Parametric geometry of numbers over a number field and extension of scalars

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slides available at: https://mysite.science.uottawa.ca/droy//

Diophantische Approximationen (online presentation)

Mathematisches Forschungsinstitut Oberwolfach,

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1.1 Exponents of approximation

Fix $n \geq 2$ and a non-zero $\boldsymbol{\xi} \in \mathbb{R}^n$. Using Euclidean norms, define:

$\omega(\boldsymbol{\xi})$ = supremum of all ω for which the conditions

$$\|\mathbf{x}\| \leq Q \quad \text{and} \quad |\mathbf{x} \cdot \boldsymbol{\xi}| \leq Q^{-\omega}$$

have a non-zero solution $\mathbf{x} \in \mathbb{Z}^n$ for arbitrarily large Q's.

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Here: $|\mathbf{x} \cdot \boldsymbol{\xi}| = \|\mathbf{x}\|\|\boldsymbol{\xi}\|\cos(\theta)$ and $\|\mathbf{x} \wedge \boldsymbol{\xi}\| = \|\mathbf{x}\|\|\boldsymbol{\xi}\|\sin(\theta)$

where $\theta \in [0, \pi/2]$ is the angle between the lines spanned by $\mathbf{x}$ and $\boldsymbol{\xi}$.

1.2 Some transference inequalities

Suppose that $\boldsymbol{\xi} \in \mathbb{R}^n$ has linearly independent coordinates over $\mathbb{Q}$.

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- Khintchine (1926-28): $\dfrac{\omega(\boldsymbol{\xi})}{(n-2)\omega(\boldsymbol{\xi}) + n - 1} \leq \lambda(\boldsymbol{\xi}) \leq \dfrac{\omega(\boldsymbol{\xi}) - n + 2}{n-1}$

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- German (2012): Spectrum of $(\widehat{\lambda}, \widehat{\omega})$ for any n.
- Marnat-Moshchevitin (2020): Spectra of the pairs $(\widehat{\lambda}, \lambda)$ and $(\widehat{\omega}, \omega)$.

2.1 Absolute Weil height

K = a fixed number field,
d = $[K : \mathbb{Q}]$ its degree,
K_v = completion of K at a place v,
d_v = $[K_v : \mathbb{Q}_v]$ its local degree,
w = a fixed place of K.

Each $|\ |_v$ is normalized to extend one of the usual absolute values on $\mathbb{Q}$.

Then the product formula $\boxed{\prod_v |a|_v^{d_v} = 1}$ holds for each non-zero $a \in K$.

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The (absolute) Weil height of a non-zero point $\mathbf{x} = (x_1, \ldots, x_n) \in K^n$ is

$$H(\mathbf{x}) = \prod_v \|\mathbf{x}\|_v^{d_v/d} \quad \text{where} \quad \|\mathbf{x}\|_v = \begin{cases} \left(\sum |x_i|_v^2\right)^{1/2} & \text{if } v \mid \infty, \\ \max |x_i|_v & \text{else.} \end{cases}$$

$H(\mathbf{x}) \geq 1$ depends only on the class of $\mathbf{x}$ in the projective space $\mathbb{P}(K^n)$.

2.2 Exponents of approximation over K

Fix $n \geq 2$ and a non-zero $\boldsymbol{\xi} \in K_w^n$. Define:

$\omega(\boldsymbol{\xi}, K, w)$ = supremum of all ω for which the conditions

$$H(\mathbf{x}) \leq Q \quad \text{and} \quad D_{\boldsymbol{\xi}}(\mathbf{x}) := |\,\mathbf{x}\cdot\boldsymbol{\xi}\,|_w^{d_w/d} \prod_{v \neq w} \|\,\mathbf{x}\,\|_v^{d_v/d} \leq Q^{-\omega}$$

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$$\lambda(\boldsymbol{\xi}, K, w) = \text{supremum of all } \lambda \text{ for which the conditions}$$
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- $D_{\boldsymbol{\xi}}(\mathbf{x})$ and $D^*_{\boldsymbol{\xi}}(\mathbf{x})$ depend only on the class of $\mathbf{x}$ in $\mathbb{P}(K^n)$.
- When $\boldsymbol{\xi} \in \mathbb{R}^n$, we get the usual exponents $\omega(\boldsymbol{\xi}, \mathbb{Q}, \infty) = \omega(\boldsymbol{\xi})$, etc.

2.3 A general result of Pierre Bel

Theorem (P. Bel, 2013)

The supremum of all numbers

$$\widehat{\lambda}\Big((1,\xi,\xi^2),K,w\Big)$$

with $\xi \in K_w$ *having* $[K(\xi):K] > 2$ *(possibly transcendental over* K*) is*

$$1/\gamma \simeq 0.618 > 1/2$$

where $\gamma = (1+\sqrt{5})/2$ *denotes the golden ratio.*

Previous cases known:

- $K = \mathbb{Q}$ and $w = \infty$: Davenport & Schmidt 1969 and R. 2003.
- $K = \mathbb{Q}$ and $w = p$: Teulié 2002, Zelo 2008 and Bugeaud 2010.

3.1 A parametric family of minima

Fix again a non-zero $\boldsymbol{\xi} \in K_w^n$.

For each $j = 1, \ldots, n$ and each $q \geq 0$, define

$L_{\boldsymbol{\xi},j}(q) =$ smallest $t \geq 0$ for which the conditions

$$H(\mathbf{x}) \leq e^t \quad \text{and} \quad D_{\boldsymbol{\xi}}(\mathbf{x}) \leq e^{t-q}$$

admit j solutions $\mathbf{x} \in K^n$ that are linearly independent over K.

Then, form the map

$$\begin{aligned} \mathbf{L}_{\boldsymbol{\xi}} : [0, \infty) &\longrightarrow \mathbb{R}^n \\ q &\longmapsto (L_{\boldsymbol{\xi},1}(q), \ldots, L_{\boldsymbol{\xi},n}(q)) \end{aligned}$$

3.2 Relationship with exponents of approximation

Lemma

Write $\widehat{\omega}(\boldsymbol{\xi})$ *for* $\widehat{\omega}(\boldsymbol{\xi}, K, w)$ *and similarly for the three other exponents. Then, we have*

$$\liminf_{q\to\infty} \frac{L_{\boldsymbol{\xi},1}(q)}{q} = \frac{1}{\omega(\boldsymbol{\xi})+1}, \qquad \limsup_{q\to\infty} \frac{L_{\boldsymbol{\xi},1}(q)}{q} = \frac{1}{\widehat{\omega}(\boldsymbol{\xi})+1},$$

$$\liminf_{q\to\infty} \frac{L_{\boldsymbol{\xi},n}(q)}{q} = \frac{\lambda(\boldsymbol{\xi})}{\lambda(\boldsymbol{\xi})+1}, \qquad \limsup_{q\to\infty} \frac{L_{\boldsymbol{\xi},n}(q)}{q} = \frac{\widehat{\lambda}(\boldsymbol{\xi})}{\widehat{\lambda}(\boldsymbol{\xi})+1}.$$

This follows from the definitions as observed by Schmidt and Summerer in 2013 for the case $K = \mathbb{Q}$ and $K_w = \mathbb{R}$.

Corollary

Knowing $\mathbf{L}_{\boldsymbol{\xi}} \colon [0,\infty) \to \mathbb{R}^n$ *up to bounded error is enough to compute the four exponents.*

3.3 Proper *n*-systems

A proper *n*-system on $[0, \infty)$ is a continous map

$$\begin{aligned} \mathbf{P} \colon [0, \infty) &\longrightarrow \mathbb{R}^n \\ q &\longmapsto (P_1(q), \ldots, P_n(q)) \end{aligned}$$

with the following properties:

(S1) $0 \leq P_1(q) \leq \cdots \leq P_n(q)$ and $P_1(q) + \cdots + P_n(q) = q$ for each $q \geq 0$;

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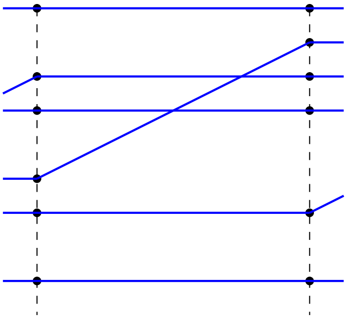
(S4) P_1 is unbounded.

Illustration for $n = 6$



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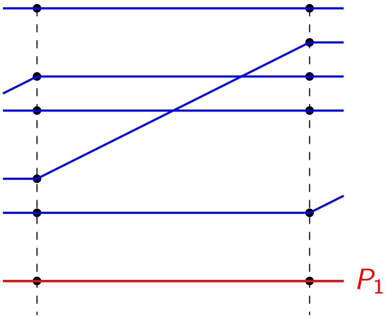

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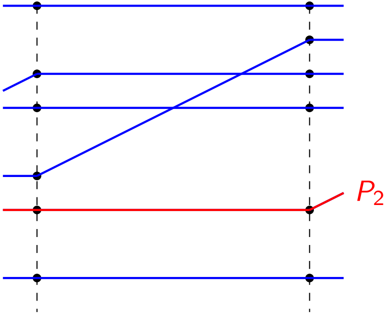

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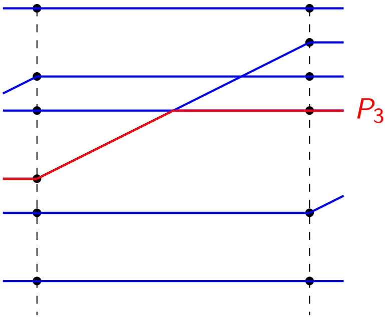

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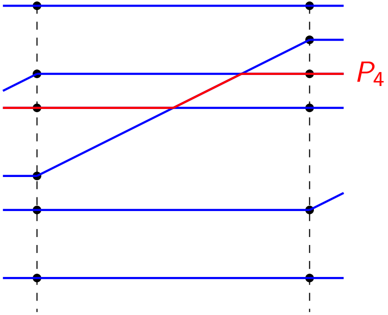

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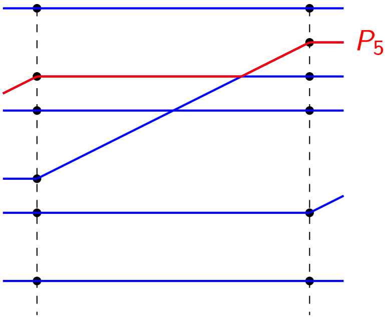

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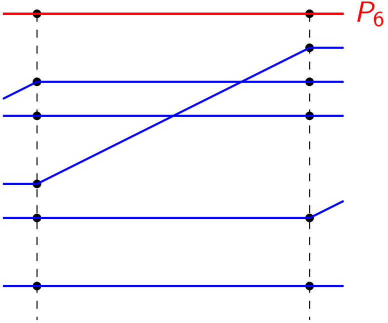

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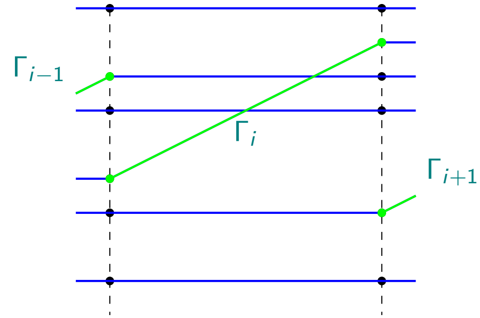

$$q_{i-1} = \sum_j P_j(q_{i-1}) \qquad q_i = \sum_j P_j(q_i)$$

3.4 First main result

Theorem A

There is a constant $c = c(K, w, n) > 0$ with the following property. For each point $\boldsymbol{\xi} \in K_w^n$ with linearly independent coordinates over K, there is a proper n-system $\mathbf{P}\colon [0, \infty) \to \mathbb{R}^n$ such that

$$\sup_{q \geq 0} \|\mathbf{L}_{\boldsymbol{\xi}}(q) - \mathbf{P}(q)\| \leq c,$$

and conversely.

In other words, the set of maps $\mathbf{L}_{\boldsymbol{\xi}}$ attached to points $\boldsymbol{\xi} \in K_w^n$ with linearly independent coordinates over K coincides with the set of proper n-systems modulo the additive group of bounded functions from $[0, \infty)$ to $\mathbb{R}^n$.

When $K = \mathbb{Q}$ and $K_w = \mathbb{R}$, this is due to Schmidt and Summerer (2013) and R. (2015).

Corollary

The set of quadruples $(\widehat{\lambda}(\boldsymbol{\xi}, K, w), \lambda(\boldsymbol{\xi}, K, w), \widehat{\omega}(\boldsymbol{\xi}, K, w), \omega(\boldsymbol{\xi}, K, w))$ *attached to points* $\boldsymbol{\xi} \in K_w^n$ *with* K*-linearly independent coordinates is the same for any choice of* K *and* w

Proof.

This is the set of points

$$\left(\frac{\underline{\varphi}_n(\mathbf{P})}{1 - \underline{\varphi}_n(\mathbf{P})}, \frac{\bar{\varphi}_n(\mathbf{P})}{1 - \bar{\varphi}_n(\mathbf{P})}, \frac{1}{\bar{\varphi}_1(\mathbf{P})} - 1, \frac{1}{\underline{\varphi}_1(\mathbf{P})} - 1 \right)$$

where $\mathbf{P} = (P_1, \ldots, P_n)$ runs through all proper n-systems, using Schmidt and Summerer's notation

$$\underline{\varphi}_j(\mathbf{P}) = \liminf_{q \to \infty} \frac{P_j(q)}{q} \quad \text{and} \quad \bar{\varphi}_j(\mathbf{P}) = \limsup_{q \to \infty} \frac{P_j(q)}{q} \quad (1 \leq j \leq n). \qquad \square$$

3.5 Example

For any $\boldsymbol{\xi} \in K_w^3$ with K-linearly independent coordinates, we have Jarník's identity

$$\frac{1}{\widehat{\lambda}(\boldsymbol{\xi}, K, w)} - 1 = \frac{1}{\widehat{\omega}(\boldsymbol{\xi}, K, w) - 1}.$$

Applying this to the result of Bel, we deduce:

Corollary

The supremum of all numbers

$$\widehat{\omega}\Big((1, \xi, \xi^2), K, w\Big)$$

with $\xi \in K_w$ having $[K(\xi) : K] > 2$ (possibly transcendental over K) is

$$\gamma^2 \simeq 2.618 > 2$$

where $\gamma = (1 + \sqrt{5})/2$ denotes the golden ratio.

4.1 Second main result

Suppose that the place w of K has relative degree one over $\mathbb{Q}$, namely that $\boxed{K_w = \mathbb{Q}_\ell}$ for the place ℓ of $\mathbb{Q}$ induced by w.

Choose a basis $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$ of K over $\mathbb{Q}$ and a point $\boldsymbol{\xi} \in K_w^n$ with linearly independent coordinates over K. Define

$$\Xi = \boldsymbol{\alpha} \otimes \boldsymbol{\xi} = (\alpha_1 \boldsymbol{\xi}, \dots, \alpha_d \boldsymbol{\xi}) \in K_w^{nd} = \mathbb{Q}_\ell^{nd}.$$

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$$\Xi = \boldsymbol{\alpha} \otimes \boldsymbol{\xi} = (\alpha_1 \boldsymbol{\xi}, \dots, \alpha_d \boldsymbol{\xi}) \in K_w^{nd} = \mathbb{Q}_\ell^{nd}.$$

Then Ξ has linearly independent coordinates over $\mathbb{Q}$ and the maps $\mathbf{L}_{\boldsymbol{\xi}}$ and $\mathbf{L}_{\Xi}$ are linked as follows.

Theorem B

With the above notation and hypotheses, we have

$$\sup_{q \geq 0} |L_{\Xi, d(i-1)+j}(dq) - L_{\boldsymbol{\xi}, i}(q)| < \infty,$$

for any $i = 1, \dots, n$ *and* $j = 1, \dots, d$.

4.2 Exponents of approximation under extension of scalars

We keep the same hypotheses.

Corollary

Writing $\widehat{\omega}(\boldsymbol{\xi})$ for $\widehat{\omega}(\boldsymbol{\xi}, K, w)$, $\widehat{\omega}(\Xi)$ for $\widehat{\omega}(\Xi, \mathbb{Q}, \ell)$, and similarly for the other exponents, we have

$$d\big(\widehat{\omega}(\boldsymbol{\xi})+1\big)=\widehat{\omega}(\Xi)+1, \qquad d\big(\omega(\boldsymbol{\xi})+1\big)=\omega(\Xi)+1,$$

$$d\Big(\frac{1}{\widehat{\lambda}(\boldsymbol{\xi})}+1\Big)=\frac{1}{\widehat{\lambda}(\Xi)}+1, \qquad d\Big(\frac{1}{\lambda(\boldsymbol{\xi})}+1\Big)=\frac{1}{\lambda(\Xi)}+1.$$

When $n=3$, Jarník's identity applies to $\boldsymbol{\xi}$ (over K), and so we obtain

$$\boxed{\frac{1}{\widehat{\lambda}(\Xi)}-(2d-1)=\frac{d^2}{\widehat{\omega}(\Xi)-(2d-1)}}$$

4.3 Last main result

We combine Theorem B with the result of Bel.

Theorem C

When $K_w = \mathbb{Q}_\ell$, *we have*

$$\sup\left\{\widehat{\lambda}\big((\boldsymbol{\alpha}, \xi\boldsymbol{\alpha}, \xi^2\boldsymbol{\alpha}), \mathbb{Q}, \ell\big) \ ; \ \xi \in \mathbb{Q}_\ell \setminus \overline{\mathbb{Q}}\right\} = \frac{1}{d\gamma^2 - 1} > \frac{1}{3d-1}.$$

Example: $K = \mathbb{Q}(\sqrt{2}) \subset \mathbb{R}$ and $\boldsymbol{\alpha} = (1, \sqrt{2})$

$$\sup\left\{\widehat{\lambda}\big(1, \sqrt{2}, \xi, \sqrt{2}\xi, \xi^2, \sqrt{2}\xi^2\big) \ ; \ \xi \in \mathbb{R} \setminus \overline{\mathbb{Q}}\right\} = \frac{1}{2\gamma^2 - 1} \simeq 0.236 > \frac{1}{5}.$$

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A question

What is $\sup\left\{\widehat{\lambda}(1, \sqrt{2}, \xi, \xi^2)\ ;\ \xi \in \mathbb{R} \setminus \overline{\mathbb{Q}}\right\}$? Is it $> 1/3$?

Note that for $\widehat{\lambda}(1, \sqrt{2}, \xi)$, it is $1/\gamma \simeq 0.618 > 1/2$ by (R. 2013).

5.1 Highlights of the proof of Theorem A

Fix some $j \in \{1, \ldots, n\}$ and some $q \geq 0$.

- $L_{\boldsymbol{\xi},j}(q)$ is defined as the smallest $t \geq 0$ for which the conditions

  $$H(\mathbf{x}) \leq e^t \quad \text{and} \quad D_{\boldsymbol{\xi}}(\mathbf{x}) \leq e^{t-q}$$

  admit j solutions $\mathbf{x} \in K^n$ that are linearly independent over K.

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When $K = \mathbb{Q}$ and $K_w = \mathbb{R}$, we may choose $\mathbf{x} \in \mathbb{Z}^n$ primitive, and so

- $L_{\boldsymbol{\xi},j}(q)$ is the smallest $t \geq 0$ for which the conditions

  $$\|\mathbf{x}\| \leq e^{t} \quad \text{and} \quad |\mathbf{x} \cdot \boldsymbol{\xi}| \leq e^{t-q}$$

  admit at least j linearly independent solutions $\mathbf{x} \in \mathbb{Z}^n$.

5.1 Highlights of the proof of Theorem A

Fix some $j \in \{1, \ldots, n\}$ and some $q \geq 0$.

- $L_{\boldsymbol{\xi},j}(q)$ is defined as the smallest $t \geq 0$ for which the conditions

$$H(\mathbf{x}) \leq e^t \quad \text{and} \quad D_{\boldsymbol{\xi}}(\mathbf{x}) \leq e^{t-q}$$

  admit j solutions $\mathbf{x} \in K^n$ that are linearly independent over K.

When $K = \mathbb{Q}$ and $K_w = \mathbb{R}$, we may choose $\mathbf{x} \in \mathbb{Z}^n$ primitive, and so

- $L_{\boldsymbol{\xi},j}(q)$ is the smallest $t \geq 0$ for which the conditions

$$\|\mathbf{x}\| \leq e^t \quad \text{and} \quad |\mathbf{x} \cdot \boldsymbol{\xi}| \leq e^{t-q}$$

  admit at least j linearly independent solutions $\mathbf{x} \in \mathbb{Z}^n$.

- So, $L_{\boldsymbol{\xi},j}(q) = \log \lambda_j(q)$ where $\lambda_j(q)$ stands for the j-th minimum of the convex body

$$\mathcal{C}_{\boldsymbol{\xi}}(q) = \Big\{ \mathbf{x} \in \mathbb{R}^n ; \|\mathbf{x}\| \leq 1 \quad \text{and} \quad |\mathbf{x} \cdot \boldsymbol{\xi}| \leq e^{-q} \Big\}.$$

In general, $L_{\boldsymbol{\xi},j}(q) = \log \lambda_j(q) + \mathcal{O}(1)$ where $\lambda_j(q)$ denotes the j-th minimum of the adelic convex body

$$\mathcal{C}_{\boldsymbol{\xi}}(q) = \Big\{ (\mathbf{x}_v) \in K_{\mathbb{A}}^n \,;\, |\mathbf{x}_w \cdot \boldsymbol{\xi}|_w \leq e^{-q} \text{ and } \| \mathbf{x}_v \|_v \leq 1 \text{ for all } v \Big\}.$$

In general, $L_{\boldsymbol{\xi},j}(q) = \log \lambda_j(q) + \mathcal{O}(1)$ where $\lambda_j(q)$ denotes the j-th minimum of the adelic convex body

$$\mathcal{C}_{\boldsymbol{\xi}}(q) = \Big\{ (\mathbf{x}_v) \in K_{\mathbb{A}}^n \,;\, |\mathbf{x}_w \cdot \boldsymbol{\xi}|_w \le e^{-q} \text{ and } \|\, \mathbf{x}_v \|_v \le 1 \text{ for all } v \Big\}.$$

1. From a point $\boldsymbol{\xi} \in K_w^n$ to an n-system $\mathbf{P}$:

- We follow Schmidt and Summerer (2013) to construct an (n, γ)-system $\tilde{\mathbf{P}}$ which approximates $\mathbf{L}_{\boldsymbol{\xi}}$;
- then use R. (2015) to approximate $\tilde{\mathbf{P}}$ by an n-system $\mathbf{P}$.

Tools

Case $K = \mathbb{Q} \subset K_w = \mathbb{R}$	General case
minima of $\mathcal{C}_{\boldsymbol{\xi}}(q) \subset \mathbb{R}^n$ w. r. to $\mathbb{Z}^n$	minima of $\mathcal{C}_{\boldsymbol{\xi}}(q) \subset K_{\mathbb{A}}^n$ w. r. to K^n
Minkowski's 2^{nd} convex body thm	MacFeat / Bombieri–Vaaler version
Mahler's compound bodies	version of E. Burger

2. From an n-system $\mathbf{P}$ to a point $\xi \in K_w^n$:

- We replace $\mathbb{Z}$ by the ring of S-integers $\mathcal{O}_S$ of K where S consists of w and all archimedean places of K.
- As in R. (2015) we construct a sequence of bases $\mathbf{x}^{(i)} = (\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)})$ of $\mathcal{O}_S^n$ that will realize up to a bounded factor the successive minima of $\mathcal{C}_{\boldsymbol{\xi}}(q)$ for $q \in [q_i, q_{i+1}]$ as prescribed by $\mathbf{P}$.

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- We move from a basis to the next one by changing one point. We ask

$$(\mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{\ell_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)}) = (\mathbf{x}_1^{(i-1)}, \dots, \widehat{\mathbf{x}_{k_{i-1}}^{(i-1)}}, \dots, \mathbf{x}_n^{(i-1)})$$

with $k_{i-1} \le \ell_i$, $k_i < \ell_i$ and

$$\mathbf{x}_{\ell_i}^{(i)} = \epsilon_i \mathbf{x}_{k_{i-1}}^{(i-1)} + \sum_{j=1}^{\ell_i - 1} a_j \mathbf{x}_j^{(i)}$$

with $\epsilon_i \in \mathcal{O}_S^*$ and $a_1, \dots, a_{\ell_i - 1} \in \mathcal{O}_S$ chosen so that

$$(\mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_{\ell_i}^{(i)})$$

is almost orthogonal at all archimedean places of K.

5.2 Highlights of the proof of Theorem B

- This is inspired by the alternative proof due to Jeff Thunder (2002) of the adelic version of Minkowski's theorem due to MacFeat and Bombieri–Vaaler.
- His proof is based on the usual Minkowski's theorem for the minima of a convex body in $\mathbb{R}^n$ with respect to $\mathbb{Z}^n$.
- Given $\boldsymbol{\xi} \in K_w^n = \mathbb{Q}_\ell^n$ and a basis $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ of K over $\mathbb{Q}$, this principle of J. Thunder allows us to relate the minima of $\mathcal{C}_{\boldsymbol{\xi}}(q)$ with respect to K^n to those of $\mathcal{C}_{\Xi}(q)$ with respect to $\mathbb{Q}^{dn}$ where

$$\Xi = \boldsymbol{\alpha} \otimes \boldsymbol{\xi} = (\alpha_1 \boldsymbol{\xi}, \ldots, \alpha_d \boldsymbol{\xi}) \in \mathbb{Q}_\ell^{dn}.$$

Thank you!