.1) that if $\sigma' \otimes \tilde{\sigma}''$ is considered as a representation of N^{j-1}, then

$$\mathrm{Res}^{N^{j-1}}_{N^j} \sigma' \otimes \tilde{\sigma}''$$

is an irreducible representation of N^j. Now it follows from the statement of [CG, Theorem 2.5.1] that for any such j, we are in the situation of part (b) of that theorem. But then [CG, Proposition 1.3.4] implies that in this case, the dimension of the coadjoint orbit of $\sigma' \otimes \tilde{\sigma}''$ considered as a representation of N^j is the same as the dimension of the coadjoint orbit of $\sigma' \otimes \tilde{\sigma}''$ considered as a representation of N^{j-1}. Consequently,

$$\dim \mathcal{O}^*_{\sigma' \hat{\otimes} \tilde{\sigma}''} = \dim \mathcal{O}^*_{\tilde{\sigma}' \otimes \tilde{\sigma}''}.$$

Now consider the map $\mathbf{q}$ such that

$$\mathbf{q} : N' \times N'' \mapsto N' \times N,$$

which is defined to be the identity map $N' \mapsto N'$ in the first component and the injection $N'' \subset N$ in the second component. Then

$$\mathrm{Res}^{N' \times N}_{\mathbf{q}(N' \times N'')} \sigma' \hat{\otimes} \tilde{\sigma}'' = \sigma' \hat{\otimes} \sigma'',$$

which is an irreducible representation. Again, a similar argument (taking a maximal chain, etc.) proves that

$$\dim \mathcal{O}^*_{\sigma' \hat{\otimes} \tilde{\sigma}''} = \dim \mathcal{O}^*_{\sigma' \hat{\otimes} \sigma''}.$$

But the coadjoint orbit $\mathcal{O}^*_{\sigma' \hat{\otimes} \sigma''}$ equals $\mathcal{O}^*_{\sigma'} \times \mathcal{O}^*_{\sigma''}$, which completes the proof. □

One can apply Lemma 4.2.2 to any H-tower group iteratively and obtain the following result.

COROLLARY 4.2.3
Let U be an H-tower group as in (3.5). Let

$$\sigma = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_k$$

be a rankable representation of U. (Recall that each ρ_j is extended from a representation of H^j with central character χ_j.) Then σ is irreducible. Moreover, as an analytic manifold, the coadjoint orbit $\mathcal{O}^*_\sigma$ has dimension $2(n_1 + \cdots + n_k)$, where

$$n_j = \frac{\dim \mathfrak{h}^j - 1}{2}.$$

5. Main theorems

5.1. Technical issues of central extensions
Let G, $\mathbf{G}$, $\mathfrak{g}$ be as in Section 1.2 and such that $\mathfrak{g}$ satisfies condition (3.1). Let P be the Heisenberg parabolic of G (see Definition 3.1.2). Suppose that $P = LN$ is the Levi decomposition of P. Therefore N is a Heisenberg group. As in Section 2.2, let ρ be the irreducible unitary representation of N with an arbitrary nontrivial central character χ. Let N_Γ be the H-tower subgroup of G constructed in Section 3.2, and assume that $\mathrm{ht}(N_\Gamma) > 1$.

This section is mainly devoted to a technical issue that arises with central extensions of G. The main result is Proposition 5.1.1. To avoid being distracted from the main point of the article, the reader may assume Proposition 5.1.1 and go on to Section 5.2.

Proposition 3.1.1 implies that the adjoint action provides a group homomorphism $[L, L] \mapsto \mathrm{Sp}(N/\mathscr{Z}(N))$. Our next aim is to somehow apply Proposition 2.3.1 and extend ρ to a representation of a larger group which at least contains N_Γ. One natural candidate may be $[P, P]$. However, when $\mathbb{F} \neq \mathbb{C}$, in order to extend the representation ρ of N to $[P, P]$ we need to know that indeed the group homomorphism $[L, L] \mapsto \mathrm{Sp}(N/\mathscr{Z}(N))$ is a composition of a group homomorphism into the *metaplectic* group; that is,

$$[L, L] \mapsto \mathrm{Mp}\big(N/\mathscr{Z}(N)\big),$$

and the projection map

$$\mathrm{Mp}\big(N/\mathscr{Z}(N)\big) \mapsto \mathrm{Sp}\big(N/\mathscr{Z}(N)\big).$$

This issue was also dealt with in [To] (see [GS, Section 3.3]). We explain it more clearly below. In fact, it can be seen that in some cases, if $\mathbf{P}_\mathbb{F} = \mathbf{L}_\mathbb{F}\mathbf{N}_\mathbb{F}$ is the Heisenberg parabolic of $\mathbf{G}_\mathbb{F}$, then $[\mathbf{L}_\mathbb{F}, \mathbf{L}_\mathbb{F}]$ does *not* act as a subgroup of the metaplectic group, and therefore extending ρ to $[\mathbf{P}_\mathbb{F}, \mathbf{P}_\mathbb{F}]$ may not be possible. An obvious example is when $\mathbf{G}_{\mathbb{F}}$ is the symplectic group. A less obvious example is the split group of type $\mathbf{F}_4$. (Actually, in both cases, the extension is possible if we use the metaplectic covers. This may be seen after some simple calculations. For real groups, one can use the results of [A], which tell us when a covering of a real simple Lie group splits over embedded root subgroups $\mathrm{SL}_2(\mathbb{R})$. However, here we do not rely on such calculations.)

To overcome this difficulty, we see below that we have to consider an appropriate finite (topological) central extension of G instead of G itself. Note that a representation of G can be considered trivially as a representation of its central extension, and we can always study this extension instead of the original G because the central extension has an H-tower subgroup too, which is identical to N_Γ and is essentially obtained by exponentiating the corresponding nilpotent subalgebra $\mathfrak{n}_\Gamma$ of $\mathfrak{g}_{\mathbb{F}}$. This follows from what was explained in Section 1.2 as well, that a universal topological central extension of $\mathbf{G}_{\mathbb{F}}$ splits over the maximal unipotent subgroup (see [Du, Chapitre II, Lemme 11] or [D, Section 1.9]). (In fact, the splitting is unique because any two splittings differ by a finite-order character, whereas any unipotent subgroup is a divisible group and therefore does not have such characters.) Therefore, as far as it concerns the main results of this work, we can substitute G with such a "good" central extension of G, without any loss of generality.

To show the existence of this "good" central extension for an arbitrary G, we first show its existence for the group $\mathbf{G}_{\mathbb{F}}$. That is to say, we show that we can find a finite topological central extension $\tilde{\mathbf{G}}_{\mathbb{F}}$ of $\mathbf{G}_{\mathbb{F}}$ in which the representation ρ of N can be extended to a larger subgroup. We warn the reader that we may (and do) think of N as a subgroup of both $\mathbf{G}_{\mathbb{F}}$ and $\tilde{\mathbf{G}}_{\mathbb{F}}$ at the same time since the latter group has a subgroup identical to the subgroup N of the former one.

Note that once we find $\tilde{\mathbf{G}}_{\mathbb{F}}$, a simple argument implies that the required extension exists for G as well. In fact, the existence of a universal topological central extension of $\mathbf{G}_{\mathbb{F}}$ implies that there is a finite topological central extension that covers both G and $\tilde{\mathbf{G}}_{\mathbb{F}}$. This extension is the one in which the extension of ρ to a larger subgroup is possible.

Next, we show that we can find $\tilde{\mathbf{G}}_{\mathbb{F}}$. Let

$$1 \longrightarrow F \xrightarrow{\hat{\imath}} \tilde{\mathbf{G}}_{\mathbb{F}} \xrightarrow{\hat{p}} \mathbf{G}_{\mathbb{F}} \longrightarrow 1$$

be a finite topological central extension of $\mathbf{G}_{\mathbb{F}}$. Let $\mathbf{P} = \mathbf{LN}$ be the Heisenberg parabolic of $\mathbf{G}$ with its usual Levi decomposition. Let $L = \hat{p}^{-1}(\mathbf{L}_{\mathbb{F}})$, and let N be the unipotent radical of $P = \hat{p}^{-1}(\mathbf{P}_{\mathbb{F}})$. The group $[\mathbf{L}, \mathbf{L}]$, the derived subgroup of $\mathbf{L}$, is a product of isotropic and anisotropic factors (see [To, Section 8.19]). Let $\mathbf{L}^{\mathrm{iso}}$ be the isotropic factor of $[\mathbf{L}, \mathbf{L}]$. Let $\mathbf{M}$ be the simply connected subgroup of $[\mathbf{L}, \mathbf{L}]$ whose Lie algebra is equal to $\mathfrak{m}$, where $\mathfrak{m}$ is the Lie algebra introduced in Definition 3.2.1. (The simple connectedness of $\mathbf{M}$ follows from [To, Section 8.19].) It follows from $\mathrm{ht}(N_\Gamma) > 1$ that $\mathbf{M} \subseteq \mathbf{L}^{\mathrm{iso}}$. Let $M = \hat{p}^{-1}(\mathbf{M}_{\mathbb{F}})$.

Recall that the adjoint action provides a map

$$M \mapsto \mathrm{Sp}\big(N/\mathscr{Z}(N)\big)$$

(see Proposition 3.1.1). Let us call the inverse image of any subgroup M' of M inside the metaplectic group $\mathrm{Mp}(N/\mathscr{Z}(N))$ the *metaplectic extension* of M'. It follows from [To, Sections 8.25, 9.3] that if the $\mathbb{F}$-points of the minimal nilpotent $\mathbf{G}$-orbit of $\mathfrak{g}$ contain a $\tilde{\mathbf{G}}_{\mathbb{F}}$-admissible orbit (see [To, Section 3.17] for the definition of an admissible orbit), then there exists a closed subgroup M' of M, which is a normal subgroup of L, such that $\hat{p}(M') \supseteq \mathbf{M}_{\mathbb{F}}^{+}$ (where $\mathbf{M}_{\mathbb{F}}^{+}$ is the subgroup of $\mathbf{M}_{\mathbb{F}}$ generated by its $\mathbb{F}$-rational unipotent elements) and the metaplectic extension of M' is trivial. (Note that, in fact, $\hat{p}(M') \supseteq \mathbf{M}_{\mathbb{F}}$ because by [To, Section 8.19], $\mathbf{M}$ is simply connected and therefore $\mathbf{M}_{\mathbb{F}}^{+} = \mathbf{M}_{\mathbb{F}}$ (see [PR, Section 7.2]).) This means that one can extend ρ to $M' \cdot N$. We keep the notation ρ for such an extension. We see below that the extension of ρ is actually uniquely determined on a slightly smaller group, and therefore using ρ to denote the extension is not ambiguous.

Tables at the end of [To] determine which groups $\mathbf{G}_{\mathbb{F}}$ have finite topological central extensions $\tilde{\mathbf{G}}_{\mathbb{F}}$ with admissible orbits. Among all $\mathbb{F}$-forms of groups $\mathbf{G}$ whose Lie algebras $\mathfrak{g}$ satisfy condition (3.1), the only groups that do not have coverings with admissible orbits are groups of Tits index $B_{r,r}$ and $B_{r,r-1}$ in the non-Archimedean case and $\mathrm{SU}(p,q)$ and $\mathrm{SO}(p,q)$ when $p+q$ is odd in the Archimedean case. In these cases, however, the metaplectic extension of $\mathbf{M}_{\mathbb{F}}$ is trivial by [L1, Lemma 2.2] and [K, Lemma 7]. Therefore if $M' = \mathbf{M}_{\mathbb{F}}$, then ρ extends to M'. This settles the issue of existence of an appropriate covering for $\mathbf{G}_{\mathbb{F}}$.

For a general G, we can take $\tilde{G}$ as a finite topological central extension that covers both G and $\tilde{\mathbf{G}}_{\mathbb{F}}$. Let M_0 be the inverse image of M' in $\tilde{G}$. Obviously, extending ρ to $M_0 \cdot N$ is possible. M_0 has a closed finite-index subgroup M_1 that is perfect (i.e., $[M_1, M_1] = M_1$). M_1 is found as follows. Define the sequences $M^{(0)} = M_0$ and $M^{(i)} = [M^{(i-1)}, M^{(i-1)}]$. Then each $M^{(i)}$ is a subgroup of M' of finite index and the indices are uniformly bounded. (In fact, one upper bound is the size of the kernel of the map $\tilde{G} \mapsto \mathbf{G}_{\mathbb{F}}$.) Therefore there is some $i_\circ$ such that $M^{(i_\circ)} = M^{(i_\circ+1)}$ (i.e., $M^{(i_\circ)}$ is perfect). Let $M_1 = \overline{M^{(i_\circ)}}$, the closure of $M^{(i_\circ)}$ inside M'. Then M_1 is perfect by [To, Section 8.11].

The extension of ρ to $M_1 \cdot N$ is unique. In fact, the projective representation of M_1 is uniquely determined by the relation

$$\rho(m)\rho(n)\rho(m^{-1}) = \rho(mnm^{-1}) \quad \text{for any } n \in N, m \in M_1,$$

and since M_1 is perfect, this projective representation lifts to a linear representation of M_1 in a unique way.

The discussion in the previous paragraph proves the following result.

PROPOSITION 5.1.1

Let $\mathbf{G}$ *be as in Section 1.2 and such that* $\mathfrak{g}$ *satisfies condition (3.1). Let* G *be a finite topological central extension of* $\mathbf{G}_{\mathbb{F}}$ *represented as in (1.1). Let* P, L, N, ρ *be as in the beginning of Section 5.1. Then there exist a finite topological central extension* $\tilde{G}$ *of* $\mathbf{G}_{\mathbb{F}}$ *which covers* G *as in*

$$1 \longrightarrow \tilde{F} \xrightarrow{\tilde{\iota}} \tilde{G} \xrightarrow{\tilde{p}} G \longrightarrow 1$$

and a (closed) subgroup $P_1 = M_1 \cdot \tilde{N}$ *of* $\tilde{G}$ *such that*

(1) $p \circ \tilde{p}(M_1) = \mathbf{M}_{\mathbb{F}}$ *and* $p \circ \tilde{p}(\tilde{N}) = \mathbf{N}_{\mathbb{F}}$*;*

(2) $\tilde{N}$ *is the unipotent radical of* $\tilde{p}^{-1}(P)$ *(i.e.,* $\tilde{p} : \tilde{N} \mapsto N$ *is an isomorphism);*

(3) *the representation* ρ *(which can naturally be thought of as a representation of* $\tilde{N}$ *too) can be extended in a unique way to a representation of* P_1*;*

(4) M_1 *is perfect (i.e.,* $[M_1, M_1] = M_1$*);*

(5) P_1 *is a normal subgroup of* $\tilde{p}^{-1}(P)$.

5.2. Mackey analysis

In this section we assume that the notation is the same as in Section 5.1. We also assume (without loss of generality) that

$$G \text{ is equal to the extension } \tilde{G} \text{ of Proposition 5.1.1, and } [G, G] \text{ is dense in } G. \tag{5.1}$$

Note that if $[G, G]$ is dense in G, then from [To, Section 8.11] it follows that, in fact, $[G, G] = G$. The assumption $[G, G] = G$ is not a crucial one. This is because one can always find a closed finite-index subgroup G_1 of G such that G_1 is perfect and $G = G_1 \cdot \mathscr{Z}(G)$, and once a representation π of G is understood on G_1, it is easy to describe it as a representation of G.

Assume that condition (5.1) holds. Let $P_1 \subset G$ be the subgroup given in the statement of Proposition 5.1.1. Then we can write $P_1 = M_1 \cdot N$, where M_1 is defined in Proposition 5.1.1 and $N \approx \tilde{N}$ is the unipotent radical of the Heisenberg parabolic P of G which appears in parts (2) and (3) of Proposition 5.1.1.

Let π be a unitary representation of G without nonzero G-invariant vectors. Consider the restriction of π to P. Recall that $\mathscr{Z}(N)$ means the center of N. By the Howe-Moore theorem [HM, Theorem 5.1], π does not have a nonzero $\mathscr{Z}(N)$-invariant vector either. Therefore, in the direct integral decomposition of π as a representation of N, the spectral measure is supported on $\hat{N}_\circ$ (see the notation introduced at the end of Section 2.2).

Elements of P_1 commute with elements of $\mathscr{Z}(N)$. Therefore, under the coadjoint action of P on the unitary characters of $\mathscr{Z}(N)$, P_1 lies within $\mathrm{stab}_P(\chi)$, the stabilizer (inside P) of any nontrivial additive unitary character χ of $\mathscr{Z}(N)$. In fact, all these stabilizers are identical groups. We denote this common stabilizer group by J.

The action of P on $\mathscr{Z}(N)$ (and also on its unitary characters) has only a finite number of orbits. Consequently, since π has no nonzero G-invariant vectors, by elementary Mackey theory (see [M2, Theorems 3.11, 3.12]) the restriction of π to P can be expressed as a finite direct sum

$$\pi_{|P} = \bigoplus_i \mathrm{Ind}_J^P \sigma_i,$$

where each σ_i is a representation of J which, when restricted to N, is a direct integral of representations isomorphic to the representation ρ_i of N (defined in Section 2.2) with some central character χ_i. However, each ρ_i extends in exactly one way to a representation of P_1. We still denote this extension by ρ_i. Again, by Mackey theory, we can write the restriction of σ_i to P_1 as

$$\sigma_{i\,|P_1} = \nu_i \otimes \rho_i,$$

where ν_i is indeed a representation of M_1 extended trivially on N to P_1. Therefore we have proved the following result.

LEMMA 5.2.1
Let G, P, L, and N be as in the beginning of Section 5.1 so that G satisfies condition (5.1). Let π be a unitary representation of G without a nonzero G-invariant vector. Then π can be written as a finite direct sum

$$\pi_{|P} = \bigoplus_i \mathrm{Ind}_J^P \sigma_i, \tag{5.2}$$

where each σ_i is a representation of J such that

$$\sigma_{i\,|P_1} = \nu_i \otimes \rho_i \tag{5.3}$$

and ν_i factors to a representation of M_1. (In other words, ν_i is trivial on N.)

Recall the construction of the H-tower subgroup $U = N_\Gamma$ (see (3.4)). In the notation of (3.5), we have $H^1 = N$. The quotient group $U_2 = N_\Gamma/N$ is identical to the H-tower subgroup that is constructed for M. By Proposition 5.1.1, the latter H-tower group should lie within M_1.

LEMMA 5.2.2
Let ρ be the extension to P_1 of the irreducible representation of N (which is also denoted by ρ) with an arbitrary nontrivial central character χ (see Section 2.2). Let N_Γ be as in (3.4). Then the restriction of the representation ρ to N_Γ is supported on rankable representations of N_Γ of rank one.

Proof
The uniqueness of extension of ρ to P_1 implies that on M_1, this extension should be identical to the extension obtained by the restriction of extension of ρ to the metaplectic group acting on $N/\mathscr{Z}(N)$. Therefore the lemma follows from Definition 4.1.1. $\square$

Notation. Let K be an arbitrary (abstract) group, and let K' be a subgroup of K. Let σ be a representation of K' on a vector space $\mathscr{H}$. Let $a \in K$. Then by σ^a we mean a representation of the subgroup $K'_a = aK'a^{-1}$ on $\mathscr{H}$ defined as follows:

$$\text{for every } x \in K'_a, \quad \sigma^a(x) = \sigma(a^{-1}xa). \tag{5.4}$$

In particular, if $K' = K$, then σ^a is a representation of K.

At this point we recall Mackey's subgroup theorem. We use it in Sections 5.4 and 5.5. Giving a precise statement would require some definitions and would distract us from the main point of this article. Therefore, for a detailed discussion, we refer the reader to [M2, Theorem 3.5] or [M1, Theorem 12.1].

THEOREM 5.2.3 (Mackey's subgroup theorem)
Let K be a locally compact group, and let K', K'' be its closed subgroups. Assume that K, K', K'' are "nice" (see the above references for details). Let σ be an irreducible representation of K'. Then the representation

$$\mathrm{Res}^K_{K''} \mathrm{Ind}^K_{K'} \sigma$$

has a direct integral decomposition supported on representations τ_v, $v \in K$, where

$$\tau_v = \mathrm{Ind}^{K''}_{K'' \cap K'_v} \mathrm{Res}^{K'_v}_{K'' \cap K'_v} \sigma^v.$$

Here

$$K'_v = vK'v^{-1},$$

and σ^v is defined as in (5.4).

The following elementary lemma, which essentially follows from Mackey theory too, helps us in Sections 5.4 and 5.5

LEMMA 5.2.4

Let $K' \subset K$ be two arbitrary locally compact groups such that K' is a closed normal subgroup of K. Let σ be a unitary representation of K which acts on the Hilbert space $\mathscr{H}$. Suppose that

$$\mathrm{Res}_{K'}^{K}\,\sigma = \sigma_1 \oplus \sigma_2,$$

where σ_1 and σ_2 are unitary representations of K' such that for any $a \in K$, we have

$$\mathrm{Hom}_{K'}(\sigma_1^a, \sigma_2) = \{0\}.$$

(Here σ_1^a is defined as in (5.4), and $\mathrm{Hom}_{K'}(\sigma_1^a, \sigma_2)$ means the space of K'-intertwining operators from σ_1^a to σ_2.) For $i \in \{1, 2\}$, let $\mathscr{H}_i$ be the closed $\sigma(K')$-invariant subspace of $\mathscr{H}$ which corresponds to the summand σ_i. Then each $\mathscr{H}_i$ is actually invariant under $\sigma(K)$.

Proof

Let $a \in K$. Then the linear operator $\mathscr{T}$ defined as

$$\mathscr{T} : \mathscr{H}_1 \mapsto \mathscr{H},$$
$$\mathscr{T}(w) = \sigma(a)w \quad \text{for all } w \in \mathscr{H}_1$$

is an element of $\mathrm{Hom}_{K'}(\sigma_1^a, \sigma)$. The orthogonal projection $\mathscr{P}$ of $\mathscr{H}$ onto $\mathscr{H}_2$ is an element of $\mathrm{Hom}_{K'}(\sigma, \sigma_2)$. By the hypothesis of the lemma, one should have $\mathscr{P} \circ \mathscr{T} = 0$. Therefore $\sigma(a)\mathscr{H}_1 \subseteq \mathscr{H}_1$. □

5.3. Statements of the main theorems

We state the main theorems in this ection and prove them in Sections 5.4 and 5.5.

Throughout this section, we assume that G, $\mathbf{G}_{\mathbb{F}}$, and $\mathfrak{g}$ are as in Section 1.2 and that $\mathfrak{g}$ satisfies condition (3.1). Moreover, we assume that N_Γ is the H-tower subgroup of G (see equality (3.4)), described as in (3.5).

THEOREM 5.3.1

Let π be a unitary representation of G. Then in the direct integral decomposition

$$\pi_{|N_\Gamma} = \int_{\hat{N}_\Gamma}^{\oplus} \tau \, d\mu(\tau),$$

the support of the spectral measure μ is inside the subset of rankable representations of N_Γ.

In the next theorem the subset of rankable representations of rank k of N_Γ is denoted by $\hat{N}_\Gamma(k)$.

THEOREM 5.3.2

Let π be a unitary representation of G on a Hilbert space $\mathscr{H}_\pi$. Consider the direct integral decomposition

$$\pi_{|N_\Gamma} = \int_{\hat{N}_\Gamma}^{\oplus} \tau \, d\mu(\tau),$$

and let $\mathscr{P}_\mu$ be the projective measure corresponding to this decomposition. Let $\hat{N}_\Gamma(\pi)$ be the support of π in this direct integral decomposition. Set

$$\mathscr{H}_\pi^j = \mathscr{P}_\mu\big(\hat{N}_\Gamma(j) \cap \hat{N}_\Gamma(\pi)\big) \cdot \mathscr{H}_\pi \quad \textit{for any } 0 \leq j \leq \mathrm{ht}(N_\Gamma).$$

Then for every j such that

$$j \in \big\{0, 1, 2, \dots, \mathrm{ht}(N_\Gamma)\big\},$$

$\mathscr{H}_\pi^j$ is a G-invariant subspace of $\mathscr{H}_\pi$. The direct sum of all these subspaces is equal to $\mathscr{H}_\pi$.

Remark. It is clear that for $\mathscr{P}_\mu(\hat{N}_\Gamma(j) \cap \hat{N}_\Gamma(\pi))$ to make sense, we need to show that the sets $\hat{N}_\Gamma(j) \cap \hat{N}_\Gamma(\pi)$ are indeed Borel subsets of $\hat{N}_\Gamma$. In fact, one can show that the set of rankable representations of a given rank of an H-tower group $U = H^1 \cdots H^t$ can be constructed with a finite number of set-theoretic operations on open and closed sets of the unitary dual of U. Here is a sketch of the proof. Any rankable representation σ is a tensor product of the form

$$\rho_1 \otimes \cdots \otimes \rho_t$$

such that $\rho_1, \dots, \rho_{\mathrm{rank}(\sigma)}$ are extensions of representations (with nontrivial central characters) of $H^1, \dots, H^{\mathrm{rank}(\sigma)}$, respectively, and the rest of the ρ_j's are trivial. The first requirement imposes open conditions on the subset of rankable representations of a given rank, whereas the second requirement imposes a closed condition.

Definition 5.3.3

A unitary representation π of G is called *pure-rank* if its restriction to N_Γ is a direct integral of rankable representations of a fixed rank. The common rank of these rankable representations is called the *rank of π*.

Although Theorem 5.3.2 is slightly stronger than Corollary 5.3.4, we state it in order to clarify the analogy between our new theory and the older one.

COROLLARY 5.3.4
Let π be an irreducible representation of G. Then π is pure-rank.

5.4. Proof of Theorem 5.3.1

In this section we prove Theorem 5.3.1. Without loss of generality, we can assume that condition (5.1) holds. Theorem 5.3.1 is proved by induction on $\mathrm{ht}(N_\Gamma)$. Let N_Γ be described as in (3.5). By the Howe-Moore theorem in [HM], when $\mathrm{ht}(N_\Gamma) = 1$, there is nothing to prove. Now let $\mathrm{ht}(N_\Gamma) > 1$. Let P, L, M, and N be as in Section 5.1. Let P_1, M_1 be as in Proposition 5.1.1. The H-tower subgroup that is associated to M is $N_2 = N_\Gamma/N$, and it lies within M_1. But $\mathrm{ht}(N_2) = \mathrm{ht}(N_\Gamma)-1$, and therefore Theorem 5.3.1 holds for M_1.

Without loss of generality, we can assume that π has no nonzero G-fixed vectors. Let σ_i's be the representations that appear in the decomposition of π using Lemma 5.2.1. By a straightforward application of Mackey's subgroup theorem (see Theorem 5.2.3), or even more directly (with only little difficulty) by writing the definition of the induced representation $\mathrm{Ind}_J^P \sigma_i$ explicitly, one can see that the representation

$$\mathrm{Res}_{P_1}^P \mathrm{Ind}_J^P \sigma_i$$

is supported on representations of the form

$$\mathrm{Res}_{P_1}^J \sigma_i^x, \quad x \in P,$$

where σ_i^x is defined as in (5.4); that is,

$$\text{for any } y \in J, \quad \sigma_i^x(y) = \sigma_i(x^{-1}yx). \tag{5.5}$$

Note that J is a normal subgroup of P because it contains $[P, P]$.

By (5.3), $\mathrm{Res}_{P_1}^J \sigma_i^x$ is unitarily equivalent to $\nu_i^x \otimes \rho_i^x$, where ν_i^x and ρ_i^x are defined similarly to (5.5). Theorem 5.3.1 follows from

(1) Lemma 5.2.2 applied to ρ_i^x;
(2) the fact that by induction hypothesis, ν_i^x is supported on rankable representations of $N_2 = N_\Gamma/N$; and
(3) Definition 4.1.1. □

5.5. Proof of Theorem 5.3.2

We now proceed toward proving Theorem 5.3.2. Without loss of generality, we can assume that condition (5.1) holds. Let P, L, N, and M be as in Section 5.1. Let P_1, M_1 be as in Proposition 5.1.1. The main idea behind the proof is that the Kirillov orbits of representations of different rank have different dimensions. We apply some basic Kirillov theory.

Recall from Section 5.3 that G, $\mathfrak{g}$, $\mathfrak{g}_{\mathbb{F}}$ are as in Section 1.2 and that $\mathfrak{g}$ satisfies condition (3.1). Let Σ, Σ^+, and Σ_B be as in Section 1.2 too. Let $\tilde{\beta}$ be the highest root of $\mathfrak{g}$ (see Section 1.2), and let β be a simple restricted root such that $(\beta, \tilde{\beta}) = 1$. Recall that

$$P_{\{\beta\}} = L_{\{\beta\}} N_{\{\beta\}} \tag{5.6}$$

is a standard parabolic subgroup of G associated to $\{\beta\}$ (see Section 1.2). Let N_Γ be the H-tower subgroup of G, and suppose that $\mathfrak{n}_\Gamma$ is its Lie algebra. Let $N_{\{\beta\}}$ be as in equation (5.6), and suppose that $\mathfrak{n}_{\{\beta\}}$ is its Lie algebra. We consider $\mathfrak{n}_\Gamma$ and $\mathfrak{n}_{\{\beta\}}$ as subalgebras of $\mathfrak{g}_{\mathbb{F}}$. Recall that for any $\gamma \in \Sigma$, $(\mathfrak{g}_{\mathbb{F}})_\gamma$ denotes the restricted root subspace of $\mathfrak{g}_{\mathbb{F}}$ associated to γ. Our next aim is to define a subgroup N_Γ^β of N_Γ in case $\mathrm{ht}(N_\Gamma) > 1$.

Let L_Γ be as in equation (3.4), with Lie algebra $\mathfrak{l}_\Gamma$ where $\mathfrak{l}_\Gamma \subset \mathfrak{g}_{\mathbb{F}}$, and let

$$\Sigma^L = \big\{\gamma \in \Sigma \,\big|\, (\mathfrak{g}_{\mathbb{F}})_\gamma \subset \mathfrak{l}_\Gamma\big\}.$$

Define

$$\mathfrak{n}_\Gamma^\beta = \bigoplus_{\gamma \in S_\Gamma} (\mathfrak{g}_{\mathbb{F}})_\gamma, \tag{5.7}$$

where

$$S_\Gamma = \big\{\gamma \in \Sigma^+ \,\big|\, (\mathfrak{g}_{\mathbb{F}})_\gamma \subset \mathfrak{n}_\Gamma \cap \mathfrak{n}_{\{\beta\}} \text{ and } \gamma - \beta \notin \Sigma^L\big\}. \tag{5.8}$$

One can see that $\mathfrak{n}_\Gamma^\beta$ is a Lie subalgebra of $\mathfrak{g}_{\mathbb{F}}$. To see this, assume that $\gamma, \gamma' \in S_\Gamma$ and $\gamma + \gamma' \in \Sigma^+$. Then since both $\mathfrak{n}_\Gamma$ and $\mathfrak{n}_{\{\beta\}}$ are Lie algebras, $(\mathfrak{g}_{\mathbb{F}})_{\gamma+\gamma'} \subset \mathfrak{n}_\Gamma \cap \mathfrak{n}_{\{\beta\}}$. Moreover, if $\gamma+\gamma'-\beta \in \Sigma^L$, then either $\gamma \in \Sigma^L$ or $\gamma' \in \Sigma^L$, which is a contradiction. Consequently, $\gamma + \gamma' \in S_\Gamma$.

The group N_Γ^β is defined as the subgroup of $N_\Gamma \cap N_{\{\beta\}}$ with Lie algebra $\mathfrak{n}_\Gamma^\beta$. (It is worth mentioning that N_Γ^β is a proper subgroup of $N_\Gamma \cap N_{\{\beta\}}$ if and only if Γ contains an element that is not orthogonal to β in Σ.)

LEMMA 5.5.1
Let $\mathrm{ht}(N_\Gamma) > 1$*, and let* $L_{\{\beta\}}$ *be as in equation (5.6). Let* $M_{\{\beta\}} = [L_{\{\beta\}}, L_{\{\beta\}}]$*, the commutator subgroup of* $L_{\{\beta\}}$*. Then* $M_{\{\beta\}}$ *normalizes* N_Γ^β*.*

Proof
It suffices to show that for any $\gamma \in S_\Gamma$, if $\gamma + \beta \in \Sigma$ (resp., $\gamma - \beta \in \Sigma$), then $\gamma + \beta \in S_\Gamma$ (resp., $\gamma - \beta \in S_\Gamma$). Note that $\gamma - \beta$ cannot be zero.

Assume that $\gamma \in S_\Gamma$ and $\gamma + \beta \in \Sigma$. Since $(\mathfrak{g}_{\mathbb{F}})_\gamma \subset \mathfrak{n}_{\{\beta\}}$, we have $(\mathfrak{g}_{\mathbb{F}})_{\gamma+\beta} \subset \mathfrak{n}_{\{\beta\}}$ too. Similarly, since $(\mathfrak{g}_{\mathbb{F}})_\gamma \subset \mathfrak{n}_\Gamma$, $\gamma + \beta \notin \Sigma^L$ and $(\gamma + \beta) - \beta = \gamma \notin \Sigma^L$, which implies that $\gamma + \beta \in S_\Gamma$.

Next, assume that $\gamma \in S_\Gamma$ and $\gamma - \beta \in \Sigma$. Note that $\gamma - \beta \in \Sigma$ implies $\gamma - \beta \in \Sigma^+$ since $\gamma \in \Sigma^+$. Again, since $(\mathfrak{g}_\mathbb{F})_\gamma \subset \mathfrak{n}_{\{\beta\}}$, we have $(\mathfrak{g}_\mathbb{F})_{\gamma-\beta} \in \mathfrak{n}_{\{\beta\}}$. Moreover, by the definition of $\mathfrak{n}_\Gamma^\beta$, $\gamma - \beta \notin \Sigma^L$ and therefore $(\mathfrak{g}_\mathbb{F})_{\gamma-\beta} \subset \mathfrak{n}_\Gamma$. Finally, if $(\gamma - \beta) - \beta \in \Sigma^L$, then $\gamma = 2\beta + \gamma_1$, where $\gamma_1 \in \Sigma^L$. This means that $(\gamma, \tilde{\beta}) = 2$ (i.e., γ is the highest root). Since any simple restricted root appears in the highest root with a positive coefficient, it follows that Σ_B consists of β and the elements of Γ. Consequently, there is only one simple restricted root outside Γ, which implies that $\mathrm{ht}(N_\Gamma) = 1$, which contradicts our assumption. □

PROPOSITION 5.5.2

Let G be as in Theorem 5.3.2, let N_Γ be the H-tower subgroup of G, and let $\mathrm{ht}(N_\Gamma) > 1$. *Let N_Γ^β be defined as in equation (5.7). Then the restriction of a rankable representation σ of rank k of N_Γ to N_Γ^β is a direct integral of irreducible representations of N_Γ^β whose attached coadjoint orbits have the same dimension equal to* $2(n_1 + \cdots + n_k - c)$, *where c is the codimension of N_Γ^β in N_Γ, and n_i's are defined as in Corollary 4.2.3.*

Proof

Throughout the proof, we assume that $\mathbb{F} = \mathbb{R}$ for simplicity. The proof for other local fields is essentially similar.

We analyze

$$\mathrm{Res}_{N_\Gamma^\beta}^{N_\Gamma}(\rho_1 \otimes \cdots \otimes \rho_k)$$

with the ρ_i's as in Definition 4.1.1. It is easy to see that $N_\Gamma^\beta \supset H^2 \cdots H^{\mathrm{ht}(N_\Gamma)}$, so when $j > 1$, the restriction of ρ_j to N_Γ^β is irreducible. It remains to understand the restriction of ρ_1 to N_Γ^β. The group $H^1 \cap N_\Gamma^\beta$ is a direct product of a Heisenberg group of dimension $2(n_1 - c) + 1$ and a c-dimensional abelian group whose Lie algebra corresponds to the direct sum of restricted root spaces $(\mathfrak{g}_\mathbb{R})_{\tilde{\beta}-\gamma}$ such that $(\gamma, \tilde{\beta}) > 0$ but $\gamma \notin S_\Gamma$. To see why these restricted root spaces form an isotropic subspace of the Heisenberg nilradical of $\mathfrak{g}_\mathbb{R}$, suppose that $\tilde{\beta} - \gamma_1, \tilde{\beta} - \gamma_2$ are given such that for any $i \in \{1, 2\}$, $(\gamma_i, \tilde{\beta}) > 0$ but $\gamma_i \notin S_\Gamma$. If $(\tilde{\beta} - \gamma_1) + (\tilde{\beta} - \gamma_2) = \tilde{\beta}$, then $\tilde{\beta} = \gamma_1 + \gamma_2$. But for any $i \in \{1, 2\}$, $\gamma_i = \beta + \gamma_i'$, where $\gamma_i' \in \Sigma^L$ or $\gamma_i' = 0$. Therefore $\tilde{\beta} = 2\beta + \gamma_1' + \gamma_2'$, which implies that Σ_B consists of β and the elements of Γ (see the proof of Lemma 5.5.1). Consequently, $\mathrm{ht}(N_\Gamma) = 1$, which is a contradiction.

Let the decomposition of $H^1 \cap N_\Gamma^\beta$ as a direct product be $H^\beta \times \mathbb{R}^c$, where H^β is the $(2(n_1 - c) + 1)$-dimensional Heisenberg group and $\mathbb{R}^c$ is the abelian factor. We denote the irreducible representation of H^β with central character χ_1 by ρ_1^β.

LEMMA 5.5.3
Under the foregoing assumptions,

$$\operatorname{Res}^{N_\Gamma}_{H^1\cap N^\beta_\Gamma} \rho_1 = \int_{\mathbb{R}^{c*}} \rho_1^\beta \hat{\otimes} \psi_s \, d\mu(s),$$

where each $\psi_s(t)$, given by $\psi_s(t) = e^{\mathrm{i}s(t)}$ for some $s \in \mathbb{R}^{c*}$, the vector-space dual of $\mathbb{R}^c$, is a unitary character of $\mathbb{R}^c$.

Proof
Clearly,

$$\operatorname{Res}^{N_\Gamma}_{H^1\cap N^\beta_\Gamma} \rho_1 = \operatorname{Res}^{H^1}_{H^\beta\times\mathbb{R}^c} \operatorname{Res}^{N_\Gamma}_{H^1} \rho_1 = \operatorname{Res}^{H^1}_{H^\beta\times\mathbb{R}^c} \rho_1.$$

The space $\mathscr{H}_\rho$ of any representation ρ of the Heisenberg group $H^1 = H_{n_1}$ introduced in Section 2.2 can be written as

$$\begin{aligned}\mathscr{H}_\rho = L^2(\mathfrak{y}) &= L^2(\operatorname{Span}_{\mathbb{R}}\{Y_1, \dots, Y_c\} \oplus \operatorname{Span}_{\mathbb{R}}\{Y_{c+1}, \dots, Y_{n_1}\}) \\ &= L^2(\mathbb{R}^c)\hat{\otimes} L^2(\operatorname{Span}_{\mathbb{R}}\{Y_{c+1}, \dots, Y_{n_1}\}) \approx \int^{\oplus}_{\mathbb{R}^{c*}} L^2_s \, d\mu(s),\end{aligned}$$

where each L^2_s is equal to $L^2(\operatorname{Span}_{\mathbb{R}}\{Y_{c+1}, \dots, Y_{n_1}\})$ on which $\mathbb{R}^c$ acts via the character $\chi_s(x) = e^{\mathrm{i}s(x)}$. In fact, L^2_s is a representation of $H^\beta \times \mathbb{R}^c$. Lemma 5.5.3 is proved. □

Next, we show that $\mathbb{R}^c \subset \mathscr{Z}(N^\beta_\Gamma)$. To see this, take a restricted root space $(\mathfrak{g}_{\mathbb{R}})_{\tilde{\beta}-\gamma}$ that lies inside $\mathbb{R}^c$. It suffices to show that for any $\gamma' \in S_\Gamma$, $(\tilde{\beta}-\gamma)+\gamma' \notin \Sigma$. But we know that $\gamma = \beta+\gamma_1$, where $\gamma_1 \in \Sigma^L$ or $\gamma_1 = 0$. Therefore $(\tilde{\beta}-\gamma)+\gamma' = \tilde{\beta}-\beta-\gamma_1+\gamma'$. Consequently, if $(\tilde{\beta}-\gamma)+\gamma' \in \Sigma$, then either $\gamma' \in \Sigma^L$ or $\gamma'-\beta = \gamma_1 \in \Sigma^L$. However, none of these is possible for γ' by the definition of S_Γ in (5.8).

We have shown that $\mathbb{R}^c \subset \mathscr{Z}(N^\beta_\Gamma)$, and the action of $\mathbb{R}^c$ on each L^2_s is by a distinct character ψ_s (see Lemma 5.5.3). Hence the restriction of ρ_1 to N^β_Γ breaks into a direct integral of a c-parameter family of irreducible representations. Consequently, the same thing happens to any rankable representation

$$\sigma = \rho_1 \otimes \cdots \otimes \rho_k.$$

By [CG, Theorem 2.5.1] and Lemma 4.2.2 applied to N^β_Γ, which is a semidirect product of $H^\beta \times \mathbb{R}^c$ and N_2, it follows that the projection of the coadjoint orbit $\mathscr{O}^*_\sigma$ onto the Lie algebra of N^β_Γ is foliated by subvarieties of codimension $2c$, which are indeed coadjoint orbits of the constituents of the rankable representations in the restriction of σ to N^β_Γ. Proposition 5.5.2 is proved. □

We now concentrate on finishing the proof of Theorem 5.3.2. This theorem is proved by induction on the height $\operatorname{ht}(N_\Gamma)$ of the H-tower group N_Γ. If $\operatorname{ht}(N_\Gamma) = 1$, then there is nothing to prove. Let $\mathrm{ht}(N_\Gamma) > 1$. Then the H-tower subgroup of M_1 is equal to N_2, where

$$N_2 = N_\Gamma / N.$$

Clearly, $\mathrm{ht}(N_2) = \mathrm{ht}(N_\Gamma) - 1$.

Without loss of generality, we can assume that π has no nonzero G-invariant vectors. Consider the decomposition of π given in (5.2). Applying induction hypothesis to M_1, which contains the H-tower subgroup N_2, we can refine this decomposition by expressing each ν_i as a direct sum of its M_1-invariant pure-rank parts (where the rank for a representation of M_1 is naturally defined with respect to N_2). Consequently, as a representation of M_1,

$$\nu_i = \nu_{i,0} \oplus \nu_{i,1} \oplus \cdots \oplus \nu_{i,\mathrm{ht}(N_2)}.$$

Here $\nu_{i,j}$ denotes the part of ν_i supported on rankable representations of N_2 of rank j. (Note that some of the $\nu_{i,j}$'s may be trivial but that this fact does not affect our proof.) Therefore we have

$$\sigma_{i\,|P_1} = (\nu_{i,0} \otimes \rho_i) \oplus \cdots \oplus (\nu_{i,\mathrm{ht}(N_2)} \otimes \rho_i).$$

Let $\mathscr{H}$ be the Hilbert space of the representation σ_i. For any $j \in \{0, 1, 2, \ldots, \mathrm{ht}(N_2)\}$, let $\mathscr{H}_j$ be the subspace of $\mathscr{H}$ which corresponds to $\nu_{i,j} \otimes \rho_i$. Our next task is to prove that each $\mathscr{H}_j$ is, in fact, invariant under $\sigma_i(J)$ (see Lemma 5.2.1). This follows from Lemma 5.2.4 once we prove the following lemma.

LEMMA 5.5.4
Let $\eta = \nu_{i,j} \otimes \rho_i$ and $\eta' = \nu_{i,j'} \otimes \rho_i$, where $j \neq j'$. Let $a \in J$. Then

$$\mathrm{Hom}_{P_1}(\eta^a, \eta') = \{0\},$$

where η^a is a representation of P_1 on $\mathscr{H}_j$ defined as in (5.4); that is,

$$\eta^a(x) = \eta(a^{-1}xa) \quad \textit{for all } x \in P_1.$$

Proof
We actually prove more, that is, that

$$\mathrm{Hom}_{N_\Gamma}(\mathrm{Res}^{P_1}_{N_\Gamma} \eta^a, \mathrm{Res}^{P_1}_{N_\Gamma} \eta') = \{0\}. \tag{5.9}$$

We claim that $\mathrm{Res}^{P_1}_{N_\Gamma} \eta^a$ is a direct integral supported on rankable representations of N_Γ of rank $j + 1$. The claim implies (5.9) because if $j \neq j'$, then it implies that $\mathrm{Res}^{P_1}_{N_\Gamma} \eta^a$ and $\mathrm{Res}^{P_1}_{N_\Gamma} \eta'$ are supported on disjoint subsets of $\hat{N}_\Gamma$. We prove this claim below.

By Lemma 5.2.2, ρ_i^a is supported on rankable representations of N_Γ of rank one. Next, we show that $\nu_{i,j}^a$ is supported on rankable representations of $N_2 = N_\Gamma/N$ of rank j.

As we know, G is a central extension of $\mathbf{G}_\mathbb{F}$. Suppose that this extension is represented as in (1.1).

First, let $a \in N$. Then $a \in N_\Gamma$, and consequently, $\mathrm{Res}_{N_\Gamma}^{P_1} \eta^a$ is unitarily equivalent to $\mathrm{Res}_{N_\Gamma}^{P_1} \eta$. Since $J \subset P$ and any element of P is a product of an element of N and an element of $p^{-1}(\mathbf{L}_\mathbb{F})$, it follows that it suffices to assume that $p(a) \in \mathbf{L}_\mathbb{F}$.

It follows from $p(a) \in \mathbf{L}_\mathbb{F}$ that $p(a)\mathbf{M}_\mathbb{F} p(a^{-1}) = \mathbf{M}_\mathbb{F}$. The group $p(P_\Gamma)$ is the $\mathbb{F}$-points of the $\mathbb{F}$-parabolic $\mathbf{P}_{\Gamma'}$ of $\mathbf{G}$. Now $\mathbf{P_m} = \mathbf{P}_{\Gamma'} \cap \mathbf{M}$ is an $\mathbb{F}$-parabolic of $\mathbf{M}$. Since $p(a) \in \mathbf{L}_\mathbb{F}$, $p(a)\mathbf{P_m} p(a^{-1})$ is another $\mathbb{F}$-parabolic of $\mathbf{M}$, and therefore by [B, Theorem 20.9], these two parabolics are conjugate under an element $p(b)$ of $\mathbf{M}_\mathbb{F}$, where $b \in M_1$. (Recall from Proposition 5.1.1 that $p(M_1) \supseteq \mathbf{M}_\mathbb{F}$.) This means that $p(ba)\mathbf{P_m} p(ba)^{-1} = \mathbf{P_m}$. If $\mathbf{U_m}$ is the unipotent radical of $\mathbf{P_m}$, then $p(ba)\mathbf{U_m} p(ba)^{-1} = \mathbf{U_m}$, which implies that $baN_2a^{-1}b^{-1} = N_2$.

Let $c = ba$. Then $c \in P$. Let $\mathfrak{n}_2 \subset \mathfrak{n}_\Gamma$ be the Lie algebra of N_2. Consider $\mathrm{Ad}^*(c)$ as a linear map from the dual of the Lie algebra of P to itself. Obviously, $\mathrm{Ad}^*(c)(\mathfrak{n}_2^*) = \mathfrak{n}_2^*$. Let τ be an irreducible unitary representation of N_2, and let $\mathcal{O}_\tau^*$ be the coadjoint orbit associated to τ. Then the coadjoint orbit associated to τ^c is $\mathrm{Ad}^*(c)(\mathcal{O}_\tau^*)$. This fact follows, for instance, from [Du, Chapitre III, Section 11]. A short proof of this fact can be obtained by an adaptation of the proof of Lemma 5.5.6.

For simplicity, let $\nu = \nu_{i,j}$ for fixed i, j. The representation $\mathrm{Res}_{N_2}^{M_1} \nu$ is supported on rankable representations of N_2 whose associated coadjoint orbits have dimension

$$n_2 + \cdots + n_{j+1}$$

(see Corollary 4.2.3 for the definition of n_i's).

Let ν^a, ν^b, ν^c be representations of M_1 defined as in (5.4). (Note that $aM_1a^{-1} = M_1$ because P_1 is normal in $p^{-1}(\mathbf{P}_\mathbb{F})$.) Since $b \in M_1$, any representation θ of M_1 is obviously unitarily equivalent to θ^b. Therefore ν^c is unitarily equivalent to ν^a. Since the action of $\mathrm{Ad}^*(c)$ on $\mathfrak{n}_2^*$ is linear, it does not change the dimension of coadjoint orbits, and therefore $\mathrm{Res}_{N_2}^{M_1} \nu^c$ is supported on unitary representations of N_2 whose associated coadjoint orbits have dimension $n_2 + \cdots + n_{j+1}$.

Since ν^c is a unitary representation of M_1, by Theorem 5.3.1 all representations in the support of its restriction to N_2 should be rankable. Therefore $\mathrm{Res}_{N_2}^{M_1} \nu^c$ is supported on rankable representations of N_2 of rank j. Now ν^a is unitarily equivalent to ν^c as a representation of M_1 and hence as a representation of N_2. Therefore ν^a is also supported on rankable representations of N_2 of rank j. Definition 4.1.1 completes the proof of our claim. The proof of Lemma 5.5.4 is complete. □

As mentioned before, Lemma 5.5.4 implies that each of the subspaces $\mathscr{H}_j$ (which corresponds to the representation $\nu_{i,j} \otimes \rho_i$ of P_1) is invariant under $\sigma_i(J)$.

LEMMA 5.5.5
There is a P-invariant direct sum decomposition of π such as

$$\pi_{|P} = \pi_1 \oplus \cdots \oplus \pi_{\mathrm{ht}(N_\Gamma)},$$

where each π_i is of pure N_Γ-rank i.

Proof
The invariance of the spaces $\mathscr{H}_j$ under $\sigma_i(J)$ means that one can actually decompose σ_i as a direct sum

$$\sigma_i = \sigma_{i,1} \oplus \sigma_{i,2} \oplus \cdots \oplus \sigma_{i,\mathrm{ht}(N_\Gamma)}$$

such that each $\sigma_{i,j}$ is a representation of J and

$$\sigma_{i,j\,|P_1} = \nu_{i,j-1} \otimes \rho_i.$$

Now an application of Mackey's subgroup theorem (see Theorem 5.2.3) immediately implies that the representation

$$\mathrm{Res}^P_{N_\Gamma} \mathrm{Ind}^P_J \sigma_{i,j}$$

is supported on rankable representations of N_Γ of rank j (see the arguments of Theorem 5.3.1, Lemma 5.5.4). Lemma 5.5.5 is proved. □

Recall from Lemma 5.5.1 that the group N_Γ^β is normalized by $M_{\{\beta\}}$. A version of the following lemma was mentioned in the proof of Lemma 5.5.4.

LEMMA 5.5.6
Let τ be a unitary representation of N_Γ^β, and let $g \in M_{\{\beta\}}$. Let τ^g be defined as in (5.4). Then the coadjoint orbits associated to τ and τ^g (in the sense of Kirillov's orbital theory) have the same dimension.

Proof
The proof follows immediately from the fact that the coadjoint orbit attached to τ^g is $\mathrm{Ad}^*(g)(\mathscr{O}^*)$, which follows from [Du, Chapitre III, Section 11]. We give a short proof of this fact for the reader's convenience. Let $\mathfrak{n}_\Gamma^\beta$ be the Lie algebra of N_Γ^β. Fix an additive character χ of $\mathbb{F}$ as done in [Mo2, Section 4]. (When $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $\chi(t) = e^{\mathrm{i}\,\mathrm{Re}(t)}$, where $\mathrm{Re}(t)$ means the real part of t, and when $\mathbb{F}$ is p-adic, χ is an unramified character given by Tate.) Let the coadjoint orbit associated to τ be $\mathscr{O}_\tau^* \subset \mathfrak{n}_\Gamma^\beta$. By Kirillov's orbital theory, we know that τ is constructed as follows. One chooses an arbitrary element $\lambda \in \mathscr{O}_\tau^*$ and a maximal subalgebra $\mathfrak{q}$ of $\mathfrak{n}_\Gamma^\beta$ subordinate to λ, which

exponentiates to a closed subgroup Q of N_Γ^β. Then

$$\tau = \mathrm{Ind}_Q^{N_\Gamma^\beta}(\chi \circ \lambda \circ \log).$$

For any $g \in M_{\{\beta\}}$, let $(\chi \circ \lambda \circ \log)^g$ be defined as in (5.4). Then

$$\tau^g = \mathrm{Ind}_{gQg^{-1}}^{N_\Gamma^\beta}(\chi \circ \lambda \circ \log)^g = \mathrm{Ind}_{gQg^{-1}}^{N_\Gamma^\beta}\big(\chi \circ \mathrm{Ad}^*(g)(\lambda) \circ \log\big).$$

Since $\mathrm{Ad}(g)(\mathfrak{q})$ is a maximal subalgebra subordinate to $\mathrm{Ad}^*(g)(\lambda)$, the coadjoint orbit attached to τ^g is

$$\mathrm{Ad}^*(g)(\mathcal{O}^*).$$

Since the action of $\mathrm{Ad}^*(g)$ is linear on $\mathfrak{n}_\Gamma^\beta$, it does not change the dimension of the coadjoint orbit. □

To prove Theorem 5.3.2, we show that each of the components π_i given in Lemma 5.5.5 is G-invariant as well. To this end, we first prove the following lemma.

LEMMA 5.5.7
The P-invariant decomposition in Lemma 5.5.5 is preserved by the action of $M_{\{\beta\}}$, where

$$M_{\{\beta\}} = [L_{\{\beta\}}, L_{\{\beta\}}]$$

and $L_{\{\beta\}}$ is the Levi component of the standard parabolic $P_{\{\beta\}}$ with β as in Proposition 5.5.2.

Proof
For any $1 \le j \le \mathrm{ht}(N_\Gamma)$, the representation

$$\mathrm{Res}_{N_\Gamma^\beta}^P \pi_j$$

is a direct integral of representations that correspond to coadjoint orbits of dimension $2(n_1 + \cdots + n_j - c)$. However, $n_1, \ldots, n_{\mathrm{ht}(N_\Gamma)} > 0$. Therefore, if we define π_j^a (for any $a \in M_{\{\beta\}}$) as in (5.4), then Lemma 5.5.6 implies that for any $j' \neq j$, the dimension of the coadjoint orbits associated to the irreducible representations of N_Γ^β in the support of $\mathrm{Res}_{N_\Gamma}^P \pi_j^a$ is different from the dimension of the coadjoint orbits associated to the irreducible representations of N_Γ^β in the support of $\mathrm{Res}_{N_\Gamma}^P \pi_{j'}$. This means that

$$\mathrm{Hom}_{N_\Gamma^\beta}(\mathrm{Res}_{N_\Gamma^\beta}^P \pi_j^a, \mathrm{Res}_{N_\Gamma^\beta}^P \pi_{j'}) = \{0\}.$$

Now we apply Lemma 5.2.4 with $K = [P_{\{\beta\}}, P_{\{\beta\}}]$, $K' = N_\Gamma^\beta$, $\sigma = \pi$, $\sigma_1 = \pi_i$, and

$$\sigma_2 = \bigoplus_{i \neq j} \pi_i.$$

It follows that each π_j is invariant under the action of $M_{\{\beta\}}$. Lemma 5.5.7 is proved. □

To finish the proof of Theorem 5.3.2, we note that the parabolic subgroup P in G is maximal; therefore the group generated by P and $M_{\{\beta\}}$ is equal to G. (This follows from the Bruhat-Tits decomposition.) The decomposition of π given in Lemma 5.5.5 is preserved by both P and $M_{\{\beta\}}$ and hence by the group generated by them. Therefore the decomposition of Lemma 5.5.5 is G-invariant. □

Remark. Let G be as in Theorem 5.3.1, and let the central extension identifying G be as in (1.1). Let N_Γ be the H-tower subgroup of the group G. Let π be a unitary representation of G of pure-rank k, where $k \leq \mathrm{ht}(N_\Gamma)$. Consider the subgroup A_k of G defined as follows. Then $A_k = p^{-1}(\mathbf{A}'_{\mathbb{F}})$, where $\mathbf{A}'$ is an $\mathbb{F}$-torus inside the maximal split $\mathbb{F}$-torus $\mathbf{A}$ whose Lie algebra is spanned by the coroots $H_{\tilde{\beta}_1}, \ldots, H_{\tilde{\beta}_k}$. Let τ be a rankable representation of N_Γ of rank k. The stabilizer S_τ of τ inside A_k is a finite subgroup of A_k. Moreover, under the action of A_k, there are only a finite number of orbits of rankable representations of rank k. By Mackey theory, the restriction of π to $A_k \ltimes N_\Gamma$ is a direct integral of representations of the form

$$\mathrm{Ind}_{S_\tau \ltimes N_\Gamma}^{A_k \ltimes N_\Gamma} \sigma_\tau, \tag{5.10}$$

where σ_τ is irreducible and $\sigma_{\tau\,|N_\Gamma} = n_\tau \tau$ for some $n_\tau \in \{1, 2, 3, \ldots, \infty\}$. Moreover, by Frobenius reciprocity, σ_τ is a subrepresentation of $\mathrm{Ind}_{N_\Gamma}^{S_\tau \ltimes N_\Gamma} \tau$. Since S_τ is a finite group, there are only a finite number of possibilities for σ_τ. Since the number of orbits (and hence stabilizers) of rankable representations of rank k under the action of A_k is finite, the number of representations of the form (5.10) is finite as well. This implies the following result.

PROPOSITION 5.5.8
Let π be a unitary representation of G of pure-rank k. Then there exists a finite family $\{\tau_1, \ldots, \tau_t\}$ of irreducible representations of $A_k \ltimes N_\Gamma$, independent of π, such that

$$\pi_{|A_k \ltimes N_\Gamma} = n_1\tau_1 + \cdots + n_t\tau_t,$$

where $n_i \in \{1, 2, 3, \ldots, \infty\}$ for each i.

6. Relation with the old theory

6.1. Outline of the old theory

In this section we show how the notion of rank defined in the past sections relates to the existing theory for classical groups. To this end, we show that for the real forms of classical groups, the two notions of rank (the one defined in [L2] and the one defined in Definition 5.3.3) are equivalent. Here we give a brief outline of the old theory. In classical cases, rank of a representation of the real semisimple group G is defined in terms of its restriction to the centers of nilradicals of maximal parabolic subgroups. One can characterize each of these parabolics with a node in the Dynkin diagram of the restricted root system in a natural way. It turns out that there is a (not necessarily unique) standard parabolic that provides the most refined information about the rank. We devote this section to exhibiting the coincidence of the two notions of rank on this parabolic. Actually, the main idea is some slight modification of the fact that the nilradical of the rank parabolic subalgebra contains a maximal isotropic subspace of each of the Heisenberg algebras in the H-tower N_Γ. Our presentation of the results follows the notation of older literature (see [Ho4], [L2], [Sc]).

The notation used in this section is chosen independently of other sections in order to simplify matters and be more coherent with older works. For simplicity, we consider only the case $\mathbb{F} = \mathbb{R}$. The general case is essentially the same and is only more technical. It is more convenient to consider classical groups of different types (in the sense of [Ho1]) separately. Here we quickly review the definition of classical groups of types I and II over a local field, but later we retain our assumption that $\mathbb{F} = \mathbb{R}$.

Let $\mathbb{F}$ be a local field, let D be a division algebra over $\mathbb{F}$ with an involution, and let V be a left vector space over D of dimension n. A classical group G is said to be of type II if $G = \mathrm{GL}_D(V)$. From now on, by $(\cdot, \cdot)$ we mean a Hermitian or skew-Hermitian sesquilinear form $(\cdot, \cdot)$ on V. A classical group G of type I is the connected component of identity of the stabilizer subgroup of $(\cdot, \cdot)$ inside $\mathrm{GL}_D(V)$. The real groups of type I which are of interest here (i.e., those that satisfy the assumptions of Proposition 3.1.1) correspond to the cases where $\mathbb{F} = \mathbb{R}$ and $D = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ with their usual involutions.

6.2. *Groups of type I*

A typical maximal parabolic of these groups can be described as follows. Take a maximal polarization inside V, that is, a maximal set of vectors

$$\{e_1, \dots, e_r, e_1^*, \dots, e_r^*\}$$

in V which satisfy

$$(e_i, e_j) = (e_i^*, e_j^*) = 0,$$
$$(e_i, e_j^*) = \delta_{i,j}.$$

In fact, r is equal to the split rank of G. For any k, let

$$X_k = \mathrm{Span}_D\{e_1, \dots, e_k\}, \qquad X_k^* = \mathrm{Span}_D\{e_1^*, \dots, e_k^*\},$$

and define V_k to be $X_k \oplus X_k^*$. Let P_k be the subgroup of G which consists of elements that leave the subspace X_k^* invariant. P_k is a parabolic subgroup, and the Levi decomposition of P_k looks like

$$P_k = \mathrm{GL}_D(X_k^*)G(V_k^{\perp})N_k, \tag{6.1}$$

where by $G(V_k^{\perp})$ we mean the stabilizer of $(\cdot, \cdot)$ as a form on $V_k^{\perp}$. Here N_k is the unipotent radical of P_k.

P_r is the parabolic that provides the most refined information about the rank (in the sense of [Ho4], [L2]).

Definition 6.2.1
Let r be the split rank of G. The parabolic P_r or its Lie algebra is called the *rank parabolic*.

The unipotent radical N_k is a two-step nilpotent simply connected Lie group, and therefore it can be identified with its Lie algebra via the exponential map. From now on, we think of any N_k through this identification, and although slightly ambiguous, we use the same notation for its Lie subgroups and their Lie algebras. This is done in order to avoid complicated notation and to keep the presentation as close to the style used in the articles of Howe and Li. We make it clear whether or not we are using a Lie group or a Lie algebra wherever necessary.

As in [L2], we have the following exact sequence of Lie algebras:

$$0 \longrightarrow ZN_k \longrightarrow N_k \longrightarrow \mathrm{Hom}_D(V_k^{\perp}, X_k^*) \longrightarrow 0,$$

where ZN_k is the center of N_k. ZN_k is isomorphic to

$$\mathrm{Hom}_D^{\mathrm{inv}}(X_k, X_k^*),$$

where $\mathrm{Hom}_D^{\mathrm{inv}}(X_k, X_k^*)$ is the $\mathbb{F}$-subspace of elements T of $\mathrm{Hom}_D(X_k, X_k^*)$ satisfying

$$\forall i, j \in \{1, 2, \ldots, k\}, \quad (Te_i, e_j) + (e_i, Te_j) = 0.$$

Thus as an $\mathbb{F}$-vector space, the Lie algebra N_k can be expressed as

$$N_k = \mathrm{Hom}_D(V_k^{\perp}, X_k^*) \oplus \mathrm{Hom}_D^{\mathrm{inv}}(X_k, X_k^*). \tag{6.2}$$

The isomorphism of $\mathrm{Hom}_D(V_k^{\perp}, X_k^*)$ into the Lie algebra N_k can be depicted as

$$\tilde{}: \mathrm{Hom}_D(V_k^{\perp}, X_k^*) \mapsto \mathrm{Hom}_D(V, V), \tag{6.3}$$
$$T \mapsto \tilde{T},$$

where $\tilde{T}$ is defined as

$$\tilde{T}v = Tv \quad \text{for } v \in V_k^{\perp},$$
$$\tilde{T}v = 0 \quad \text{for } v \in X_k^*,$$
$$\tilde{T}v = T^t v \quad \text{for } v \in X_k.$$

Here $T^t \in \mathrm{Hom}_D(X_k, V_k^{\perp})$ is defined uniquely by

$$\forall\, v \in V_k^{\perp},\ x \in X_k, \quad (Tv, x) + (v, T^t x) = 0.$$

It turns out that the Heisenberg parabolic P of G is P_{k_1}, where $k_1 = 2$ for $G = \mathrm{SO}_{p,q}$ and $k_1 = 1$ for all other classical cases under consideration. Let $P = LN$ be the Levi decomposition of P. Let M be the appropriate simple isotropic factor of $[L, L]$; that is, we drop the redundant factor of $[L, L]$ which, in (6.1), corresponds to $\mathrm{GL}_D(X_{k_1}^*)$. Let $\mathfrak{m}$ be the Lie algebra of M. For any k, define

$$Y_k = \mathrm{Span}\{e_k, \dots, e_r\}, \qquad Y_k^* = \mathrm{Span}\{e_k^*, \dots, e_r^*\}.$$

The center of the nilradical of the rank parabolic of $\mathfrak{m}$ is identical to

$$\mathrm{Hom}_D^{\mathrm{inv}}(Y_{k_1+1}, Y_{k_1+1}^*).$$

The Lie algebra $\mathrm{Hom}_D^{\mathrm{inv}}(Y_{k_1+1}, Y_{k_1+1}^*)$ acts on the Lie algebra N_{k_1} through the adjoint action of $\mathfrak{m}$. By Theorem 3.1.1, this action is trivial on ZN_{k_1}. The following simple lemma describes this action more explicitly.

LEMMA 6.2.2
Let $X \in \mathrm{Hom}_D^{\mathrm{inv}}(Y_{k+1}, Y_{k+1}^*)$, and let $Y \in \mathrm{Hom}_D(V_k^{\perp}, X_k^*)$. Then the adjoint action of $\mathfrak{m}$ on $\mathfrak{n} = N_k$ is described as

$$\mathrm{ad}_X(\tilde{Y}) = -(\tilde{YX}),$$

where $\tilde{YX}$ is defined as in (6.3).

Remark. Note that we think of $-YX$ as an element of $\mathrm{Hom}_D(V_k^{\perp}, X_k^*)$ which is zero on $V_r^{\perp}$ and Y_{k+1}^*.

The restriction of $(\cdot,\cdot)$ to $V_r^{\perp}$ is a definite form. Without loss of generality, we may assume that the form is positive definite. Let $\{f_1, \dots, f_{n-2r}\}$ be an orthonormal basis for $V_c = V_r^{\perp}$; consequently, $V_c = \mathrm{Span}_D\{f_1, \dots, f_{n-2r}\}$. One can see that $\mathrm{Hom}_D(V_k^{\perp}, X_k^*)$ is equal to

$$\mathrm{Hom}_D(Y_{k+1}, X_k^*) \oplus \mathrm{Hom}_D(Y_{k+1}^*, X_k^*) \oplus \mathrm{Hom}_D(V_c, X_k^*). \tag{6.4}$$

We consider the direct sum decomposition (6.4) inside N_k (see (6.2)). We observe the following lemma.

LEMMA 6.2.3
In the direct sum decomposition (6.4), the third summand commutes with the first two summands, and the adjoint action of $\mathrm{Hom}_D^{\mathrm{inv}}(Y_{k+1}, Y_{k+1}^*)$ *on the third summand is trivial.*

Now take $k = k_1$. Then the direct sum

$$\mathrm{Hom}_D(Y_{k_1+1}, X_{k_1}^*) \oplus \mathrm{Hom}_D(Y_{k_1+1}^*, X_{k_1}^*) \oplus \mathrm{Hom}_D^{\mathrm{inv}}(X_{k_1}, X_{k_1}^*) \tag{6.5}$$

is a Lie subalgebra of N_{k_1} and also a Heisenberg algebra with a polarization given by the first two summands in (6.5). We denote the Lie algebra in (6.5) (and also its corresponding Lie group) by $\overline{N}_{k_1}$. The Lie bracket when restricted to the polarization is described as follows.

LEMMA 6.2.4
Let $X \in \mathrm{Hom}_D(Y_{k_1+1}, X_{k_1}^*)$*, and let* $Y \in \mathrm{Hom}_D(Y_{k_1+1}^*, X_{k_1}^*)$*. Then* $[\tilde{X}, \tilde{Y}]$ *is an element of* $\mathrm{Hom}_D^{\mathrm{inv}}(X_{k_1}, X_{k_1}^*)$ *given by*

$$[\tilde{X}, \tilde{Y}] = XY^t - YX^t,$$

where $X^t \in \mathrm{Hom}_D(X_{k_1}, Y_{k_1+1}^*)$ *and* $Y^t \in \mathrm{Hom}_D(X_{k_1}, Y_{k_1+1})$ *are uniquely determined as follows:*

$$\forall\, i \le k_1 \text{ and } \forall\, j > k_1, \quad (X^t e_i, e_j) + (e_i, Xe_j) = 0,$$
$$\forall\, i \le k_1 \text{ and } \forall\, j > k_1, \quad (Y^t e_i, e_j^*) + (e_i, Ye_j^*) = 0.$$

Proof
The proof follows immediately from $[\tilde{X}, \tilde{Y}] = \tilde{X}\tilde{Y} - \tilde{Y}\tilde{X}$, where $\tilde{X}$ and $\tilde{Y}$ are defined as in (6.3). □

The adjoint action of $\mathrm{Hom}_D^{\mathrm{inv}}(Y_{k_1+1}, Y_{k_1+1}^*)$ on $\overline{N}_{k_1}$ is given by Lemma 6.2.2. This action normalizes $\overline{N}_{k_1}$ and takes $\mathrm{Hom}_D(Y_{k_1+1}^*, X_{k_1}^*)$ to $\mathrm{Hom}_D(Y_{k_1+1}, X_{k_1}^*)$.

At this point we come back to nilpotent groups and their representations. Consider an irreducible representation ρ_1 of the Heisenberg group N_{k_1} with (nontrivial) central character χ_1. From the orthogonal decomposition obtained in (6.5), it follows that the restriction of ρ_1 to the group $\overline{N}_{k_1}$ decomposes into a direct integral of representations of this latter Heisenberg group with the same central character. We study the restriction of a rankable representation of rank one of the H-tower unipotent radical of G to its subgroup

$$\overline{N}_{k_1} \rtimes \mathrm{Hom}_D^{\mathrm{inv}}(Y_{k_1+1}, Y_{k_1+1}^*).$$

This restriction is a direct integral of representations of the latter group obtained by extending the irreducible representation of $\overline{N}_{k_1}$ with central character χ_1 to $\overline{N}_{k_1} \rtimes \mathrm{Hom}_D^{\mathrm{inv}}(Y_{k_1+1}, Y_{k_1+1}^*)$, as suggested by Proposition 2.3.1. This is because Lemma 6.2.3 implies that as subspaces of N_{k_1},

$$[\mathrm{Hom}_D^{\mathrm{inv}}(Y_{k_1+1}, Y_{k_1+1}^*), \mathrm{Hom}_D(V_c, X_{k_1}^*)] = \{0\},$$

and the Weil representation is functorial (see [Ho4, (1.15)] for a precise meaning of functoriality).

Since $\mathrm{Hom}_D^{\mathrm{inv}}(Y_1, Y_1^*)$ is an abelian Lie group, any unitary representation of this group can be described as a direct integral of unitary characters. $\mathrm{Hom}_D^{\mathrm{inv}}(Y_1, Y_1^*)$ is isomorphic to some $\mathbb{R}^p$, so its group of unitary characters can be naturally identified to $\mathrm{Hom}_D^{\mathrm{inv}}(Y_1^*, Y_1)$ via the $\mathbb{F}$-bilinear form

$$\beta(A, B) = \mathrm{tr}(AB). \tag{6.6}$$

Definition 6.2.5
The rank of a character of $\mathrm{Hom}_D^{\mathrm{inv}}(Y_1, Y_1^*)$ is the $\mathbb{F}$-rank of the element in $\mathrm{Hom}_D^{\mathrm{inv}}(Y_1^*, Y_1)$ which corresponds to it via the bilinear form in (6.6).

Definition 6.2.6 (see [Ho4], [L2])
A unitary representation π of G is said to have rank k if and only if $\pi_{|\mathrm{Hom}_D^{\mathrm{inv}}(Y_1, Y_1^*)}$ decomposes into a direct integral of characters of rank equal to k.

The following proposition is a key result of this section.

PROPOSITION 6.2.7
The restriction of a rankable representation of rank one to

$$\mathrm{Hom}_D^{\mathrm{inv}}(Y_1, Y_1^*)$$

is supported on characters whose rank is equal to k_1.

Proof
The polarization for the group $\overline{N}_{k_1}$ has the structure of a D-vector space. Therefore, similarly to (2.5), we can realize the representation ρ_1 of $\overline{N}_{k_1}$ on

$$L^2\big(\mathrm{Hom}_D(Y_{k_1+1}^*, X_{k_1}^*)\big)$$

and then extend it to $\mathrm{Hom}_D^{\mathrm{inv}}(Y_{k_1+1}, Y^*_{k_1+1})$. In Lemma 6.2.8, we denote elements of the Lie algebras by $X, Y, \dots$ and the corresponding elements in the Lie groups by $e^X, e^Y, \dots$.

LEMMA 6.2.8
The action of the extension of ρ_1 is described as follows.

(1) *For any $X \in \mathrm{Hom}_D(Y_{k_1+1}, X^*_{k_1})$, $Y \in \mathrm{Hom}_D(Y^*_{k_1+1}, X^*_{k_1})$,*

$$\big(\rho_1(e^X)f\big)(Y) = \chi_1(e^{[Y,X]})f(Y).$$

(2) *For any $X \in \mathrm{Hom}_D^{\mathrm{inv}}(Y_{k_1+1}, Y^*_{k_1+1})$, $Y \in \mathrm{Hom}_D(Y^*_{k_1+1}, X^*_{k_1})$,*

$$\big(\rho_1(e^X)f\big)(Y) = \chi_1(e^{(1/2)[Y,YX]})f(Y).$$

(3) *For any $X \in \mathrm{Hom}_D^{\mathrm{inv}}(X_{k_1}, X^*_{k_1})$, $Y \in \mathrm{Hom}_D(Y^*_{k_1+1}, X^*_{k_1})$,*

$$\big(\rho_1(e^X)f\big)(Y) = \chi_1(e^X)f(Y).$$

Proof
This is an almost immediate consequence of the Schrödinger model for the realization of the Weil representation (see [Ho6]). □

Let $X \in \mathrm{Hom}_D^{\mathrm{inv}}(Y_{k_1+1}, Y^*_{k_1+1})$, and let $Y \in \mathrm{Hom}_D(Y^*_{k_1+1}, X^*_{k_1})$. For all $i \leq k_1$ and $j > k_1$,

$$\begin{aligned}\big(e_j, (YX)^t e_i\big) &= -(YXe_j, e_i) = (Xe_j, Y^t e_i)\\ &= -(e_j, XY^t e_i) = (e_j, -XY^t e_i),\end{aligned}$$

which implies that $(YX)^t = -XY^t$. Thus the equation in Lemma 6.2.8(2) can be simplified as

$$\big(\rho_1(X)f\big)(Y) = \chi_1(e^{-YXY^t})f(Y).$$

We would like to have a single formula instead of parts (1), (2), and (3) of Lemma 6.2.8. To this end, we define the linear operator $S = S(Y)$ such that

$$S : Y_1^* \mapsto X^*_{k_1}$$

by

$$\begin{aligned}Se_i^* &= e_i^* \quad \text{if } i \leq k_1,\\ Se_i^* &= Ye_i^* \quad \text{if } i > k_1.\end{aligned}$$

We have the following lemma.

LEMMA 6.2.9
Let $X \in \mathrm{Hom}_D^{\mathrm{inv}}(Y_1, Y_1^*)$. *Let* f *be a function such that*

$$f \in L^2\big(\mathrm{Hom}_D(Y_{k_1+1}^*, X_{k_1}^*)\big).$$

Then

$$\rho_1(e^X)f(Y) = \chi_1(e^{-SXS^t})f(Y),$$

where S^t *is defined as in Lemma 6.2.4.*

Proof
Applying $(e_i, Se_j^*) + (S^t e_i, e_j^*) = 0$, one can see that for any $i \le k_1$,

$$S^t e_i = -e_i + Y^t e_i,$$

and thus for any $X \in \mathrm{Hom}_D(Y_{k_1+1}, X_{k_1}^*)$,

$$\begin{aligned} -SXS^t e_i &= -S\tilde{X}(-e_i + Y^t e_i) \\ &= (-S)(-X^t)e_i - SXY^t e_i \\ &= YX^t e_i - XY^t e_i = [Y, X]e_i, \end{aligned}$$

which proves $-SXS^t = [Y, X]$.

For any $X \in \mathrm{Hom}_D^{\mathrm{inv}}(Y_{k_1+1}, Y_{k_1+1}^*)$, we have

$$-SXS^t e_i = -S\tilde{X}(-e_i + Y^t e_i) = -SXY^t e_i = -YXY^t e_i,$$

which proves $-SXS^t = -YXY^t$.

Finally, when $X \in \mathrm{Hom}_D^{\mathrm{inv}}(X_{k_1}, X_{k_1}^*)$,

$$-SXS^t e_i = -SX(-e_i + Y^t e_i) = SXe_i = Xe_i,$$

which proves $-SXS^t = X$. This completes the proof. □

To complete the proof of Proposition 6.2.7, note that via the identification described in (6.6), the character $\chi(e^X) = \chi_1(e^{-S^t XS})$ corresponds to an element

$$-\frac{a}{k_1} S^t VS, \quad a \in i\mathbb{R} - \{0\}, \tag{6.7}$$

where

$$V : X_{k_1}^* \mapsto X_{k_1}$$

is defined as

$$Ve_l^* = (-1)^{l+1} e_{k_1-l+1}^* \quad \text{for any } 1 \le l \le k_1.$$

It is easy to see that (6.7) is an element of $\mathrm{Hom}_D^{\mathrm{inv}}(Y_1^*, Y_1)$ of rank k_1, even when its domain is restricted to X_{k_1}. □

The following theorem shows that in groups of type I, the two notions of rank are essentially the same.

THEOREM 6.2.10
Let π be an irreducible representation of a classical group G of type I. Let N_Γ be the H-tower subgroup of G (see (3.4)).

- *Assume that the rank of π in the sense of Definition 6.2.6 is less than* $\mathrm{ht}(N_\Gamma) \times k_1$. *Then π has rank k in the sense of Definition 5.3.3 if and only if π has rank kk_1 in the sense of Definition 6.2.6.*
- *If the rank of π in the sense of Definition 5.3.3 is equal to* $\mathrm{ht}(N_\Gamma)$, *then the rank of π in the sense of Definition 6.2.6 is* $\mathrm{ht}(N_\Gamma) \times k_1$ *or higher.*

Proof
Since π is supported on rankable representations, the only thing we have to show is that a rankable representation of rank k (in the sense of Definition 4.1.1), when restricted to $\mathrm{Hom}_D^{\mathrm{inv}}(Y_1^*, Y_1)$, decomposes as a direct integral of characters of rank kk_1. By Proposition 6.2.7, this is true when $k = 1$.

Next, consider a rankable representation ρ of rank $k > 1$, say,

$$\rho = \rho_1 \otimes \cdots \otimes \rho_k.$$

Elementary properties of tensor product imply that the restriction of ρ to $\mathrm{Hom}_D^{\mathrm{inv}}(Y_1, Y_1^*)$ is a direct integral of characters of the form

$$\phi_1 \cdot \phi_2,$$

where the characters ϕ_1 and ϕ_2 are constituents of the direct integral decomposition of ρ_1 and $\rho_2 \otimes \cdots \otimes \rho_k$ when restricted to $\mathrm{Hom}_D^{\mathrm{inv}}(Y_1, Y_1^*)$, respectively. But if ϕ_i ($i \in \{1, 2\}$) corresponds to $A_i \in \mathrm{Hom}_D^{\mathrm{inv}}(Y_1^*, Y_1)$ via (6.6), then $\phi_1 \cdot \phi_2$ corresponds to $A_1 + A_2$. Since $\rho_2 \otimes \cdots \otimes \rho_k$, and therefore any possible ϕ_2 is a trivial representation when restricted to N_{k_1}, any possible A_2 is really an element of $\mathrm{Hom}_D^{\mathrm{inv}}(Y_{k_1+1}^*, Y_{k_1+1})$ which is extended trivially on X_{k_1} to Y_1^*. However, at the end of the proof of Proposition 6.2.7, it was shown that any such character ϕ_1 corresponds to some element of $\mathrm{Hom}_D^{\mathrm{inv}}(Y_1^*, Y_1)$ which is of rank k_1 even when its domain is restricted to $X_{k_1}^*$. It is now easy to show that we have

$$\mathrm{rank}(A_1 + A_2) = \mathrm{rank}(A_1) + \mathrm{rank}(A_2) = k_1 + \mathrm{rank}(A_2).$$

An induction on k completes the proof. □

Example
Let $G = \mathrm{SO}(6,6)$. Let π be an irreducible unitary representation of G. Then the rank of π in the sense of Definition 6.2.6 can be 0, 2, 4, or 6. The rank of π in the sense of Definition 5.3.3 can be 0, 1, or 2. The following chart shows how the ranks correspond to each other:

Definition 6.2.6	Definition 5.3.3
0	0
2	1
4	2
6	2

Now let $G = \mathrm{SO}(5,11)$. Then we have a similar chart for the rank of π:

Definition 6.2.6	Definition 5.3.3
0	0
2	1
4	2

Therefore the correspondence of ranks may or may not be one-to-one. It is an easy exercise to determine in which cases the correspondence is actually one-to-one.

6.3. $\mathrm{SL}_{l+1}(\mathbb{R})$
Let $G = \mathrm{SL}_{l+1}(\mathbb{R})$, the group of linear transformations on the $(l+1)$-dimensional vector space V with a fixed basis

$$\{e_1, \ldots, e_{l+1}\}.$$

Set $r = \lfloor (l+1)/2 \rfloor$. For any k, let P_k be the maximal parabolic of G which is represented by matrices of the form

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix},$$

where

$$A \in \mathrm{GL}_k(\mathbb{R}), \qquad C \in \mathrm{GL}_{l+1-k}(\mathbb{R}), \qquad \text{and} \qquad B \in \mathrm{M}_{k\times(l+1-k)}(\mathbb{R}).$$

Therefore

$$P_k = S\big(\mathrm{GL}_k(\mathbb{R}) \times \mathrm{GL}_{l+1-k}(\mathbb{R})\big) \cdot N_k,$$

where

$$N_k = \mathrm{Hom}\big(\mathrm{Span}(\{e_{k+1}, \ldots, e_{l+1}\}), \mathrm{Span}(\{e_1, \ldots, e_k\})\big).$$

The parabolic of G which gives the most refined rank is P_r, and henceforth we focus our attention on P_r. Let

$$X_k = \mathrm{Span}(\{e_2, \ldots, e_k\}), \qquad Y_k = \mathrm{Span}(\{e_k, \ldots, e_l\}).$$

The Heisenberg parabolic subgroup of G is $P_1 \cap P_l$, and a polarization of the Lie algebra of its unipotent radical N is a direct sum of

$$\mathrm{Hom}(X_r, \mathbb{R}e_1) \oplus \mathrm{Hom}(\mathbb{R}e_{l+1}, Y_{r+1})$$

and

$$\mathrm{Hom}(Y_{r+1}, \mathbb{R}e_1) \oplus \mathrm{Hom}(\mathbb{R}e_{l+1}, X_r).$$

The second summand lies inside the nilradical of the Lie algebra of P_r. Its center is isomorphic to

$$\mathrm{Hom}(\mathbb{R}e_{l+1}, \mathbb{R}e_1).$$

As before, we are interested in the description of the restriction of a representation ρ_1 of N to

$$\mathrm{Hom}\big(\mathrm{Span}(\{e_{r+1}, \dots, e_{l+1}\}), \mathrm{Span}(\{e_1, \dots, e_r\})\big).$$

We identify the dual of

$$\mathrm{Hom}\big(\mathrm{Span}(\{e_{r+1}, \dots, e_{l+1}\}), \mathrm{Span}(\{e_1, \dots, e_r\})\big)$$

with itself via the bilinear form

$$\beta(X, Y) = \mathrm{tr}(X^t Y).$$

The rank of a unitary character of

$$\mathrm{Hom}\big(\mathrm{Span}(\{e_{r+1}, \dots, e_{l+1}\}), \mathrm{Span}(\{e_1, \dots, e_r\})\big)$$

is defined to be the rank of the linear transformation which corresponds to it via the bilinear form β.

We write any

$$Y \in \mathrm{Hom}(X_r, \mathbb{R}e_1) \oplus \mathrm{Hom}(\mathbb{R}e_{l+1}, Y_{r+1})$$

naturally as $Y = Y_1 \oplus Y_2$. Define

$$Y_1^+ \in \mathrm{Hom}\big(\mathrm{Span}(\{e_1, \dots, e_r\}), \mathbb{R}e_1\big)$$

by

$$\begin{aligned} Y_1^+ e_1 &= e_1, \\ Y_1^+ e_j &= Y_1 e_j \quad \text{for any } j > 1. \end{aligned}$$

Similarly,

$$Y_2^+ \in \mathrm{Hom}\big(\mathbb{R}e_{l+1}, \mathrm{Span}(\{e_{r+1}, \ldots, e_{l+1}\})\big)$$

is defined as

$$Y_2^+ e_{l+1} = -Y_2 e_{l+1} + e_{l+1}.$$

We can prove the following version of Lemma 6.2.9.

LEMMA 6.3.1
Let ρ_1 be a representation of N with central character χ_1 realized on

$$\mathscr{H} = L^2\big(\mathrm{Hom}(X_r, \mathbb{R}e_1) \oplus \mathrm{Hom}(\mathbb{R}e_{l+1}, Y_{r+1})\big),$$

as in Section 2.1.

For any

$$X \in \mathrm{Hom}\big(\mathrm{Span}(\{e_{r+1}, \ldots, e_{l+1}\}), \mathrm{Span}(\{e_1, \ldots, e_r\})\big),$$

we have

$$\rho_1(X) f(Y) = \chi_1(e^{Y_1^+ X Y_2^+}) f(Y).$$

Proof
We write Y as $Y_1 \oplus Y_2$ according to the polarization given in the statement of the lemma. Based on the Schrödinger model, if $X \in \mathrm{Hom}(Y_{r+1}, X_r)$, then the action of X on the function f at a point Y is multiplication by the character

$$\chi_1(e^{-(1/2)[Y, YX]}) = \chi_1(e^{-Y_1 X Y_2}).$$

If $X = X_1 \oplus X_2 \in \mathrm{Hom}(Y_{r+1}, \mathbb{R}e_1) \oplus \mathrm{Hom}(\mathbb{R}e_{l+1}, X_r)$, then $[X, Y] = -X_1 Y_2 + Y_1 X_2$ and the action of X is by the character

$$\chi_1(e^{-X_1 Y_2 + Y_1 X_2}),$$

and if X belongs to the center of N, then clearly the action is by the character

$$\chi_1(e^X).$$

The statement of the lemma follows by a simple calculation. □

One can see that by the duality provided via the bilinear form β, the restriction of a representation of rank one to the nilradical of P_r is a direct integral of characters that correspond to linear operators of the form $Y_2^+ Y_1^+$, which have rank one (in the usual sense) even when the domain is restricted to $\mathbb{R}e_{l+1}$. Proof of the following theorem (which shows the equivalence of the two notions of rank) is similar to that of Theorem 6.2.10.

THEOREM 6.3.2
Let $G = \mathrm{SL}_{l+1}(\mathbb{R})$. Then the restriction of a pure-rank representation of G of rank k (in the sense of Definition 5.3.3) to the abelian nilradical of P_r is supported on unitary characters of rank k.

Remark. Note that when l is even, the maximum rank of the unitary characters is $l/2$, which is the same as the height of the H-tower subgroup of G. However, for odd l, the maximum rank of unitary characters exceeds the height of the H-tower group by one.

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