The Ottawa Workshop, Lecture 1:

Basic Symmetries and Representations

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Infinite Exchangeable Sequences

Say that the random elements $\xi_1, \xi_2, \ldots$ are *exchangeable* if their joint distribution is invariant under arbitrary permutations:

$$(\xi_{k_1}, \ldots, \xi_{k_n}) \stackrel{d}{=} (\xi_1, \ldots, \xi_n)$$

for any set of distinct integers $k_1, \ldots, k_n > 0$.

♣ *(de Finetti) An infinite sequence of random elements, taking values in a Borel space, is exchangeable iff it is mixed i.i.d.*

Here *mixed i.i.d.* means that the joint distribution of the variables ξ_k is a mixture of distributions of i.i.d. sequences. Thus, the ξ_k are i.i.d. on S if their joint distribution is an infinite product measure $\mu^\infty = \mu \otimes \mu \otimes \cdots$ for some distribution μ on S. Now take an arbitrary probability distribution ν on the space of measures μ, and form the mixture

$$P\{(\xi_1, \xi_2, \ldots) \in \cdot\} = \int \mu^\infty \, \nu(d\mu).$$

Finite Exchangeable Sequences

de Finetti's theorem fails for finite sequences. Here we have instead:

♣ *A finite sequence of random elements* $\xi_1, \ldots, \xi_n$ *is exchangeable iff it is a mixture of urn sequences.*

An *urn sequence* is a sequence of random elements obtained by random sampling without replacement from a finite set: Let an urn contain n tickets labeled $a_1, \ldots, a_n$. Draw the tickets, one by one, and record their labels $\xi_1, \ldots, \xi_n$. The joint distribution depends only on the counting measure $\pi = \sum_k \delta_{a_k}$. Now form a mixture over such distributions $\pi^{(n)}/n!$:

$$P\{(\xi_1, \ldots, \xi_n) \in \cdot\} = \int \pi^{(n)} \nu(d\pi)/n!$$

Contractable Sequences

A random sequence $\xi_1, \xi_2, \ldots$ is said to be *contractable* (contraction invariant) if all subsequences have the same distribution:

$$(\xi_{k_1}, \xi_{k_2}, \ldots) \stackrel{d}{=} (\xi_1, \xi_2, \ldots)$$

for any positive integers $k_1 < k_2 < \cdots$. Exchangeable sequences are clearly contractable. The converse is not so obvious:

♣ *(Ryll-Nardzewski) An infinite sequence of random elements $\xi_1, \xi_2, \ldots$ in a space S is contractable iff it is exchangeable. Thus, when S is Borel, the ξ_k are contractable iff they are mixed i.i.d.*

This improves on de Finetti's theorem. Both statements fail for finite sequences.

Rotatable Sequences

Imposing further conditions on our exchangeable sequences yields mixtures of i.i.d. sequences of special kinds. Say that the random variables $\xi_1, \xi_2, \ldots$ are *rotatable* (rotation invariant), if for every n, the distribution of the random vector $(\xi_1, \ldots, \xi_n)$ is spherically symmetric. Any rotatable sequence is clearly exchangeable.

♣ *(Schoenberg, Freedman) An infinite sequence of random variables $\xi_1, \xi_2, \ldots$ is rotatable iff it is mixed i.i.d. centered Gaussian. In other words,*

$$(\xi_1, \xi_2, \ldots) \stackrel{d}{=} \sigma(\eta_1, \eta_2, \ldots),$$

where $\eta_1, \eta_2, \ldots$ are i.i.d. $N(0,1)$ and σ is an independent random variable ≥ 0.

Again, this clearly fails for finite sequences.

Discrete-Time Symmetries

For finite or infinite sequences of random elements, we have the following basic symmetries, listed here together with the associated classes of transformations:

stationary	*shifts*
contractable	*contractions*
exchangeable	*permutations*
rotatable	*rotations*

Note that each property in the table is stronger than the previous one. Many other symmetries are conceivable, but these are the basic ones, and they are intimately related in many ways. Their study leads to a unified theory, which is the focus of these lectures.

Symmetric Increments

In continuous time, we may define the corresponding symmetries in terms of the *increments*. Then, for a process X on $\mathbb{R}_+$ or $[0,1]$, we fix any $h > 0$ and define

$$\xi_k = X_{kh} - X_{(k-1)h}, \quad k = 1, 2, \ldots$$

Say that X is *contractable* (has contractable increments) if $X_0 = 0$ and the sequence of increments $\xi_1, \xi_2, \ldots$ is contractable for every $h > 0$. The definitions of exchangeable and rotatable processes are similar. In addition, we may impose various regularity conditions, such as continuity in probability.

Pathwise Symmetries

It is often more convenient to introduce, for any times $a < b$, the *contraction*

$$\tilde{X}_t^{a,b} = \begin{cases} X_t, & t \le a, \\ X_a + X_{t+b-a} - X_b, & t > a, \end{cases}$$

and say that X is contractable if $\tilde{X}^{a,b} \stackrel{d}{=} X$ for all $a < b$. The contraction amounts to removing the path on $(a, b]$ and gluing together the remaining paths on $[0, a]$ and (b, ∞).

To define exchangeability, we consider instead, for any $a < b$, the *transposition*

$$\tilde{X}_t^{a,b} = \begin{cases} X_{t+a} - X_a, & t \le b - a, \\ X_b - X_a + X_{t-b+a}, & t \in (b - a, b], \\ X_t, & t > b. \end{cases}$$

This amounts to interchanging the order of the paths on $[0, a]$ and $(a, b]$.

Functional Symmetries

For rotatability, we may require instead that the *Wiener functional*

$$Xf = \int f_t \, dX_t$$

has the same distribution for all $f \in L^2$ with $\|f\|_2 = 1$.

For random measures or point processes ξ on $I = \mathbb{R}_+$ or $[0,1]$, we may also define exchangeability in terms of *measure-preserving transformations*. Thus, letting λ denote Lebesgue measure on I, we say that ξ is exchangeable if, for any measurable function $f\colon I \to I$,

$$\lambda \circ f^{-1} = \lambda \quad \Longrightarrow \quad \xi \circ f^{-1} \stackrel{d}{=} \xi.$$

The same condition can be used to define the notion of λ-*symmetry*, for any random measure ξ on a diffuse measure space $(S, \mathcal{S}, \lambda)$. We may also consider point processes on S with marks in an arbitrary measurable space $(K, \mathcal{K})$.

Simple Point Processes

Here is a de Finetti-type theorem for simple point processes:

♣ *A simple point process on* $\mathbb{R}_+$ *or* $[0, 1]$ *is exchangeable iff it is a mixed Poisson or binomial process based on* λ.

Letting P_ρ be the distribution of a Poisson process on $\mathbb{R}_+$ with the constant rate $\rho \geq 0$, we form the mixture

$$P\{\xi \in \cdot\} = \int P_\rho \, \nu(\rho),$$

where ν is a distribution on $\mathbb{R}_+$.

To form a *binomial process* based on λ and κ, let $\gamma_1, \gamma_2, \ldots$ be i.i.d. random variables with distribution λ and define

$$\xi = \sum_{k \leq \kappa} \delta_{\gamma_k}$$

For a *mixed* binomial process, take κ to be an independent random variable.

Continuous Processes

The following is a de Finetti-type theorem for continuous processes:

♣ *A continuous process* X *on* $I = \mathbb{R}_+$ *or* $[0,1]$, *taking values in* $\mathbb{R}^d$, *is exchangeable iff*

$$X_t = \alpha t + \sigma B_t, \quad t \in I,$$

where B *is a Brownian motion or bridge, respectively, and* (α, σ) *is an independent pair of random elements.*

Here α is a random vector in $\mathbb{R}^d$ and σ is a random $d \times d$ matrix.

Rotations and Contractions

It is interesting to compare with the result in the rotatable case:

♣ *(Freedman) A measurable process X on $I = \mathbb{R}_+$ or $[0,1]$, taking values in $\mathbb{R}^d$, is rotatable iff*

$$X_t = \sigma B_t, \quad t \in I,$$

where B is a Brownian motion on I and σ is an independent random matrix.

The last two results, together with Ryll-Nardzewski's theorem, yield a surprising connection between the basic symmetries. Here a process X on $\mathbb{R}_+$ with stationary increments is said to be *centered* if

$$\lim_{t\to\infty} t^{-1} X_t = 0 \text{ a.s.}$$

♣ *For a centered, continuous process on $\mathbb{R}_+$ taking values in $\mathbb{R}^d$, the contractable, exchangeable, and rotatable properties are all equivalent.*

Contractable Processes on $\mathbb{R}_+$

Now turn to the general continuous-time counterpart of de Finetti's theorem:

♣ *(Bühlmann) A right-continuous process X on $\mathbb{R}_+$, taking values in $\mathbb{R}^d$, is contractable (hence exchangeable) iff it is a mixture of Lévy processes.*

A *Lévy process* is a right-continuous process X with stationary, independent increments and $X_0 = 0$. Its distribution is determined by the triple $(\alpha, \sigma\sigma', \nu)$, where α is the *drift vector*, $\sigma\sigma'$ is the *diffusion matrix*, and ν is the *Lévy measure* on $\mathbb{R}^d \setminus \{0\}$ governing the jumps of X. Recall the *Lévy representation*

$$\begin{aligned} X_t \;&=\; \alpha t + \sigma B_t + \int_0^t \int_{|x|\le 1} x\,(\eta - E\eta)(ds\,dx) \\ &\quad + \int_0^t \int_{|x|>1} x\,\eta(ds\,dx), \end{aligned}$$

where B is a Brownian motion and η is an independent Poisson process with $E\eta = \lambda \otimes \nu$.

Exchangeable Processes on [0,1]

The last result fails for processes on $[0,1]$. Here we have instead:

♣ *A right-continuous process X on $[0,1]$, taking values in $\mathbb{R}^d$, is exchangeable iff it has a representation*

$$X_t = \alpha t + \sigma B_t + \textstyle\sum_j \beta_j (1\{\tau_j \le t\} - t),$$

where B is a Brownian bridge, $\tau_1, \tau_2, \ldots$ are independent of B and i.i.d. $U(0,1)$, and the set of vectors α, σ, and $\beta_1, \beta_2, \ldots$ is independent of B and (τ_j) with $\sum_j |\beta_j|^2 < \infty$. The representing series then converges a.s., uniformly on $[0,1]$.

For non-random coefficients α, σ, and $\beta_1, \beta_2, \ldots$, the distribution of X is clearly determined by the triple $(\alpha, \sigma\sigma', \beta)$, where $\beta = \sum_j \delta_{\beta_j}$.

The Ottawa Workshop, Lecture 2:

Convergence and Approximation

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Directing Random Measures

de Finetti's theorem can be stated in the following stronger conditional form:

♣ *For any infinite, contractable sequence* $\xi = (\xi_1, \xi_2, \ldots)$ *in a Borel space* S*, there exists an a.s. unique random probability measure* μ *on* S *such that*

$$P[\xi \in \cdot \mid \mu] = \mu^\infty \quad a.s.$$

We call μ the *directing random measure* of ξ. It is equivalent to condition on the shift or permutation invariant σ-fields $\mathcal{I}$ or $\mathcal{E}$, or on the tail σ-field $\mathcal{T}$. Taking expected values gives

$$P\{\xi \in \cdot\} = E\mu^\infty = \int m^\infty \nu(dm),$$

where ν is the distribution of μ. Thus, the present condition implies that ξ is mixed i.i.d.

For suitable spaces S, we may recover μ from ξ via the law of large numbers:

$$n^{-1} \sum_{k \le n} \delta_{\xi_k} \xrightarrow{w} \mu \text{ a.s.}$$

Directing Random Elements

All the previous representations have similar conditional versions. In each case, there exist some *directing random elements*, a.s. determined by the sequence or process, such that an associated conditioning yields a unique integral representation of the distribution in terms of extreme symmetric distributions. We list some examples of directing random elements:

μ — infinite sequences

π — finite sequences

ρ — simple point processes on $\mathbb{R}_+$

κ — simple point processes on $[0, 1]$

$\sigma\sigma'$ — rotatable sequences or processes

$(\alpha, \sigma\sigma')$ — continuous processes

$(\alpha, \sigma\sigma', \nu)$ — processes on $\mathbb{R}_+$

$(\alpha, \sigma\sigma', \beta)$ — processes on $[0, 1]$

Uniqueness and Continuity

In all the previous cases, the distribution of the symmetrically distributed sequence or process determines and is determined by the distribution of the associated set of directing random elements. The correspondence is even continuous in a suitable sense. For example, we have for de Finetti sequences:

♣ *Let ξ and $\xi_1, \xi_2, \ldots$ be infinite exchangeable sequences in a Polish space S, and let μ and $\mu_1, \mu_2, \ldots$ denote the associated directing random measures. Then*

$$\xi_n \xrightarrow{d} \xi \qquad \Longleftrightarrow \qquad \mu_n \xrightarrow{wd} \mu$$

On the left we have convergence in distribution in the infinite product space S^∞, whereas the right we have convergence in distribution in the measure space $\mathcal{M}_1(S)$, equipped with the topology of weak convergence. This is equivalent to weak convergence in the measure spaces $\mathcal{M}_1(S^\infty)$ and $\mathcal{M}_1(\mathcal{M}_1(S))$, respectively.

Approximation in Distribution

We may also connect the different representations by limit theorems of various kind. For example, we have the following approximation of finite exchangeable sequences by infinite ones:

♣ *Let* $\xi_1, \xi_2, \ldots$ *be exchangeable sequences of finite lengths* $m_1, m_2, \ldots \to \infty$, *and let* $\pi_1, \pi_2, \ldots$ *denote the associated directing point processes. Also consider an infinite exchangeable sequence* ξ *directed by* μ. *Then*

$$\xi_n \xrightarrow{d} \xi \qquad \Longleftrightarrow \qquad m_n^{-1} \pi_n \xrightarrow{wd} \mu$$

Similar continuity and convergence theorems hold in the other cases considered so far. In general, however, one must be careful with the choice of topology in the space of directing elements. The difficulties are similar to those arising in the statements of classical limit theorems for triangular arrays.

Sequences and Processes

The following table summarizes the directing random elements of exchangeable sequences or processes on a finite or infinite interval:

	finite	*infinite*
discrete	π	μ
continuous	$(\alpha, \sigma\sigma', \beta)$	$(\alpha, \sigma\sigma', \nu)$

The four fields in the table are related by five limit theorems, in addition to continuity theorems within each field. This gives totally nine limit theorems, some of which extend the classical limit theorems for triangular arrays and for sampling from a finite population.

Asymptotically Invariant Sampling

For a sequence $\tau = (\tau_1, \tau_2, \ldots)$ in $\mathbb{R}$ and a process X on $\mathbb{R}$, consider the *sampled sequence*

$$X \circ \tau = (X_{\tau_1}, X_{\tau_2}, \ldots).$$

Say that the random sequences

$$\tau_n = (\tau_{n1}, \tau_{n2}, \ldots), n \in \mathbb{N},$$

are *asymptotically invariant (in distribution)*, if for any $k \in \mathbb{N}$ and $r_1, \ldots, r_k \in \mathbb{R}$,

$$\begin{aligned} \|\mathcal{L}(\tau_{n1} + r_1, \ldots, \tau_{nk} + r_k) \\ -\mathcal{L}(\tau_{n1}, \ldots, \tau_{nk})\| \to 0. \end{aligned}$$

♣ *Let $X = (X_t)$ be a stationary process in a Polish space S with invariant σ-field $\mathcal{I}_X$, and let $\xi = (\xi_j)$ be an infinite exchangeable sequence in S with directing measure $\mu = P[X_0 \in \cdot | \mathcal{I}_X]$. Consider some asymptotically invariant sequences $\tau_n = (\tau_{nj})$ in $\mathbb{R}$ and the associated the sampled sequences $X \circ \tau_n$ in S. Then $X \circ \tau_n \xrightarrow{d} \xi$.*

Strong Comparison

For an exchangeable process X in $\mathbb{R}$ with directing triple $(\alpha, \sigma^2, \beta)$, define the *index* ρ_X in $[0,2]$ by

$$\rho_X = \inf\Big\{c \geq 0;\ \textstyle\sum_j |\beta_j|^c < \infty\Big\}$$

♣ *For any exchangeable process X on $[0,1]$, there exist some jointly exchangeable processes $\tilde{X}$ and Y with $X = \tilde{X} + Y$ a.s., where $\tilde{X}$ is mixed Lévy with the same directing triple as X and Y has vanishing diffusion term and index*

$$\rho_Y \leq \frac{\rho_X}{1 + \frac{1}{2}\,\rho_X}$$

Iterating results of this type, we may extend many path properties known for Lévy processes to the much larger class of exchangeable processes on $[0,1]$.

Rényi Stability

Define the *shift operators* θ_n by

$$\theta_n(\xi_1, \xi_2, \ldots) = (\xi_{n+1}, \xi_{n+2}, \ldots).$$

♣ *(Rényi/Révész) Let* $\xi = (\xi_n)$ *be an infinite exchangeable sequence in a Polish space* S *with directing random measure* μ. *Then*

(i) $P[\theta_n \xi \in \cdot \mid \xi_1, \ldots, \xi_n] \xrightarrow{w} \mu^\infty$ *a.s.*,

(ii) $E[\eta;\, \theta_n \xi \in \cdot] \xrightarrow{w} E\eta\mu^\infty$, $\eta \in L^1$.

Recall that $\mu^\infty = P[\xi \in \cdot | \mu]$. In (ii), choosing $\tilde{\xi} \stackrel{d}{=} \xi$ with $\tilde{\xi} \perp\!\!\!\perp_\mu \xi$ (conditional independence), we get

$$E[\eta;\, \theta_n \xi \in \cdot] \xrightarrow{w} E[\eta;\, \tilde{\xi} \in \cdot], \quad \eta \in L^1(\xi).$$

For $\eta = 1_A$ with $A \in \sigma(\xi)$, this amounts to the *stable convergence* $\theta_n \xi \to \tilde{\xi}$.

Weak Subsequence Principle

For any sequences $\xi = (\xi_1, \xi_2, \ldots)$ and $p = (p_1, p_2, \ldots)$ with all $p_n \in \mathbb{N}$, define

$$\xi \circ p = (\xi_{p_1}, \xi_{p_2}, \ldots).$$

♣ *(Dacunha-Castelle, Aldous) For any tight random sequence $\xi = (\xi_n)$ in a Polish space S, there exist a subsequence p of $\mathbb{N}$ and a random distribution μ on S such that*

$$E[\eta;\, \theta_n(\xi \circ p) \in \cdot\,] \xrightarrow{w} E\eta\mu^\infty, \quad \eta \in L^1.$$

Letting $\zeta = (\zeta_n)$ be exchangeable in S and directed by μ with $\zeta \perp\!\!\!\perp_\mu \xi$, we get

$$E[\eta;\, \theta_n(\xi \circ p) \in \cdot\,] \xrightarrow{w} E[\eta;\, \zeta \in \cdot\,], \quad \eta \in L^1(\xi),$$

which implies the Rényi stable convergence $\theta_n(\xi \circ p) \to \zeta$.

Strong Subsequence Principle

One might hope to prove that, for any tight random sequence $\xi = (\xi_n)$ in a suitable metric space (S, ρ), there exist a subsequence p of $\mathbb{N}$ and an exchangeable sequence $\zeta = (\zeta_n)$ in S such that $\rho(\xi_{p_n}, \zeta_n) \to 0$ a.s. Unfortunately, this statement is false, and the best we can do is the following:

♣ *(Berkes, Péter) Let $\xi = (\xi_n)$ be a tight random sequence in a separable and complete metric space (S, ρ). Then for any $\varepsilon > 0$, there exist a subsequence p of $\mathbb{N}$ and an exchangeable sequence ζ in S such that*

$$E[\rho(\xi_{p_n}, \zeta_n) \wedge 1] \le \varepsilon, \quad n \in \mathbb{N}.$$

Such results can be used to prove that every "reasonable" limit theorem for i.i.d. sequences remains true, in a conditional form and under suitable moment conditions, for some subsequence of an arbitrary random sequence ξ.

The Ottawa Workshop, Lecture 3:

Martingales and Predictable Sampling

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Basic Definitions

A discrete *filtration* $\mathcal{F}$ on a probability space is a sequence of σ-fields $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$. A random sequence ξ is said to be $\mathcal{F}$-*adapted* if ξ_n is $\mathcal{F}_n$-measurable for every n. The filtration *induced* by ξ is given by $\mathcal{G}_n = \sigma(\xi_1, \ldots, \xi_n)$ for all n.

An $\mathcal{F}$-*optional (stopping) time* τ is a random variable in $\overline{\mathbb{Z}}_+$ such that $\{\tau = n\} \in \mathcal{F}_n$ for all n. We say that τ is $\mathcal{F}$-*predictable* if $\{\tau = n\} \in \mathcal{F}_{(n-1)_+}$ for all n, so that $(\tau - 1)_+$ is optional.

A random sequence ξ is said to be $\mathcal{F}$-*contractable* or $\mathcal{F}$-*exchangeable* if it is $\mathcal{F}$-adapted and such that, for every n, the shifted sequence $\theta_n\xi$ is conditionally contractable or exchangeable given $\mathcal{F}_n$. This clearly holds for the induced filtration when ξ is contractable or exchangeable in the unqualified sense.

Strong Stationarity and Prediction

For an $\mathcal{F}$-adapted sequence ξ in a Borel space S, the associated *prediction sequence* $\mu = (\mu_n)$ is the measure-valued sequence given by

$$\mu_n = P[\theta_n \xi \in \cdot | \mathcal{F}_n], \quad n \in \mathbb{Z}_+.$$

The following result gives a connection between three of our basic symmetry properties.

♣ *For any infinite, $\mathcal{F}$-adapted random sequence ξ with prediction sequence μ, these conditions are equivalent:*

(i) ξ *is $\mathcal{F}$-contractable,*

(ii) ξ *is $\mathcal{F}$-exchangeable,*

(iii) $\theta_\tau \xi \overset{d}{=} \xi$ *for any $\mathcal{F}$-optional time $\tau < \infty$,*

(iv) μ *is a measure-valued $\mathcal{F}$-martingale.*

The *strong stationarity* in (iii) should be compared with the corresponding unqualified notion, where the relation $\theta_n \xi \overset{d}{=} \xi$ is required only for non-random n.

Local Prediction

Under additional hypotheses on ξ and $\mathcal{F}$, we may replace the strong stationarity and martingale criterion of the previous theorem by some weaker conditions.

♣ *Let ξ be an infinite, stationary random sequence with induced filtration $\mathcal{F}$, taking values in a Borel space S. Then these conditions are equivalent:*

(i) ξ *is $\mathcal{F}$-contractable,*

(ii) ξ *is $\mathcal{F}$-exchangeable,*

(iii) $\xi_\tau \overset{d}{=} \xi_1$ *for any $\mathcal{F}$-predictable time $\tau < \infty$,*

(iv) $\mu_k = P[\xi_{k+1}|\mathcal{F}_k]$ *is a measure-valued $\mathcal{F}$-martingale.*

Strong Reflection Property

For finite sequences, the shifts of the previous results need to be replaced by reflections. Say that the random sequence $\xi = (\xi_1, \ldots, \xi_n)$ satisfies the *strong reflection property* if

$$(\xi_{\tau+1}, \ldots, \xi_n) \stackrel{d}{=} (\xi_n, \ldots, \xi_{\tau+1})$$

for any optional time τ in $[0, n)$.

♣ *Let ξ be a finite, $\mathcal{F}$-adapted sequence in a Borel space S. Then ξ is $\mathcal{F}$-exchangeable iff it satisfies the strong reflection property.*

This idea is especially useful in a continuous-time setting, to study exchangeable random sets in $[0, 1]$.

Continuous-Time Shifts

With the symmetries now defined in terms of the increments, we need to modify the notions of the previous theorems. The $\mathcal{F}$*-prediction process* is now defined by

$$\mu_t = P[\,\theta_t X - X_t \in \cdot \mid \mathcal{F}_t\,], \quad t \geq 0,$$

and we say that X has $\mathcal{F}$*-stationary increments* if $\theta_\tau X - X_\tau \stackrel{d}{=} X$ for every $\mathcal{F}$-optional time τ, now defined by the condition $\{\tau \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$. The *shift operators* θ_t are now given by $(\theta_t X)_s = X_{s+t}$ for all s and t.

♣ *Let* $X = (X_t)$ *be an* $\mathcal{F}$*-adapted,* $\mathbb{R}^d$*-valued, right-continuous process on* $\mathbb{R}_+$ *with* $X_0 = 0$*, and let* $\mu = (\mu_t)$ *denote the associated* $\mathcal{F}$*-prediction process. Then these conditions are equivalent:*

(i) X *is* $\mathcal{F}$*-contractable,*

(ii) X *is* $\mathcal{F}$*-exchangeable,*

(iii) X *has* $\mathcal{F}$*-stationary increments,*

(iv) μ *is a measure-valued* $\mathcal{F}$*-martingale.*

Reverse Martingale Criterion

For any random sequence $\xi = (\xi_1, \xi_2, \ldots)$ in a space S, the associated *empirical distributions* are given by

$$\eta_n = n^{-1} \sum_{k \le n} \delta_{\xi_k}, \quad n \in \mathbb{N},$$

where δ_x denotes a unit mass at x.

♣ *Let* $\xi = (\xi_k)$ *be a finite or infinite random sequence with empirical distributions* $\eta_1, \eta_2, \ldots$. *Then* ξ *is exchangeable iff the* η_k *form a reverse, measure-valued martingale.*

The reverse martingale property is given by

$$E[\eta_m f | \mathcal{T}_n] = \eta_n f \ \text{ a.s.}, \quad m < n,$$

for any bounded, measurable function f, where the *tail filtration* $\mathcal{T}_1 \supset \mathcal{T}_2 \supset \cdots$ is given by

$$\mathcal{T}_n = \sigma(\eta_n, \eta_{n+1}, \ldots), \quad n \in \mathbb{N}.$$

Predictable Skipping and Sampling

The defining property of contractable sequences remains valid for sub-sequences involving predictable times:

♣ *For any finite or infinite $\mathcal{F}$-contractable sequence $\xi = (\xi_j)$ and any $\mathcal{F}$-predictable times $\tau_1 < \ldots < \tau_k$ in the index set of ξ, we have*

$$(\xi_{\tau_1}, \ldots, \xi_{\tau_k}) \stackrel{d}{=} (\xi_1, \ldots, \xi_k).$$

Similarly, the defining property of exchangeable sequences remains valid for permutations involving predictable times:

♣ *For any finite or infinite $\mathcal{F}$-exchangeable sequence $\xi = (\xi_j)$ and any a.s. distinct $\mathcal{F}$-predictable times $\tau_1, \ldots, \tau_k$ in the index set of ξ, we have*

$$(\xi_{\tau_1}, \ldots, \xi_{\tau_k}) \stackrel{d}{=} (\xi_1, \ldots, \xi_k).$$

Gambling Puzzle

♡ In a casino, you watch a roulette game where only red and black may occur, each with probability $\frac{1}{2}$ (no slots for the bank). At a suitable time, based on your previous observations, you choose to bet \$100 in the next round. If the outcome is red, you win \$200 back, otherwise you lose your bet. Here your average gain is clearly zero, regardless of strategy, so the gambling is totally pointless.

♡ Now compare with the corresponding card game: From a well-shuffled card deck you pick the cards one by one, without replacement, and observe their colors, red or black. At a suitable moment, you choose to bet \$100 on the next card. If red, you win \$200, otherwise you lose. How should you play to maximize your average gain? *Hint:* At every stage, you know the proportion of red cards in the remaining deck.

Sojourns and Maxima

The following classical result from fluctuation theory, once regarded as deep and surprising, follows easily from the predictable sampling theorem:

♣ *(Sparre-Andersen) Let* $\xi_1, \ldots, \xi_n$ *be exchangeable random variables, and put* $S_k = \xi_1 + \ldots + \xi_k$ *for all* $k \le n$*. Then*

$$\sum_{k \le n} 1\{S_k > 0\} \stackrel{d}{=} \min\left\{k \ge 0;\, S_k = \max_{j \le n} S_j\right\}$$

The left-hand side gives the amount of time that the random walk (S_k) stays positive, while the right-hand side gives the moment of the first maximum. The statement may be used to give a short proof of the third and most difficult arcsine law for Brownian motion. (The other two are elementary.)

Decoupling Identities

Say that a random sequence $\eta = (\eta_k)$ is $\mathcal{F}$-*predictable* for some discrete filtration $\mathcal{F}$, if η_k is $\mathcal{F}_{(k-1)_+}$-measurable for all k. The following results extend the classical *Wald identities* for i.i.d. sequences:

♣ *Consider two sequences $\xi = (\xi_k)$ and $\eta = (\eta_k)$ of random variables, where ξ is $\mathcal{F}$-i.i.d. and η is $\mathcal{F}$-predictable, and suppose that $S_m = \sum_k \eta_k^m$ is non-random for every $m < n$. Introduce a sequence $\tilde{\eta} \stackrel{d}{=} \eta$ with $\tilde{\eta} \perp\!\!\!\perp \xi$. Then, under suitable moment conditions,*

$$E\Big(\sum\nolimits_k \xi_k \eta_k\Big)^n = E\Big(\sum\nolimits_k \xi_k \tilde{\eta}_k\Big)^n$$

For finite exchangeable sequences, slightly stronger conditions are needed:

♣ *The previous result remains true for $\mathcal{F}$-exchangeable urn sequences ξ, provided we assume S_m to be non-random even for $m = n$.*

Product Moments

The previous results can be extended to suitable product moments. Here we consider sequences $\xi = (\xi_{jk})$ and $\eta = (\eta_{jk})$ in $\mathbb{R}^d$ and define

$$S_J = \sum_{k\geq 1} \prod_{j\in J} \eta_{jk}, \quad J \subset \{1, \ldots, d\}.$$

♣ *Consider two random sequences $\xi = (\xi_k)$ and $\eta = (\eta_k)$ in $\mathbb{R}^d$, where ξ is $\mathcal{F}$-i.i.d. and η is $\mathcal{F}$-predictable, and suppose that S_J is non-random for every proper subset $J \subset \{1, \ldots, d\}$. Introduce a sequence $\tilde{\eta} \stackrel{d}{=} \eta$ with $\tilde{\eta} \perp\!\!\!\perp \xi$. Then, under suitable moment conditions,*

$$E \prod_{j\leq d} \sum_{k\geq 1} \xi_{jk}\eta_{jk} = E \prod_{j\leq d} \sum_{k\geq 1} \xi_{jk}\tilde{\eta}_{jk}$$

This remains true for $\mathcal{F}$-exchangeable urn sequences ξ in $\mathbb{R}^d$, provided we assume S_J to be non-random even for $J = \{1, \ldots, d\}$.

Palm Measure Invariance

For a random measure ξ on a Borel space S with σ-finite intensity measure $E\xi$, the associated *Palm distributions* Q_s are given by

$$Q_s(A) = \frac{E[\xi(ds);\, \xi \in A]}{E\xi(ds)}, \quad s \in S.$$

When ξ is a simple point process, they allow the interpretation

$$Q_s(A) = P[\,\xi \in A \,|\, \xi\{s\} = 1\,], \quad s \in S,$$

and we define the *reduced Palm distributions* by

$$Q'_s(A) = Q_s\{\mu;\, \mu - \delta_s \in A\}, \quad s \in S.$$

♣ *Let ξ be a simple point process on a Borel space S with diffuse, σ-finite intensity measure $\lambda = E\xi$. Then ξ is λ-symmetric, hence a mixed Poisson or binomial process based on λ, iff the reduced Palm distributions Q'_s of ξ can be chosen to be independent of s.*

Uniform and Poisson Sampling

The range of an increasing Lévy process is a regenerative set in $\mathbb{R}_+$. Similarly, an increasing, exchangeable process X on $[0,1]$ with $X_1 = 1$ generates an *exchangeable random set* Ξ on $[0,1]$. Define the *local time* L of Ξ as the right-continuous inverse of X.

♣ *Let Ξ be an exchangeable random set in $[0,1]$ with local time L, and let $\tau_1, \tau_2, \dots$ be i.i.d. $U(0,1)$ and independent of Ξ. Let $\sigma_1, \sigma_2, \dots$ be the distincts values of $L_{\tau_1}, L_{\tau_2}, \dots$. Then the σ_n are again i.i.d. $U(0,1)$.*

Compare with the following (easy) result for regenerative sets on $\mathbb{R}_+$.

♣ *Let Ξ be a regenerative set in $\mathbb{R}_+$ with local time L, and let $\tau_1, \tau_2, \dots$ form an independent homogeneous Poisson process on $\mathbb{R}_+$. Then the distinct values of $L_{\tau_1}, L_{\tau_2}, \dots$ will again form homogeneous Poisson process on $\mathbb{R}_+$.*

The Ottawa Workshop, Lecture 4:

Semi-Martingales and Integral Criteria

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Olav Kallenberg
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Basic Definitions

Given a right-continuous, complete filtration $\mathcal{F} = (\mathcal{F}_t)$, we define a *special semimartingale* as a right-continuous, adapted process X in $\mathbb{R}^d$ admitting a decomposition $M + \hat{X}$, where M is a local martingale and $\hat{X}$ is a predictable process of locally finite variation starting at 0.

There is a further decomposition $M = X^c + X^d$ into a continuous and a purely discontinuous local martingale. We may also introduce the *jump point process* of X on $\mathbb{R}_+ \times (\mathbb{R}^d \setminus \{0\})$, given by

$$\xi = \sum_{t>0} \delta_{t,\Delta X_t}$$

The following processes are called the *local characteristics* of X:

$$\begin{array}{rcl} \hat{X} & — & \text{the compensator of } X \\ [X^c] & — & \text{the covariation matrix of } X^c \\ \hat{\xi} & — & \text{the compensator of } \xi \end{array}$$

Martingale Properties

The following result gives the basic martingale properties of contractable and exchangeable processes on $[0,1]$.

♣ *Let X be a contractable, $\mathbb{R}^d$-valued process on $\mathbb{Q} \cap [0,1]$ with $E|X_t| < \infty$ for all t. Then X extends to a special semimartingale on* $[0,1]$ *with local characteristics $[X^c]$, $\hat{X}$, and $\hat{\xi}$, where*

(i) $[X^c]$ *is a.s. linear,*

(ii) $\hat{X} = M \cdot \lambda$ *for a martingale M in $\mathbb{R}^d$,*

(iii) $\hat{\xi} = \lambda \otimes \eta$ *for a martingale η in $\mathcal{M}(\mathbb{R}^d)$.*

Furthermore, the martingales in (ii) *and* (iii) *are given by*

$$M_t = \frac{E[X_1 - X_t | \mathcal{F}_t]}{1-t}, \quad \eta_t = \frac{E[\xi_1 - \xi_t | \mathcal{F}_t]}{1-t}$$

The measure $\lambda \otimes \eta$ in (iii) should be understood in the sense of composition of kernels.

Martingale Criteria

Under various additional hypotheses, the previous martingale properties essentially characterize the exchangeability of X.

♣ *Let X be a locally L^1-bounded, special semimartingale on $\mathbb{R}_+$ with $X_0 = 0$, having stationary increments and induced filtration $\mathcal{F}$. Suppose that the local characteristics of X are absolutely continuous and admit martingale densities. Then X is exchangeable.*

An important special case is when the local characteristics are linear. Then no stationarity need to be assumed.

♣ *Let X be a uniformly integrable, special semimartingale on $[0, 1]$ with jump point process ξ such that $X_0 = 0$ and the end values X_1, $[X^c]_1$, and ξ_1 are non-random. Then X is exchangeable iff its local characteristics are absolutely continuous and admit martingale densities on $(0, 1)$.*

Norm Relations

For real semimartingales X on $[0,1)$, define

$$X_t^* = \sup_{s \le t} |X_s|, \quad t \in [0,1],$$

$$\gamma_X = ([X]_1 + X_1^2)^{1/2}$$

♣ *Let X be an exchangeable process on* $[0,1]$.

(i) *For fixed $t > 0$ and $p > 0$,*

$$\|X_t\|_p \asymp \|X_t^*\|_p \asymp \|\gamma_X\|_p$$

(ii) *As $t \to 0$ for fixed $p > 0$,*

$$t^{1/(p \wedge 1)} \|\gamma_X\|_p \lesssim \|X_t\|_p \asymp \|X_t^*\|_p \lesssim t^{1/(p \vee 2)} \|\gamma_X\|_p$$

Similar results hold for contractable semimartingales, as well as for summation processes based on exchangeable or contractable sequences. All bounds are sharp, though those in (ii) can be improved for increasing processes X.

Predictable Mapping

Given an $\mathbb{R}^d$-valued semimartingale X on $I = [0,1]$ or $\mathbb{R}_+$ and an I-valued, predictable process V on I, we define the process $X \circ V^{-1}$ by

$$(X \circ V^{-1})_t = \int_I 1\{V_s \le t\}\, dX_s, \quad t \in I,$$

provided these stochastic integrals exist. If $X_t = \xi[0,t]$ for some random measure ξ, then $X \circ V^{-1}$ is the process corresponding to $\xi \circ V^{-1}$. Say that V is λ-*preserving* if $\lambda \circ V^{-1} = \lambda$ a.s.

♣ *Let X be an $\mathcal{F}$-exchangeable process on $[0,1]$ or $\mathbb{R}_+$ and let V be an $\mathcal{F}$-predictable, λ-preserving process from I to I. Then*

$$X \circ V^{-1} \stackrel{d}{=} X$$

This is the continuous-time counterpart of the predictable sampling theorem.

Predictable Contraction

For any predictable set A with $\lambda A \geq h > 0$, define the associated *contraction* $C_A X$ by

$$(C_A X)_t = \int_0^{\tau_t} 1_A(s)\, dX_s, \quad t \in [0, h],$$

where $\tau = (\tau_t)$ is the right-continuous inverse of the process $\lambda_A(s) = \lambda(A \cap [0, s])$. The *selection integral* above exists for contractable processes X, even if X is not a semi-martingale.

♣ *Let X be an $\mathcal{F}$-contractable process on $I = [0, 1]$ or $\mathbb{R}_+$ and consider an $\mathcal{F}$-predictable random set A in I with $\lambda A \geq h$. Then*

$$C_A X \overset{d}{=} X \ \text{ on } \ [0, h].$$

This is the continuous-time counterpart of the optional skipping theorem.

Invariance of Stable Processes

Stronger invariance properties hold for stable Lévy processes. Here is a simple case:

♣ *Let X be a strictly p-stable Lévy process, and consider two predictable processes $U \geq 0$ and V, where U^p is locally integrable. Suppose that*

$$(U^p \cdot \lambda) \circ V^{-1} = \lambda \ \ a.s.$$

Then

$$(U \cdot X) \circ V^{-1} \stackrel{d}{=} X.$$

In particular, we can use such results to derive time-change representations of stable integrals:

♣ *Let X be symmetric p-stable and let V be predictable and X-integrable. Then there exists a process $Y \stackrel{d}{=} X$ such that*

$$V \cdot X = Y \circ (|V|^p \cdot \lambda) \ \ a.s.$$

Mapping of Optional Times

Consider an optional time τ with a random mark κ in some space K, such that the measure $\xi = \delta_{\tau,\kappa}$ is adapted to a filtration $\mathcal{F}$. Let μ be the distribution of (τ, κ). When $\mathcal{F}$ is induced by ξ, the compensator η is determined by μ and τ, and the inverse map gives the restriction of μ to $[0, \tau] \times K$, expressed in terms of η.

For general $\mathcal{F}$, the same inverse mapping, given by a Doléans differential equation, yields the *discounted compensator* ζ of (τ, κ), a random sub-probability measure on $[0, \tau] \times K$.

♣ *Let* (τ, κ) *be a marked optional time with discounted compensator* ζ*, and let* V *be a predictable mapping of* $\mathbb{R}_+ \times K$ *into* $[0, 1]$ *such that* $\zeta \circ V^{-1} \leq \lambda$ *a.s. Then* $V(\tau, \kappa)$ *is* $U(0, 1)$*.*

Similar maps of *orthogonal* pairs

$$(\tau_1, \kappa_1),\ (\tau_2, \kappa_2),\ \ldots$$

yield i.i.d. $U(0, 1)$ random variables.

Decoupling Identities

Say that a process X on $\mathbb{R}_+$ is $\mathcal{F}$-*Lévy* if it is an $\mathcal{F}$-adapted Lévy process such that $\theta_t X - X_t \perp\!\!\!\perp \mathcal{F}_t$ for every $t \geq 0$.

♣ *Let X and V be processes on $\mathbb{R}_+$, where X is $\mathcal{F}$-Lévy and V is $\mathcal{F}$-predictable, and suppose that $I_m = \int_0^\infty V_s^m ds$ is non-random for every $m < n$. Introduce a process $\tilde{V} \stackrel{d}{=} V$ with $\tilde{V} \perp\!\!\!\perp X$. Then, under suitable moment conditions,*

$$E\Big(\int_0^\infty V\,dX\Big)^n = E\Big(\int_0^\infty \tilde{V}\,dX\Big)^n$$

An exchangeable process on $[0,1]$ is said to be *extreme* if its directing triple $(\alpha, \sigma\sigma', \beta)$ is non-random.

♣ *The previous result remains true for extreme, $\mathcal{F}$-exchangeable processes on $[0,1]$, provided we assume I_m to be non-random even for $m = n$.*

Product Moments

For $\mathbb{R}^d$-valued processes V on $\mathbb{R}_+$, we define

$$I_J = \int_0^\infty \prod_{j\in J} V_j(s)\,ds, \quad J \subset \{1,\ldots,d\}.$$

♣ *Let X and V be $\mathbb{R}^d$-valued processes on $\mathbb{R}_+$, where X is $\mathcal{F}$-Lévy and V is $\mathcal{F}$-predictable, and suppose that I_J is non-random for every proper subset $J \subset \{1,\ldots,d\}$. Introduce a process $\tilde{V} \stackrel{d}{=} V$ with $\tilde{V} \perp\!\!\!\perp X$. Then, under suitable moment conditions,*

$$E \prod_{j\le d} \int_0^\infty V_j\,dX_j = E \prod_{j\le d} \int_0^\infty \tilde{V}_j\,dX_j.$$

This remains true for extreme, $\mathcal{F}$-exchangeable processes X on $[0,1]$, provided we assume I_J to be non-random even for $J = \{1,\ldots,d\}$.

Growth Rates

Martingale methods, too technical to explain here, can be used to derive path properties of exchangeable and related processes. The following statements extend some classical results for Lévy processes. Such results are also accessible by coupling.

♣ *Let X be an exchangeable process on $[0,1]$ with characteristics $(\alpha, 0, \beta)$, and consider an even, continuous function $f : \mathbb{R} \to \mathbb{R}_+$ with $f_0 = 0$ such that $\beta f < \infty$ a.s.*

(i) *If f is convex, f' is concave on $\mathbb{R}_+$ with $f'_0 = 0$, and $c > 1$, then as $t \to 0$*

$$\frac{X_t}{f^{-1}(t|\log t|^c)} \to 0 \ \ a.s., \qquad \frac{X_t}{f^{-1}(t)} \xrightarrow{P} 0.$$

(ii) *If X is increasing of pure jump type and f is concave on $\mathbb{R}_+$, then as $t \to 0$*

$$\frac{X_t}{f^{-1}(t)} \to 0 \ \ a.s.$$

The Ottawa Workshop, Lecture 5:

Exchangeable and Contractable Arrays

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Coding of Contractable Sequences

Here is a functional form of de Finetti's theorem:

♣ *Let* $X = (X_n)$ *be an infinite random sequence in a Borel space* S. *Then* X *is contractable iff there exist a measurable function* $f : [0,1]^2 \to S$ *and some i.i.d.* $U(0,1)$ *random variables* α *and* $\xi_1, \xi_2, \ldots$ *such that a.s.*

$$X_n = f(\alpha, \xi_n), \quad n \in \mathbb{N}$$

Note that f is not unique and that the construction of α and $\xi_1, \xi_2, \ldots$ may require an extension of the basic probability space.

To recover the standard form of de Finetti's theorem we note that, conditionally on α, the X_n are i.i.d. with distribution

$$\mu = P[f(\alpha, \xi_1) \in \cdot \mid \alpha]$$

Two-Dimensional Symmetries

Consider a doubly infinite array

$$X = (X_{ij};\, i, j \in \mathbb{N})$$

of random elements in a Borel space S. We say that X is *separately exchangeable* if

$$(X_{p_i, q_j}) \overset{d}{=} (X_{ij})$$

for any permutations $p = (p_i)$ and $q = (q_j)$ of $\mathbb{N}$ and *jointly exchangeable* if

$$(X_{p_i, p_j}) \overset{d}{=} (X_{ij})$$

for any permutation $p = (p_i)$ of $\mathbb{N}$. The latter condition is clearly the weakest.

Even weaker is the notion of *joint contractability*, where $(X_{p_i, p_j}) = (X_{ij})$ is required for every subsequence $p = (p_i)$ of $\mathbb{N}$. Note that separate exchangeability and contractability are equivalent, by Ryll-Nardzewski's theorem.

Coding of Exchangeable Arrays

Here is a coding representation of separately exchangeable arrays:

♣ *(Aldous, Hoover) A doubly infinite random array* $X = (X_{ij})$ *in a Borel space* S *is separately exchangeable iff there exist a measurable function* $f : [0,1]^4 \to S$ *and some i.i.d.* $U(0,1)$ *random variables* $\alpha, \xi_i, \eta_j, \zeta_{ij}$, $i, j \in \mathbb{N}$, *such that a.s.*

$$X_{ij} = f(\alpha, \xi_i, \eta_j, \zeta_{ij}), \quad i, j \in \mathbb{N}$$

More generally, we have the following result for jointly exchangeable arrays:

♣ *(Hoover) A doubly infinite random array* $X = (X_{ij})$ *in a Borel space* S *is jointly exchangeable iff there exist a measurable function* $f : [0,1]^4 \to S$ *and some i.i.d.* $U(0,1)$ *random variables* α, ξ_i, $\zeta_{ij} = \zeta_{ji}$, $i, j \in \mathbb{N}$, *such that a.s.*

$$X_{ij} = f(\alpha, \xi_i, \xi_j, \zeta_{ij}), \quad i, j \in \mathbb{N}$$

Coding of Contractable Arrays

A doubly infinite array $X = (X_{ij})$ is clearly jointly contractable iff the same property holds for the triangular array

$$Y_{ij} = (X_{ij}, X_{ji}, X_{ii}, X_{jj}), \quad i < j \text{ in } \mathbb{N}.$$

It is then enough to consider contractable arrays X on the *lower triangular* index set

$$\Delta_2 = \{(i,j) \in \mathbb{N}^2;\, i < j\}$$

This is equivalent to considering arrays indexed by subsets $J \subset \mathbb{N}$ of cardinality 2.

♣ *Let X be a random array X on Δ_2 with values in a Borel space S. Then X is jointly contractable iff there exist a measurable function $f\colon [0,1]^4 \to S$ and some i.i.d. $U(0,1)$ random variables $\alpha, \xi_i, \zeta_{ij}$, $i,j \in \mathbb{N}$, $i < j$, such that a.s.*

$$X_{ij} = f(\alpha, \xi_i, \xi_j, \zeta_{ij}), \quad i < j \;\; \textit{in } \mathbb{N}$$

Extension of Contractable Arrays

Reflecting Δ_2 in the main diagonal gives the *upper triangular* index set

$$\Delta_2' = \{(i, j) \in \mathbb{N}^2;\ i > j\},$$

and together the two sets form the non-diagonal part $\mathbb{N}^{(2)}$ of $\mathbb{N}^2$. Since $\mathbb{N}^{(2)}$ is closed under joint permutations, it makes sense to consider jointly exchangeable arrays on $\mathbb{N}^{(2)}$.

Comparing the previous representations yields:

♣ *A random array X on Δ_2 is jointly contractable iff it can be extended to a jointly exchangeable array Y on $\mathbb{N}^{(2)}$.*

In other words, for such an X there exists a jointly exchangeable array Y on $\mathbb{N}^{(2)}$ such that a.s.

$$X_{ij} = Y_{ij}, \quad i < j \text{ in } \mathbb{N}.$$

The required extension is not unique. No direct proof is known.

Equivalent Representations

We consider only contractable arrays, the other cases being similar. Let α, α', ξ_i, ξ_i', ζ_{ij}, ζ_{ij}' be i.i.d. $U(0,1)$.

♣ *Two measurable functions $f, f' : [0,1]^4 \to S$ can be used to represent the same contractable array on Δ_2 iff either of the following equivalent conditions is fulfilled:*

(i) *There exist some measurable functions $g_0, g_0', g_1, g_1', g_2, g_2'$ between suitable spaces, each preserving λ in the highest order argument, such that a.s.*

$$\begin{aligned} & f(g_0(\alpha), g_1(\alpha, \xi_i), g_1(\alpha, \xi_j), g_2(\alpha, \xi_i, \xi_j, \zeta_{ij})) \\ & \quad = f'(g_0'(\alpha), g_1'(\alpha, \xi_i), g_1'(\alpha, \xi_j), g_2'(\alpha, \xi_i, \xi_j, \zeta_{ij})) \end{aligned}$$

(ii) *There exist some measurable functions g_0, g_1, g_2 between suitable spaces, each mapping λ^2 to λ in the highest order arguments, such that a.s.*

$$\begin{aligned} & f(\alpha, \xi_i, \xi_j, \zeta_{ij}) \\ & \quad = f'(g_0(\alpha, \alpha'), g_1(\alpha, \alpha', \xi_i, \xi_i'), g_1(\alpha, \alpha', \xi_j, \xi_j'), \\ & \qquad\qquad g_2(\alpha, \alpha', \xi_i, \xi_i', \xi_j, \xi_j', \zeta_{ij}, \zeta_{ij}')) \end{aligned}$$

Set-Indexed Arrays

For the study of d-dimensional, jointly contractable arrays, we may choose as our index set the *tetrahedral* set

$$\Delta_d = \{(k_1, \ldots, k_d);\ k_1 < \cdots < k_d\}.$$

Identifying the elements of Δ_d with subsets $J \subset \mathbb{N}$ of cardinality d, it is equivalent to consider contractable arrays $X = (X_J)$ indexed by $\tilde{\mathbb{N}}$, the class of finite subsets of $\mathbb{N}$.

For the purpose of coding, we need to consider so called *U-arrays* $\xi = (\xi_J)$, consisting of i.i.d. $U(0,1)$ random variables ξ_J, $J \in \tilde{\mathbb{N}}$. We also need to introduce the associated finite subarrays

$$\hat{\xi}_J = (\xi_I;\ I \subset J), \quad J \in \tilde{\mathbb{N}}$$

General Contractable Arrays

We may now state a functional representation of contractable arrays on $\tilde{\mathbb{N}}$.

♣ *Let X be a random array on $\tilde{\mathbb{N}}$ with values in a Borel space S. Then X is contractable iff there exist a measurable function f between suitable spaces and a U-array ξ on $\tilde{\mathbb{N}}$ such that a.s.*

$$X_J = f(\hat{\xi}_J), \quad J \in \tilde{\mathbb{N}}$$

As before, this yields an extension theorem for jointly contractable arrays on Δ_d. Here we write $\mathbb{N}^{(d)}$ for the non-diagonal part of $\mathbb{N}^d$.

♣ *A random array on Δ_d is jointly contractable iff it can be extended to a jointly exchangeable array on $\mathbb{N}^{(d)}$.*

General Exchangeable Arrays

Here we consider the index set

$$\overline{\mathbb{N}} = \bigcup_{d=0}^{\infty} \mathbb{N}^{(d)},$$

consisting of all finite sequences $k = (k_1, \ldots, k_d)$ in $\mathbb{N}$ with distinct elements. Letting $\tilde{k}$ denote the corresponding set $\{k_1, \ldots, k_d\}$, we write for any array ξ on $\tilde{\mathbb{N}}$

$$\hat{\xi}_k = (\xi_I;\, I \subset \tilde{k}), \quad k \in \overline{\mathbb{N}},$$

where the elements need to be enumerated consistently according to the order within k.

♣ *Let X be a random array on $\overline{\mathbb{N}}$ with values in a Borel space S. Then X is jointly exchangeable iff there exist a measurable function f between suitable spaces and a U-array ξ on $\tilde{\mathbb{N}}$ such that*

$$X_k = f(\hat{\xi}_k), \quad k \in \overline{\mathbb{N}}$$

Symmetric Random Measures

The notions of separate or joint exchangeability or contractability extend immediately to random measures. The jointly contractable random measures are considered, most naturally, on the tetrahedral index sets

$$\Delta_d = \{(r_1, \ldots, r_d) \in \mathbb{R}_+^d;\, r_1 < \cdots < r_d\}$$

♣ *(Casukhela) A random measure on* Δ_d *is jointly contractable iff it can be extended to a jointly exchangeable random measure on* $\mathbb{R}_+^d$.

Though this holds in any dimension d, explicit representations of exchangeable random measures are known only in two dimensions.

Exchangeable Point Processes

For separately and jointly exchangeable random measures on $\mathbb{R}_+^2$, general representations are known involving independent Poisson processes and U-arrays. Here we consider only simple point processes, the general case being similar but more complicated:

♣ *If ξ is a simple point process on $\mathbb{R}_+^2$, then*

(i) *ξ is extreme separately exchangeable iff*

$$\begin{aligned}\xi &= \zeta + \textstyle\sum_{ij}\alpha_{ij}(\delta_{\sigma_i} \otimes \delta_{\tau_j}) \\ &\quad + \textstyle\sum_i(\delta_{\sigma_i} \otimes \eta_i) + \sum_j(\eta'_j \otimes \delta_{\tau_j}),\end{aligned}$$

(ii) *ξ is extreme jointly exchangeable iff*

$$\begin{aligned}\xi &= \zeta + \tilde{\zeta}' + \textstyle\sum_{ij}\alpha_{ij}(\delta_{\tau_i} \otimes \delta_{\tau_j}) \\ &\quad + \textstyle\sum_i(\delta_{\tau_i} \otimes \eta_i) + \sum_j(\eta'_j \otimes \delta_{\tau_j}),\end{aligned}$$

for suitably independent arrays (α_{ij}), Poisson processes ζ, ζ', η_i, η'_j, and sequences $(\sigma_i), (\tau_j)$.

Coding of Separately Exchangeable Point Processes

A more precise description of the previous distributions and dependencies is in terms of coding:

♣ *A simple point process* ξ *on* $\mathbb{R}_+^2$ *is extreme separately exchangeable iff*

$$\begin{aligned}\xi \;&=\; \textstyle\sum_k \delta_{a\rho_k} + \displaystyle\sum_{i,j} f(\vartheta_i, \vartheta'_j, \zeta_{ij})(\delta_{\tau_i} \otimes \delta_{\tau'_j}) \\ &\quad + \textstyle\sum_{i,k}(\delta_{\tau_i} \otimes \delta_{g(\vartheta_i)\sigma_{ik}}) + \sum_{j,k}(\delta_{g'(\vartheta'_j)\sigma'_{jk}} \otimes \delta_{\tau'_j})\end{aligned}$$

for a constant $a \geq 0$*, some measurable functions* f, g, g'*, some independent, unit rate Poisson processes* $\{(\tau_i, \vartheta_i)\}$*,* $\{(\tau'_i, \vartheta'_i)\}$*,* $\{\rho_k\}$ *on* $\mathbb{R}_+^2$ *and* $\{\sigma_{ik}\}$*,* $\{\sigma'_{jk}\}$ *on* $\mathbb{R}_+$*, and some independent i.i.d.* $U(0,1)$ *r.v.'s* ζ_{ij}*.*

Coding of Jointly Exchangeable Point Processes

The corresponding representation in the jointly exchangeable case is even more complicated:

♣ *A simple point process* ξ *on* $\mathbb{R}_+^2$ *is extreme jointly exchangeable iff*

$$\begin{aligned}\xi \;=\; & \textstyle\sum_k \left(l(\eta_k)\delta_{\rho_k,\rho'_k} + l'(\eta_k)\delta_{\rho'_k,\rho_k}\right) \\ & + \textstyle\sum_{i,j} f(\vartheta_i,\vartheta_j,\zeta_{ij})\delta_{\tau_i,\tau_j} \\ & + \textstyle\sum_{j,k}\left(g(\vartheta_j,\chi_{jk})\delta_{\tau_j,\sigma_{jk}} + g'(\vartheta_j,\chi_{jk})\delta_{\sigma_{jk},\tau_{jk}}\right)\end{aligned}$$

for some measurable functions f, g, g', l, l', *some independent, unit rate Poisson processes* $\{(\tau_j,\vartheta_j)\}$, $\{(\sigma_{jk},\chi_{jk})\}$ *on* $\mathbb{R}_+^2$ *and* $\{(\rho_k,\rho'_k,\eta_k)\}$ *on* $\mathbb{R}_+^3$, *and some independent i.i.d.* $U(0,1)$ *r.v.'s* $\zeta_{ij} = \zeta_{ji}$.

Symmetric Partitions

A random partition of $\mathbb{N}$ into disjoint subsets $A_1, A_2, \ldots$ is said to be *exchangeable* if its distribution is invariant under permutations of $\mathbb{N}$. More precisely, writing

$$X_{ij} = \textstyle\sum_k 1\{i, j \in A_k\}, \quad i, j \in \mathbb{N},$$

we say that (A_k) is exchangeable iff the array $X = (X_{ij})$ is jointly exchangeable. Here is the celebrated *paintbox* representation:

♣ *(Kingman) A random partition (A_k) of $\mathbb{N}$ is exchangeable iff there exist some exchangeable random variables $\xi_1, \xi_2, \ldots$ such that a.s.*

$$X_{ij} = 1\{\xi_i = \xi_j\}, \quad i, j \in \mathbb{N}$$

The result extends to arbitrary symmetries of $\mathbb{N}$, defined in terms of families $\mathcal{T}$ of injective transformations $p\colon \mathbb{N} \to \mathbb{N}$. It even holds for suitably *marked* partitions.

The Ottawa Workshop, Lecture 6:

Rotatable Arrays and Functionals

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Multivariate Rotations

An array $U = (U_{ij})$ on $\mathbb{N}^2$ is said to be *orthogonal* if there exists an $n \in \mathbb{N}$ such that U is an orthogonal matrix on $\{1, \ldots, n\}^2$ and $U_{ij} = \delta_{ij}$ for $i \vee j > n$. For a random array $X = (X_{ij})$ on $\mathbb{N}^2$, we say that X is *separately rotatable* if the transformed array

$$Y_{ij} = \sum_{h,k} U_{ih} V_{jk} X_{hk}, \quad i, j \in \mathbb{N},$$

has the same distribution as X for any orthogonal arrays U and V, and *jointly rotatable* if the same condition holds with $U = V$.

To simplify the notation, write the definition of Y as $Y = (U \otimes V)X$ and let $\mathcal{O}$ be the class of orthogonal arrays on $\mathbb{N}^2$. Then X is separately rotatable if

$$(U \otimes V)X \stackrel{d}{=} X, \quad U, V \in \mathcal{O},$$

and jointly rotatable if

$$U^{\otimes 2} X \stackrel{d}{=} X, \quad U \in \mathcal{O}.$$

Two-Dimensional Arrays

By a *G-array* we mean an indexed set of i.i.d. $N(0,1)$ random variables. In the separately rotatable case, we have the representation:

♣ *(Aldous) An array X on $\mathbb{N}^2$ is separately rotatable iff*

$$X_{ij} = \sigma\zeta_{ij} + \textstyle\sum_k \alpha_k\, \xi_{ki}\, \eta_{kj}, \quad i,j \in \mathbb{N},$$

for some independent G-arrays $(\xi_{ki}), (\eta_{kj}), (\zeta_{ij})$ and an independent collection of r.v.'s σ and α_k satisfying $\sum_k \alpha_k^2 < \infty$.

In the jointly rotatable case, we have instead:

♣ *An array X on $\mathbb{N}^2$ is jointly rotatable iff*

$$\begin{aligned} X_{ij} &= \rho\delta_{ij} + \sigma\zeta_{ij} + \sigma'\zeta_{ji} \\ &\quad + \textstyle\sum_{h,k} \alpha_{hk}\,(\xi_{ki}\,\xi_{kj} - \delta_{ij}\delta_{hk}), \quad i,j \in \mathbb{N}, \end{aligned}$$

for some independent G-arrays $(\xi_{ki}), (\zeta_{ij})$ and an independent collection of r.v.'s $\rho, \sigma, \sigma', \alpha_{hk}$ satisfying $\sum_{h,k} \alpha_{hk}^2 < \infty$.

Random Sheets

A *random sheet* on $\mathbb{R}_+^2$ is a continuous process X such that $X(s,t) = 0$ for $s \wedge t = 0$. Separate or joint rotatability may be defined as before in terms of the two-dimensional *increments*

$$\begin{aligned}\Delta_{h,k}X(s,t) &= X(s+h,t+k) - X(s+h,t)\\ &\quad -X(s,t+k) + X(s,t).\end{aligned}$$

A *Brownian sheet* is a random sheet X on $\mathbb{R}_+^2$ with independent increments such that $\Delta_{h,k}X(s,t)$ is $N(0,hk)$. Equivalently, we may introduce *Gaussian white noise* η as a centered Gaussian process on $L^2(\mathbb{R}_+^2)$ with

$$\mathrm{Cov}(\eta h, \eta k) = \langle h, k \rangle, \quad h, k \in L^2(\mathbb{R}_+^2),$$

and define X as an a.s. continuous version of the process

$$X(s,t) = \eta([0,s] \times [0,t]), \quad s, t \geq 0.$$

Rotatable Random Sheets

Here is a representation of separately rotatable random sheets on $\mathbb{R}_+^2$.

♣ *A random sheet X on $\mathbb{R}_+^2$ is separately rotatable iff, a.s. for all $s, t \geq 0$,*

$$X(s,t) = \sigma Z(s,t) + \textstyle\sum_k \alpha_k \, B_k(s) \, C_k(t),$$

for a Brownian sheet Z, some independent Brownian motions B_k, C_k, and an independent set of r.v.'s σ and α_k with $\sum_k \alpha_k^2 < \infty$.

For jointly rotatable sheets we have instead:

♣ *A random sheet X on $\mathbb{R}_+^2$ is jointly rotatable iff, a.s. for all $s, t \geq 0$,*

$$\begin{aligned} X(s,t) &= \rho(s \wedge t) + \sigma Z(s,t) + \sigma' Z(t,s) \\ &\quad + \textstyle\sum_{h,k} \alpha_{hk} (B_h(s) B_k(t) - \delta_{hk}(s \wedge t)), \end{aligned}$$

for a Brownian sheet Z, some independent Brownian motions B_k, and an independent set of r.v.'s ρ, σ, σ', and α_{hk} with $\sum_{h,k} \alpha_{hk}^2 < \infty$.

Functional Notation

The previous representations become more transparent when written in functional form, for separately rotatable sheets as

$$X = \sigma Z + \textstyle\sum_k \alpha_k (B_k \otimes C_k),$$

and for jointly rotatable ones as

$$X = \lambda_D + \sigma Z + \sigma' \tilde{Z} + \textstyle\sum_{h,k} \alpha_{hk} (B_h \otimes B_k).$$

Here products such as $B_h \otimes B_k$ are finitely additive random set functions on the rectangles in $\mathbb{R}_+^2$, defined by

$$\begin{aligned}(B_h \otimes B_k)&([0,s] \times [0,t]) \\ &= B_h(s) B_k(t) - \delta_{hk}(s \wedge t),\end{aligned}$$

where the last centering term is motivated by

$$E B_h(s) B_k(t) = \delta_{hk}(s \wedge t).$$

The term λ_D represents Lebesgue measure along the main diagonal D of $\mathbb{R}_+^2$. Finally, $\tilde{Z}$ is the reflection of Z in D.

Rotatable Random Functionals

To unify the representations of rotatable sequences and processes, fix any σ-finite measure with infinite support on a space S. A real-valued process X on $L^2(\mu)$ is called a *continuous linear random functional (CLRF)* if

$$X(af + bg) = aXf + bXg \ \text{ a.s.}$$

for all $f, g \in L^2$ and $a, b \in \mathbb{R}$, and

$$\|f_n\|_2 \to 0 \quad \Rightarrow \quad Xf_n \xrightarrow{P} 0.$$

Say that X is *rotatable* if

$$\|f\|_2 = \|g\|_2 \quad \Rightarrow \quad Xf \overset{d}{=} Xg$$

A *G-process* on $L^2(\mu)$ is a centered Gaussian process η with $E(\eta f)^2 = \|f\|_2^2$.

♣ *A CLRF X on $L^2(\mu)$ is rotatable iff there exist a G-process η and an independent random variable $\sigma \geq 0$ such that*

$$X = \sigma\eta \ \textit{ a.s.}$$

Tensor Products

Turning to an abstract Hilbert space setting, we note that any separable Hilbert space H can be identified with $L^2(\mu)$ for some σ-finite measure space (S, μ), where μ has infinite support iff H is infinite-dimensional. For any Hilbert spaces $H_k = L^2(S_k, \mu_k)$, $k \le n$, their *tensor product* is given by

$$H_1 \otimes \cdots \otimes H_n = L^2(\mu_1 \otimes \cdots \otimes \mu_n),$$

or simply $\bigotimes_k H_k = L^2(\bigotimes_k \mu_k)$. If $H_k = H$ for all k, we write $\bigotimes_k H_k = H^{\otimes n}$.

A *unitary operator* U on H is an invertible linear isometry. If U_k is unitary on H_k for every $k \le n$, there exists a unique unitary operator $\bigotimes_k U_k$ on $\bigotimes_k H_k$ satisfying

$$(U_1 \otimes \cdots \otimes U_n)(f_1 \otimes \cdots \otimes f_n) = U_1 f_1 \otimes \cdots \otimes U_n f_n$$

for any elements $f_k \in H_k$, where

$$(f_1 \otimes \cdots \otimes f_n)(s_1, \ldots, s_n) = f_1(s_1) \cdots f_n(s_n).$$

If $H_k = H$ and $U_k = U$ for all k, we write $\bigotimes_k U_k = U^{\otimes n}$.

Multivariate Rotations

For any CLRF X and unitary operator U on a Hilbert space H, we define the CLRF $X \circ U$ on H by

$$(X \circ U)f = X(Uf), \quad f \in H.$$

A CLRF X on $H^{\otimes n}$ is said to be *separately rotatable* if

$$X \circ (U_1 \otimes \cdots \otimes U_n) \stackrel{d}{=} X$$

for any unitary operators $U_1, \ldots, U_n$ on H, and *jointly rotatable* if

$$X \circ U^{\otimes n} \stackrel{d}{=} X$$

for any unitary operator U on H. Note that the latter condition is the weakest. Separate rotatability makes sense even for CLRF's on arbitrary tensor products $\bigotimes_k H_k$. However, this generalization is illusory, since all separable, infinite-dimensional Hilbert spaces are isomorphic.

Multiple Stochastic Integrals

The basic examples of separately or jointly rotatable CLRF's are the *multiple Wiener-Itô integrals*, defined as follows:

♣ *If* $\eta_1, \ldots, \eta_n$ *are independent G-processes on some Hilbert spaces* $H_1, \ldots, H_n$, *there exists an a.s. unique CLRF* $\bigotimes_k \eta_k$ *on* $\bigotimes_k H_k$ *such that, for any elements* $f_k \in H_k$,

$$(\eta_1 \otimes \cdots \otimes \eta_n)(f_1 \otimes \cdots \otimes f_n) = \eta_1 f_1 \cdots \eta_n f_n$$

♣ *For any G-process* η *on a Hilbert space* H *and every* $n \in \mathbb{N}$, *there exists an a.s. unique CLRF* $\eta^{\otimes n}$ *on* $H^{\otimes n}$ *such that, for any orthogonal elements* $f_k \in H$,

$$\eta^{\otimes n}(f_1 \otimes \cdots \otimes f_n) = \eta f_1 \cdots \eta f_n$$

Note that $\bigotimes_k \eta_k$ is separately rotatable and $\eta^{\otimes n}$ is jointly rotatable. More generally, we may define CLRF's on $\bigotimes_k H_k^{\otimes m_k}$ of the form $\bigotimes_k \eta_k^{\otimes m_k}$.

Separately Rotatable Functionals

For any $d \in \mathbb{N}$, let $\mathcal{P}_d$ be the set of partitions π of $\{1, \ldots, d\}$ into non-empty subsets J. Write $H^{\otimes J} = \bigotimes_{j \in J} H$ and $H^{\otimes \pi} = \bigotimes_{J \in \pi} H$.

♣ *A CLRF X on $H^{\otimes d}$ is extreme separately rotatable iff a.s.*

$$Xf = \sum_{\pi \in \mathcal{P}_d} \Big(\bigotimes_{J \in \pi} \eta_J \Big)(\alpha_\pi \otimes f), \quad f \in H^{\otimes d},$$

for some independent G-processes η_J on $H \otimes H^{\otimes J}$, $\emptyset \neq J \subset \{1, \ldots, d\}$, and elements $\alpha_\pi \in H^{\otimes \pi}$, $\pi \in \mathcal{P}_d$.

For any ONB $h_1, h_2, \ldots$ in H, we may define a separately rotatable array on $\mathbb{N}^d$ by

$$X_{k_1, \ldots, k_d} = X(h_{k_1} \otimes \cdots \otimes h_{k_d}).$$

Conversely, any separately rotatable array can be represented in this form. The representation of X has the equivalent basis form

$$X_k = \sum_{\pi \in \mathcal{P}_d} \sum_{l \in \mathbb{N}^\pi} \alpha_l^\pi \prod_{J \in \pi} \eta^J_{k_J, l_J}, \quad k \in \mathbb{N}^d$$

Jointly Rotatable Functionals

Here we introduce the class $\mathcal{O}_d$ of partitions of $\{1,\ldots,d\}$ into *ordered* sets $k=(k_1,\ldots,k_r)$ of size $|k|=r$.

♣ *A CLRF X on H^d is extreme jointly rotatable iff a.s.*

$$Xf = \sum_{\pi\in\mathcal{O}_d} \Big(\bigotimes_{k\in\pi} \eta_{|k|}\Big)(\alpha_\pi \otimes f), \quad f \in H^{\otimes d},$$

for some independent G-processes η_k on $H^{\otimes(k+1)}$, $k\le d$, and elements $\alpha_\pi\in H^{\otimes\pi}$, $\pi\in\mathcal{O}_d$.

Note that the integrals $\otimes_k \eta_{|k|}$ depend on the order within each sequence k. Though the representation may again be restated in basis form, for an associated array (X_k) on $\mathbb{N}^d$, not every jointly rotatable array has such a representation. The formulas also become more complicated in this case, as evident from the representation of Wiener-Itô integrals in terms of Hermite polynomials.

Separately Contractable Sheets

Given a random sheet X on $\mathbb{R}_+^d$, we consider a decomposition

$$X_t = \sum_{J \in 2^d} X_{t_J}^J \prod_{i \notin J} t_i, \quad t \in \mathbb{R}_+^d,$$

where each X^J is a suitably centered process on $\mathbb{R}_+^J$. Then X is separately contractable iff the family (X^J) is separately rotatable. Define $\hat{\mathcal{P}}_d = \cup_J \mathcal{P}_J$, where $\mathcal{P}_J$ is the class of partitions of $J \subset \{1, \ldots, d\}$. For $\pi \in \mathcal{P}_J$, write $\pi^c = J^c$.

♣ *A random sheet X on $\mathbb{R}_+^d$ is extreme separately contractable iff a.s.*

$$X_t = \sum_{\pi \in \hat{\mathcal{P}}_d} (\lambda^{\pi^c} \otimes \bigotimes_{J \in \pi} \eta_J)(\alpha_\pi \otimes [0,t]), \quad t \in \mathbb{R}_+^d,$$

for some independent G-processes η_J on $H \otimes L^2(\lambda^J)$ and elements $\alpha_\pi \in H^{\otimes \pi}$.

This may be transformed into a similar representation of separately exchangeable random sheets on $[0,1]^d$.

Jointly Exchangeable and Contractable Sheets

Put $\hat{\mathcal{O}}_d = \cup_J \mathcal{O}_J$ and, for any $\pi \in \hat{\mathcal{O}}_d$, define the vectors $\hat{t}_\pi$ by $\hat{t}_{\pi,J} = \min_{j\in J}$, $J \in \pi$.

♣ *A random sheet X on $\mathbb{R}_+^d$ is extreme jointly exchangeable iff, a.s. for all $t \in \mathbb{R}_+^d$,*

$$X_t = \sum_{\pi\in\mathcal{P}_d} \sum_{\kappa\in\hat{\mathcal{O}}_\pi} (\lambda^{\pi^c} \otimes \bigotimes_{k\in\kappa} \eta_{|k|})(\alpha_{\pi,\kappa} \otimes [0, \hat{t}_\pi])$$

for some independent G-processes η_m on $H \otimes L^2(\lambda^m)$, $m \le d$, and elements $\alpha_{\pi,\kappa} \in H^{\otimes\kappa}$, $\kappa \in \hat{\mathcal{O}}_\pi$, $\pi \in \mathcal{P}_d$.

A similar, even more complicated representation holds for jointly contractable sheets on $\mathbb{R}_+^d$. The problem of characterizing jointly exchangeable sheets on $[0,1]^d$ remains open.

Probabilistic Symmetries

Ottawa Lectures, May 15-17, 2006

1. Basic symmetries and representations
2. Convergence and approximation
3. Martingales and predictable sampling
4. Semi-martingales and integral criteria
5. Exchangeable and contractable arrays
6. Rotatable arrays and functionals

1928–30	*de Finetti*	*exchangeable events*
1936	*Doob*	*first optional skipping theorem*
1937	*de Finetti*	*exchangeable random variables*
1944	*Wald*	*first decoupling identities*
1951	*Itô*	*multiple Wiener integrals*
1955	*Hewitt/Savage*	*abstract theory and spaces*
1957	*Ryll-Nardzewski*	*contractable sequences*
1960	*Bühlmann*	*exchangeable processes on* R_+
1962–3	*Freedman*	*rotatable sequences and processes*
1963	*Rényi/Révész*	*stable convergence*
1973	*OK*	*processes on* $[0,1]$*, weak convergence*
1973–5	*Papangelou/OK*	*Palm measure invariance*
1975	*Grigelionis*	*first semi-martingale criteria*
1975–7	*Dacunha-C/Aldous*	*weak subsequence principle*
1978	*Kingman*	*exchangeable partitions*
1979–81	*Aldous/Hoover*	*exchangeable and rotatable arrays*
1980–90	*Diaconis/Freedman*	*special mixtures*
1982	*OK*	*semi-martingale criteria*
1986	*Berkes/Péter*	*strong subsequence principle*
1988	*OK*	*predictable sampling and mapping*
1989	*OK*	*decoupling identities*
1990	*OK*	*exchangeable measures on* R_+^2
1992	*OK*	*contractable arrays*
1992–04	*Ivanoff/Weber*	*array convergence and sampling*
1995	*OK*	*multivariate rotations*