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# A note on a theorem of Ljunggren and the Diophantine equations $x^2 - kxy^2 + y^4 = 1, 4$

By

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**Abstract.** Let $D$ denote a positive nonsquare integer. Ljunggren has shown that there are at most two solutions in positive integers $(x, y)$ to the Diophantine equation $x^2 - Dy^4 = 1$, and that if two such solutions $(x_1, y_1), (x_2, y_2)$ exist, with $x_1 < x_2$, then $x_1 + y_1^2\sqrt{D}$ is the fundamental unit $\varepsilon_D$ in the quadratic field $\mathbb{Q}(\sqrt{D})$, and $x_2 + y_2^2\sqrt{D}$ is either $\varepsilon_D^2$ or $\varepsilon_D^4$. The purpose of this note is twofold. Using a recent result of Cohn, we generalize Ljunggren's theorem. We then use this generalization to completely solve the Diophantine equations $x^2 - kxy^2 + y^4 = 1, 4$.

**1. Introduction.** There have been many papers in the literature which consider the problem of determining the arithmetic structure of terms which appear in binary linear recurrences. The reader should refer to [12] for a detailed history on this. In a series of papers, Ljunggren considered the particular problem of determining which terms are perfect squares for the particular sequences that arise from Pellian equations of the form $aX^2 - bY^2 = 1$. In particular, one of the more well known results is the following, which appeared in [9].

**Theorem A** (Ljunggren, 1942)**.** *For a positive nonsquare integer $D$ there are at most two solutions to the Diophantine equation $X^2 - DY^4 = 1$. If two solutions exist, and $\varepsilon_D$ denotes the fundamental unit in the quadratic field $\mathbb{Q}(\sqrt{D})$, then they are given by $X + Y^2\sqrt{D} = \varepsilon_D, \varepsilon_D^2$ or by $\varepsilon_D, \varepsilon_D^4$, with the latter case occurring for only finitely many $D$.*

The purpose of this note is twofold. Using a recent result of Cohn [5], we prove a generalization of Theorem A. We then use this generalization to completely solve the Diophantine equation given in the title. In what follows let $T + U\sqrt{D}$ denote the minimal solution in positive integers to the Pell equation $x^2 - Dy^2 = 1$, and for $k \geqq 1$ let $T_k + U_k\sqrt{D} = (T + U\sqrt{D})^k$. The following theorem generalizes Theorem A.

**Theorem 1.** *Let $D$ be a nonsquare positive integer with $D \notin \{1785, 7140, 28560\}$. Then there are at most two positive indices $k$ for which $U_k = 2^\delta y^2$ with $y$ an integer and $\delta = 0$ or $1$. If two solutions $k_1 < k_2$ exist, then $k_1 = 1$ and $k_2 = 2$, and provided that $D \neq 5$, $T + U\sqrt{D}$ is the fundamental unit in $\mathbb{Q}(\sqrt{D})$, or its square. For $D \in \{1785, 7140, 28560\}$, the only solutions to $U_k = 2^\delta y^2$ are $k = 1$, $k = 2$, and $k = 4$.*

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McDaniel and Ribenboim (see [10], [11]) have recently proved general results on squares, double-squares, and square-classes in binary linear recurrences. Their results apply to binary linear recurrences defined by odd parameters. In Theorem 1, we consider the inherently more difficult problem of determining squares and double-squares in sequences with an even first parameter. Namely, the sequence $\{U_k\}$ satisfies the recurrence relation

$$U_{k+1} = (2T)U_k - U_{k-1}$$

for $k \geqq 1$. The methods of McDaniel and Ribenboim do not apply here, and we are forced to appeal to the main result of [1], whose proof is based on the theory of linear forms in the logarithms of algebraic numbers.

The following is an immediate corollary to Theorem 1, which will be used in the proof of Theorem 2.

**Corollary 1.** *For $k = 169$, the only positive integer solutions to $x^2 - (k^2-1)y^4 = 1$ are $(x,y) = (169,1), (6525617281, 6214)$.*

*For $k > 1$ and $k \neq 169$, the only positive integer solution $(x,y)$ to $x^2 - (k^2-1)y^4 = 1$ is $(x,y) = (k,1)$, unless $k = 2v^2$ for some integer $v$, in which case $(x,y) = (8v^4-1, 2v)$ is the only other solution.*

*For $k > 1$ there is no positive integer solutions $(x,y)$ to $x^2 - (k^2-1)y^4 = 4$, unless $k = v^2$ for some integer $v$, in which case $(x,y) = (4v^4-2, 2v)$ is the only solution.*

Let $k$ denote a positive integer. In [6], Cusick determined all integer solutions to the Diophantine equation $x^4 - kx^2y^2 + y^4 = 1$. This was generalized by Cohn in [4], wherein he determined all solutions to the equation $x^2 - kxy^2 + y^4 = c$ for $c \in \{\pm 1, \pm 2, \pm 4\}$, with the assumption that $k$ is odd for the particular cases $c = 1$ and $c = 4$. Using the above results, we finish off the remaining cases.

**Theorem 2.** *Let $k$ be an even positive integer.*

1. *The only solutions to $x^2 - kxy^2 + y^4 = 1$ in non-negative integers $(x,y)$ are $(k,1), (1,0), (0,1)$, unless either $k$ is a perfect square, in which case there are also the solutions $(1, \sqrt{k}), (k^2-1, \sqrt{k})$, or $k = 338$ in which there are the solutions $(x,y) = (114243, 6214), (13051348805, 6214)$.*
2. *The only solution in non-negative integers $x, y$ to the equation $x^2 - kxy^2 + y^4 = 4$ is $(x,y) = (2,0)$, unless $k = 2v^2$ for some integer $v$, in which case there are also the solutions $(2, \sqrt{2k}), (2k^2-2, \sqrt{2k})$.*

**2. Preliminary results.** In this section we present some notation and some preliminary results to be used in the proofs of the theorems.

For a nonsquare positive integer $D$ we let $\varepsilon_D$ denote the fundamental unit in the quadratic field $\mathbb{Q}(\sqrt{D})$. Let $D = e^2 d$, with $e$ an integer and $d$ a positive squarefree integer. Then $\varepsilon_D = \dfrac{a + b\sqrt{d}}{2}$, where $a$ and $b$ are positive integers with the same parity, and satisfy $a^2 - db^2 = (-1)^\alpha 4$, where $\alpha \in \{0,1\}$. Define $\lambda_D = \lambda_d$ to be the minimal solution $u + v\sqrt{d}$ to

$X^2 - dY^2 = 1$, with $u$ and $v$ positive integers. Then $\lambda_D = (\varepsilon_D)^c$, where

$$c = \begin{cases} 1 & \text{if } a \text{ and } b \text{ are even and } \alpha = 0 \\ 2 & \text{if } a \text{ and } b \text{ are even and } \alpha = 1 \\ 3 & \text{if } a \text{ and } b \text{ are odd and } \alpha = 0 \\ 6 & \text{if } a \text{ and } b \text{ are odd and } \alpha = 1. \end{cases}$$

The proof of Theorem 1 relies on the following recent result of Cohn from [5]. Our statement of his result is slightly refined, but nevertheless follows from the proof given in [5].

**Lemma 1** (Cohn, 1997)**.** *Let $D$ be a nonsquare positive integer. If the equation $X^4 - DY^2 = 1$ is solvable in positive integers $X, Y$, then either $X^2 + Y\sqrt{D} = \lambda_D$ or $\lambda_D^2$. Solutions to $X^4 - DY^2 = 1$ arise from both $\lambda_D$ and $\lambda_D^2$ only for $D \in \{1785, 7140, 28560\}$.*

Lemma 1 implies the following refinement of Theorem A, which we state as a lemma for use in the proof of Theorem 1.

**Lemma 2.** *There are at most two solutions to the equation $X^2 - DY^4 = 1$. If there are two solutions, and $\varepsilon_D$ denotes the fundamental unit in $\mathbb{Q}(\sqrt{D})$, then they are given by $X + Y^2\sqrt{D} = \varepsilon_D, \varepsilon_D^4$ for $D \in \{1785, 28560\}$, and $X + Y^2\sqrt{D} = \varepsilon_D, \varepsilon_D^2$ otherwise.*

Proof. If there exist two indices $k_1$ and $k_2$ for which $U_{k_1}$ and $U_{k_2}$ are squares, then by Theorem A, $(k_1, k_2) = (1, 4)$ or $(k_1, k_2) = (1, 2)$. If there are integers $x$ and $y$ such that $U_1 = x^2$ and $U_4 = y^2$, then since $U_4 = 2T_2U_2$, there exist integers $w$ and $z$ such that either $(T_2, U_2) = (w^2, 2z^2)$, or $(T_2, U_2) = (2w^2, z^2)$. The latter case is not possible, since it would imply the existence of three solutions to $X^2 - DY^4 = 1$, contradicting Theorem A. In the former case, since $2z^2 = U_2 = 2T_1U_1$, there are integers $u$ and $v > 1$ such that $T_1 = v^2$ and $U_1 = u^2$. We thus have solutions to $X^4 - DY^2 = 1$ arising from both $\varepsilon_D$ and $\varepsilon_D^2$. By Lemma 1, we deduce that $D \in \{1785, 7140, 28560\}$, and since $U_1 = 2$ for $D = 7140$, we have finally that $D \in \{1785, 28560\}$.

The following was proved in [8].

**Lemma 3** (Ljunggren, 1942)**.** *The only positive integer solutions to the equation $X^2 - 2Y^4 = -1$ are $(X, Y) = (1, 1), (239, 13)$.*

The following is the main result from [2].

**Lemma 4** (Bumby, 1967)**.** *The only positive integer solutions to the equation $3X^4 - 2Y^2 = 1$ are $(X, Y) = (1, 1), (3, 11)$.*

The following is a special case of Corollary 1.3 in [1] .

**Lemma 5.** *If $T_k = 2x^2$ for some integer $x$, then $k = 1$.*

**3. Proofs.**

Proof of Theorem 1. If one of the equations $x^2 - Dy^4 = 1$, $x^2 - 4Dy^4 = 1$ is not solvable, then the result follows from Lemma 2 applied to $4D$ and $D$ respectively. Therefore we may assume that both of these equations are solvable. Let $k$ and $l$ be indices for which $U_k = z^2$ and $U_l = 2w^2$. It follows from the binomial theorem that not both of $k$ and $l$ are odd.

Assume first that $k$ and $l$ are both even. We will show that this leads to $D \in \{1785, 7140, 28560\}$. Let $l = 2m$, then there are integers $u > 1$ and $v$ such that $T_m = u^2$ and $U_m = v^2$. Then by Lemma 1, either $m = 1$ or $m = 2$. Also, by Lemma 2, and the fact that $k$ is even, either $(k, m) = (2, 1)$, $(k, m) = (4, 1)$ and $D \in \{1785, 28560\}$, or else $k = m$. The first case is not possible since it would imply $k = l = 2$, and this contradicts the assumed forms of $U_k$ and $U_l$. Thus, for $D \notin \{1785, 28560\}$, we have that $k = m$, and furthermore, the only possibility is $k = m = 2$. Since $U_2 = 2T_1U_1$, there are positive integers $a, b$ for which either $(T_1, U_1) = (a^2, 2b^2)$ or $(T_1, U_1) = (2a^2, b^2)$. From the identity $T_2 = 2T_1^2 - 1$, these two possibilities yield the respective equations $u^2 = 2a^4 - 1$ or $u^2 = 8a^4 - 1$. The equation $u^2 = 8a^4 - 1$ is not solvable modulo 4. By Lemma 3, the only positive integer solution to the equation $u^2 = 2a^4 - 1$, with $u > 1$, is $u = 239$ and $a = 13$. Therefore, $T_1 = 169$, and $U_1 = 2b^2$ for some integer $b$. The only choice for $b$ is $b = 1$, which results in $D = 7140$.

We can assume that $k$ and $l$ are of opposite parity. First assume that $l$ is even, $l = 2m$ say, and that $k$ is odd. Thus, we have that $U_{2m} = 2w^2$. From the identity $U_{2m} = 2T_mU_m$, and the fact that $\gcd(T_m, U_m) = 1$, it follows that there are integers $u$ and $v$ such that $T_m = u^2$ and $U_m = v^2$. By Lemma 1, either $m = 1$ or $m = 2$, and $T_1 + U_1\sqrt{D} = \lambda_D$. Furthermore, by Lemma 2, either $k = m$ or $k = 1, m = 2$. If $k = m$, then since $k$ is odd and $m = 1$ or 2, we have that $k = 1$ and $l = 2$, which is our desired result. On the other hand, if $k = 1$ and $m = 2$, then $l = 4$, and we have that $U_4 = 2w^2$, $U_2 = v^2$, and $T_2 = u^2$. As in the previous paragraph, this leads to $D = 7140$.

Now assume that $l$ is odd and that $k$ is even, $k = 2m$. Therefore, $U_{2m} = 2T_mU_m = z^2$, and it follows that there are integers $u$ and $v$ such that either $(T_m, U_m) = (2u^2, v^2)$ or $(T_m, U_m) = (u^2, 2v^2)$. In the first case Lemma 2 implies that $(m, k) = (1, 2)$, since $U_m$ and $U_k$ are both squares. Therefore $U_1$ is a square, and $2^{2\alpha}$ properly divides $U_1$ for some integer $\alpha \geqq 0$. Since $U_l = 2w^2$, $2^{2\beta+1}$ properly divides $U_l$ for some integer $\beta \geqq 0$. From the fact that $l$ is odd, the binomial theorem exhibits that the same power of 2 divides $U_1$ and $U_l$, thus leading to a contradiction. In the case that $(T_m, U_m) = (u^2, 2v^2)$, Lemma 1 shows that $m = 1$ or $m = 2$, and that $T_1 + U_1\sqrt{D} = \lambda_D$. Also, by Lemma 2 applied to $4D$, either $m = l$ or $(l, m) = (1, 2)$. The former possibility leads to $l = 1$ and $k = 2$, which is the desired result. The latter possibility implies that $k = 4$, and that $T_2 = u^2$, $U_2 = 2v^2$. Since $U_2 = 2T_1U_1$, there are integers $a$ and $b$ such that $T_1 = a^2$, and $U_1 = b^2$. Therefore, $u^2 = T_2 = 2T_1^2 - 1 = 2a^4 - 1$, and by Lemma 3, it follows that $T_1 = 169$, and hence that $D = 1785$ or $D = 28560$.

It remains to prove that for $D \neq 5$, $T + U\sqrt{D} = T_1 + U_1\sqrt{D}$ is the fundamental unit $\varepsilon_D$ in $\mathbb{Q}(\sqrt{D})$, or its square. Let $T + U\sqrt{D} = \varepsilon_d^c$, then we need to prove that $c = 1$ or $c = 2$, where $c$ is defined in Section 2.

Let $D = l^2d$ with $d$ squarefree. Let $\lambda_d = t + u\sqrt{d}$, and for $k \geqq 1$, define $\lambda_d^k = t_k + u_k\sqrt{d}$. Then $T + U\sqrt{D} = \lambda_d^r = t_r + u_r\sqrt{d}$ for some integer $r$, and $u_{ir} = lU_i$ for each $i \geqq 1$. We

assume now that $U_1 = 2^{\delta_1}x^2$ and $U_2 = 2^{\delta_2}y^2$ for some integers $x$ and $y$. Then $u_r = 2^{\delta_1}lx^2$ and $u_{2r} = 2^{\delta_2}ly^2$. Since $u_{2r} = 2t_r u_r$, it follows that $t_r = z^2$ or $2z^2$ for some integer $z$. By Lemma 2 and Lemma 5, either $r = 1$ or $r = 2$. This implies that $c$ divides 12. We wish to show that 4 does not divide $c$. If 4 divides $c$, then $r = 2$ and $N(\varepsilon_d) = -1$, and so there are integers $V > 1$ and $W$ such that $V^2 - W^2 d = -1$ with $t_2 + u_2\sqrt{d} = \lambda_d^2 = (V + W\sqrt{d})^4$. Since $r = 2$, Lemma 5 shows that $t_2 = z^2$. Therefore, $t_2 = z^2 = 8V^4 + 8V^2 + 1$, and as it was shown in [8] that this equation implies $V = 0$, we have a contradiction. Therefore $c$ divides 6, and to complete the proof of the theorem, we need to show that 3 does not divide $c$.

Assume that 3 divides $c$. Then $T + U\sqrt{D}$ is the cube of a unit in $\mathbb{Q}(\sqrt{D})$ of the form $\dfrac{a + b\sqrt{D}}{2}$, where $a$ and $b$ are odd, and $a^2 - b^2 D = 4$. Moreover, $T = a\left(\dfrac{a^2 - 3}{2}\right)$ is odd, and so either $T + U\sqrt{D} = X^2 + Y^2\sqrt{D}$ or $T + U\sqrt{D} = X^2 + 2Y^2\sqrt{D}$, i.e. $T$ is not of the form $2X^2$. It follows that $a(a^2 - 3) = 2X^2$. If $\gcd(a, a^2 - 3) = 1$, then since $a$ is odd, $a = A^2$ and $a^2 - 3 = 2B^2$ for some integers $A, B$, which is not possible by considering this last equation modulo 8. Therefore $\gcd(a, a^2 - 3) = 3$, and there are integers $A, B$ for which $a = 3A^2$ and $a^2 - 3 = 6B^2$, which results in the equation $3A^4 - 2B^2 = 1$. By Lemma 4, the only positive integer solutions are $(A, B) = (1, 1)$ and $(A, B) = (3, 11)$. This shows that either $a = 3$ and $a = 27$. The case $a = 3$ yields $D = 5$, which we have excluded. The case $a = 27$ yields that either $D = 29$ or $D = 725$. It is easily checked that the hypotheses in Theorem 1 are not satisfied for both of these values of $D$.

Proof of Corollary 1. The particular case $k = 169$ is easily verified for both equations, and so we assume that $k > 1$ and $k \neq 169$. The minimal solution of $x^2 - (k^2 - 1)y^2 = 1$ is $k + \sqrt{k^2 - 1}$. For $i \geqq 1$ define $T_i + U_i\sqrt{k^2 - 1} = (k + \sqrt{k^2 - 1})^i$. There is always the solution $(x, y) = (k, 1)$ to $x^2 - (k^2 - 1)y^4 = 1$, and so by Theorem 1, if there is another solution, it must come from $T_2 + U_2\sqrt{k^2 - 1} = 2k^2 - 1 + 2k\sqrt{k^2 - 1}$, i.e. $(x, y) = (2k^2 - 1, \sqrt{2k})$. This entails that $2k$ is a perfect square, and hence that $k = 2v^2$ for some integer $v$. This gives $(x, y) = (8v^4 - 1, 2v)$.

We note that if $k$ is odd, then the minimal solution to $x^2 - \left(\dfrac{k^2 - 1}{4}\right)y^2 = 1$ is $(x, y) = (k, 2)$, from which it follows that for $k$ even or odd, any solution to $x^2 - (k^2 - 1)y^2 = 4$ has both $x$ and $y$ even. Now let $(x, y)$ be a positive integer solution to $x^2 - (k^2 - 1)y^4 = 4$, then $x$ and $y$ are even, and $(u, v) = (x/2, y/2)$ is a positive integer solution to $u^2 - 4(k^2 - 1)v^4 = 1$, and hence there is a positive integer $i$ for which $U_i = 2v^2$. By Theorem 1, since $U_1 = 1$ is already a square, $i = 2$. Therefore $u + 2v^2\sqrt{k^2 - 1} = T_2 + U_2\sqrt{k^2 - 1} = 2k^2 - 1 + 2k\sqrt{k^2 - 1}$, and hence $k = v^2$. This leads to the solution $(x, y) = (4v^4 - 2, 2v)$ to the equation $x^2 - (k^2 - 1)y^4 = 4$. This completes the proof.

Proof of Theorem 2. Let $k = 2s$, then we can rewrite the equation $x^2 - kxy^2 + y^4 = 1$ as

$$(x - sy^2)^2 - (s^2 - 1)y^4 = 1.$$

Aside from the trivial solution $(x, y) = (1, 0)$, Corollary 1 implies that the only solutions are $y = 1$, $x - sy^2 = \pm s$, unless $s = 2v^2$ for some integer $v$, in which case there is also the

solutions $y = 2v$ and $x - sy^2 = \pm(8v^4 - 1)$, or $k = 338$. In either case, the solutions listed in Corollary 1 lead to the solutions given in Theorem 2.

The equation $x^2 - kxy^2 + y^4 = 4$ can be rewritten as

$$(x - sy^2)^2 - (s^2 - 1)y^4 = 4.$$

Corollary 1 shows that, aside from the trivial solution $(x, y) = (2, 0)$, there is no solution in positive integers unless $s = v^2$ for some integer $v$, in which case $y = 2v$ and $x - sy^2 = \pm 4v^4 - 2$. It follows that $k = 2v^2$, $y = \sqrt{2k}$, and either $x = 2$ or $x = 2k^2 - 2$.

**4. Final remarks.** In order to get information on which indices $i$ have the property that $U_i$ is either a square or twice a square, Theorem 1 requires the assumption that two solutions exist. There is very little known in the case that only one solution exists.

It is a simple matter to construct an infinite family of discriminants $D$ for which $U_3$ is a perfect square. Let $k > 1$ be an integer satisfying $k \equiv \pm 1 \ (mod\ 6)$. For such an integer $k$ define integers $a_k, b_k, D_k$ by

$$2a_k + b_k\sqrt{3} = (2 + \sqrt{3})^k, \quad D_k = \frac{a_k^2 - 1}{9}.$$

By our choice of $k$, $D_k$ is an integer, and $a_k + 3\sqrt{D_k}$ is the minimal solution to the Pell equation $X^2 - D_kY^2 = 1$. Upon cubing this minimal solution, it is easy to verify that $U_3 = (3b_k)^2$.

The following is likely true, as it follows from an effective form of the abc conjecture. We omit the details.

**Conjecture.** *Let $D$ be a positive nonsquare integer such that $D \notin \{1785, 7140, 28560\}$. Let $T + U\sqrt{D}$ denote the minimal solution to $X^2 - DY^2 = 1$, and $T_i + U_i\sqrt{D} = (T + U\sqrt{D})^i$ for $i \geqq 1$. If $i \geqq 4$, then $U_i$ is neither a square, nor twice a square.*

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