# An example in multi-parametric geometry of numbers

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Diophantine approximation and related fields

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*Dedicated to Bertrand Russell*
*for his commitment to peace*

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# General framework (dimension 3)

Let $A = (a_{i,j}) \in \mathrm{GL}_3(\mathbb{R})$ and let $\mathbf{q} = (q_1, q_2, q_3) \in \mathbb{R}^3$. Consider the parallelepiped

$$\mathcal{C}_A(\mathbf{q}) \,:\, \begin{cases} |a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3| \le e^{-q_1} \\ |a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3| \le e^{-q_2} \\ |a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3| \le e^{-q_3} \end{cases}$$

For each $i = 1, 2, 3$, the logarithm of its $i$-th minimum with respect to $\mathbb{Z}^3$ is the smallest $t \in \mathbb{R}$, denoted $L_{A,i}(\mathbf{q})$ such that

$$e^t\mathcal{C}_A(\mathbf{q}) \,:\, \begin{cases} |a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3| \le e^{t-q_1} \\ |a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3| \le e^{t-q_2} \\ |a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3| \le e^{t-q_3} \end{cases}$$

contains at least $i$ linearly independent points $\mathbf{x}$ in $\mathbb{Z}^3$. We form the map

$$\begin{array}{rcl} \mathbf{L}_A \,:\, \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ \mathbf{q} & \longmapsto & (L_{A,1}(\mathbf{q}), L_{A,2}(\mathbf{q}), L_{A,3}(\mathbf{q})) \end{array}$$

? Determine $\mathbf{L}_A$ up to bounded error on $\mathbb{R}^3$.

? Characterize the set of all maps $\mathbf{L}_A$ modulo bounded functions on $\mathbb{R}^3$.

Since $\log \mathrm{vol}(\mathcal{C}_A(\mathbf{q})) = -(q_1 + q_2 + q_3) + \mathcal{O}_A(1)$, Minkowski's convex body theorem gives

$$L_{A,1}(\mathbf{q}) + L_{A,2}(\mathbf{q}) + L_{A,3}(\mathbf{q}) = q_1 + q_2 + q_3 + \mathcal{O}_A(1).$$

Since $L_{A,1}(\mathbf{q}) \leq L_{A,2}(\mathbf{q}) \leq L_{A,3}(\mathbf{q})$, we deduce that

$$L_{A,1}(\mathbf{q}) \leq (q_1 + q_2 + q_3)/3 + \mathcal{O}_A(1).$$

### Example

Let $K$ be a totally real cubic number field and let $A = (\sigma_i(\omega_j))$ where

- $(\omega_1, \omega_2, \omega_3)$ is a basis of the ring of integers of $K$,
- $(\sigma_1, \sigma_2, \sigma_3)\colon K \to \mathbb{R}^3$ is the canonical embedding of $K$.

Then

$$\sup_{\mathbf{q} \in \mathbb{R}^3} \max_{1 \leq i \leq 3} |L_{A,i}(\mathbf{q}) - (q_1 + q_2 + q_3)/3| < \infty.$$

# Specific framework

- We restrict to $A = \begin{pmatrix} 1 & 0 & 0 \\ \xi_1 & -1 & 0 \\ \xi_2 & 0 & -1 \end{pmatrix}$ where $\boldsymbol{\xi} = (1, \xi_1, \xi_2) \in \mathbb{R}^3$.
- Since $e^t \mathcal{C}_{\boldsymbol{\xi}}(q_1, q_2, q_3) = \mathcal{C}_{\boldsymbol{\xi}}(q_1 - t, q_2 - t, q_3 - t)$, we may fix $q_1 = 0$. (The standard normalization is $q_1 + q_2 + q_3 = 0$.)

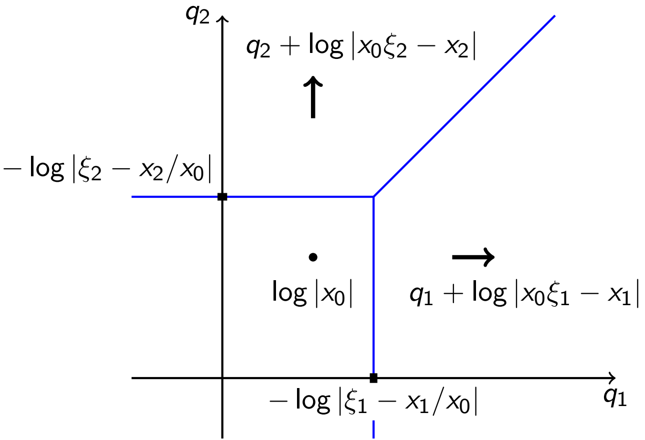
- Thus, we work with $\mathcal{C}_{\boldsymbol{\xi}}(q_1, q_2)$ : $\begin{cases} |x_0| \leq 1 \\ |x_0\xi_1 - x_1| \leq e^{-q_1} \\ |x_0\xi_2 - x_2| \leq e^{-q_2} \end{cases}$
- $L_{\boldsymbol{\xi},i}(q_1, q_2)$ = smallest $t \in \mathbb{R}$ for which the conditions

$$|x_0| \leq e^t, \quad |x_0\xi_1 - x_1| \leq e^{t-q_1}, \quad |x_0\xi_2 - x_2| \leq e^{t-q_2}$$

admit at least $i$ linearly independent solutions $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3$.

The trajectory of a non-zero point $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3$ is the map $L_{\mathbf{x}} \colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$\begin{aligned} L_{\mathbf{x}}(q_1, q_2) &= \text{the smallest } t \text{ such that } \mathbf{x} \in e^t \mathcal{C}_{\boldsymbol{\xi}}(q_1, q_2) \\ &= \max\{\log|x_0|, q_1 + \log|x_0\xi_1 - x_1|, q_2 + \log|x_0\xi_2 - x_2|\}. \end{aligned}$$




(if $x_0 \neq 0$)

# Basic tool

We denote by

$$\Phi\colon \mathbb{R}^3 \to \{(t_0, t_1, t_2) \in \mathbb{R}^3\,;\ t_0 \le t_1 \le t_2\}$$

the map that puts the coordinates of a point in non-decreasing order.

Let $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2$ and let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{Z}^3$ be **linearly independent**.

We have $\quad \mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) \le \Phi(L_{\mathbf{x}_1}(\mathbf{q}), L_{\mathbf{x}_2}(\mathbf{q}), L_{\mathbf{x}_3}(\mathbf{q})) \quad$ coordinate-wise,

$$\Rightarrow \ \left\|\Phi(L_{\mathbf{x}_1}(\mathbf{q}), L_{\mathbf{x}_2}(\mathbf{q}), L_{\mathbf{x}_3}(\mathbf{q})) - \mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q})\right\|_\infty \le \sum_{i=1}^{3} L_{\mathbf{x}_i}(\mathbf{q}) - \sum_{i=1}^{3} L_{\boldsymbol{\xi},i}(\mathbf{q})$$

$$= \sum_{i=1}^{3} L_{\mathbf{x}_i}(\mathbf{q}) - (q_1 + q_2) + \mathcal{O}_{\boldsymbol{\xi}}(1).$$

**Goal:** Find $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{Z}^3$ such that the last expression is $\mathcal{O}_{\boldsymbol{\xi}}(1)$

# Example

Let $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, $\mathbf{e}_3 = (0,0,1)$ denote the elements of the canonical basis of $\mathbb{Z}^3$. We find

$$\begin{aligned} L_{\mathbf{e}_1}(\mathbf{q}) &= \max\{0, q_1 + \log|\xi_1|, q_2 + \log|\xi_2|\}, \\ L_{\mathbf{e}_2}(\mathbf{q}) &= \max\{-\infty, q_1, q_2 - \infty\} = q_1, \\ L_{\mathbf{e}_3}(\mathbf{q}) &= \max\{-\infty, q_1 - \infty, q_2\} = q_2. \end{aligned}$$

If $q_1 \le 0$ and $q_2 \le 0$, then $L_{\mathbf{e}_1}(\mathbf{q}) = \mathcal{O}_{\boldsymbol{\xi}}(1)$, so

$$\sum_{i=1}^{3} L_{\mathbf{e}_i}(\mathbf{q}) = q_1 + q_2 + \mathcal{O}_{\boldsymbol{\xi}}(1).$$

Thus:

If $q_1 \le 0$ and $q_2 \le 0$, then $\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) = \Phi(q_1, q_2, 0) + \mathcal{O}_{\boldsymbol{\xi}}(1)$.

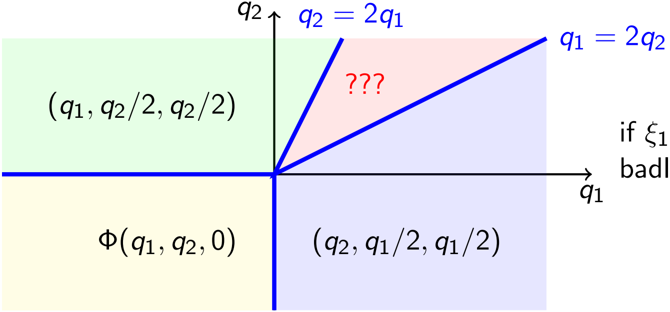
Suppose that $\xi_1$ is badly approximable

$$\text{i.e.} \quad \nu(\xi_1) := \liminf_{n\to\infty} n\|n\xi_1\| > 0,$$

then, for $\mathbf{q} = (q_1, q_2)$ with $q_1 \geq 0$ and $q_2 \leq q_1/2$, we have

$$\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) = (q_2, q_1/2, q_1/2) + \mathcal{O}_{\boldsymbol{\xi}}(1).$$

**Proof.** Take $\mathbf{x}_1 = (n, m, [n\xi_2])$, $\mathbf{x}_2 = (n', m', [n'\xi_2])$ and $\mathbf{x}_3 = (0, 0, 1)$ where $m/n$ and $m'/n'$ are consecutive convergents of $\xi_1$ with $\log n = q_1/2 + \mathcal{O}_{\xi_1}(1)$.




if $\xi_1$ and $\xi_2$ are badly appoximable

## Littlewood's conjecture

For any $\boldsymbol{\xi} = (1, \xi_1, \xi_2) \in \mathbb{R}^3$ and any $b > 0$, there exists $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2$ with $0 \leq q_1/2 \leq q_2 \leq 2q_1$ such that

$$L_{\boldsymbol{\xi},1}(\mathbf{q}) \leq (q_1 + q_2)/3 - b.$$

## Goal of this talk

Compute $\mathbf{L}_{\boldsymbol{\xi}}(q_1, q_2)$ up to bounded error in the region where $q_1 \geq 0$ and $q_2 \leq q_1$ for $\boldsymbol{\xi} = (1, \xi, \xi^2)$ where $\xi$ is an extremal real number.

The plan is to describe

I the particular extremal numbers $\xi$,

II the approximation function (2-parameter 3-system),

III the triple of points which approximate the three minima.

# Part I. The numbers.

## I.1. Notation.

- The norm $\|A\|$ of a matrix $A$ with real coefficients is the maximum of the absolute values of its entries.
- We identify each $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3$ with the symmetric matrix

$$\mathbf{x} = \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{Z}).$$

Then, for $\xi \in \mathbb{R}$, we have

$$\|(\xi, -1)\mathbf{x}\| = \max\{|x_0\xi - x_1|, \, |x_1\xi - x_2|\},$$

and

$$|x_0\xi^2 - x_2| = |x_1\xi - x_2| + \mathcal{O}(|x_0\xi - x_1|).$$

# I.2. Fibonacci sequences

**Definition.** A Fibonacci sequence in a monoid $\mathcal{M}$ is a sequence $(w_i)_{i\geq 1}$ in $\mathcal{M}$ such that $w_{i+2} = w_{i+1}w_i$ for each $i \geq 1$.

**Example 1.** The set $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ is a monoid under addition.
We denote by $(F_i)_{i\geq 1} = (1, 2, 3, 5, 8, \dots)$ the usual Fibonacci sequence in $\mathbb{N}$ with $F_1 = 1$, $F_2 = 2$ and $F_{i+2} = F_{i+1} + F_i$ for each $i \geq 1$.

**Example 2.** Let $E^* =$ monoid of words on an alphabet $E$.
The Fibonacci sequence in $\{a, b\}^*$ starting with $w_1 = a$ and $w_2 = ab$ has

$$w_3 = aba,\ w_4 = abaab,\ w_5 = abaababa,\ \dots$$

It converges to the Fibonacci word on $a, b$,

$$f_{a,b} := w_\infty := \lim_{i\to\infty} w_i = abaababaabaab\dots$$

The map $|\ |\colon \{a, b\}^* \to \mathbb{N}$ sending a word $w$ to its length $|w|$ is a morphism of monoids. We have $|w_i| = F_i$ for each $i \geq 1$.

## I.3. Extremal numbers

Set $\boxed{\gamma = (1+\sqrt{5})/2 \simeq 1.618} \quad \Rightarrow \quad 1/\gamma = \gamma - 1 \simeq 0.618$

**Theorem (Davenport and Schmidt, 1969)**

*Let $\xi \in \mathbb{R}$ such that $1, \xi, \xi^2$ are linearly independent over $\mathbb{Q}$. There exists $c_1 = c_1(\xi) > 0$ such that,* **for arbitrarily large values of $X$**, *the conditions*

$$|x_0| \leq X, \quad |x_0\xi - x_1| \leq c_1 X^{-1/\gamma}, \quad |x_0\xi^2 - x_2| \leq c_1 X^{-1/\gamma}$$

*have* **no non-zero solution** $\mathbf{x} = (x_0, x_1, x_2)$ *in $\mathbb{Z}^3$.*

**Corollary (Davenport and Schmidt, 1969)**

*For $\xi \in \mathbb{R}$ as above, there exists $c_2 = c_2(\xi) > 0$ such that*

$$|\xi - \alpha| \leq c_2 H(\alpha)^{-\gamma-1}$$

*for infinitely many algebraic integers $\alpha$ of degree at most 3 over $\mathbb{Q}$.*

Here, $H(\alpha) =$ maximum of the absolute values of the coefficients of the irreducible polynomial of $\alpha$ over $\mathbb{Z}$.

**Definition.** We say that a real number $\xi$ is **extremal** if $1, \xi, \xi^2$ are linearly independent over $\mathbb{Q}$ and if there exists a constant $c > 0$ such that, **for each large enough $X$**, the conditions

$$|x_0| \le X, \quad |x_0\xi - x_1| \le cX^{-1/\gamma}, \quad |x_0\xi^2 - x_2| \le cX^{-1/\gamma}$$

admit **a non-zero solution** $\mathbf{x} = (x_0, x_1, x_2)$ in $\mathbb{Z}^3$.

### Theorem (R. 2003)

*The set of extremal real numbers is countably infinite.*

- These numbers are transcendental over $\mathbb{Q}$.
- If $\xi$ is extremal and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$, then $A.\xi := \dfrac{a\xi + b}{c\xi + d}$ is extremal.
- **Open problem:** Are all extremal real numbers badly approximable?

# I.4. Extremal numbers of $\mathrm{SL}_2(\mathbb{Z})$-type [R. 2008]

- Let $M \in \mathrm{SL}_2(\mathbb{Z})$ with ${}^tM \neq \pm M$.
- For each $i \geq 1$, let $\mathbf{x}_i = \begin{pmatrix} x_{i,0} & x_{i,1} \\ x_{i,1} & x_{i,2} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ be symmetric.

Suppose that, for each $i \geq 1$,

- $\mathbf{x}_{i+2} = \mathbf{x}_{i+1} M_{i+1} \mathbf{x}_i = \mathbf{x}_i M_i \mathbf{x}_{i+1}$ where $M_i = \begin{cases} M & \text{if } i \text{ is even,} \\ {}^tM & \text{if } i \text{ is odd,} \end{cases}$
- $\|\mathbf{x}_{i+2}\| \asymp \|\mathbf{x}_{i+1}\| \, \|\mathbf{x}_i\| \to \infty$.

Then we have $\|\mathbf{x}_{i+1}\| \asymp \|\mathbf{x}_i\|^\gamma$ and there is an extremal $\xi \in \mathbb{R}$, said of $\mathrm{SL}_2(\mathbb{Z})$-type, such that

$$\|(\xi, -1)\mathbf{x}_i\| \asymp \|\mathbf{x}_i\|^{-1} \asymp \|\mathbf{x}_{i+1}\|^{-1/\gamma}.$$

- The products $W_i = \mathbf{x}_i M_i$ form a Fibonacci sequence in $\mathrm{SL}_2(\mathbb{Z})$, i.e.

$$W_{i+2} = W_{i+1} W_i \quad \text{for each } i \geq 1.$$

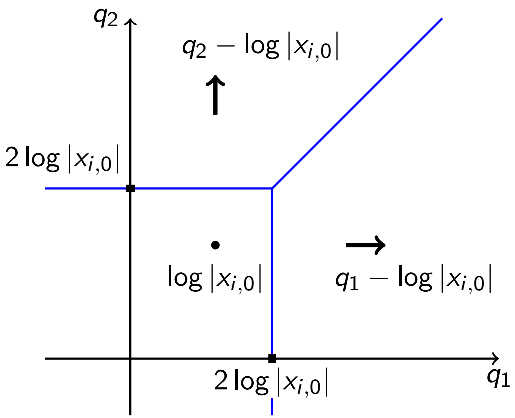
- The number $\xi$ is badly approximable.

## I.5. Trajectory of $\mathbf{x}_i$

It can be shown that, for each $i \geq 1$,

$$|x_{i,0}\xi - x_{i,1}| \asymp |x_{i,0}\xi^2 - x_{i,2}| \asymp |x_{i,0}|^{-1},$$

$$\Rightarrow \boxed{L_{\mathbf{x}_i}(q_1, q_2) = \max\{\log|x_{i,0}|, q_1 - \log|x_{i,0}|, q_2 - \log|x_{i,0}|\} + \mathcal{O}_\xi(1).}$$



# I.6. Choice of $\xi$

## Objective

We want to estimate $\mathbf{L}_{\boldsymbol{\xi}}$ for points $\boldsymbol{\xi} = (1, \xi, \xi^2)$ where $\xi$ is an extremal real number of $\mathrm{SL}_2(\mathbb{Z})$-type with associated matrix

$$M = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}.$$

## Some properties of such $\xi$

- There exists $c = c(\xi) > 0$ such that $|\xi - \alpha| \geq cH(\alpha)^{-\gamma-1}$ for each algebraic integer $\alpha$ of degree $\leq 3$ [R. 2003].
- Similar results for approximation by algebraic integers of degree $\leq 4$ and trace 0, etc [R.-Zelo 2011].
- $\liminf_{n\to\infty} n\|n\xi\| = 1/3$ is largest possible for a non-quadratic irrational number [R. 2011].

# A specific example

For $M$ as above, the sequence $(\mathbf{x}_i)_{i\geq 1}$ in $\mathrm{SL}_2(\mathbb{Z})$ starting with

$$\mathbf{x}_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},\ \mathbf{x}_2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix},\ \mathbf{x}_3 = \begin{pmatrix} 29 & 17 \\ 17 & 10 \end{pmatrix},\ \mathbf{x}_4 = \begin{pmatrix} 433 & 254 \\ 254 & 149 \end{pmatrix},\ \dots$$

produces an extremal $\xi \in \mathbb{R}$ such that

$$\|(\xi, -1)\mathbf{x}_i\| \asymp \|\mathbf{x}_i\|^{-1} \asymp \|\mathbf{x}_{i+1}\|^{-1/\gamma}.$$

Setting $\mathbf{1} = (1,1)$ and $\mathbf{2} = (2,2)$, its continued fraction expansion is

$$\xi = [0, 1, 1, 2, 2, 1, 1, 2, 2, 2, 2, 1, 1, 2, 2, \dots] = [0, \mathbf{1}, f_{\mathbf{2},\mathbf{1}}]$$

where $f_{\mathbf{2},\mathbf{1}}$ is the Fibonacci word on $\mathbf{2}$ and $\mathbf{1}$.

# Part II. The function.

## II.1. The maps $\iota$ and $\alpha$.

**Definition.** Set

$$\mathcal{F} = \{0, F_1, F_2, F_3, \dots\} = \{0, 1, 2, 3, 5, 8, 13, 21, \dots\}.$$

We define a map $\iota \colon \mathbb{N} \to \mathbb{N}$ by

$$\iota(x) = \begin{cases} x & \text{if } x \in \mathcal{F}, \\ x - 2F_{k-2} & \text{if } F_k < x < F_{k+1} \text{ for some } k \geq 3. \end{cases}$$

We also define $\alpha \colon \mathbb{N} \to \mathcal{F}$ by

$$\alpha(x) = \lim_{k \to \infty} \iota^k(x)$$

where $\iota^k$ denotes the $k$-th iterate of $\iota$.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\iota(x)$ | 0 | 1 | 2 | 3 | 2 | 5 | 2 | 3 | 8 | 3 | 4 | 5 | 6 | 13 |
| $\alpha(x)$ | 0 | 1 | 2 | 3 | 2 | 5 | 2 | 3 | 8 | 3 | 2 | 5 | 2 | 13 |

## II.2. The sets $\mathcal{N}_\ell$

**Definition.** For each integer $\ell \geq 1$, we set
$$\mathcal{N}_\ell = \{x \in \mathbb{N}\,;\, \alpha(x) \geq F_\ell\}.$$

Set $F_0 = 1$. For $\ell \geq 2$, we find

$$\mathcal{N}_\ell = \{F_\ell,\ F_{\ell+1},\ F_{\ell+1} + F_{\ell-1},\ F_{\ell+2},\ F_{\ell+2} + F_{\ell-2},\ F_{\ell+2} + F_\ell,\ F_{\ell+3}, \dots\}$$

**Theorem**

*Let $\ell \geq 3$ and write $\mathcal{N}_\ell = \{x_1 < x_2 < x_3 < \dots\}$.*

- *Then $x_1 = F_\ell$ and the sequence of differences*

$$(x_2 - x_1,\ x_3 - x_2,\ x_4 - x_3, \dots) = (F_{\ell-1},\ F_{\ell-1},\ F_{\ell-2},\ F_{\ell-2},\ F_{\ell-1},\ F_{\ell-1}, \dots)$$

  *is the infinite Fibonacci word on $(F_{\ell-1},\ F_{\ell-1})$ and $(F_{\ell-2},\ F_{\ell-2})$.*

- *For each $i \geq 1$, we have $\alpha(x_i) = F_\ell$ if and only if $i = 1$ or $i > 1$ and $(x_{i-1}, x_i, x_{i+1})$ is not an arithmetic progression.*

# II.3. Illustration

We find

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x \in \mathcal{N}_3$ | 3 | 5 | 7 | 8 | 9 | 11 | 13 | 15 | 17 | 18 | 19 | 21 | 23 | 24 | 26 |
| diff. | | 2 | 2 | 1 | 1 | 2 | 2 | 2 | 2 | 1 | 1 | 2 | 2 | 1 | 1 |
| $x \in \mathcal{N}_4$ | | 5 | | 8 | | 11 | 13 | 15 | | 18 | | 21 | | 24 | |
| diff. | | | | 3 | | 3 | 2 | 2 | | 3 | | 3 | | 3 | |
| $x \in \mathcal{N}_5$ | | | | 8 | | | 13 | | | 18 | | 21 | | 24 | |

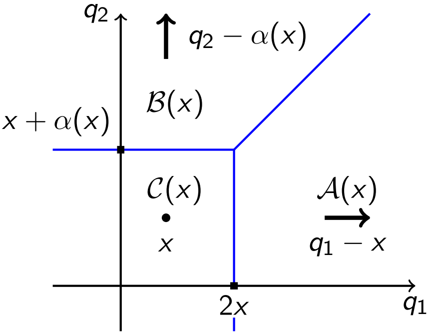
# II.4. The ideal trajectories $P_x$

For each $x \in \mathbb{N}$, we define $P_x \colon \mathbb{R}^2 \to \mathbb{R}^3$ by

$$P_x(q_1, q_2) = \Phi(q_1 - x, q_2 - \alpha(x), x)$$

We set

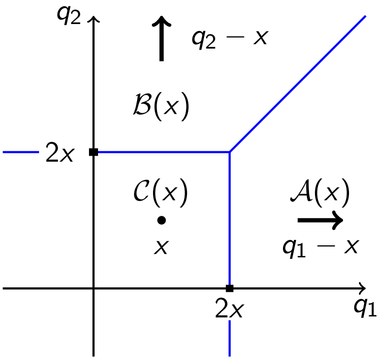
- $\mathcal{A}(x) = \{\mathbf{q} \in \mathbb{R}^2\,;\, P_x(\mathbf{q}) = q_1 - x\}$,
- $\mathcal{B}(x) = \{\mathbf{q} \in \mathbb{R}^2\,;\, P_x(\mathbf{q}) = q_2 - \alpha(x)\}$,
- $\mathcal{C}(x) = \{\mathbf{q} \in \mathbb{R}^2\,;\, P_x(\mathbf{q}) = x\}$.



For example, if $x \in \mathcal{F} = \{0, 1, 2, 3, 5, 8, \dots\}$, then $\alpha(x) = x$, so

$$P_x(q_1, q_2) = \Phi(q_1 - x, q_2 - x, x)$$

We find




In particular,

$$\mathcal{A}(0) = \{(q_1, q_2) \in \mathbb{R}^2;\ q_1 \geq \max\{0, q_2\}\}.$$

## II.5. The map $\mathbf{P}\colon \mathcal{A}(0) \to \mathbb{R}^3$

- Let $y \in \mathbb{N} \setminus \mathcal{F}$. Write $\alpha(y) = F_\ell$. Then, $\ell \geq 2$ and there are unique $x, z \in \mathcal{N}_\ell$ such that $x < y < z$ are consecutive in $\mathcal{N}_\ell$. The set

$$\mathrm{Cell}(y) := \mathcal{A}(x) \cap \mathcal{B}(y) \cap \mathcal{C}(z)$$

  is a compact polygon with 4 or 5 sides. For each $\mathbf{q} = (q_1, q_2) \in \mathrm{Cell}(y)$, we define

$$\mathbf{P}(\mathbf{q}) = \Phi(P_x(\mathbf{q}), P_y(\mathbf{q}), P_{\mathbf{z}}(\mathbf{q})) = \Phi(q_1 - x, q_2 - \alpha(y), z).$$

- Let $x \in \mathbb{N}$. Then

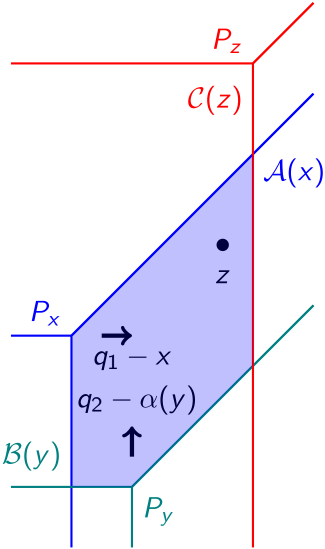
$$\mathrm{Trap}(x) := \mathcal{A}(x) \cap \mathcal{C}(x+1)$$

  is a trapeze unbound with two unbounded vertical sides. For each $\mathbf{q} = (q_1, q_2) \in \mathrm{Trap}(x)$, we define

$$\mathbf{P}(\mathbf{q}) = \Phi(P_x(\mathbf{q}), q_2 - 1, P_{x+1}(\mathbf{q})) = \Phi(q_1 - x, q_2 - 1, x + 1).$$

- These polygons have no interior point in common and cover $\mathcal{A}(0)$. The definition of $\mathbf{P}$ match at common boundary points yielding a continous map $\mathbf{P}\colon \mathcal{A}(0) \to \mathbb{R}^3$.

# Typical cell : $\text{Cell}(y) = \mathcal{A}(x) \cap \mathcal{B}(y) \cap \mathcal{C}(z)$

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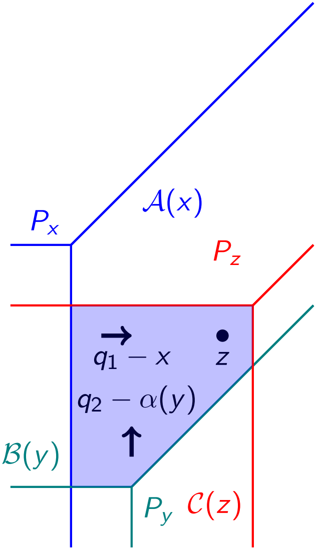
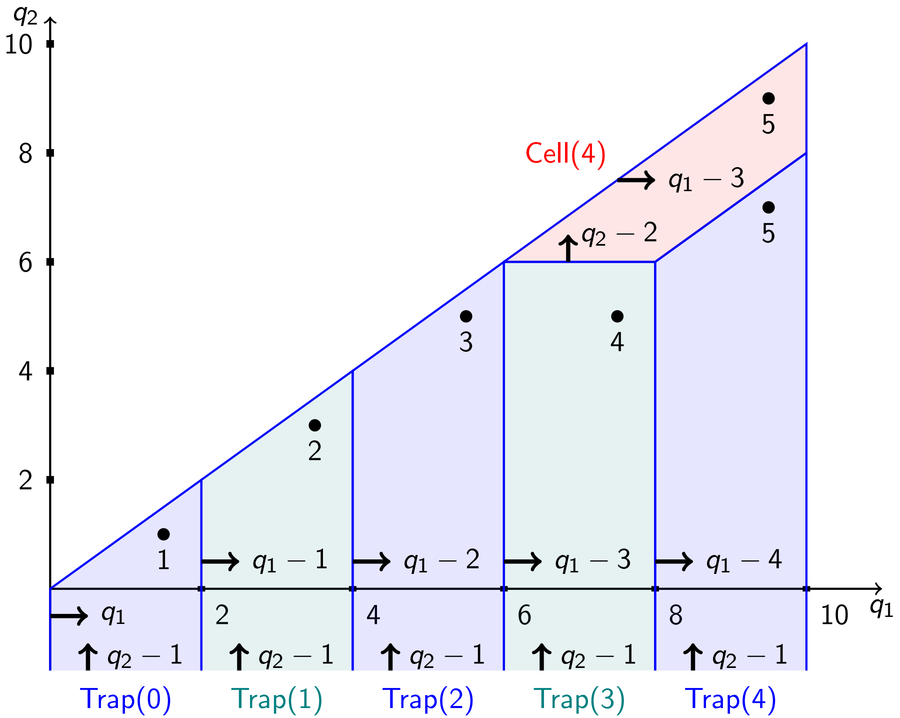
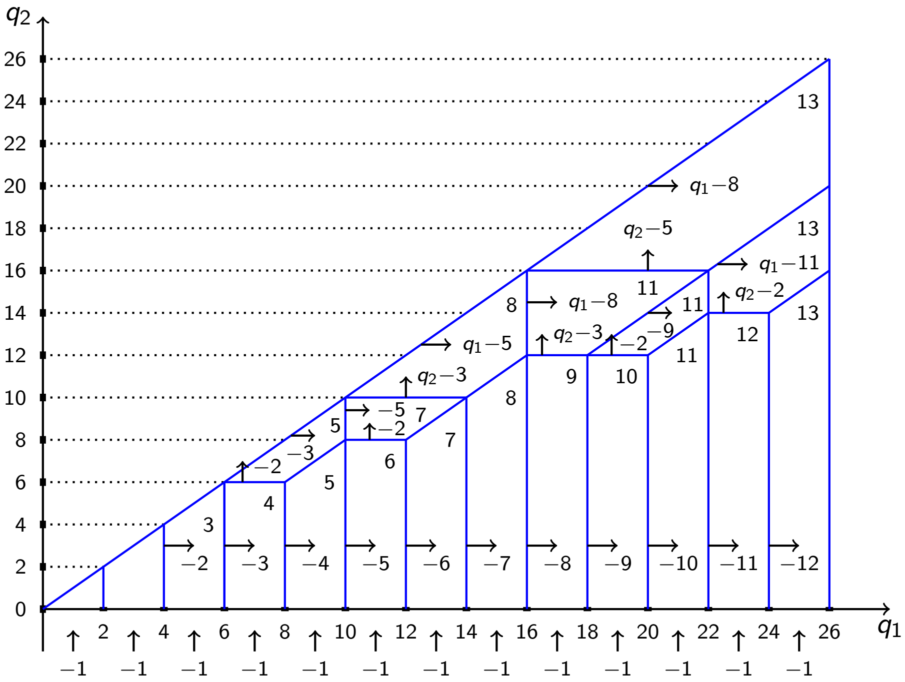


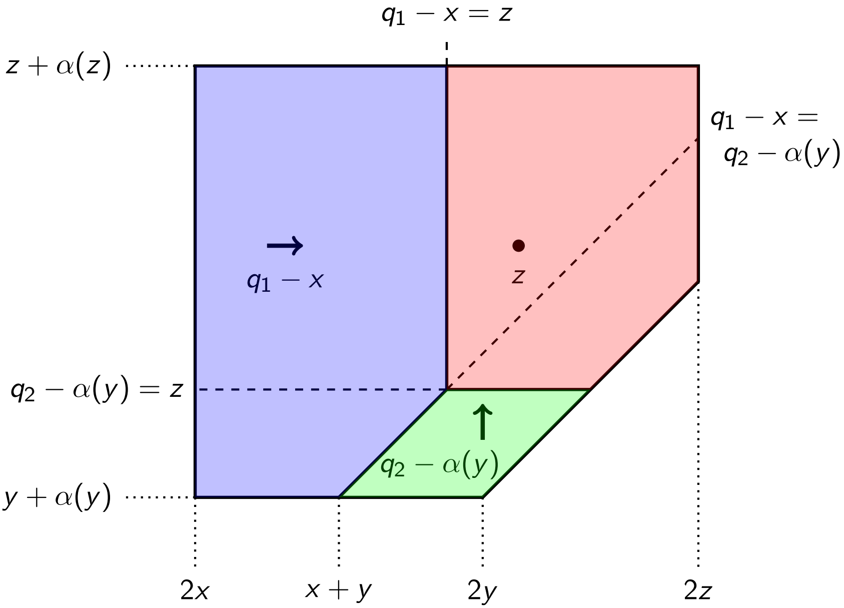


$q_2$
10
8
6
4
2
$q_1$
2
4
6
8
10
Cell(4)
$q_1 - 3$
$q_2 - 2$
5
5
3
4
2
1
$q_1 - 1$
$q_1 - 2$
$q_1 - 3$
$q_1 - 4$
$q_1$
$q_2 - 1$
$q_2 - 1$
$q_2 - 1$
$q_2 - 1$
$q_2 - 1$
Trap(0)
Trap(1)
Trap(2)
Trap(3)
Trap(4)



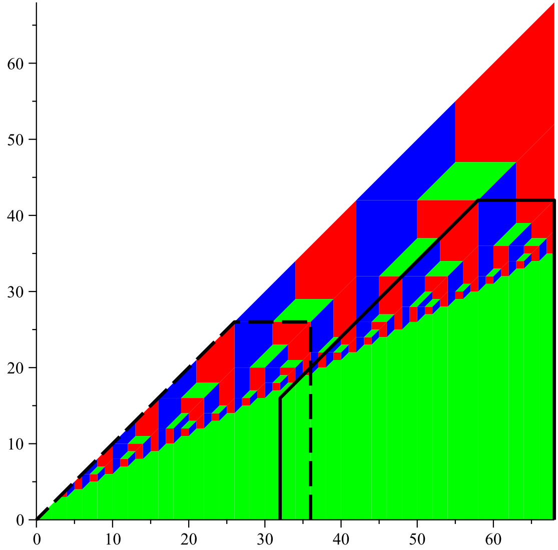
$q_2$
26
24
22
20
18
16
14
12
10
8
6
4
2
0
$q_1$
2
4
6
8
10
12
14
16
18
20
22
24
26
−1
−1
−1
−1
−1
−1
−1
−1
−1
−1
−1
−1
−1
−2
−3
−4
−5
−6
−7
−8
−9
−10
−11
−12
3
4
5
5
6
7
7
8
8
9
10
11
11
11
12
13
13
13
−2
−3
−5
−2
$q_2-3$
$q_1-5$
$q_1-8$
$q_2-3$
−2
−9
$q_2-5$
$q_1-8$
$q_1-11$
$q_2-2$

## II.7. Graph of $P_1$ inside Cell$(y) = \mathcal{A}(x) \cap \mathcal{B}(y) \cap \mathcal{C}(z)$



# Graph of $P_1$

blue: gradient $(1,0)$, green: gradient $(0,1)$, red: gradient $(0,0)$.



# II.8. Properties

**Theorem**

- *For each integer* $k \geq 3$*, each* $(q_1, q_2) \in \mathcal{A}(0) \cap \mathcal{C}(F_{k-1} + F_{k-3})$*, and each* $j = 1, 2, 3$*, we have*

$$P_j(q_1 + 4F_{k-2}, q_2 + 2F_{k-2}) = P_j(q_1, q_2) + 2F_{k-2}.$$

- *For each* $\mathbf{q} \in \mathcal{A}(0)$*, we have*

$$\mathbf{P}(\gamma\mathbf{q}) = \gamma\mathbf{P}(\mathbf{q}) + \mathcal{O}(1).$$

# II.9. Main result

**Theorem**

*Let* $\xi$ *be the particular extremal real number defined in Part I. There exists* $\rho > 0$ *such that the point* $\boldsymbol{\xi} = (1, \xi, \xi^2)$ *satisfies*

$$\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) = \rho \mathbf{P}(\rho^{-1}\mathbf{q}) + \mathcal{O}(1)$$

*for each* $\mathbf{q} = (q_1, q_2) \in \mathcal{A}(0)$.

# Part III. The points

## III.1. The Fibonacci sequence in $\mathrm{SL}_2(\mathbb{Z})$ attached to $\xi$

Recall that the sequence of symmetric matrices $(\mathbf{x}_i)_{i\geq 1}$ in $\mathrm{SL}_2(\mathbb{Z})$ attached to $\xi$ is given by

$$\mathbf{x}_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \ \mathbf{x}_2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}, \ \mathbf{x}_{i+2} = \mathbf{x}_{i+1} M_{i+1} \mathbf{x}_i \quad (i \geq 1)$$

where

$$M_i = \begin{pmatrix} 3 & (-1)^i \\ (-1)^{i+1} & 0 \end{pmatrix} \quad (i \geq 1).$$

Set $W_i = x_i M_i$ for each $i \geq 1$. We find

$$W_1 = \begin{pmatrix} 7 & -2 \\ 4 & -1 \end{pmatrix}, \ W_2 = \begin{pmatrix} 12 & 5 \\ 7 & 3 \end{pmatrix}, \ W_{i+2} = W_{i+1} W_i \quad (i \geq 1).$$

So $(W_i)_{i\geq 1}$ is a Fibonacci sequence in $\mathrm{SL}_2(\mathbb{Z})$. We extend it to a Fibonacci sequence $(W_i)_{i\geq 0}$ starting with

$$W_0 = W_2 W_1^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

# III.2. The morphism associated to $\xi$

We denote by $\varphi\colon \{a,b\}^* \to \mathrm{SL}_2(Z)$ the morphism of monoids with

$$\varphi(a) = W_1 = \begin{pmatrix} 7 & -2 \\ 4 & -1 \end{pmatrix} \quad \text{and} \quad \varphi(b) = W_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$\varphi(w_i) = W_i \quad (i \geq 1),$$

where $(w_i)_{i\geq 1} = (a,\ ab,\ aba, ababa, \dots)$.

For each non-empty word $v$ in $\{a,b\}^*$, we set

$$M(v) = \begin{cases} M & \text{if } v \text{ ends in b,} \\ {}^tM & \text{if } v \text{ ends in a.} \end{cases}$$

For each $i \geq 1$, we have $M(w_i) = M_i$ and so

$$\varphi(w_i)M(w_i)^{-1} = W_i M_i^{-1} = \mathbf{x}_i$$

is a symmetric matrix.

# III.3. The points $\mathbf{x}(v)$

For each non-empty prefix $v$ of $w_\infty = abaab\ldots$, we denote by

$$\mathbf{x}(v) = \begin{pmatrix} x_0(v) & x_1(v) \\ x_1(v) & x_2(v) \end{pmatrix}$$

the symmetric matrix in $\mathrm{Mat}_{2\times 2}(\mathbb{Z})$ which

- has the same first column as $\varphi(v)M(v)^{-1}$ and
- satisfies $|x_1(v)\xi - x_2(v)| < 1/2$.

This specifies $x_2(v)$ uniquely as $\xi \notin \mathbb{Q}$.

For the first non-empty prefixes $v$ of $w_\infty = abaababa\ldots$, we find

$$\mathbf{x}(a) = \mathbf{x}_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{x}(ab) = \mathbf{x}_2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix},$$

$$\mathbf{x}(aba) = \mathbf{x}_3 = \begin{pmatrix} 29 & 17 \\ 17 & 10 \end{pmatrix}, \quad \mathbf{x}(abaa) = \begin{pmatrix} 179 & 105 \\ 105 & 62 \end{pmatrix}.$$

# III.4. Recurrence relations

For each integer $\ell \geq 1$, we set $\mathcal{W}_\ell = \{v \in ]\epsilon, w_\infty[\,;\ |v| \in \mathcal{N}_\ell\}$.

For $\ell \geq 3$, we have

$$\mathcal{N}_\ell = \{F_\ell, F_{\ell+1}, F_{\ell+1} + F_{\ell-1}, F_{\ell+2}, F_{\ell+2} + F_{\ell-2}, \dots\},$$

thus

$$\mathcal{W}_\ell = \{w_\ell, w_{\ell+1}, w_{\ell+1}w_{\ell-1}, w_{\ell+2}, w_{\ell+2}w_{\ell-2}, \dots\}.$$

**Theorem**

*There exists $\ell_0 \geq 4$ such that, for any $\ell \geq \ell_0$ and any triple of consecutive words $u < v < w$ of $\mathcal{W}_\ell$, the points $\mathbf{x}(u)$, $\mathbf{x}(v)$, $\mathbf{x}(w)$ are linearly independent if and only if $v \notin \mathcal{W}_{\ell+1}$. More precisely,*

(i) *if $v \in \mathcal{W}_{\ell+1}$, then $|v| - |u| = |w| - |v| = F_k$ with $k \in \{\ell-2, \ell-1\}$ and*

$$\mathbf{x}(w) = t_k \mathbf{x}(v) \pm \mathbf{x}(u) \quad \textit{where} \quad t_k = \mathrm{trace}(W_k)\,;$$

(ii) *if $v \notin \mathcal{W}_{\ell+1}$, then $\det(\mathbf{x}(u), \mathbf{x}(v), \mathbf{x}(w)) = \pm 2$.*

# III.5. The trajectory of $\mathbf{x}(v)$

## Proposition

*There exists $\rho > 0$ such that, for any $v \in ]\epsilon, w_\infty[$, we have*

(i) $\log x_0(v) = \rho|v| + \mathcal{O}(1)$,

(ii) $\log |x_0(v)\xi - x_1(v)| = -\rho|v| + \mathcal{O}(1)$,

(iii) $\log |x_0(v)\xi^2 - x_2(v)| = -\rho\alpha(|v|) + \mathcal{O}(1)$,

*and so*

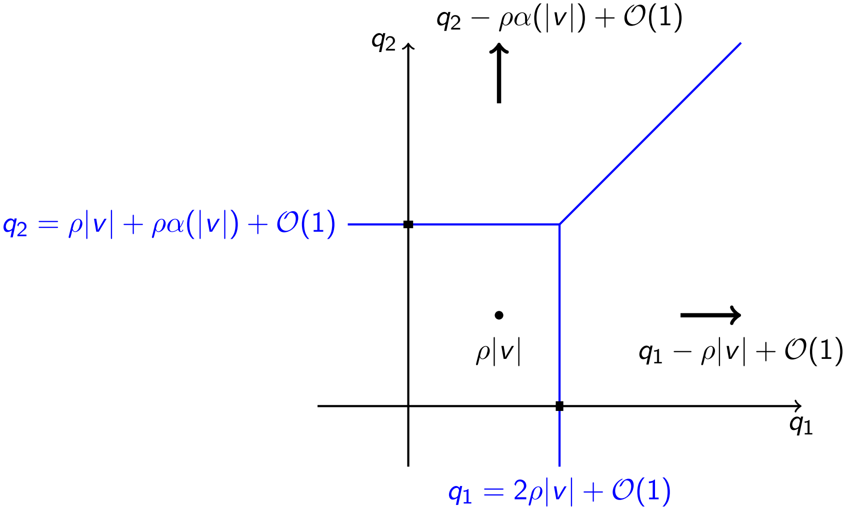
$$L_{\mathbf{x}(v)}(\mathbf{q}) = \rho P_{|v|}(\rho^{-1}\mathbf{q}) + \mathcal{O}(1) \quad (\mathbf{q} \in \mathbb{R}^2).$$

## Sketch of proof.

- We first show the existence of $\rho$ such that (i) holds.
- Then (ii) follows from the fact that $x_1(v)/x_0(v)$ is a convergent of $\xi$ in reduced form, for each $v \in ]\epsilon, w_\infty[$.
- Finally, (iii) is proved using the recurrence relations.

□

# Graph of $L_{\mathbf{x}(v)}$



## III.6. Proof of the main result

**Theorem**

*We have* $\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) = \rho \mathbf{P}(\rho^{-1}\mathbf{q}) + \mathcal{O}_{\xi}(1)$ *for each* $\mathbf{q} \in \mathcal{A}(0)$.

**Sketch of proof**

- It suffices to show this on $\rho\mathrm{Cell}(y)$ for each $y \in \mathbb{N} \setminus \mathcal{F}$ with $\alpha(y) = F_\ell \geq F_{\ell_0}$.
- We have $\mathrm{Cell}(y) = \mathcal{A}(x) \cap \mathcal{B}(y) \cap \mathcal{C}(z)$ with $x < y < z$ consecutive in $\mathcal{N}_\ell$.
- Write $x = |u|$, $y = |v|$ and $z = |w|$ for consecutive words $u < v < w$ in $\mathcal{W}_\ell$. Since $v \notin \mathcal{W}_{\ell+1}$, the points $\mathbf{x}(u)$, $\mathbf{x}(v)$ and $\mathbf{x}(w)$ are linearly independent.
- For $\mathbf{q} = (q_1, q_2) \in \rho\mathrm{Cell}(y)$, we find

$$L_{\mathbf{x}(u)}(\mathbf{q}) + L_{\mathbf{x}(v)}(\mathbf{q}) + L_{\mathbf{x}(w)}(\mathbf{q}) = q_1 + q_2 + \mathcal{O}_\xi(1),$$

$$\Rightarrow \ \mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) = \Phi(L_{\mathbf{x}(u)}(\mathbf{q}), L_{\mathbf{x}(v)}(\mathbf{q}), L_{\mathbf{x}(w)}(\mathbf{q})) + \mathcal{O}_\xi(1) = \rho\mathbf{P}(\rho^{-1}\mathbf{q}) + \mathcal{O}_\xi(1).$$

# Thank you !

# Example of computation

- Write $A \equiv B$ when $A, B \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ have the same first column.
- Then $AM^{-1} \equiv -A\,{}^t M^{-1}$ for each $A \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ (specific to $M$).

We have $w_{\ell+1} w_{\ell-1} = w_\ell w_{\ell-1}^2$, thus

$$\begin{aligned}
\varphi(w_{\ell+1} w_{\ell-1}) &= W_\ell W_{\ell-1}^2 \\
&= W_\ell \big(t_{\ell-1} W_{\ell-1} - I\big) \qquad \text{by Cayley-Hamilton Theorem,} \\
&= t_{\ell-1} W_{\ell+1} - W_\ell, \\
\Rightarrow \ \mathbf{x}(w_{\ell+1} w_{\ell-1}) &\equiv \varphi(w_{\ell+1} w_{\ell-1}) M_{\ell-1}^{-1} \\
&= t_{\ell-1} W_{\ell+1} M_{\ell+1}^{-1} - W_\ell\,{}^t M_\ell^{-1} \equiv t_{\ell-1} \mathbf{x}_{\ell+1} + \mathbf{x}_\ell \quad \text{symmetric !}
\end{aligned}$$

and $\|(\xi, -1)(t_{\ell-1}\mathbf{x}_{\ell+1} + \mathbf{x}_\ell)\| \ll \|\mathbf{x}_\ell\|^{-1}$, thus

$$\mathbf{x}(w_{\ell+1} w_{\ell-1}) = t_{\ell-1}\mathbf{x}_{\ell+1} + \mathbf{x}_\ell \ \text{ if } \ \ell \gg 1.$$