Simultaneous rational approximation and the Markoff spectrum

Damien Roy

Université d'Ottawa, Département de mathématiques

September 3, 2008

The Lagrange spectrum

Definition

For each $\xi \in \mathbb{R}$, define

$$\nu(\xi) := \liminf_{\mathbb{N}^* \ni q \to \infty} q \| q\xi \|$$

where $\| \ \|$ = distance to a nearest integer.

The Lagrange spectrum

Definition

For each $\xi \in \mathbb{R}$, define

$$\nu(\xi) := \liminf_{\mathbb{N}^* \ni q \to \infty} q\|q\xi\|$$

where $\| \; \| =$ distance to a nearest integer.

Examples: $\nu\left((\sqrt{5}-1)/2\right) = 1/\sqrt{5}$, $\nu\left(\sqrt{2}-1\right) = 1/\sqrt{8}$, ...

The Lagrange spectrum

Definition

For each $\xi \in \mathbb{R}$, define

$$\nu(\xi) := \liminf_{\mathbb{N}^* \ni q \to \infty} q \| q\xi \|$$

where $\| \ \|$ = distance to a nearest integer.

Examples: $\nu\left((\sqrt{5}-1)/2\right) = 1/\sqrt{5}$, $\nu\left(\sqrt{2}-1\right) = 1/\sqrt{8}$, ...

Definition

$\mathcal{L} := \{\nu(\xi)\,;\ \xi \in \mathbb{R}\}$ is called the *Lagrange* spectrum

The Lagrange spectrum

Definition

For each $\xi \in \mathbb{R}$, define

$$\nu(\xi) := \liminf_{\mathbb{N}^* \ni q \to \infty} q\|q\xi\|$$

where $\| \ \|$ = distance to a nearest integer.

Examples: $\nu\left((\sqrt{5}-1)/2\right) = 1/\sqrt{5}$, $\nu\left(\sqrt{2}-1\right) = 1/\sqrt{8}$, ...

Definition

$\mathcal{L} := \{\nu(\xi)\,;\, \xi \in \mathbb{R}\}$ is called the *Lagrange* spectrum

Property: $\nu\left(\dfrac{a\xi+b}{c\xi+d}\right) = \nu(\xi)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$.

Link with continued fractions

Assume that $\xi \notin \mathbb{Q}$ (otherwise $\nu(\xi) = 0$).

Write $\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = [a_0, a_1, a_2, \dots]$

Link with continued fractions

Assume that $\xi \notin \mathbb{Q}$ (otherwise $\nu(\xi) = 0$).

Write $\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = [a_0, a_1, a_2, \dots]$

$$\Rightarrow \quad \nu(\xi) = \liminf_{k \to \infty} \frac{1}{[a_{k+1}, a_{k+2}, \dots] + [0, a_k, \dots, a_1]}$$

Link with continued fractions

Assume that $\xi \notin \mathbb{Q}$ (otherwise $\nu(\xi) = 0$).

Write $\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = [a_0, a_1, a_2, \dots]$

$$\Rightarrow \quad \nu(\xi) = \liminf_{k \to \infty} \frac{1}{[a_{k+1}, a_{k+2}, \dots] + [0, a_k, \dots, a_1]}$$

Corollary

$\nu(\xi) > 0 \iff (a_k)_{k \geq 0}$ *is a bounded sequence.*

Link with continued fractions

Assume that $\xi \notin \mathbb{Q}$ (otherwise $\nu(\xi) = 0$).

Write $\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = [a_0, a_1, a_2, \dots]$

$$\Rightarrow \quad \nu(\xi) = \liminf_{k\to\infty} \frac{1}{[a_{k+1}, a_{k+2}, \dots] + [0, a_k, \dots, a_1]}$$

Corollary

$\nu(\xi) > 0 \iff (a_k)_{k\geq 0}$ *is a bounded sequence.*

- This is a tool for investigating the Lagrange spectrum $\mathcal{L}$.

The Markoff equation

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2$$

The Markoff equation

$$m^2 + m_1^2 + m_2^2 \;=\; 3mm_1m_2$$

Solutions $(m, m_1, m_2) \in (\mathbb{N}^*)^3$ with $m \geq m_1,\ m_2$ form a tree :

- $(1, 1, 1)$
  - $(2, 1, 1)$
    - $(5, 1, 2)$
      - $(13, 1, 5)$
        - $(34, 1, 13)$
        - $(194, 13, 5)$
      - $(29, 5, 2)$
        - $(433, 5, 29)$
        - $(169, 29, 2)$

$m^2 + m_1^2 + m_2^2 \;=\; 3mm_1m_2$

The recurrence is

$$(m,\ m_1,\ m_2)$$

$$(3mm_1 - m_2,\ \underbrace{m_1,\ m}_{\substack{\text{(last two coordinates}\\ \text{are ``inverted'')}}}) \qquad\qquad (3mm_2 - m_1,\ \underbrace{m,\ m_2}_{\substack{\text{(all coordinates in}\\ \text{decreasing order)}}})$$

$m^2 + m_1^2 + m_2^2 \ = \ 3mm_1m_2$

- Fix (m, m_1, m_2) in the tree. Then: $\gcd(m, m_1, m_2) = 1$.

$m^2 + m_1^2 + m_2^2 \ = \ 3mm_1m_2$

- Fix (m, m_1, m_2) in the tree. Then: $\gcd(m, m_1, m_2) = 1$.
- Choose $k \in \mathbb{Z}$ such that

$$k \equiv \frac{m_2}{m_1} \mod m \quad \text{and} \quad 0 \leq k < m.$$

Then $\ell := \dfrac{k^2+1}{m} \in \mathbb{Z}$.

$m^2 + m_1^2 + m_2^2 = 3mm_1m_2$

- Fix (m, m_1, m_2) in the tree. Then: $\gcd(m, m_1, m_2) = 1$.
- Choose $k \in \mathbb{Z}$ such that

$$k \equiv \frac{m_2}{m_1} \mod m \quad \text{and} \quad 0 \le k < m.$$

  Then $\ell := \dfrac{k^2+1}{m} \in \mathbb{Z}$.

- The quadratic form

$$\begin{aligned} F :&= mx^2 + (3m - 2k)xy + (\ell - 3k)y^2 \\ &= m(x - \beta y)(x - \beta' y) \end{aligned}$$

  has equivalent roots $\beta \sim \beta'$ (under $\mathrm{GL}_2(\mathbb{Z})$) with

$$\nu(\beta) = \nu(\beta') = \frac{1}{\sqrt{9 - 4m^{-2}}} > \frac{1}{3}.$$

Theorem (Markoff, 1871)

$\nu(\xi) > \frac{1}{3} \iff$ *ξ is equivalent to the roots of the quadratic form attached to a Markoff triple (m, m_1, m_2)*

Theorem (Markoff, 1871)

$\nu(\xi) > \dfrac{1}{3} \iff$ *ξ is equivalent to the roots of the quadratic form attached to a Markoff triple (m, m_1, m_2)*

Corollary

$\mathcal{L} \cap (1/3, \infty) = \{\mu_1 > \mu_2 > \mu_3 > \dots\}$ *with* $\lim \mu_i = 1/3$.

$\nu(\xi) = \mu_i \iff \xi$ *belongs to a finite set of equivalence classes.*

(…conjecturally just one.)

Theorem (Markoff, 1871)

$$\nu(\xi) > \frac{1}{3} \iff$$ *ξ is equivalent to the roots of the quadratic form attached to a Markoff triple (m, m_1, m_2)*

Corollary

$\mathcal{L} \cap (1/3, \infty) = \{\mu_1 > \mu_2 > \mu_3 > \dots\}$ *with* $\lim \mu_i = 1/3$.

$\nu(\xi) = \mu_i \iff \xi$ *belongs to a finite set of equivalence classes.*

(...conjecturally just one.)

Theorem (Markoff, 1871)

There are uncountably many $\xi \in \mathbb{R}$ such that $\nu(\xi) = 1/3$.

Connections

Work of Harvey Cohn provides link with

1955: fundamental domains of genus one in the upper half plane

1971: geodesics on a perforated torus

1972: primitive words in F_2

Extremal numbers

$$\gamma := \frac{1+\sqrt{5}}{2} \simeq 1.618\,, \quad \frac{1}{\gamma} = \gamma - 1 \simeq 0.618\,, \quad \gamma^2 = \gamma + 1 \simeq 2.618$$

Extremal numbers

$$\gamma := \frac{1+\sqrt{5}}{2} \simeq 1.618\,, \quad \frac{1}{\gamma} = \gamma - 1 \simeq 0.618\,, \quad \gamma^2 = \gamma + 1 \simeq 2.618$$

Definition

We say that $\xi \in \mathbb{R}$ is *extremal* if 1, ξ, ξ^2 are linearly independent over $\mathbb{Q}$ and there exists a constant $c_1 > 0$ such that

$$|x_0| \le X, \quad |x_0\xi - x_1| \le c_1 X^{-1/\gamma}, \quad |x_0\xi^2 - x_2| \le c_1 X^{-1/\gamma}$$

has a solution $(x_0, x_1, x_2) \in \mathbb{Z}^3 \setminus \{0\}$ for each sufficiently large X.

Extremal numbers

$$\gamma := \frac{1+\sqrt{5}}{2} \simeq 1.618\,, \quad \frac{1}{\gamma} = \gamma - 1 \simeq 0.618\,, \quad \gamma^2 = \gamma + 1 \simeq 2.618$$

Definition

We say that $\xi \in \mathbb{R}$ is *extremal* if $1, \xi, \xi^2$ are linearly independent over $\mathbb{Q}$ and there exists a constant $c_1 > 0$ such that

$$|x_0| \le X, \quad |x_0\xi - x_1| \le c_1 X^{-1/\gamma}, \quad |x_0\xi^2 - x_2| \le c_1 X^{-1/\gamma}$$

has a solution $(x_0, x_1, x_2) \in \mathbb{Z}^3 \setminus \{0\}$ for each sufficiently large X.

- Davenport and Schmidt, 1969: there exists no such ξ if $c_1 X^{-1/\gamma}$ is replaced by $o(X^{-1/\gamma})$.

Extremal numbers

$$\gamma := \frac{1+\sqrt{5}}{2} \simeq 1.618\,, \quad \frac{1}{\gamma} = \gamma - 1 \simeq 0.618\,, \quad \gamma^2 = \gamma + 1 \simeq 2.618$$

Definition

We say that $\xi \in \mathbb{R}$ is *extremal* if $1, \xi, \xi^2$ are linearly independent over $\mathbb{Q}$ and there exists a constant $c_1 > 0$ such that

$$|x_0| \le X, \quad |x_0\xi - x_1| \le c_1 X^{-1/\gamma}, \quad |x_0\xi^2 - x_2| \le c_1 X^{-1/\gamma}$$

has a solution $(x_0, x_1, x_2) \in \mathbb{Z}^3 \setminus \{0\}$ for each sufficiently large X.

- Davenport and Schmidt, 1969: there exists no such ξ if $c_1 X^{-1/\gamma}$ is replaced by $o(X^{-1/\gamma})$.
- R., 2004: There are countably many extremal numbers.

Properties (R., 2004)

Let $\xi \in \mathbb{R}$ be extremal.

(i) $\dfrac{a\xi + b}{c\xi + d}$ *is extremal for all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q})$

Properties (R., 2004)

Let $\xi \in \mathbb{R}$ be extremal.

(i) $\dfrac{a\xi + b}{c\xi + d}$ *is extremal for all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q})$

(ii) *There are infinitely many quadratic numbers $\alpha \in \mathbb{R}$ such that*

$$|\xi - \alpha| \le c_2 H(\alpha)^{-2\gamma^2}$$

and any quadratic α has $|\xi - \alpha| \ge c_3 H(\alpha)^{-2\gamma^2}$.

Properties (R., 2004)

Let $\xi \in \mathbb{R}$ be extremal.

(i) $\dfrac{a\xi + b}{c\xi + d}$ *is extremal for all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q})$

(ii) *There are infinitely many quadratic numbers $\alpha \in \mathbb{R}$ such that*

$$|\xi - \alpha| \le c_2 H(\alpha)^{-2\gamma^2}$$

and any quadratic α has $|\xi - \alpha| \ge c_3 H(\alpha)^{-2\gamma^2}$.

(iii) $\left|\xi - \dfrac{p}{q}\right| \ge c_4 q^{-2}(1 + \log q)^{-c_5}$ *for all* $\dfrac{p}{q} \in \mathbb{Q},\ q \ge 1.$

Properties (R., 2004)

Let $\xi \in \mathbb{R}$ be extremal.

(i) $\dfrac{a\xi + b}{c\xi + d}$ *is extremal for all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q})$

(ii) *There are infinitely many quadratic numbers $\alpha \in \mathbb{R}$ such that*

$$|\xi - \alpha| \leq c_2 H(\alpha)^{-2\gamma^2}$$

and any quadratic α has $|\xi - \alpha| \geq c_3 H(\alpha)^{-2\gamma^2}$.

(iii) $\left|\xi - \dfrac{p}{q}\right| \geq c_4 q^{-2}(1 + \log q)^{-c_5}$ *for all* $\dfrac{p}{q} \in \mathbb{Q}$, $q \geq 1$.

Definition

$$\mathcal{P} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})\,;\; ad - bc \neq 0\,,\, \gcd(a, b, c, d) = 1 \right\}$$

is a group under $A * B = \gcd(AB)^{-1} AB$.

Theorem (R. 2004)

A real number ξ is extremal

$\Longleftrightarrow$ *there exists a sequence of symmetric matrices* $\mathbf{x}_i = \begin{pmatrix} x_{i,0} & x_{i,1} \\ x_{i,1} & x_{i,2} \end{pmatrix}$ *in $\mathcal{P}$ and there exists $M \in \mathcal{P}$ with ${}^tM \neq \pm M$ such that*

Theorem (R. 2004)

A real number ξ is extremal

$\Longleftrightarrow$ *there exists a sequence of symmetric matrices* $\mathbf{x}_i = \begin{pmatrix} x_{i,0} & x_{i,1} \\ x_{i,1} & x_{i,2} \end{pmatrix}$ *in* $\mathcal{P}$ *and there exists* $M \in \mathcal{P}$ *with* ${}^tM \neq \pm M$ *such that*

1) $\mathbf{x}_{i+2} = \mathbf{x}_{i+1} * M_{i+1} * \mathbf{x}_i$ *where* $M_i = \begin{cases} M & \text{if } i \text{ is even,} \\ {}^tM & \text{else,} \end{cases}$

Theorem (R. 2004)

A real number ξ is extremal

$\Longleftrightarrow$ *there exists a sequence of symmetric matrices* $\mathbf{x}_i = \begin{pmatrix} x_{i,0} & x_{i,1} \\ x_{i,1} & x_{i,2} \end{pmatrix}$ *in* $\mathcal{P}$ *and there exists* $M \in \mathcal{P}$ *with* ${}^tM \neq \pm M$ *such that*

1) $\mathbf{x}_{i+2} = \mathbf{x}_{i+1} * M_{i+1} * \mathbf{x}_i$ *where* $M_i = \begin{cases} M & \text{if } i \text{ is even,} \\ {}^tM & \text{else,} \end{cases}$
2) $\|\mathbf{x}_{i+1}\| \asymp \|\mathbf{x}_i\|^\gamma$ *and* $\lim_{i\to\infty} \|\mathbf{x}_i\| = \infty$,
3) $\|(\xi, -1)\mathbf{x}_i\| \asymp \|\mathbf{x}_i\|^{-1}$,

Theorem (R. 2004)

A real number ξ *is extremal*

$\Longleftrightarrow$ *there exists a sequence of symmetric matrices* $\mathbf{x}_i = \begin{pmatrix} x_{i,0} & x_{i,1} \\ x_{i,1} & x_{i,2} \end{pmatrix}$ *in* $\mathcal{P}$ *and there exists* $M \in \mathcal{P}$ *with* ${}^tM \neq \pm M$ *such that*

1) $\mathbf{x}_{i+2} = \mathbf{x}_{i+1} * M_{i+1} * \mathbf{x}_i$ *where* $M_i = \begin{cases} M & \text{if } i \text{ is even,} \\ {}^tM & \text{else,} \end{cases}$
2) $\|\mathbf{x}_{i+1}\| \asymp \|\mathbf{x}_i\|^\gamma$ *and* $\lim_{i\to\infty} \|\mathbf{x}_i\| = \infty$,
3) $\|(\xi, -1)\mathbf{x}_i\| \asymp \|\mathbf{x}_i\|^{-1}$,
4) $1 \leq |\det(\mathbf{x}_i)| \ll 1$.

Theorem (R. 2004)

A real number ξ is extremal

$\Longleftrightarrow$ *there exists a sequence of symmetric matrices* $\mathbf{x}_i = \begin{pmatrix} x_{i,0} & x_{i,1} \\ x_{i,1} & x_{i,2} \end{pmatrix}$ *in $\mathcal{P}$ and there exists $M \in \mathcal{P}$ with ${}^tM \neq \pm M$ such that*

1) $\mathbf{x}_{i+2} = \mathbf{x}_{i+1} * M_{i+1} * \mathbf{x}_i$ *where* $M_i = \begin{cases} M & \text{if } i \text{ is even,} \\ {}^tM & \text{else,} \end{cases}$
2) $\|\mathbf{x}_{i+1}\| \asymp \|\mathbf{x}_i\|^{\gamma}$ *and* $\lim_{i\to\infty} \|\mathbf{x}_i\| = \infty$,
3) $\|(\xi, -1)\mathbf{x}_i\| \asymp \|\mathbf{x}_i\|^{-1}$,
4) $1 \leq |\det(\mathbf{x}_i)| \ll 1$.

Then $W_i := \mathbf{x}_i * M_i$ is a Fibonacci sequence in $\mathcal{P}$:

$$W_{i+2} = W_{i+1} * W_i \quad \text{for all } i \geq 1.$$

Fibonacci continued fractions

Definition

- $E^* :=$ monoid of words on $\mathbb{N} \setminus \{0\}$
- Given any non-commuting words w_1, w_2 in E^* the Fibonacci sequence $(w_i)_{i \geq 1}$ defined recursively by $w_{i+1} = w_i w_{i-1}$ for all $i \geq 2$ converges to an infinite ultimately non-periodic word w_∞ called a *generalized Fibonacci word* on $\mathbb{N} \setminus \{0\}$.

Fibonacci continued fractions

Definition

- E^* := monoid of words on $\mathbb{N} \setminus \{0\}$
- Given any non-commuting words w_1, w_2 in E^* the Fibonacci sequence $(w_i)_{i\geq 1}$ defined recursively by $w_{i+1} = w_i w_{i-1}$ for all $i \geq 2$ converges to an infinite ultimately non-periodic word w_∞ called a *generalized Fibonacci word* on $\mathbb{N} \setminus \{0\}$.

Example

$$w_1 = 3,\ \ w_2 = (1,2) \implies w_3 = (1,2,3),\ w_4 = (1,2,3,1,2),\ \dots$$
$$\text{converges to } w_\infty = (1,2,3,1,2,1,2,3,\dots).$$

Fibonacci continued fractions

Definition

- E^* := monoid of words on $\mathbb{N} \setminus \{0\}$
- Given any non-commuting words w_1, w_2 in E^* the Fibonacci sequence $(w_i)_{i \geq 1}$ defined recursively by $w_{i+1} = w_i w_{i-1}$ for all $i \geq 2$ converges to an infinite ultimately non-periodic word w_∞ called a *generalized Fibonacci word* on $\mathbb{N} \setminus \{0\}$.

Example

$$w_1 = 3, \; w_2 = (1,2) \implies w_3 = (1,2,3), \; w_4 = (1,2,3,1,2), \ldots$$

converges to $w_\infty = (1,2,3,1,2,1,2,3,\ldots)$.

Theorem (R. 2008)

A real number ξ is extremal with associated sequence $(W_i)_{i \geq 1}$ in $GL_2(\mathbb{Z}) \iff$ the continued fraction expansion of ξ coincides with a generalized Fibonacci word on $\mathbb{N} \setminus \{0\}$, up to its first terms.

A special class of extremal numbers

Definition

For each $a = 1, 2, 3, \dots$, let $\mathcal{E}_a$ (resp. $\mathcal{E}_a^+$) denote the set of extremal numbers

- with associated sequence $(\mathbf{x}_i)_{i \geq 1}$ in $\mathrm{GL}_2(\mathbb{Z})$ (resp. in $\mathrm{SL}_2(\mathbb{Z})$)
- and associated matrix $M = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}$.

A special class of extremal numbers

Definition

For each $a = 1, 2, 3, \dots$, let $\mathcal{E}_a$ (resp. $\mathcal{E}_a^+$) denote the set of extremal numbers

- with associated sequence $(\mathbf{x}_i)_{i\geq 1}$ in $\mathrm{GL}_2(\mathbb{Z})$ (resp. in $\mathrm{SL}_2(\mathbb{Z})$)
- and associated matrix $M = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}$.

Theorem (R., 2003)

Let $\xi \in \mathcal{E}_a$ for some $a \geq 1$. Then

$$|\xi - \alpha| \geq c_6 H(\alpha)^{-\gamma^2}$$

for any cubic algebraic integer α.

$\Longrightarrow$ Davenport and Schmidt 1969 general result of approximation to real numbers by cubic algebraic integers is optimal.

Link with Markoff's equation

Fix $\xi \in \mathcal{E}_a^+$ for some integer $a \geq 1$.

Let $\begin{pmatrix} x_{i,0} & x_{i,1} \\ x_{i,1} & x_{i,2} \end{pmatrix}$ $(i \geq 1)$ be the associated sequence of symmetric matrices in $\mathrm{SL}_2(\mathbb{Z})$.

Then, for each $i \geq 1$, we have

1) $x_{i+2,0}^2 + x_{i+1,0}^2 + x_{i,0}^2 = a x_{i+2,0} x_{i+1,0} x_{i,0}$
2) $x_{i+3,0} = a x_{i+2,0} x_{i+1,0} - x_{i,0}$,
3) $x_{i+2,1} \equiv (-1)^i \frac{x_{i,0}}{x_{i+1,0}} \mod x_{i+2,0}$.

Relation 1) comes from Fricke's identity (ref. [Cohn, 1957]).

Link with Markoff's equation

Fix $\xi \in \mathcal{E}_a^+$ for some integer $a \geq 1$.

Let $\begin{pmatrix} x_{i,0} & x_{i,1} \\ x_{i,1} & x_{i,2} \end{pmatrix}$ $(i \geq 1)$ be the associated sequence of symmetric matrices in $\mathrm{SL}_2(\mathbb{Z})$.

Then, for each $i \geq 1$, we have

1) $x_{i+2,0}^2 + x_{i+1,0}^2 + x_{i,0}^2 = a x_{i+2,0} x_{i+1,0} x_{i,0}$
2) $x_{i+3,0} = a x_{i+2,0} x_{i+1,0} - x_{i,0}$,
3) $x_{i+2,1} \equiv (-1)^i \frac{x_{i,0}}{x_{i+1,0}} \mod x_{i+2,0}$.

Relation 1) comes from Fricke's identity (ref. [Cohn, 1957]).

- 1) is impossible unless $a = 1$ or $a = 3$ (Hurwitz, 1907)
- Case $a = 1$ also leads to a contradiction.

Link with Markoff's equation

Fix $\xi \in \mathcal{E}_a^+$ for some integer $a \geq 1$.

Let $\begin{pmatrix} x_{i,0} & x_{i,1} \\ x_{i,1} & x_{i,2} \end{pmatrix}$ $(i \geq 1)$ be the associated sequence of symmetric matrices in $\mathrm{SL}_2(\mathbb{Z})$.

Then, for each $i \geq 1$, we have

1) $x_{i+2,0}^2 + x_{i+1,0}^2 + x_{i,0}^2 = a x_{i+2,0} x_{i+1,0} x_{i,0}$
2) $x_{i+3,0} = a x_{i+2,0} x_{i+1,0} - x_{i,0}$,
3) $x_{i+2,1} \equiv (-1)^i \frac{x_{i,0}}{x_{i+1,0}} \mod x_{i+2,0}$.

Relation 1) comes from Fricke's identity (ref. [Cohn, 1957]).

- 1) is impossible unless $a = 1$ or $a = 3$ (Hurwitz, 1907)
- Case $a = 1$ also leads to a contradiction.
- Thus $a = 3$ and $(x_{i+2,0}, x_{i+1,0}, x_{i,0})$ is a solution of Markoff's equation for each $i \geq 1$.

Main result

Theorem

There is a bijection between the positive solutions of the Markoff equation except $(1, 1, 1)$ *and the extremal numbers of* $\mathcal{E}_3^+$ *in the interval* $(0,\ 1/2)$*. It is given by*

$$\xi \longmapsto \textit{initial triple } (x_{3,0} > x_{2,0} > x_{1,0}) \textit{ from a zig-zag path}$$

Starting from $(2, 1, 1)$

(2,1,1)

(5,1,2)

(29,5,2)

(433,5,29)

(37666,433,29)

$\vdots$

Main result

Theorem

There is a bijection between the positive solutions of the Markoff equation except $(1,1,1)$ *and the extremal numbers of* $\mathcal{E}_3^+$ *in the interval* $(0,\ 1/2)$*. It is given by*

$$\xi \longmapsto \text{initial triple } (x_{3,0} > x_{2,0} > x_{1,0}) \text{ from a zig-zag path}$$

Starting from $(2,1,1)$

(2,1,1)

(5,1,2)

(29,5,2)

(433,5,29)

(37666,433,29)

⋮

Starting from $(5,1,2)$

(5,1,2)

(13,1,5)

(194,13,5)

(7553,13,194)

⋮

Work in progress

- All extremal numbers ξ in $\mathcal{E}_3^+$ have $\nu(\xi) = 1/3$.
- Good hopes to show that they are the only extremal numbers with $\nu(\xi) = 1/3$.
- Connections with the works of Bugeaud-Laurent and Fischler.

Associated tree of symmetric matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 13 & 5 \\ 5 & 2 \end{pmatrix} \qquad \begin{pmatrix} 29 & 12 \\ 12 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 34 & 13 \\ 13 & 5 \end{pmatrix} \quad \begin{pmatrix} 194 & 75 \\ 75 & 29 \end{pmatrix} \quad \begin{pmatrix} 433 & 179 \\ 179 & 74 \end{pmatrix} \quad \begin{pmatrix} 169 & 70 \\ 70 & 29 \end{pmatrix}$$

Associated tree of palindromes: