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ON A SEQUENCE OF INTEGERS ARISING FROM SIMULTANEOUS PELL EQUATIONS

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Abstract: We define a sequence of squarefree positive integers which arise naturally in the context of the solvability of a family of simultaneous Pell equations. It is proved that, apart from an explicitly given finite subset, each integer in this sequence has at least eight prime factors.
Keywords:

1. Introduction

In [5], Ono studied the solvability of the system of simultaneous Pell equations

$$x^2 - dy^2 = 1,\ z^2 - 2dy^2 = 1. \tag{1.1}$$

Subsequently, an investigation into those squarefree positive integers d for which (1.1) has solutions in positive integers x, y, z was pursued by the second author in [6], and more recently by Cao et.al. in [1]. It was proved in [6] that for a given positive integer d, equation (1.1) has at most one solution in positive integers x, y, z. Several necessary conditions for the solvability of (1.1) were given in [6] as well. One of the conditions given involved the number of prime factors of d. In particular, it was shown that if (1.1) is solvable, then apart from a short list of values, d must have at least five distinct odd prime factors. In [1], this particular question was pursued more rigorously, and it was proved that apart from a short list of values, the solvability of (1.1) implies that d must have at least seven distinct prime factors.
The purpose of the present paper is to improve on the result of [1]. In particular, we prove the following

Theorem 1.1. *Assume that d is a squarefree positive integer, and that equation* (1.1) *is solvable in positive integers x, y, z. Then, except for*

$$d \in \{6, 210, 1785, 60639, 184030, 915530, 14066106\},$$

d has at least eight prime factors.

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2. Preliminary Results

Let $\alpha = 1 + \sqrt{2}$, and for $k \geq 1$, define sequences $\{T_k\}, \{U_k\}$ by

$$\alpha^k = T_k + U_k\sqrt{2}.$$

The main tool to be used in the proof of Theorem 1 is the location of squares in the sequences $\{T_k\}$ and $\{U_k\}$. The difficulty in establishing these results is dealing with the case that k is odd, which is tantamount to solving the two Diophantine equations $x^2 - 2y^4 = -1$ and $x^4 - 2y^2 = -1$. These equations were both solved completely by Ljunggren, in [3] and [4] respectively. Consequently, we have the following

Lemma 2.1. *The only square in the sequence $\{T_k\}$ is $T_1 = 1$, and the only squares in the sequence $\{U_k\}$ are $U_1 = 1$ and $U_7 = 169$.*

We will frequently make use of some well known facts concerning terms in the sequences $\{T_k\}, \{U_k\}$. For instance, for any $k \geq 1$, $U_{2k} = T_k U_k$, and if $k > 0$ is odd, then $T_{2k} = 4U_k^2 - 1$. Moreover, if l is a positive integer and k is odd, then T_l divides T_{kl}, whereas for any positive integers k, l, U_l divides U_{kl}. The reader is referred to the comprehensive article by Lehmer [2] on this subject.

3. The Proof

Let us first define the sequence $\{d_n\}$ which we wish to study. Subtracting twice the first equation in (1.1) from the second shows that

$$z^2 - 2x^2 = -1,$$

and hence $z = T_{2n-1}$ for some integer $n \geq 1$. It follows that $(T_{2n-1}^2 - 1)/2 = dy^2$, where it will always be assumed that d is squarefree, provided $n > 1$. For $n \geq 2$, define sequences $\{d_n\}$ and $\{y_n\}$, with d_n squarefree, by

$$d_n y_n^2 = (T_{2n-1}^2 - 1)/2.$$

As noted in [6], for $n \geq 1$,

$$d_n y_n^2 = 4T_n T_{n-1} U_n U_{n-1}. \tag{3.1}$$

This identity will form the basis to prove the existence of eight prime factors of d_n. The proof of Theorem 1 will be separated into cases.

Case 1: $n \equiv 0, 1 \pmod 4$
In the case $n = 4k$ for some integer k, we have that

$$d_n y_n^2 = 4T_{4k} T_{4k-1} U_{4k} U_{4k-1},$$

while if $n = 4k + 1$, then

$$d_n y_n^2 = 4T_{4k} T_{4k+1} U_{4k} U_{4k+1}.$$

The argument below will prove the result in the case $n = 4k$. In the case $n = 4k+1$, the same argument as the one below, with n replaced by $n+1$ gives the desired result.

Subcase 1a: $n = 16k + 8$ for some integer $k \geq 0$.
In this case, equation (3.1) becomes

$$d_n y_n^2 = 32 T_{16k+8} T_{16k+7} U_{16k+7} U_{2k+1} T_{2k+1} T_{4k+2} T_{8k+4},$$

where the terms in the product on the right hand side are pairwise coprime. By Lemma 1, the only terms in the product which can be squares are U_{2k+1} for $k = 0$ and $k = 3$, T_{2k+1} for $k = 0$, and U_{16k+7} for $k = 0$. Thus, for n different from 8 and 56, d_n has at least 8 prime factors.

Subcase 1b: $n = 16k$ for some integer $k > 0$.
In this case, equation (3.1) becomes

$$d_n y_n^2 = 64 T_{16k} T_{16k-1} U_{16k-1} U_k T_k T_{2k} T_{4k} T_{8k},$$

where the terms in the product on the right hand side are pairwise coprime. By Lemma 1, the only terms in the product which can be squares are U_k for $k = 1$ and $k = 7$, and T_k for $k = 1$. Thus, for n different from 16 and 112, d_n has at least 8 prime factors.

Subcase 1c: $n = 8k + 4$ for some integer $k \geq 0$.
In this case, equation (3.1) becomes

$$d_n y_n^2 = 16 T_{8k+4} T_{8k+3} U_{8k+3} U_{2k+1} T_{2k+1} T_{4k+2}, \tag{3.2}$$

where the terms in the product on the right hand side are pairwise coprime. By Lemma 1, the only terms in the product which can be squares are U_{2k+1} for $k = 0$ and $k = 3$, and T_{2k+1} for $k = 0$. This leaves the exceptional values $n = 4$ and $n = 28$, which will be dealt with later. For all other n in this subcase, we again separate the situation into two cases, depending on whether 3 divides n.
Assume first that 3 does not divide n. This implies that 3 properly divides T_{4k+2}. We will use the identity

$$T_{4k+2} = 4U_{2k+1}^2 - 1 = (2U_{2k+1} - 1)(2U_{2k+1} + 1)$$

to show that, apart from a few exceptional values, T_{4k+2} is divisible by at least three distinct primes, each of which divide T_{4k+2} exactly to an odd power.
In order to show this, we need to find all integer solutions to

$$2U_{2k+1} \pm 1 = \mu z^2, \quad (\mu \in \{1, 3\}).$$

The equations $2U_{2k+1} \pm 1 = z^2$ can be rewritten as

$$x^2 - 2((z^2 \pm 1)/2)^2 = -1,$$

which can be brought into the Weierstrass forms

$$Y^2 = X^3 \pm 4X^2 - 4X,$$

with $Y = 4x$ and $X = 2z^2$. We used MAGMA to find the integer points on these two curves. This results in the two solutions $X = 2$, $z = 1$, $U_{2k+1} = 1$, $k = 0$, and hence $n = 4$, and $X = 18$, $z = 3$, $U_{2k+1} = 5$, $k = 1$, and hence $n = 12$. These two values of n will be considered at the end of this subcase.
The equations $2U_{2k+1} \pm 1 = 3z^2$ can be rewritten as

$$x^2 - 2((3z^2 \pm 1)/2)^2 = -1,$$

which can be brought into the Weierstrass forms

$$Y^2 = X^3 \pm 12X^2 - 36X,$$

with $Y = 36xz$ and $X = 18z^2$. We used MAGMA to find the integer points on these two curves. This results in the solution $X = 18$, $z = 1$, $U_{2k+1} = 1$, $k = 0$, and hence $n = 4$.
For all other values of n in this subcase, with 3 not dividing n, we see that both $2U_{2k+1} + 1$ and $2U_{2k+1} - 1$ are divisible by an odd prime different from 3 to an odd power. Therefore, T_{4k+2} is divisible by 3 and these two primes all to an odd power. It follows that there are at least 8 primes dividing the right hand side of (3.2) to an odd power.
Now consider the case that 3 divides $n = 8k + 4$. This implies that $k \geq 2$. We will again use the fact that

$$d_n y_n^2 = 16T_{8k+4}T_{8k+3}U_{8k+3}U_{2k+1}T_{2k+1}(2U_{2k+1} + 1)(2U_{2k+1} - 1), \tag{3.3}$$

where as before, each pair of terms in the product are pairwise coprime. In this case however, $n = 12l$ for some integer l, and hence we use the identity

$$T_{12l} = T_{4l}(4T_{4l}^2 - 3). \tag{3.4}$$

Since 3 divides $T_{4k+2} = T_{6l}$, we know that 3 does not divide T_{12l}, and hence the two factors on the right hand side of (3.4) are coprime. These same two factors are evidently not perfect squares, and hence $T_{12l} = T_{8k+4}$ is divisible by two primes to an odd power. Since $k \geq 2$, no factors in (3.3) can be squares, and hence the product is divisible by at least eight distinct primes, each to an odd power.
The exceptional values to consider in this case are $n = 4, 12, 28$.
In the case $n = 4k + 1$, the above arguments show that the exceptional values are $n = 5, 9, 13, 17, 29, 57, 113$.

Case 2: $n \equiv 2, 3 \pmod 4$
In the case $n = 4k + 2$ for some integer k, we have that

$$d_n y_n^2 = 4T_{4k+2}T_{4k+1}U_{4k+2}U_{4k+1}, \tag{3.5}$$

while if $n = 4k + 3$, then

$$d_n y_n^2 = 4T_{4k+3}T_{4k+3}U_{4k+2}U_{4k+3}.$$

The argument below will deal with the case $n = 4k + 2$, as essentially the same argument will prove the desired result in the case that $n = 4k + 3$.
In this case, equation (3.5) shows that

$$d_n y_n^2 = 8(2U_{2k+1} + 1)(2U_{2k+1} - 1)T_{4k+1}U_{4k+1}U_{2k+1}T_{2k+1}, \tag{3.6}$$

where as before, any two factors on the right hand side are pairwise coprime.
Assume first that $k \equiv 0, 2 \pmod 3$. Then 3 does not divide $4k + 2$, and hence $3 = T_2$ divides T_{4k+2} exactly to an odd power. Also, as argued in the previous case, neither of the factors $2U_{2k+1} + 1$ or $2U_{2k+1} - 1$ are a square, or three times a square, provided $k > 1$. Therefore, for $k > 1$, each factor on the right hand side of (3.6) contributes at least one prime factor to d_n, while $(2U_{2k+1} + 1)(2U_{2k+1} - 1)$ contributes three prime factors. This shows that d_n is divisible by at least eight distinct prime factors, apart from the exceptional values $n = 2, 6$.
Assume now that $k \equiv 1 \pmod 3$. This implies that 3 divides $2k + 1$, say $2k + 1 = 3l$. Therefore,

$$T_{2k+1} = T_{3l} = T_l(4T_l^2 + 3).$$

Since l is odd, 3 does not divide T_l, and hence $\gcd(T_l, 4T_l^2 + 3) = 1$. For $l > 1$, neither of T_l or $4T_l^2 + 3$ are squares, and hence T_{2k+1} is divisible by at least two primes to an odd power. Therefore, referring again to (3.6), we see that except for $k = 0, 1, 3$, there are at least eight distinct prime factors dividing d_n. The exceptional values of n are $n = 2, 6, 14$.
This same argument applies to the case $n = 4k + 3$, and the exceptional values arising from the analysis of this case are $n = 3, 7, 15$.
In all, the exceptional cases include the following values for n

$$2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 28, 29, 56, 57, 112, 113.$$

After checking all of the corresponding values d_n for the stated property, we obtain the list in the theorem.

4. Comments and Remarks

Let $\omega(n)$ be the number of distinct prime factors of the positive integer n. We would like to suggest the following conjecture.

Conjecture 4.1. If K is any fixed constant, there there are only finitely many quadruples of integers (x, y, z, d) with

$$x^2 - dy^2 = 1, \qquad z^2 - 2dy^2 = 1$$

and furthermore d squarefree with $\omega(d) < K$.

We back up this conjecture with the following heuristic. The counting function up to X of the set of positive integers m of the form d, where $\omega(d) \leq K$ is, by a theorem of Landau, $\sim c_K X(\log\log X)^{K-1}/\log X$, where c_K is a positive constant depending on K. Using this result and Abel's summation formula one proves that the counting function up to X of the set of positive integers m of the form du^2, where u is an integer and d is squarefree and has $\omega(d) \leq K$ is $\sim d_K X(\log\log X)^{K-1}/(\log X)$ for some other constant d_K depending on K. Thus, the expectation that a positive integer m has this form is $\ll_K (\log\log m)^{K-1}/(\log m)$. We now apply this heuristic to the numbers $m = T_n$ and $m = U_n$ appearing in equation (3.1), assuming that $\omega(d_n) \leq K$. Thus, the expectation that T_n is of the form du^2 with some squarefree d having at most K prime factors is $\ll_K (\log\log T_n)^{K-1}/\log(T_n) \ll (\log n)^{K-1}/n$. The same applies to U_n. Assuming that these events are independent, we deduce that the expectation that equation (3.1) holds with some squarefree integer d_n having at most K prime factors is $\ll_K (\log n)^{2(K-1)}/n^2$. Since the series

$$\sum_{n\geq 1} \frac{(\log n)^{2(K-1)}}{n^2}$$

is convergent, we conclude that there should be only finitely many values of n with the desired property.

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