

Simultaneous rational approximation and the Markoff spectrum

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The Lagrange spectrum

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For each $\xi \in \mathbb{R}$, define

$$\nu(\xi) := \liminf_{\mathbb{N}^* \ni q \rightarrow \infty} q \|q\xi\|$$

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Property: $\nu\left(\frac{a\xi + b}{c\xi + d}\right) = \nu(\xi)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z})$.

Link with continued fractions

Assume that $\xi \notin \mathbb{Q}$ (otherwise $\nu(\xi) = 0$).

$$\text{Write } \xi = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}} = [a_0, a_1, a_2, \dots]$$

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Corollary

$$\nu(\xi) > 0 \iff (a_k)_{k \geq 0} \text{ is a bounded sequence.}$$

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- This is a tool for investigating the Lagrange spectrum \mathcal{L} .

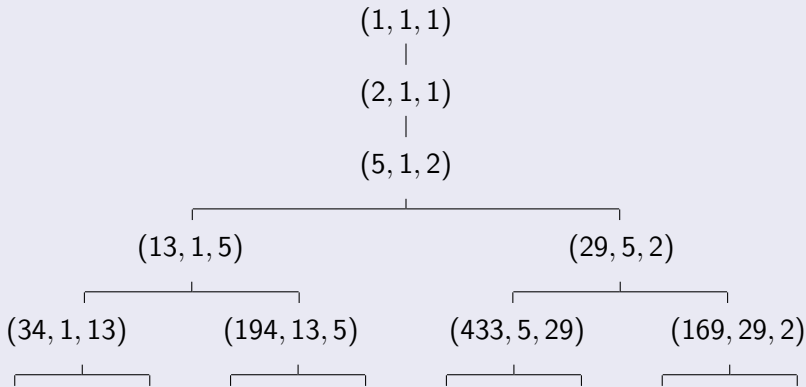
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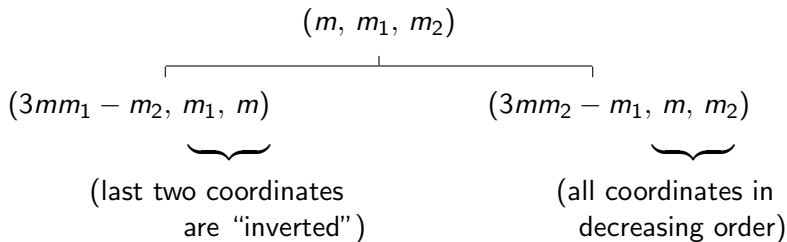
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Solutions $(m, m_1, m_2) \in (\mathbb{N}^*)^3$ with $m \geq m_1, m_2$ form a tree :



$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2$$

The recurrence is



$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2$$

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Then $\ell := \frac{k^2 + 1}{m} \in \mathbb{Z}$.

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- The quadratic form

$$\begin{aligned} F &:= mx^2 + (3m - 2k)xy + (\ell - 3k)y^2 \\ &= m(x - \beta y)(x - \beta' y) \end{aligned}$$

has equivalent roots $\beta \sim \beta'$ (under $\text{GL}_2(\mathbb{Z})$) with

$$\nu(\beta) = \nu(\beta') = \frac{1}{\sqrt{9 - 4m^{-2}}} > \frac{1}{3}.$$

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Corollary

$$\mathcal{L} \cap (1/3, \infty) = \{\mu_1 > \mu_2 > \mu_3 > \dots\} \text{ with } \lim \mu_i = 1/3.$$

$$\nu(\xi) = \mu_i \iff \xi \text{ belongs to a finite set of equivalence classes.}$$

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Theorem (Markoff, 1871)

There are uncountably many $\xi \in \mathbb{R}$ such that $\nu(\xi) = 1/3$.

Work of Harvey Cohn provides link with

1955: fundamental domains of genus one in the upper half plane

1971: geodesics on a perforated torus

1972: primitive words in F_2

Extremal numbers

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Definition

We say that $\xi \in \mathbb{R}$ is *extremal* if $1, \xi, \xi^2$ are linearly independent over \mathbb{Q} and there exists a constant $c_1 > 0$ such that

$$|x_0| \leq X, \quad |x_0\xi - x_1| \leq c_1 X^{-1/\gamma}, \quad |x_0\xi^2 - x_2| \leq c_1 X^{-1/\gamma}$$

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- R., 2004: There are countably many extremal numbers.

Properties (R., 2004)

Let $\xi \in \mathbb{R}$ be extremal.

(i) $\frac{a\xi + b}{c\xi + d}$ is extremal for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q})$

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- (ii) There are infinitely many quadratic numbers $\alpha \in \mathbb{R}$ such that

$$|\xi - \alpha| \leq c_2 H(\alpha)^{-2\gamma^2}$$

and any quadratic α has $|\xi - \alpha| \geq c_3 H(\alpha)^{-2\gamma^2}$.

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(iii) $\left| \xi - \frac{p}{q} \right| \geq c_4 q^{-2} (1 + \log q)^{-c_5}$ for all $\frac{p}{q} \in \mathbb{Q}$, $q \geq 1$.

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Definition

$$\mathcal{P} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}_{2 \times 2}(\mathbb{Z}); ad - bc \neq 0, \gcd(a, b, c, d) = 1 \right\}$$

is a group under $A * B = \gcd(AB)^{-1} AB$.

Theorem (R. 2004)

A real number ξ is extremal

\iff *there exists a sequence of symmetric matrices*

$\mathbf{x}_i = \begin{pmatrix} x_{i,0} & x_{i,1} \\ x_{i,1} & x_{i,2} \end{pmatrix}$ *in \mathcal{P} and there exists $M \in \mathcal{P}$ with*

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1) $\mathbf{x}_{i+2} = \mathbf{x}_{i+1} * M_{i+1} * \mathbf{x}_i$ where $M_i = \begin{cases} M & \text{if } i \text{ is even,} \\ {}^t M & \text{else,} \end{cases}$

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- 2) $\|\mathbf{x}_{i+1}\| \asymp \|\mathbf{x}_i\|^\gamma$ and $\lim_{i \rightarrow \infty} \|\mathbf{x}_i\| = \infty$,
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Then $W_i := \mathbf{x}_i * M_i$ is a Fibonacci sequence in \mathcal{P} :

$$W_{i+2} = W_{i+1} * W_i \quad \text{for all } i \geq 1.$$

Fibonacci continued fractions

Definition

- E^* := monoid of words on $\mathbb{N} \setminus \{0\}$
- Given any non-commuting words w_1, w_2 in E^* the Fibonacci sequence $(w_i)_{i \geq 1}$ defined recursively by $w_{i+1} = w_i w_{i-1}$ for all $i \geq 2$ converges to an infinite ultimately non-periodic word w_∞ called a *generalized Fibonacci word* on $\mathbb{N} \setminus \{0\}$.

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Example

$w_1 = 3, w_2 = (1, 2) \implies w_3 = (1, 2, 3), w_4 = (1, 2, 3, 1, 2), \dots$
converges to $w_\infty = (1, 2, 3, 1, 2, 1, 2, 3, \dots)$.

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Theorem (R. 2008)

A real number ξ is extremal with associated sequence $(W_i)_{i \geq 1}$ in $GL_2(\mathbb{Z}) \iff$ the continued fraction expansion of ξ coincides with a generalized Fibonacci word on $\mathbb{N} \setminus \{0\}$, up to its first terms.

A special class of extremal numbers

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For each $a = 1, 2, 3, \dots$, let \mathcal{E}_a (resp. \mathcal{E}_a^+) denote the set of extremal numbers

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Theorem (R., 2003)

Let $\xi \in \mathcal{E}_a$ for some $a \geq 1$. Then

$$|\xi - \alpha| \geq c_6 H(\alpha)^{-\gamma^2}$$

for any cubic algebraic integer α .

\implies Davenport and Schmidt 1969 general result of approximation to real numbers by cubic algebraic integers is optimal.

Link with Markoff's equation

Fix $\xi \in \mathcal{E}_a^+$ for some integer $a \geq 1$.

Let $\begin{pmatrix} x_{i,0} & x_{i,1} \\ x_{i,1} & x_{i,2} \end{pmatrix}$ ($i \geq 1$) be the associated sequence of symmetric matrices in $SL_2(\mathbb{Z})$.

Then, for each $i \geq 1$, we have

$$1) \quad x_{i+2,0}^2 + x_{i+1,0}^2 + x_{i,0}^2 = ax_{i+2,0}x_{i+1,0}x_{i,0}$$

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Relation 1) comes from Fricke's identity (ref. [Cohn, 1957]).

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- Case $a = 1$ also leads to a contradiction.
- Thus $a = 3$ and $(x_{i+2,0}, x_{i+1,0}, x_{i,0})$ is a solution of Markoff's equation for each $i \geq 1$.

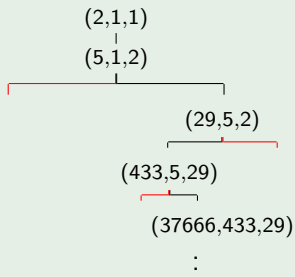
Main result

Theorem

There is a bijection between the positive solutions of the Markoff equation except $(1, 1, 1)$ and the extremal numbers of \mathcal{E}_3^+ in the interval $(0, 1/2)$. It is given by

$$\xi \longmapsto \text{initial triple } (x_{3,0} > x_{2,0} > x_{1,0}) \text{ from a zig-zag path}$$

Starting from $(2, 1, 1)$



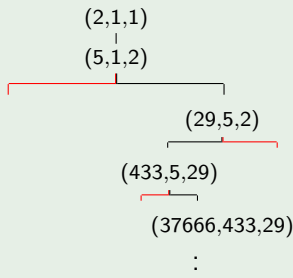
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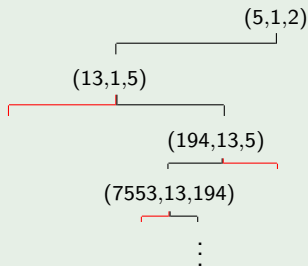
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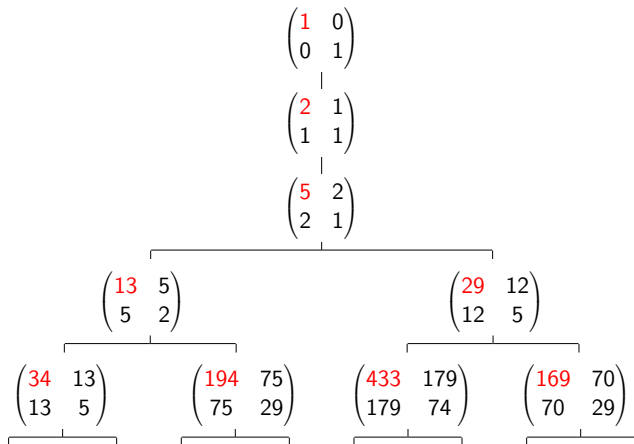


Starting from $(5, 1, 2)$



- All extremal numbers ξ in \mathcal{E}_3^+ have $\nu(\xi) = 1/3$.
- Good hopes to show that they are the only extremal numbers with $\nu(\xi) = 1/3$.
- Connections with the works of Bugeaud-Laurent and Fischler.

Associated tree of symmetric matrices



Associated tree of palindromes:

