A GENERALIZATION OF A THEOREM OF BUMBY.

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Abstract. Bumby proved that the only positive integer solutions to the quartic Diophantine equation $3X^4 - 2Y^2 = 1$ are $(X, Y) = (1, 1), (3, 11)$. In this paper we use the hypergeometric method of Thue to prove that for each integer $m \geq 1$, the only positive integers solutions to the Diophantine equation $(m^2+m+1)X^4 - (m^2+m)Y^2 = 1$ are $(X, Y) = (1, 1), (2m+1, 4m^2+4m+3)$.

1. Introduction

In [2], Bumby devised a very clever argument involving arithmetic in the quartic number field $\mathbb{Q}(\sqrt{-2}, \sqrt{-3})$ to prove that the only positive integer solutions X, Y of the quartic Diophantine equation

$$3X^4 - 2Y^2 = 1$$

are $(X, Y) = (1, 1), (3, 11)$. Equations of the type

$$aX^4 - bY^2 = 1 \tag{1.1}$$

havebeen studied for some time, most notably in the work of Ljunggren [5], who determined all integer solutions under the condition that the associated Pellian equation $aX^2 - bY^2 = 4$ is solvable in odd integers X, Y.

There has been renewed interest (for example see [1], [4], [8], and [9]) in Diophantine equations of this type. In particular, there is an effort to remove the hypothesis of Ljunggren's theorem, and thereby solve (1.1) for any pair of positive integers a, b for which the associated Pellian equation $aX^2 - bY^2 = 1$ is solvable in nonzero integers X, Y. In particular, Chen and Voutier [3] have improved upon a theorem of Ljunggren by showing that for $d > 2$, the equation $X^2 - dY^4 = -1$, which is of the above type, has at most one solution in positive integers x, y, and if such a solution exists, then $(X, Y) = (x, y^2)$ is the minimal solution of the associated Pell equation $X^2 - dY^2 = -1$.

As detailed in the introduction of a recent paper (see [7]), in order to determine the integer solutions to an equation of the form $aX^4 - bY^2 = 1$, it sufficient to consider the case that a and b differ by 1. That is, it is sufficient to determine the set of integer solutions to Diophantine equations of the type

$$(t + 1)X^4 - tY^2 = 1. \tag{1.2}$$

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We begin by stating a conjecture on the set of integer solutions to (1.2), which will provide further motivation for the main result of this paper.

Conjecture 1.1. *Let $t > 1$ denote a positive integer. Then the only positive integer solution to*

$$(t+1)X^4 - tY^2 = 1 \tag{1.3}$$

is $(X,Y) = (1,1)$, unless $t = m^2 + m$ for some positive integer m, in which case there is also the solution $(X,Y) = (2m+1, 4m^2+4m+3)$.

For $k \geq 0$, define a sequence of polynomials $\{V_{2k+1}(t)\}$ by

$$(\sqrt{t+1} + \sqrt{t})^{2k+1} = V_{2k+1}(t)\sqrt{t+1} + U_{2k+1}(t)\sqrt{t}.$$

For a given integer $t \geq 1$, a positive integer solution (X,Y) to the quartic Diophantine equation $(t+1)X^4 - tY^2 = 1$ is equivalent to an index $k \geq 0$ for which $X^2 = V_{2k+1}(t)$. In [7], the authors showed that for all $k \geq 1$, the equation $X^2 = V_{4k+1}(t)$ has no solutions in positive integers X, t. The primary method used was Thue's hypergeometric method applied to a related parametric family of quartic Thue equations. Moreover, it was shown that problems with this method arise for the remaining equation $X^2 = V_{4k+3}(t)$.

The purpose of the present paper is to generalize Bumby's result by showing that the hypergeometric method can be applied to solve the equation $X^2 = V_{4k+3}(t)$ in the particular case that $t = m^2+m$, which corresponds to the subfamily of equations in Conjecture 1.1 for which two positive integer solutions to $(t+1)X^4 - tY^2 = 1$ exist.

Theorem 1.2. *Let $m \geq 1$ denote a positive integer. Then the only positive integer solutions to*

$$(m^2+m+1)X^4 - (m^2+m)Y^2 = 1 \tag{1.4}$$

are $(X,Y) = (1,1)$ and $(X,Y) = (2m+1, 4m^2+4m+3)$.

The strategy of the paper is as follows. In Proposition 2.1 of [7] it was shown that a positive integer solution to $(t+1)X^4 - tY^2 = 1$ gives rise to a solution to a Thue equation of the form

$$x^4 + 4tx^3y - 6tx^2y^2 - 4t^2xy^3 + t^2y^4 = t_0^2, \tag{1.5}$$

where $1 \leq t_0 \leq \sqrt{t}$, and t_0 is a divisor of t. It follows that x/y is very close to one of the roots of the quartic polynomial

$$p_t(x) = x^4 + 4tx^3 - 6tx^2 - 4t^2x + t^2, \tag{1.6}$$

which are labeled as $\beta^{(i)}$, $i = 1,2,3,4$. It was shown in [7] that a solution to $X^2 = V_{4k+3}(t)$ forces x/y to be close to either $\beta^{(1)}$ or $\beta^{(2)}$, (these roots will be given explicitly below). The hypergeometric method will then be used to obtain an effective measure of approximation for these two roots, showing that no such rational number x/y can exist.

Notation Throughout the paper t is a parameter arising from the family of quartic equations $(t+1)X^4 - tY^2 = 1$, and m is a separate parameter arising from the subfamily of quartic equations $(m^2+m+1)X^4 - (m^2+m)Y^2 = 1$. In other words,

t and m are related by the equation $t = m^2 + m$ when t is specified to be of this particular form.

2. An Effective Measure of Approximation

Let us start by recalling some notation.

Notation 2.1. For positive integers n and r, we put

$$X_{n,r}(X) = {}_2F_1(-r, -r - 1/n; 1 - 1/n; X), \tag{2.1}$$

where ${}_2F_1$ denotes the classical hypergeometric function. We use $X^*_{n,r}$ to denote the homogeneous polynomials derived from these polynomials, so that

$$X^*_{n,r}(X, Y) = Y^r X_{n,r}(X/Y) \tag{2.2}$$

In Proposition 2.1 of [7], it was shown a positive integer solution of equation (1.2) gives rise to a solution to a Thue equation. For reference purposes, we recall here Proposition 2.1 of [7]

Proposition 2.2. Let $t > 1$ be a positive integer. If (X, Y) is a positive integer solution to (1.4) other than $(1, 1)$, then there is an integer solution (x, y) to the Thue equation

$$x^4 + 4tx^3y - 6tx^2y^2 - 4t^2xy^3 + t^2y^4 = t_0^2 \tag{2.3}$$

where t_0 divides t and $t_0 \leq \sqrt{t}$.

In order to apply the hypergeometric method, one requires good rational approximations to the roots $\beta^{(i)}, i = 1, 2, 3, 4$ of the polynomial

$$p_t(x) = x^4 + 4tx^3 - 6tx^2 - 4t^2x + t^2, \tag{2.4}$$

which are given explicitly by

(2.5)

$$\beta^{(1)} = \frac{\sqrt{t}}{\tau}(1 + \rho),\ \beta^{(2)} = \frac{\sqrt{t}}{\tau}(1 - \rho),\ \beta^{(3)} = (-\tau + \rho)\sqrt{t},\ \beta^{(4)} = -(\tau + \rho)\sqrt{t},$$

where $\tau = \sqrt{t+1} + \sqrt{t}$ and $\rho = \sqrt{\tau^2 + 1}$.

Now we consider $t = m^2 + m$. We also need some inequalities for the location of the roots. We obtain

(2.6)

$$\begin{aligned}
m + 1 + \tfrac{1}{8m^2} + \tfrac{1}{16m^3} &< \beta^{(1)} < m + 1 + \tfrac{1}{8m^2} + \tfrac{1}{8m^3}, \\
-m - \tfrac{1}{8m^2} + \tfrac{1}{16m^3} &< \beta^{(2)} < -m - \tfrac{1}{8m^2} + \tfrac{1}{8m^3}, \\
\tfrac{1}{4} - \tfrac{5}{64m^2} + \tfrac{1}{16m^3} &< \beta^{(3)} < \tfrac{1}{4} - \tfrac{5}{64m^2} + \tfrac{5}{64m^3}, \\
-4m^2 - 4m - \tfrac{5}{4} + \tfrac{21}{64m^2} - \tfrac{21}{64m^3} &< \beta^{(4)} < -4m^2 - 4m - \tfrac{5}{4} + \tfrac{21}{64m^2} - \tfrac{5}{16m^3}.
\end{aligned}$$

In this section we will apply the hypergeometric method to obtain effective measures of approximation to the two roots $\beta^{(1)}$ and $\beta^{(2)}$. Because of the relation $\beta^{(1)}\beta^{(2)} = -t$, we will only need to deal with one of the roots, say $\beta^{(1)}$.

Let us recall the following results in [7] that are very useful to apply the hypergeometric method.

Lemma 2.3. *Let α_1, α_2, c_1 and c_2 be complex numbers with $\alpha_1 \neq \alpha_2$. For $n \geq 2$, we define the following polynomial*

$$a(X) = \frac{n^2-1}{6}(\alpha_1-\alpha_2)(X-\alpha_2), \qquad c(X) = \frac{n^2-1}{6}\alpha_1(\alpha_1-\alpha_2)(X-\alpha_2),$$
$$b(X) = \frac{n^2-1}{6}(\alpha_2-\alpha_1)(X-\alpha_1), \qquad d(X) = \frac{n^2-1}{6}\alpha_2(\alpha_2-\alpha_1)(X-\alpha_1),$$
$$u(X) = -c_2(X-\alpha_2)^n \quad \text{and} \quad z(X) = c_1(X-\alpha_1)^n.$$

Putting $\lambda = (\alpha_1-\alpha_2)^2/4$, for any positive integer r, we define

$$\begin{aligned}(\sqrt{\lambda})^r A_r(X) &= a(X)X^*_{n,r}(z,u) + b(X)X^*_{n,r}(u,z) \text{ and}\\ (\sqrt{\lambda})^r B_r(X) &= c(X)X^*_{n,r}(z,u) + d(X)X^*_{n,r}(u,z).\end{aligned}$$

Then, for any root β of $P(X) = z(X) - u(X)$, the polynomial

$$C_r(X) = \beta A_r(X) - B_r(X)$$

is divisible by $(X-\beta)^{2r+1}$.

Proof. This is a simplified version of Lemma 2.1 from [3]. □

Lemma 2.4. *With the above notation, put $w(x) = z(x)/u(x)$ and write $w(x) = \mu e^{i\varphi}$ with $\mu \geq 0$ and $-\pi < \varphi \leq \pi$. Put $w(x)^{1/n} = \mu^{1/n}e^{i\varphi/n}$.*

(i) *For any non-zero $x \in \mathbf{C}$ such that $w = w(x)$ is not a negative real number or zero,*

$$\begin{aligned}\left(\sqrt{\lambda}\right)^r C_r(x) &= \left\{\beta\left(a(x)w(x)^{1/n} + b(x)\right) - \left(c(x)w(x)^{1/n} + d(x)\right)\right\} X_{n,r}(u,z)\\ &\quad - (\beta a(x) - c(x))\, u(x)^r R_{n,r}(w),\end{aligned}$$

with

$$R_{n,r}(w) = \frac{\Gamma(r+1+1/n)}{r!\Gamma(1/n)} \int_1^w ((1-t)(t-w))^r t^{1/n-r-1}\, dt,$$

where the integration path is the straight line from 1 to w.
(ii) *Let $w = e^{i\varphi}, 0 < \varphi < \pi$ and put $\sqrt{w} = e^{i\varphi/2}$. Then*

$$|R_{n,r}(w)| \leq \frac{n\Gamma(r+1+1/n)}{r!\Gamma(1/n)}\varphi\left|1-\sqrt{w}\right|^{2r}.$$

Proof. This is Lemma 2.5 of [3]. □

Lemma 2.5. *Let u, w and z be as above. Then*

$$\left|X^*_{n,r}(u,z)\right| \leq 4|u|^r \frac{\Gamma(1-1/n)r!}{\Gamma(r+1-1/n)}\left|1+\sqrt{w}\right|^{2r-2}.$$

Proof. This is Lemma 2.6 of [3]. □

Lemma 2.6. *Let $N_{4,r}$ be the greatest common divisor of the numerators of the coefficients of $X_{4,r}(1-2x)$ and let $D_{4,r}$ be the least common multiple of the denominators of the coefficients of $X_{4,r}(x)$. Then the polynomial $(D_{4,r}/N_{4,r})\, X_{4,r}(1-2x)$ has integral coefficients.*
Moreover, $N_{4,r} = 2^r$ and

$$D_{4,r}\frac{\Gamma(3/4)r!}{\Gamma(r+3/4)} < 0.8397 \cdot 5.342^r \qquad \text{and} \qquad D_{4,r}\frac{\Gamma(r+5/4)}{\Gamma(1/4)r!} < 0.1924 \cdot 5.342^r.$$

Proof. This is Lemma 3.4 from [7]. □

Lemma 2.7. *Let $\alpha_1, \alpha_2, A_r(X), B_r(X)$ and $P(X)$ be defined as in Lemma 3.1 and let a, b, c and d be complex numbers satisfying $ad - bc \neq 0$. Define*

$$K_r(X) = aA_r(X) + bB_r(X) \quad \text{and} \quad L_r(X) = cA_r(X) + dB_r(X).$$

If $(x - \alpha_1)(x - \alpha_2) P(x) \neq 0$, then

$$K_{r+1}(x) L_r(x) \neq K_r(x) L_{r+1}(x),$$

for all $r \geq 0$.

Proof. This is Lemma 2.7 of [3]. □

Lemma 2.8. *Let $\theta \in \mathbf{R}$. Suppose that there exist $k_0, l_0 > 0$ and $E, Q > 1$ such that for all $r \in \mathbf{N}$, there are rational integers p_r and q_r with $|q_r| < k_0 Q^r$ and $|q_r\theta - p_r| \leq l_0 E^{-r}$ satisfying $p_r q_{r+1} \neq p_{r+1} q_r$. Then for any rational integers p and q with $|q| \geq 1/(2l_0)$, we have*

$$\left| \theta - \frac{p}{q} \right| > \frac{1}{c|q|^{\kappa+1}}, \text{ where } c = 2k_0 Q (2l_0 E)^{\kappa} \text{ and } \kappa = \frac{\log Q}{\log E}.$$

Proof. This is Lemma 2.8 from [3]. □

For the remainder of this section, we shall assume that t is a fixed integer greater than 204. We shall also simplify our notation here to reflect the fact that we have $n = 4$. We shall use R_r and X_r instead of $R_{4,r}$ and $X_{4,r}$.

We now determine the quantities defined in the Lemma 2.3. Put

$$\alpha_1 = \sqrt{-t},\ \alpha_2 = -\sqrt{-t},\ c_1 = \left(1 + \sqrt{-t}\right)/2,\ c_2 = \left(1 - \sqrt{-t}\right)/2, \tag{2.7}$$

then

$$P(X) = X^4 + 4tX^3 - 6tX^2 - 4t^2X + t^2. \tag{2.8}$$

We will henceforth refer to this polynomial as $p_t(X)$, as in (2.4), and by abuse of notation, we will define the bivariate polynomial $p_t(X, Y) = Y^4 p_t(X/Y)$.

We define also

$$\tau = \sqrt{t} + \sqrt{t+1} \text{ and } \rho = \sqrt{\tau^2 + 1}. \tag{2.9}$$

for any positive integer t.

The preliminary results above will now be used in order to obtain an effective measure of approximation to $\beta^{(1)}$. By (2.6) and Lemma 2.3, we want to choose x close to $m + 1$, and by Lemma 2.4, it is useful to choose x so that

$$\beta^{(1)} = \frac{c(x)w(x)^{1/4} + d(x)}{a(x)w(x)^{1/4} + b(x)}. \tag{2.10}$$

For this purpose we will indeed select $x = m + 1$, and we set

$$\eta = 1 + i\sqrt{m^2 + m}\left(4m^2 + 4m + 3\right). \tag{2.11}$$

It follows that

$$w = w\,(m+1) = \frac{-1 + i\left(4m^2+4m+3\right)\sqrt{m^2+m}}{1 + i\,(4m^2+4m+3)\sqrt{m^2+m}} = -\frac{\overline{\eta}}{\eta}, \tag{2.12}$$

and so

$$w^{1/4} = \frac{1+i\tau}{\rho} \cdot \overline{\frac{m+1-i\sqrt{m^2+m}}{m+1+i\sqrt{m^2+m}}}. \tag{2.13}$$

Using the fact that $\rho^2 = \tau^2 + 1$, one can check that

$$a(m+1) = -5(m+1)\left[m - i\sqrt{m^2+m}\right] = \overline{b(m+1)}, \tag{2.14}$$

and

$$c(m+1) = -5m\,(m+1)\left[m+1+i\sqrt{m^2+m}\right] = \overline{d(m+1)}. \tag{2.15}$$

It follows that

$$\beta^{(1)} = \frac{c(m+1)w^{1/4} + d(m+1)}{a(m+1)w^{1/4} + b(m+1)}. \tag{2.16}$$

Therefore, the first term in the expression for $(-t)^{r/2}C_r(m+1)$ in Lemma 2.4 disappears.

We now construct our sequence of rational approximations to $\beta^{(1)}$.

By Lemma 2.3, Lemma 2.4, we have that $\lambda = -t$, and moreover

$$\begin{aligned}(-t)^{r/2}A_r(m+1) &= a(m+1)X_r^*(z(m+1),u(m+1)) \\ &\quad + b(m+1)X_r^*(u(m+1),z(m+1)), \\ (-t)^{r/2}B_r(m+1) &= c(m+1)X_r^*(z(m+1),u(m+1)) \\ &\quad + d(m+1)X_r^*(u(m+1),z(m+1)), \\ (-t)^{r/2}C_r(m+1) &= -\left(\beta^{(1)}a(m+1) - c(m+1)\right)[u(m+1)]^r R_r(w).\end{aligned} \tag{2.17}$$

These quantities will form the basis for our approximations. We first eliminate some common factors. One can check that

$$u(m+1) = -\frac{1}{2}(m+1)^2\left[1+i\sqrt{m^2+m}\left(4m^2+4m+3\right)\right] = -\frac{1}{2}(m+1)^2\eta = -\overline{z(m+1)}, \tag{2.18}$$

and

$$\frac{z(m+1)}{u(m+1)} = 1 - \frac{2}{\eta} \quad \text{and} \quad \frac{u(m+1)}{z(m+1)} = 1 - \frac{2}{\overline{\eta}}. \tag{2.19}$$

Using (2.2), (2.18), and (2.19), we obtain

$$\begin{aligned}X_r^*(z(m+1),u(m+1)) &= [u(m+1)]^r X_r\left(\tfrac{z(m+1)}{u(m+1)}\right) \\ &= (-1)^r \tfrac{1}{2^r}(m+1)^{2r}\eta^r X_r\left(1 - \tfrac{2}{\eta}\right)\end{aligned} \tag{2.20}$$

and

$$X_r^*(u(m+1), z(m+1)) = [z(m+1)]^r X_r\left(\frac{u(m+1)}{z(m+1)}\right) = \frac{1}{2^r}(m+1)^{2r}\overline{\eta}^r X_r\left(1-\frac{2}{\overline{\eta}}\right). \tag{2.21}$$

After some routine manipulations, we find that

(2.22)

$$(-t)^{r/2}A_r(m+1) = \frac{-5(m+1)^{2r+1}N_{4,r}}{2^r D_{4,r}}\left\{\frac{D_{4,r}}{N_{4,r}}\left[(-1)^r\left(m-i\sqrt{m^2+m}\right)\eta^r X_r\left(1-\frac{2}{\eta}\right) + \left(m+i\sqrt{m^2+m}\right)\overline{\eta}^r X_r\left(1-\frac{2}{\overline{\eta}}\right)\right]\right\}$$

$$(-t)^{r/2}B_r(m+1) = \frac{-5m(m+1)^{2r+1}N_{4,r}}{2^r D_{4,r}}\left\{\frac{D_{4,r}}{N_{4,r}}\left[(-1)^r\left(m+1+i\sqrt{m^2+m}\right)\eta^r X_r\left(1-\frac{2}{\eta}\right) + \left(m+1-i\sqrt{m^2+m}\right)\overline{\eta}^r X_r\left(1-\frac{2}{\overline{\eta}}\right)\right]\right\}.$$

By Lemma 2.6, the quantities inside the braces can be expressed as

$$(-1)^r(e-f\sqrt{-t}) \pm (e-f\sqrt{-t}),$$

where e and f are rational integers, and recalling from Lemma 2.6 that $N_{4,r} = 2^r$, considering the cases of r being even or odd separately, we find that

$$P_r = \frac{m^{[(r-2)/2]}D_{4,r}B_r(m+1)}{10(m+1)^{[3r/2+1]}} \quad \text{and} \quad Q_r = \frac{m^{[(r-2)/2]}D_{4,r}A_r(m+1)}{10(m+1)^{[3r/2+1]}} \tag{2.23}$$

are rational integers. We note for future reference that if r is even, then P_r will be divisible by t.

The numbers in (2.23) are those that will be used as the rational approximations to $\beta^{(1)}$. We have

$$Q_r\beta^{(1)} - P_r = S_r, \tag{2.24}$$

where

$$S_r = \frac{m^{[(r-2)/2]}D_{4,r}C_r(m+1)}{10(m+1)^{[3r/2+1]}}. \tag{2.25}$$

We want to show that these are good approximations, and we do this by estimating $|P_r|, |Q_r|$ and $|S_r|$ from above. As

$$\frac{m+1-i\sqrt{m^2+m}}{m+1+i\sqrt{m^2+m}} = \frac{1}{2m+1}\left[1-2i\sqrt{m^2+m}\right]$$

then (2.13) becomes

$$w^{1/4} = \frac{1}{(2m+1)\rho}\left[\left(1+2\tau\sqrt{t}\right)+i\left(\tau-2\sqrt{t}\right)\right] \tag{2.26}$$

and we have

$$w^{1/2} = \frac{1}{(2m+1)^2\rho^2}\left[\left(1+2\tau\sqrt{t}\right)^2-\left(\tau-2\sqrt{t}\right)^2+2i\left(1+2\tau\sqrt{t}\right)\left(\tau-2\sqrt{t}\right)\right]. \tag{2.27}$$

Using asymptotic expressions, we obtain

$$|1+\sqrt{w}| \leq 2 - \frac{1}{64m^6} + \frac{3}{64m^7} < 2, \quad \text{for } m \geq 1, \tag{2.28}$$

and hence

$$\left|u(m+1)\left(1+\sqrt{w}\right)^2\right| \leq 8m^5 + 28m^4 + 41m^3 + \frac{65}{2}m^2 + \frac{215}{16}m + \frac{53}{32} \leq 8.7m^5, \tag{2.29}$$

for $m \geq 42$. Using the expressions for $a(m+1), b(m+1), c(m+1)$, and $d(m+1)$, one can see that

$$|a(m+1)| = |b(m+1)| = 5(m+1)\sqrt{m(2m+1)}, \tag{2.30}$$

and

$$|c(m+1)| = |d(m+1)| = 5m(m+1)\sqrt{(m+1)(2m+1)}. \tag{2.31}$$

By (2.17), (2.28), (2.29), (2.30), (2.31), Lemma 2.5, and the triangle inequality, we have, for $m \geq 42$, that

$$\begin{aligned} t^{r/2}|A_r(m+1)| &\leq 2\,|a(m+1)|\,\left|X^*_{4,r}\left(u(m+1), z(m+1)\right)\right| \\ &\leq 8\tfrac{\Gamma(3/4)r!}{\Gamma(r+3/4)}\,|a(m+1)|\,|u(m+1)|^r\,|1+\sqrt{w}|^{2r-2} \\ &\leq 10\tfrac{\Gamma(3/4)r!}{\Gamma(r+3/4)}\,(m+1)\sqrt{m(2m+1)}\left(8.7m^5\right)^r \\ t^{r/2}|B_r(m+1)| &\leq 2\,|c(m+1)|\,\left|X^*_{4,r}\left(u(m+1), z(m+1)\right)\right| \\ &\leq 8\tfrac{\Gamma(3/4)r!}{\Gamma(r+3/4)}\,|c(m+1)|\,|u(m+1)|^r\,|1+\sqrt{w}|^{2r-2} \\ &\leq 10\tfrac{\Gamma(3/4)r!}{\Gamma(r+3/4)}\,m(m+1)\sqrt{(m+1)(2m+1)}\left(8.7m^5\right)^r \end{aligned} \tag{2.32}$$

Now we use (2.23) and Lemma 2.6 to obtain

$$\begin{aligned} |Q_r| &< D_{4,r}\tfrac{\Gamma(3/4)r!}{\Gamma(r+3/4)}\tfrac{(m+1)^2\sqrt{m(2m+1)}m^{[(r-2)/2]}}{(m+1)^{[3r/2+1]}(m(m+1))^{r/2}}\left(8.7m^5\right)^r \\ &< \tfrac{0.8397}{m}\sqrt{m(2m+1)}\left(46.4754m^3\right)^r \\ &< 1.195\left(46.4754m^3\right)^r, \end{aligned} \tag{2.33}$$

because

$$\frac{m^{[(r-2)/2]}}{(m+1)^{[3r/2+1]}(m(m+1))^{r/2}} < \frac{1}{m(m+1)^{2r+1}} < \frac{1}{(m+1)m^{2r+1}}$$

and

$$\frac{1}{m}\sqrt{m(2m+1)} < 1.423$$

for $m \geq 42$.
Similarly, one can obtain

$$|P_r| < 0.8397\sqrt{(m+1)(2m+1)}\left(46.4754m^3\right)^r, \tag{2.34}$$

for $m \geq 42$. By (2.17) and Lemma 2.4, we have

(2.35)

$$\begin{aligned} t^{r/2}|C_r(m+1)| &= \left|\beta^{(1)}a(m+1)-c(m+1)\right| \, |u(m+1)|^r \, |R_r(w(m+1))| \\ &\leq \left|\beta^{(1)}a(m+1)-c(m+1)\right| \, |u(m+1)|^r \, \tfrac{4\Gamma(r+5/4)}{r!\Gamma(1/4)} \, \varphi \, |1-\sqrt{w}|^{2r} \\ &\leq |a(m+1)| \left|\beta^{(1)} - \tfrac{m[m+1+i\sqrt{m^2+m}]}{m-i\sqrt{m^2+m}}\right| \, \tfrac{4\Gamma(r+5/4)}{r!\Gamma(1/4)} \, \varphi \, \left|u(m+1)\,(1-\sqrt{w})^2\right|^r, \end{aligned}$$

for $m \geq 42$.

With φ as in Lemma 2.4, it can be shown that $2\varphi/\pi \leq \sin\varphi$ and

$$\sin\varphi = \operatorname{Im} w = \frac{2\sqrt{m^2+m}(4m^2+4m+3)}{1+(m^2+m)(4m^2+4m+3)^2}.$$

From our estimates for the $\beta^{(i)}$'s, we know that $m+1 < \beta^{(1)} < m+5/4$, and so we use the triangle inequality and asymptotic expressions to obtain

$$\varphi \left|\beta^{(1)} - \frac{m[m+1+i\sqrt{m^2+m}]}{m-i\sqrt{m^2+m}}\right| < \frac{1.556}{m^2},$$

for $m \geq 42$. As above, we use asymptotic expressions to obtain

$$\left|u(m+1)\left(1-\sqrt{w(m+1)}\right)^2\right| \leq \frac{1.1}{32m}. \tag{2.36}$$

From these results, (2.25), Lemma 2.6 and because

$$\frac{m^{[(r-2)/2]}}{(m+1)^{[3r/2+1]}(m(m+1))^{r/2}} < \frac{1}{m(m+1)^{2r+1}},$$

one can see that

(2.37)

$$\begin{aligned} |S_r| &\leq \tfrac{2(m+1)\sqrt{m(2m+1)}}{(m+1)^{2r}} D_{4,r} \tfrac{\Gamma(r+5/4)}{r!\Gamma(1/4)} \varphi \left|\beta^{(1)} - \tfrac{m[m+1+i\sqrt{m^2+m}]}{m-i\sqrt{m^2+m}}\right| \left|u(m+1)\,(1-\sqrt{w})^2\right|^r, \\ &\leq \tfrac{2}{m}\sqrt{m(2m+1)} D_{4,r} \tfrac{\Gamma(r+5/4)}{r!\Gamma(1/4)} \varphi \left|\beta^{(1)} - \tfrac{m[m+1+i\sqrt{m^2+m}]}{m-i\sqrt{m^2+m}}\right| \left|\tfrac{u(m+1)}{(m+1)^2}\,(1-\sqrt{w})^2\right|^r \\ &\leq \tfrac{2(1.556)(1.423)}{m^2}(0.1924) \left|\tfrac{5.342}{(m+1)^2}\left(\tfrac{1.1}{32m}\right)\right|^r \\ &\leq \tfrac{0.8521}{m^2} \left|\left(\tfrac{0.184}{m^3}\right)\right|^r. \end{aligned}$$

Note also that since $\beta^{(1)}\beta^{(2)} = -t$, we have

$$tQ_r + \beta^{(2)}P_r = -\beta^{(2)}S_r. \tag{2.38}$$

We now apply Lemma 2.8 to prove the following theorem.

Theorem 2.9. *Suppose that $m \geq 42$. Define*

$$\kappa = \frac{\log\left(46.4754m^3\right)}{\log\left(m^3/0.184\right)}.$$

For $j = 1$ and 2, and for any rational integers p and q, we have

$$\left|p - \beta^{(j)} q\right| > \frac{1}{c_j |q|^\kappa} \tag{2.39}$$

for $|q| \geq 1$, where

$$c_1 = 111.08 m^3 (9.27m)^\kappa \quad \textit{and} \quad c_2 = 5097.53 m^5 \left(50.13 m^3\right)^\kappa.$$

Proof In each case we will apply Lemma 2.7 and Lemma 2.8. First notice that $P_r Q_{r+1} - P_{r+1} Q_r$ is a non-zero multiple of

$$A_{r+1}(m+1) B_r(m+1) - A_r(m+1) B_{r+1}(m+1).$$

Applying Lemma 2.7, with $a = d = 1, b = c = 0$ and $x = m+1$, we see that $P_r Q_{r+1} \neq P_{r+1} Q_r$.

For $\beta^{(1)}$ and using (2.24), we put $p_r = P_r$ and $q_r = Q_r$. For $m \geq 42$, from (2.33), and (2.37), we can take $k_0 = 1.195, l_0 = \frac{0.8521}{m^2}, E = \frac{m^3}{0.184}$ and $Q = 46.4754 m^3$. Hence we can use c_1 for the quantity c in Lemma 2.8.
For $\beta^{(2)}$ and using (2.38), we take advantage of the fact that P_{2r} is divisible by t. In this case let $p_r = -Q_{2r}$ and $q_r = \frac{P_{2r}}{t}$. Since $-m-1 < \beta^{(2)} < -m$, so we have

$$\left|\frac{\beta^{(2)}}{t} S_{2r}\right| < \frac{0.8521}{m^3}\left(\frac{0.184}{m^3}\right)^{2r} < \frac{0.8521}{m^3}\left(\frac{0.034}{m^6}\right)^{r}.$$

Also from (2.34), we obtain

$$\left|\frac{P_{2r}}{t}\right| < \frac{0.8397\sqrt{(m+1)(2m+1)}}{m^2+m}\left(46.4754 m^3\right)^{2r} < \frac{1.18}{m}\left(2159.97 m^6\right)^r.$$

Therefore, we put $k_0 = \frac{1.18}{m}, l_0 = \frac{0.8521}{m^3}, E = \frac{m^6}{0.034}$ and $Q = 2159.97 m^6$. Here κ is the same as in the case of $\beta^{(1)}$ and we can use c_2 for the quantity c in Lemma 2.8. Since l_0 is larger in this case, the same lower bound for $|q|$ remains valid.

3. Proof of Theorem 2.1

We have just used the hypergeometric method to determine how close a rational number x/y *can* possibly be to one of the roots of $P(X)$. Let us now estimate how close such a rational number *must* be in order that (x, y) is a solution of (1.5). As noted before the closest root to x/y must be either $\beta^{(1)}$ or $\beta^{(2)}$. We denote by $p_t(x, y)$ the polynomial in equation (1.5). We assume also that $m \geq 42$, since for all smaller positive integer values of m, we have verified Conjecture 1.1 using a SIMATH's program *faintp* on the curves $Y^2 = X^3 - (m^2+m)^2(m^2+m+1)X$, and doublechecked this computation using KANT's program *ThueSolve* on all Thue equations of the form given in (1.5).

We begin by proving a lower bound for $|y|$ in terms of m. We do this as follows, which is essentially Runge's method. For each of $1 \leq k \leq 24$, we compute the Puiseux expansions at infinity of the algebraic function $z(m)$ defined by $z^2 = V_{4k+3}(m^2+m)$ in order to obtain, for each k, a positive integer r_k and integer polynomials $f_{4k+3}(m), g_{4k+3}(m)$ with the property that

$$2^{2r_k} V_{4k+3}(m^2+m) = \left(f_{4k+3}(m)\right)^2 + g_{4k+3}(m),$$

with $2\deg f_{4k+3}(m) = \deg V_{4k+3}(m^2+m) = 4k+2$, and $\deg g_{4k+3}(m) = 2k$. We verified that each of the polynomials $g_{4k+3}(m)$ has no positive integer roots. We then noticed that $|f_{4k+3}(m)| > |g_{4k+3}(m)|$ for $m > 0$, which is a much stronger condition than required. We remark that if one could prove that this property holds for all $k \geq 1$, this would yield a completely different proof of Theorem 1.2. In any case, it follows from the above properties that each of the equations $z^2 = V_{4k+3}(m^2+m)$, $(1 \leq k \leq 24)$, has no solutions in positive integers (z, m).

Now, using equation (2.2) in [7], it is readily verified that for $m \geq 42$ and $k \geq 3$, one has

$$.9\tau^{k-1} < V_k \leq 1.1\tau^{k-1}. \tag{3.1}$$

In order to prove a lower bound for $|y|$, we refer to the proof of Proposition 2.1 in [7]. A modification of the proof with V_{4k+3} in place of V_{4k+1}, using the relation

$$V_{4k+3} = V_{2k+1}^2 + V_{2k+2}^2$$

shows that either $y = H$, where

$$\sqrt{V_{4k+3}} + V_{2k+1} = 2t_2H^2 \text{ if } t_1 \leq t_2,$$

or $y = G$, where

$$\sqrt{V_{4k+3}} - V_{2k+1} = 2t_1G^2 \text{ if } t_2 \leq t_1.$$

We will deal only with the latter case, as the former can be dealt with in the same way, and actually produces a larger lower bound for $|y|$.
It is easy to see that

$$\sqrt{V_{4k+3}} - V_{2k+1} = \frac{V_{2k+2}}{\sqrt{(V_{2k+1}/V_{2k+2})^2 + 1} + (V_{2k+1}/V_{2k+2})},$$

and so from (3.1), we deduce that

$$2t_1y^2 > (1/4)\tau^{2k+1} > (1/4)2^{2k+1}(\sqrt{t})^{2k+1}.$$

Since $t_1 \leq t$, $m < \sqrt{t}$, and $k \geq 25$, we finally deduce that

$$|y| > 2^{24}m^{24}. \tag{3.2}$$

We now estimate how close x/y must be to $\beta^{(1)}$ and $\beta^{(2)}$, and from (3.2) we can evidently assume that $|y| \geq 4$. Let us assume first that (x, y) is a solution of equation (1.5) with x/y closest to $\beta^{(1)}$. In this case, $\left|x - \beta^{(1)}y\right| \leq t^{1/4}$, otherwise $|p_t(x, y)| > t$, and so (x/y) is greater than $\beta^{(1)} - t^{1/4}/4$. Therefore,

$$\left|\frac{x}{y} - \beta^{(2)}\right| > \beta^{(1)} - \frac{m^{.5}}{4} - \beta^{(2)} > 2m - \frac{m^{.5}}{4} + 1 - \frac{1}{4m^{.5}},$$

by our estimates for the size of the roots.

Similarly, we also have that

$$\left|\frac{x}{y} - \beta^{(3)}\right| > m - \frac{m^{.5}}{4} + \frac{3}{4} + \frac{9}{64m^2}, \left|\frac{x}{y} - \beta^{(4)}\right| > 4m^2 + 5m - \frac{m^{.5}}{4} + \frac{9}{4} - \frac{-13}{64m^2},$$

and upon combining the above, assuming that $m \geq 42$,

$$\prod_{i \neq 1}\left|\frac{x}{y} - \beta^{(i)}\right| > 7.9m^4.$$

Therefore, if $|p_t(x,y)| = t_0^2 \le t$, with $t = m^2+m$ and $m \ge 42$, then

$$\left| \frac{x}{y} - \beta^{(1)} \right| < \frac{y^{-4}}{7.9m^2}. \tag{3.3}$$

Equation (3.3) shows that if x/y is closest to $\beta^{(1)}$ and $|y| \ge 4$, then x/y must be a convergent in the continued fraction expansion of $\beta^{(1)}$, since the right-hand side of (1.5) is less than $1/(2y^2)$ for such values of y.

If the closest root to x/y is $\beta^{(2)}$, then $\left|x - \beta^{(2)}y\right| < t^{.25}$, and so x/y must be less than $\beta^{(2)} + t^{.25}/4$. Therefore,

$$\left| \frac{x}{y} - \beta^{(1)} \right| > \beta^{(1)} - \frac{m^{.5}}{4} - \beta^{(2)} > 2m - \frac{m^{.5}}{4} + 1 - \frac{1}{4m^2},$$

and we also have that

$$\left| \frac{x}{y} - \beta^{(3)} \right| > m - \frac{m^{.5}}{4} + \frac{13}{64m^2}, \left| \frac{x}{y} - \beta^{(4)} \right| > 4m^2 + 5m + -\frac{m^{.5}}{4} + \frac{9}{4} - \frac{13}{64m^2}$$

We similarly conclude that for $t = m^2 + m$ and $m \ge 42$,

$$\prod_{i \ne 2} \left| \frac{x}{y} - \beta^{(i)} \right| > 7.9m^4,$$

and also, if $|p_t(x,y)| = t_0^2 \le t$, then

$$\left| \frac{x}{y} - \beta^{(2)} \right| < \frac{|y|^{-4}}{7.9m^2}. \tag{3.4}$$

As before, we deduce that if x/y is closest to $\beta^{(2)}$ and $|y| \ge 4$, then x/y must be a convergent in the continued fraction expansion of $\beta^{(2)}$.

By Theorem 2.9, (3.3) and (3.4), if (x,y) is any further solution of equation (1.5), arising from the equation $z^2 = V_{4n+3}(t)$, then x/y is a convergent to either $\beta^{(1)}$ and

$$|y|^{3-\kappa} < \frac{111.08m^3}{7.9m^2} (9.27m)^{\kappa}, \tag{3.5}$$

or a convergent to $\beta^{(2)}$ and

$$|y|^{3-\kappa} < \frac{5097.53m^5}{7.9m^2} \left(50.13m^3\right)^{\kappa}, \tag{3.6}$$

provided that $m \ge 42$.

Combining (3.5) with the lower bound for $|y|$ in (3.2) shows that $m \le 1$, while combining (3.6) with (3.2) shows that $m \le 1$.

This completes the proof of Theorem 1.2.

References

[1] M.A. Bennett and P.G. Walsh, *The Diophantine equation* $b^2X^4 - dY^2 = 1$. Proc. A.M.S. **127** (1999), 3481-3491.

[2] R.T. Bumby. *The Diophantine equation* $3x^4 - 2y^2 = 1$. Math. Scand. **21** (1967), 144-148.

[3] J.H. Chen and P.M. Voutier. *A complete solution of the Diophantine equation* $x^2+1 = dy^4$ *and a related family of quartic Thue equations.* J. Number Theory **62** (1997), 71-99.

[4] J.H.E. Cohn. *The Diophantine equation* $x^4 - Dy^2 = 1$ *II.* Acta Arith. **78** (1997), 401-403.

[5] W. Ljunggren. *On the Diophantine equation* $Ax^4 - By^2 = C(C = 1, 4)$. Math. Scand. **21** (1967), 149-158.

[6] A. Thue. *Ein Fundamentaltheorem zur Bestimmung von Annäherungswerten aller Wurzeln gewisser ganzer Funktionen*, J. Reine Angew. Math. **138** (1910), 96-108.

[7] A. Togbé, P.M. Voutier, and P.G. Walsh. *Solving a family of Thue equations with an application to the equation* $x^2 - dy^4 = 1$. Preprint.

[8] P.G. Walsh. *Diophantine equations of the form* $aX^4 - bY^2 = \pm 1$. In *Algebraic Number Theory and Diophantine Analysis*, Proceedings of the International Conference in Graz 1998, Walter de Gruyter, Berlin 2000, p. 531-554.

[9] P.G. Walsh. *A note on Ljunggren's theorem about the Diophantine equation* $aX^2 - bY^4 = 1$. Comptes Rendues Math. Rep. Acad. Sci. Canada. **20** (1998), 113-119.

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