SIMULTANEOUS QUADRATIC EQUATIONS WITH FEW OR NO SOLUTIONS

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Abstract. In this paper, we completely solve the simultaneous Diophantine equations

$$x^2 - az^2 = 1, \;\; y^2 - bz^2 = 1$$

provided the positive integers a and b satisfy $b-a \in \{1, 2, 4\}$. Further, we show that these equations possess at most one solutions in positive integers (x, y, z) if $b - a$ is a prime or prime power, under mild conditions. Our approach is (relatively) elementary in nature and relies upon classical results of Ljunggren on quartic Diophantine equations.

1. Introduction

A number of recent papers (see e.g. [1], [4], [5], [6], [9], [13], [14], [18], [19]) have discussed the solvability in integers of systems of simultaneous Pell equations of the form

$$x^2 - az^2 = 1, \;\; y^2 - bz^2 = 1, \tag{1.1}$$

where a and b are distinct nonsquare positive integers. Since (1.1) generically defines a curve of genus one, such results are analogous to finding integral points on a given model of an elliptic curve over $\mathbb{Q}$. It follows from work of Siegel [16] that (1.1) has finitely many integer solutions, upper bounds for the size of which may be deduced from the theory of linear forms in logarithms of algebraic numbers, à la Baker [2]. Indeed, in [4], the first author, sharpening work of Masser and Rickert [12], proved that such a system possesses at most three solutions in positive integers (x, y, z). Further, there are infinite families (a, b) for which (1.1) has at least two positive solutions (see e.g [5]).

The starting point of this paper is the minimal case of (1.1), with $a = 2$ and $b = 3$. In 1918, Rignaux [15] used an elementary argument (based, essentially, on Fermat's method of infinite descent) to show that (1.1) has, in this situation, no positive solutions. Much more recently, Rickert [14] gave another proof of this result, via the theory of Padé approximation to binomial functions. As remarked by Ono [13], however, the existence of only trivial solutions to (1.1) for $a = 2$ and $b = 3$ is a consequence of the related elliptic curve

$$y^2 = x(x + 2)(x + 3)$$

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having Mordell-Weil rank zero over $\mathbb{Q}$. In the general situation where $b-a=1$, though, this last argument may be insufficient to imply the nonexistence of positive solutions to (1.1). One may (say via Ian Connell's computer package APECS) check that roughly half of the elliptic curves of the form

$$y^2 = x(x+a)(x+a+1)$$

with $2 \leq a \leq 100$ have positive rank (e.g. $a = 6, 10, 17$, etc). On the other hand, in this paper we prove

Theorem 1.1. *If a and b are positive integers with $b-a \in \{1,2,4\}$, then the system of equations (1.1) has no solutions in positive integers (x,y,z) unless one of the following situations occur:*

(i) $(a,b) = (u^2-1, u^2+1)$ *for some integer u, in which case there is the one solution $(x,y,z) = (2u^2-1, 2u^2+1, 2u)$.*
(ii) $(a,b) = (u^2-2, u^2+2)$ *for some integer u, in which case there is the one solution $(x,y,z) = (u^2-1, u^2+1, u)$.*

In particular, there are no positive solutions if $b-a=1$.

In the more general setting where $b-a$ is divisible by at most one prime, we are able to obtain the slightly less precise result:

Theorem 1.2. *Suppose that a and b are nonsquare positive integers with $b-a=p^k$ for p prime and $k \in \mathbb{N}$. Then (1.1) has at most one solution in positive integers (x,y,z) with $\gcd(x,y)$ not divisible by p.*

An immediate consequence of this is

Corollary 1.3. *If a and b are nonsquare positive integers with $b-a$ prime, then (1.1) has at most one solution in positive integers (x,y,z).*

To see how Corollary 1.3 follows from Theorem 1.2, observe that if $b-a=p$, then (1.1) implies that $y^2-x^2=pz^2$. If p divides both x and y, then p also divides z, contradicting (1.1). This result is sharp in the sense that, given an integer $n=2$ or $n \geq 4$, we may find integers a and b for which (1.1) is solvable with $b-a=n$. In fact, for $n=2k+1$, $k \geq 2$, then with $b=k^2+2k$, $a=k^2-1$, (1.1) has the solution $(x,y,z)=(k,k+1,1)$. Similarly, if $n=2k$, then with $b=k^2+k$ and $a=k^2-k$, (1.1) has the solution $(x,y,z)=(2k-1,2k+1,2)$. We know of no pair (a,b) with $b-a=3$ for which (1.1) is solvable in positive integers.

One motivation for studying equations (1.1) for which $b-a$ has few prime factors is that the parameterized families of (a,b) for which (1.1) is known to have at least two solutions, when viewed as polynomials in one of the parameters m, have the property that $b-a$ factors over $\mathbb{Z}[m]$ into many irreducible, pairwise relatively prime polynomials of low degree (see e.g. [5] for a description of these families). It follows that $\omega(b-a)$ should grow quite rapidly for such pairs (a,b) (here, $\omega(n)$ denotes the number of distinct prime factors of n). One may, in fact, readily show, for these families, that $\omega(b-a) \geq 4$ unless $a=m^2-1$ and $b=n^2-1$ with

$$n = 16m^5 - 16m^3 + 3m,$$

whereby

$$b-a = 8m^2(2m^2-1)^2(8m^4-8m^2+1).$$

In this case, we have $\omega(b-a) = 3$ provided $m = 2, 8, 256$ or 512. Possibly, these are the only examples of (a, b) for which $\omega(b-a) \leq 3$ and (1.1) possesses two positive solutions.

As a final remark, we note that, with a modicum of effort, the elementary approach we take in proving Theorems 1.1 and 1.2 may be extended to treat the cases where $b - a = 2^j p^k$, for $j, k \in \mathbb{N}$ and p prime.

2. Preliminary Results

The proofs of Theorem 1.1 and Theorem 1.2 rely on the following sharpening of Ljunggren's classical result on the equation $X^2 - DY^4 = 1$ (see [11]), proved in [20].

Lemma 2.1. *Let D be a nonsquare positive integer, let $T + U\sqrt{D}$ denote the fundamental solution to $X^2 - DY^2 = 1$, and let $T_k + U_k\sqrt{D} = (T + U\sqrt{D})^k$ for $k \geq 1$. If there are two solutions $k_1 < k_2$ to the equation $U_k = 2^\delta Z^2$, with $\delta \in \{0, 1\}$ and Z a positive integer, then except for $D \in \{1785, 4 \cdot 1785, 16 \cdot 1785\}$, the two solutions are precisely $k_1 = 1$ and $k_2 = 2$. For the three exceptional values given, there is a third solution $k_3 = 4$.*

A simple consequence of this is the following

Lemma 2.2. *If D is a nonsquare positive integer with $D \not\equiv 3 \pmod 4$ and $D \notin \{1785, 4 \cdot 1785, 16 \cdot 1785\}$, then the Diophantine equation $x^2 - Dy^4 = 1$ has at most one solution in positive integers x and y.*

Proof. From Lemma 2.1, if $D \notin \{1785, 4 \cdot 1785, 16 \cdot 1785\}$ and $x^2 - Dy^4 = 1$ has two distinct positive solutions, then, in the notation of Lemma 2.1, we have $U = u^2$ and $2TU = v^2$ for integers u and v. It follows that $T = 2w^2$ for $w \in \mathbb{Z}$ and so, since $T^2 - DU^2 = 1$, we have $4w^4 - Du^4 = 1$ whereby $D \equiv 3 \pmod 4$. □

Regarding the related equation $X^2 - DY^4 = 4$, we will have need of a result of Cohn, which appeared as Theorem 3 in [7]:

Lemma 2.3. *Let D be a nonsquare positive integer for which the equation $X^2 - DY^2 = 4$ has a solution in odd integers. Let $t + u\sqrt{D}$ denote the smallest such solution, and for $k \geq 1$, define*

$$\frac{t_k + u_k\sqrt{D}}{2} = \left(\frac{t + u\sqrt{D}}{2}\right)^k.$$

The equation $u_k = x^2$ has only the solutions $k = 1$ if u is a square, $k = 2$ if t and u are squares, and the possible solution $k = 3$ if $u = 3B^2$ for some integer B.

Finally, to deal with the special values $D \in \{1785, 4 \cdot 1785, 16 \cdot 1785\}$ described in Lemma 2.1, we utilize

Lemma 2.4. *If a and b are positive integers such that $ab \in \{1785, 4 \cdot 1785, 16 \cdot 1785\}$, then (1.1) has no solutions in positive integers (x, y, z) unless*

$$(a, b) \in \{(2, 14280), (6, 1190), (68, 105), (168, 170)\}.$$

In each of these cases, there is precisely one positive solution, corresponding to $z = 2, 2, 4$ and 26, respectively.

Proof. This lemma follows from a standard combination of bounds for linear forms in logarithms of (three) algebraic numbers, with lattice basis reduction (in this case, the lemma of Baker and Davenport [3] suffices). The reader is directed to [1] and [5] for details. We note that the main result of [1] implies that the number of positive solutions to (1.1) is at most one, provided $\max\{a, b\} \leq 200$. □

3. Proof of Theorem 1.1

From Lemma 2.4, we may henceforth assume that

$$ab \notin \{1785, 4 \cdot 1785, 16 \cdot 1785\}. \tag{3.1}$$

Suppose, first, that (x, y, z) is a positive solution to (1.1) where $b-a = 1$. It follows that $y^2 - x^2 = z^2$, and $\gcd(y-x, y+x) = 1$ or 2. If $\gcd(y-x, y+x) = 1$, then there exist positive integers A and B such that $z = AB$, $y - x = A^2$ and $y + x = B^2$. Thus $y = \frac{A^2+B^2}{2}$ and upon substituting y and z into the second equation in (1.1) and simplifying, we obtain

$$\left(\frac{A^2 + (1-2b)B^2}{2}\right)^2 - b(b-1)B^4 = 1. \tag{3.2}$$

Similarly, if $\gcd(y - x, y + x) = 2$, we may find positive integers A and B with $z = 2AB$, $y - x = 2A^2$ and $y + x = 2B^2$, whence $y = A^2 + B^2$ and (from (1.1)),

$$\left(A^2 + (1-2b)B^2\right)^2 - 4b(b-1)B^4 = 1. \tag{3.3}$$

Now the fundamental solution to $X^2 - b(b-1)Y^2 = 1$ is given by

$$\epsilon = (2b-1) + 2\sqrt{b(b-1)}$$

and so, defining T_k and U_k by

$$T_k + U_k\sqrt{b(b-1)} = \epsilon^k,$$

the equation $U_k = 2^\delta B^2$ has a solution $(k, \delta, B) = (1, 1, 1)$. Inequality 3.1 and Lemma 2.1 therefore imply that all possible solutions to the equation $U_k = 2^\delta B^2$ have $k \in \{1, 2\}$. But, if $k = 1$, we have $\delta = B = 1$ and so (3.3) implies that

$$\left(A^2 + (1-2b)\right)^2 = 4b(b-1) + 1$$

whence

$$A^2 + (1-2b) = \pm(2b-1).$$

It follows that $A^2 = 0$ or $A^2 = 4b - 2$, both of which contradict the fact that A is a positive integer. If we have a solution with $k = 2$, then since

$$T_2 + U_2\sqrt{b(b-1)} = (2(2b-1)^2 - 1) + 4(2b-1)\sqrt{b(b-1)},$$

it follows that $2^\delta B^2 = 4(2b-1)$. Arguing modulo 4 implies that $\delta = 0$, whereby, substituting $4(2b-1)$ for B^2 in (3.2) yields

$$\left(\frac{A^2 - (4b-2)^2}{2}\right)^2 = \left(8b^2 - 8b + 1\right)^2.$$

We conclude that

$$A^2 - (4b-2)^2 = \pm(16b^2 - 16b + 2) \equiv 2 \pmod 4,$$

which gives the desired contradiction.

Next assume that $b - a = 2$. In this case, $y^2 - x^2 = 2z^2$ and so there are integers A and B such that $z = 2AB$, $y \pm x = 4A^2$ and $y \mp x = 2B^2$. Therefore $y = B^2 + 2A^2$,

and upon substituting y and z into the second equation in (1.1) and simplifying, we obtain

$$\left(B^2 + (2-2b)A^2\right)^2 - 4b(b-2)A^4 = 1. \tag{3.4}$$

The minimal solution to $X^2 - b(b-2)Y^2 = 1$ is $(b-1) + \sqrt{b(b-2)}$ and so we define $T_k + U_k\sqrt{b(b-2)} = ((b-1) + \sqrt{b(b-2)})^k$. It follows that the equation $U_k = 2^\delta A^2$ has the solution $(k, \delta, A) = (1, 0, 1)$ and so Lemma 2.1 implies that $U_k = 2A^2$ possesses a solution only (possibly) for $k = 2$. For such a solution, we have $2A^2 = U_2 = 2T_1U_1 = 2(b-1)$ and so $(a, b) = (A^2 - 1, A^2 + 1)$. Substituting $A^2 = b - 1$ into (3.4), we find that $B = 1$, and so it is easily deduced that the only solution in positive integers is $(x, y, z) = (2A^2 - 1, 2A^2 + 1, 2A)$. Conversely, for any integer A, equations (1.1) with $(a, b) = (A^2 - 1, A^2 + 1)$ has the positive integer solution $(x, y, z) = (2A^2 - 1, 2A^2 + 1, 2A)$.

Now assume that $b - a = 4$. In this situation, $y^2 - x^2 = 4z^2$ and so there are positive integers A and B such that $z = AB$, $y \pm x = 2A^2$ and $y \mp x = 2B^2$. In both cases, $y = A^2 + B^2$. Substituting y and z into the second equation in (1.1) and simplifying, one therefore obtains

$$(2A^2 + (2-b)B^2)^2 - b(b-4)B^4 = 4. \tag{3.5}$$

First, consider the case that b is even, $b = 2b_0$ say. Then (3.5) becomes

$$\left(A^2 + (1-b_0)B^2\right)^2 - b_0(b_0-2)B^4 = 1.$$

From an analysis similar to that in the previous paragraph, it follows that there is a solution in positive A, B only if $b_0 = 2u^2 + 1$ for some integer u. Hence $(a, b) = ((2u)^2 - 2, (2u)^2 + 2)$, with the only solution to (1.1) being $(x, y, z) = ((2u)^2 - 1, (2u)^2 + 1, 2u)$.

If b is odd, then the equation $X^2 - b(b-4)Y^2 = 4$ is solvable in odd integers $(X, Y) = (b-2, 1)$. Therefore, by Lemma 2.3, the only possible solutions to (3.5) arise from the minimal solution $\frac{b-2+\sqrt{b(b-4)}}{2}$ or its square $\frac{(b^2-4b+2)+(b-2)\sqrt{b(b-4)}}{2}$ (i.e. we have either $B^2 = 1$ or $B^2 = b - 2$). In the first instance, from (3.5) we have that $2A^2 + (2-b) = \pm(b-2)$ forcing either $A = 0$, which is not possible, or $b = A^2 + 2$. In the second case, substituting $B^2 = b - 2$ into (3.5), we have

$$2A^2 - (b-2)^2 = \pm(b^2 - 4b + 2).$$

The choices of signs lead to either $A^2 = (b-2)^2 - 1$, a contradiction since A is positive, or to $A = 1$. Therefore, in any case, $(a, b) = (u^2 - 2, u^2 + 2)$ for some positive integer u, and the only solution in positive integers to (1.1) is $(x, y, z) = (u^2 - 1, u^2 + 1, u)$. This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Let us now suppose that $b - a = p^k$ for p prime and $k \in \mathbb{N}$. We first take $p = 2$ (whereby, from Theorem 1.1, we may assume that $k \geq 3$). It follows that $y^2 - x^2 = 2^k z^2$ and so, since $\gcd(x, y)$ is odd, by assumption, there exist integers A and B, B odd, such that $z = AB$, $y \pm x = 2^{k-1}A^2$ and $y \mp x = 2B^2$. Therefore, $y = B^2 + 2^{k-2}A^2$ and so upon substituting y and z into the second equation in (1.1) and simplifying, one obtains

$$B^4 + 2^{k-1}A^2B^2 + 2^{2k-4}A^4 - bA^2B^2 = 1. \tag{4.1}$$

Note that since $k \geq 3$ and B is odd, it follows that 4 divides bA^2. Multiplying (4.1) by 4 and completing the square therefore yields

$$\left(2B^2 + (2^{k-1} - b)A^2\right)^2 - b(b - 2^k)A^4 = 4. \tag{4.2}$$

Assume first that 4 does not divide b. From the above remark, A is even, say $A = 2A_0$, whereby equation (4.2) becomes

$$\left(B^2 + (2^k - 2b)A_0^2\right)^2 - 4b(b - 2^k)A_0^4 = 1. \tag{4.3}$$

We may thus apply Lemma 2.2 to conclude that (4.3) has at most one positive solution (again, we are assuming (3.1)).

Now suppose that 4 divides b and hence a. If we write $b = 4b_0$ and $a = 4a_0$, then $b_0 - a_0 = 2^{k-2}$ and the number of solutions to (1.1) is bounded by the number of solutions to (1.1) with (a, b) replaced by (a_0, b_0) (a solution (x, y, z) in the first case corresponds to a solution $(x, y, 2z)$ in the second). The fact that there is at most one solution to (1.1) now follows inductively by the result of the previous paragraph, together with Theorem 1.1.

Suppose now that p is an odd prime. From (1.1), we have that $y^2 - x^2 = p^k z^2$ and so, since we assume that p does not divide both x and y, we must have either

$$y \pm x = p^k A^2,\ y \mp x = B^2, z = AB,$$

with A and B odd, or

$$y \pm x = 2p^k A^2,\ y \mp x = 2B^2, z = 2AB.$$

In the first case, $y = \frac{1}{2}(B^2 + p^k A^2)$ and, upon substituting this into the second equation in (1.1) and completing the square, we find

$$\left(\frac{B^2 - (a+b)A^2}{2}\right)^2 - abA^4 = 1. \tag{4.4}$$

The second case yields $y = B^2 + p^k A^2$ and, in a similar manner, we obtain

$$\left(B^2 - (a+b)A^2\right)^2 - 4abA^4 = 1. \tag{4.5}$$

In either case, we may clearly suppose that ab is not a square.

Assume that there are two distinct solutions (x_1, y_1, z_1) and (x_2, y_2, z_2) in positive integers to (1.1), with, say $z_2 > z_1$ and satisfying the condition of Theorem 1.1. Then, from (4.4) and (4.5), there are two units in $\mathbb{Q}(\sqrt{ab})$ of one of the following forms:

$$\left|\frac{B^2 - (a+b)A^2}{2}\right| + A^2\sqrt{ab} \ \text{ where } \ A \ \text{ odd}$$

or

$$\left|C^2 - (a+b)D^2\right| + 2D^2\sqrt{ab}.$$

If $\epsilon = T + U\sqrt{ab}$ denotes the fundamental solution to $X^2 - abY^2 = 1$, then it is readily deduced from Lemma 2.1, since T and U are positive, that

$$\epsilon = \pm\left(\frac{B^2 - (a+b)A^2}{2}\right) + A^2\sqrt{ab} = u^2 + A^2\sqrt{ab},$$

and

$$\epsilon^2 = \pm\left(C^2 - (a+b)D^2\right) + 2D^2\sqrt{ab},$$

where $z_1 = AB$ with A and B odd and $z_2 = 2CD$. From the choices of signs, we are left with four cases to consider, which we will treat in turn. Three of these prove to be straightforward, while the fourth requires considerably more effort.

First, suppose that

$$\epsilon = \frac{B^2 - (a+b)A^2}{2} + A^2\sqrt{ab},$$

and

$$\epsilon^2 = C^2 - (a+b)D^2 + 2D^2\sqrt{ab},$$

whereby

$$C^2 - (a+b)D^2 = 2\left(\frac{B^2 - (a+b)A^2}{2}\right)^2 - 1$$

and

$$2D^2 = 2A^2\left(\frac{B^2 - (a+b)A^2}{2}\right).$$

Upon simplifying and using the fact that

$$\frac{B^2 - (a+b)A^2}{2} = u^2,$$

we obtain

$$C^2 + 1 = B^2\left(\frac{B^2 - (a+b)A^2}{2}\right) = u^2B^2.$$

This leads to $u = 1$, $\epsilon = 1$ and hence $A = 0$, a contradiction.

Next, let

$$\epsilon = \frac{B^2 - (a+b)A^2}{2} + A^2\sqrt{ab}$$

and

$$\epsilon^2 = (a+b)D^2 - C^2 + 2D^2\sqrt{ab}.$$

Recall that $\epsilon = u^2 + A^2\sqrt{ab}$. Since one of a or b is even, u is odd, and hence

$$B^2 - (a+b)A^2 = 2u^2 \equiv 2 \pmod 8.$$

Since A and B are odd, it follows that $a + b \equiv 7 \pmod 8$. Now

$$(a+b)D^2 - C^2 = 2u^4 - 1 \equiv 1 \pmod 8$$

and since $D = uA$ is odd, it follows that $C^2 \equiv 6 \pmod 8$, which is clearly not possible.

If

$$\epsilon = \frac{(a+b)A^2 - B^2}{2} + A^2\sqrt{ab}$$

and

$$\epsilon^2 = (a+b)D^2 - C^2 + 2D^2\sqrt{ab},$$

then an argument similar to the one given for the first situation implies that $C^2 - 1 = B^2u^2$, forcing $Bu = 0$, a contradiction.

We must work a little harder to rule out the fourth possibility. Assume henceforth that

$$\epsilon = \frac{(a+b)A^2 - B^2}{2} + A^2\sqrt{ab} = u^2 + A^2\sqrt{ab} \tag{4.6}$$

and

$$\epsilon^2 = C^2 - (a+b)D^2 + 2D^2\sqrt{ab}. \tag{4.7}$$

Recall that (x_1, y_1, z_1) and (x_2, y_2, z_2) are distinct positive solutions to (1.1), with, say, $z_2 > z_1$ (whence $z_1 = AB$ and $z_2 = 2CD$). It follows that we may write

$$z_i = \frac{\alpha^{j_i} - \alpha^{-j_i}}{2\sqrt{a}} = \frac{\beta^{k_i} - \beta^{-k_i}}{2\sqrt{b}}$$

where α and β are the fundamental solutions to the equations $x^2 - az^2 = 1$ and $y^2 - bz^2 = 1$, respectively, and j_i and k_i are positive integers, for $1 \leq i \leq 2$. Recall, also, that A, B and hence z_1 are odd, while z_2 is even. It follows from standard results on the 2-divisibility of terms in linear recurrence sequences (see e.g. [10]) that j_2 and k_2 are necessarily even.

To obtain our desired contradiction, we consider the quantity

$$\frac{z_1^2}{z_2} = \frac{A^2B^2}{2CD} = \frac{\left(\alpha^{j_1} - \alpha^{-j_1}\right)^2}{2\sqrt{a}\left(\alpha^{j_2} - \alpha^{-j_2}\right)} = \frac{\left(\beta^{k_1} - \beta^{-k_1}\right)^2}{2\sqrt{b}\left(\beta^{k_2} - \beta^{-k_2}\right)}. \tag{4.8}$$

Now, from (4.6) and (4.7) and the fact that ϵ and ϵ^2 are units in $\mathbb{Q}(\sqrt{ab})$ of norm one, we may readily show that

$$0 < A^2\left(\sqrt{b} - \sqrt{a}\right)^2 - B^2 < \frac{1}{A^2\sqrt{ab}}, \tag{4.9}$$

$$0 < C - \left(\sqrt{b} + \sqrt{a}\right)D < \frac{1}{8D^3\sqrt{ab}\left(\sqrt{b} + \sqrt{a}\right)} \tag{4.10}$$

and

$$0 < D^2 - \sqrt{ab}A^4 < \frac{1}{2A^2\sqrt{ab}}. \tag{4.11}$$

Combining these inequalities with (4.8), we find that

$$\kappa\frac{\left(\sqrt{b} - \sqrt{a}\right)^2}{2\sqrt{ab}\left(\sqrt{b} + \sqrt{a}\right)} < \frac{z_1^2}{z_2} < \frac{\left(\sqrt{b} - \sqrt{a}\right)^2}{2\sqrt{ab}\left(\sqrt{b} + \sqrt{a}\right)} \tag{4.12}$$

where

$$\kappa = \left(1 - \frac{1}{A^4\sqrt{ab}\left(\sqrt{b} - \sqrt{a}\right)^2}\right)\left(1 - \frac{1}{2A^2D^2\sqrt{ab}}\right)\left(1 - \frac{1}{8CD^3\sqrt{ab}\left(\sqrt{b} + \sqrt{a}\right)}\right).$$

Using (4.9), (4.10) and (4.11), we have that

$$\kappa > \left(1 - \frac{1}{A^2B^2\sqrt{ab}}\right)\left(1 - \frac{1}{2A^6ab}\right)\left(1 - \frac{1}{8A^8(ab)^{3/2}\left(\sqrt{b} + \sqrt{a}\right)^2}\right)$$

and so $b - a \geq 3$, $A \geq 1$ and $B \geq 1$ imply that $\kappa > 0.64$.

Next, observe that (4.8) also implies the inequalities

$$\left(1 - \beta^{-2}\right)^2\frac{\beta^{2k_1 - k_2}}{2\sqrt{b}} < \frac{z_1^2}{z_2} < \frac{\beta^{2k_1 - k_2}}{2\sqrt{b}}.$$

Combining these with (4.12) yields

$$\kappa\frac{\left(\sqrt{b}-\sqrt{a}\right)^2}{\sqrt{a}\left(\sqrt{b}+\sqrt{a}\right)} < \beta^{2k_1-k_2} < \left(1-\beta^{-2}\right)^{-2}\frac{\left(\sqrt{b}-\sqrt{a}\right)^2}{\sqrt{a}\left(\sqrt{b}+\sqrt{a}\right)}. \tag{4.13}$$

Since $\beta > 2\sqrt{b}$ and k_2 is even, it follows that $2k_1 - k_2 \in \{-2, 0\}$. In fact, if $2k_1 - k_2 \geq 2$, then $\beta^{2k_1-k_2} \geq \beta^2 > 4b$, which contradicts the second inequality in (4.13). Similarly, if $2k_1 - k_2 \leq -4$, then $\beta^{2k_1-k_2} \leq \beta^{-4} \leq \left(16b^2\right)^{-1}$, contradicting the first inequality in (4.13).

Suppose that $2k_1 - k_2 = 0$. Since $\beta \geq 3+\sqrt{8}$ and $\kappa > 0.64$, it follows from (4.13) that

$$0.94 < \frac{\left(\sqrt{b}-\sqrt{a}\right)^2}{\sqrt{a}\left(\sqrt{b}+\sqrt{a}\right)} < 1.57,$$

whereby we may readily show that $8.53a < b < 13.87a$. Since (4.8) implies that

$$\kappa\frac{\left(\sqrt{b}-\sqrt{a}\right)^2}{\sqrt{b}\left(\sqrt{b}+\sqrt{a}\right)} < \alpha^{2j_1-j_2} < \left(1-\alpha^{-2}\right)^{-2}\frac{\left(\sqrt{b}-\sqrt{a}\right)^2}{\sqrt{b}\left(\sqrt{b}+\sqrt{a}\right)},$$

the inequality $\alpha \geq 2+\sqrt{3}$ yields

$$0.20 < \alpha^{2j_1-j_2} < 0.49,$$

a contradiction, since j_2 is even and, again, $\alpha \geq 2+\sqrt{3}$.

Finally, let us suppose that $2k_1 - k_2 = -2$. Define the sequences $\{V_n\}$ and $\{W_n\}$ by $V_n + W_n\sqrt{b} = \beta^n$ for $n \in \mathbb{N}$, so that $W_{k_i} = z_i$ for $1 \leq i \leq 2$. It follows, since A divides D, that A divides $2CD = z_2 = W_{k_2}$, while A also divides $W_{2k_1} = 2V_{k_1}W_{k_1} = 2V_{k_1}AB$. By the divisibility properties of the sequence $\{W_n\}$, $\gcd(W_{k_2}, W_{2k_1}) = W_{\gcd(k_2,2k_1)} = W_2$, and hence A divides W_2. But since W_{k_1} is odd and divisible by A, it follows that A divides W_1.

If $z_1 \neq W_1$, then $k_1 > 1$ whereby, since W_{k_1} and hence k_1 are odd, we have $k_1 \geq 3$. It follows that

$$AB = W_{k_1} \geq W_3 = \left(4V_1^2-1\right)W_1 > \left(4W_1^2b-1\right)W_1 \geq \left(4A^2b-1\right)A,$$

where the last inequality is a consequence of the fact that A divides W_1. This contradicts (4.9) and so we conclude that $k_1 = 1$ and thus $k_2 = 4$. Since $V_1 \geq 2$, we have

$$W_4 = 4\left(2V_1^2-1\right)V_1W_1 \geq 7V_1^3W_1 > 7W_1^4b\sqrt{b} \geq 7A^4b\sqrt{b}.$$

On the other hand, (4.10) and (4.11) show that

$$W_4 = 2CD < 2\left(\sqrt{b}+\sqrt{a}\right)\sqrt{ab}A^4 + \frac{\sqrt{b}+\sqrt{a}}{A^2\sqrt{ab}} + \frac{1}{4D^2\sqrt{ab}\left(\sqrt{b}+\sqrt{a}\right)}$$

whence

$$W_4 < 4A^4b\sqrt{b}+2.$$

This contradiction completes the proof of Theorem 1.2.

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