Diophantine equations, Diophantine approximation, and geometry of numbers

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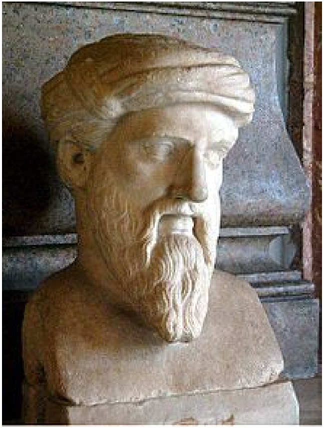
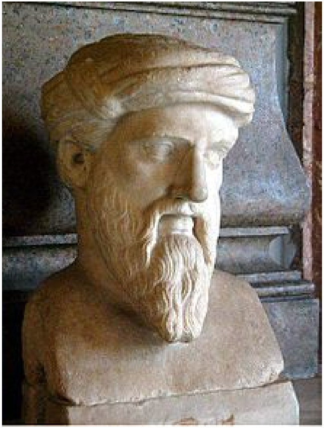
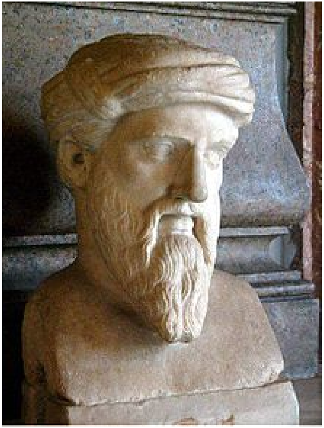




Pythagoras
(-570BC to -495BC)



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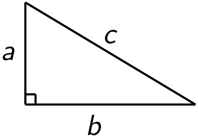
Pythagorean theorem:



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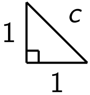
1. The triangle 

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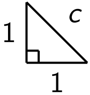
1. The triangle 

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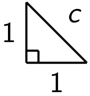
1. The triangle 

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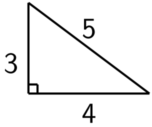

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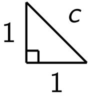
1. The triangle 

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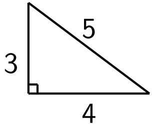

$$5^2 + 12^2 = 13^2, \ \ldots$$

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   has infinitely many solutions for each d: Lagrange (1768)

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i.e. $\dfrac{a}{b}$ is a very good rational approximation to $\sqrt{2}$.

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are $\dfrac{p_n}{q_n} = (a_0, a_1, \dots, a_n) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{\ddots}{+\cfrac{1}{a_n}}}} \in \mathbb{Q}$.

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Theorem (Legendre)

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- $(1, 2) = 1 + \frac{1}{2} = \frac{3}{2}$ is a convergent of $\sqrt{2}$: $\boxed{3^2 = 2 \times 2^2 + 1}$
- $(1, 2, 2) = 1 + \cfrac{1}{2 + \cfrac{1}{2}} = \frac{7}{5}$ is a convergent of $\sqrt{2}$:

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Example: $a^2 = 2b^2 + 1 \leadsto \sqrt{2} = (1, 2, 2, 2, \dots)$

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- $(1,2,2) = 1 + \dfrac{1}{2 + \dfrac{1}{2}} = \dfrac{7}{5}$ is a convergent of $\sqrt{2}$: $\boxed{7^2 = 2 \times 5^2 - 1}$
- $(1,2,2,2) = \dfrac{17}{12}$ is a convergent of $\sqrt{2}$:

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Theorem (Legendre)

$$\left|\frac{a}{b} - \xi\right| \leq \frac{1}{2b^2} \implies \frac{a}{b} \textit{ is a convergent of } \xi$$

$\implies$ The solutions of a Pell equation $a^2 = db^2 + 1$ come from convergents $\frac{a}{b}$ of $\sqrt{d}$.

Example: $a^2 = 2b^2 + 1 \rightsquigarrow \sqrt{2} = (1, 2, 2, 2, \dots)$

- $(1,2) = 1 + \frac{1}{2} = \frac{3}{2}$ is a convergent of $\sqrt{2}$: $\boxed{3^2 = 2 \times 2^2 + 1}$
- $(1,2,2) = 1 + \cfrac{1}{2 + \cfrac{1}{2}} = \frac{7}{5}$ is a convergent of $\sqrt{2}$: $\boxed{7^2 = 2 \times 5^2 - 1}$
- $(1,2,2,2) = \frac{17}{12}$ is a convergent of $\sqrt{2}$: $\boxed{17^2 = 2 \times 12^2 + 1}$

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(Norway, 1863-1922)

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A Thue equation is an equation of the form

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We search for solutions $(x, y) \in \mathbb{Z}^2$.

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Does there exist such good approximations to $\sqrt[3]{2}$? How many are they ?

Thue's theorem (1909)

Let α be an algebraic number of degree $d \geq 3$. For each $\mu > 1 + \frac{d}{2}$, there exists a constant $C > 0$ such that

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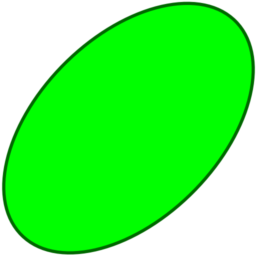

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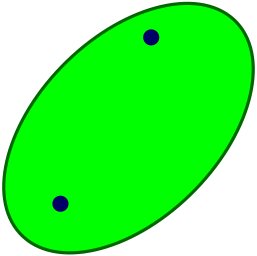

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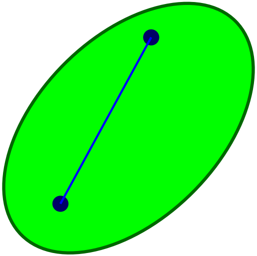

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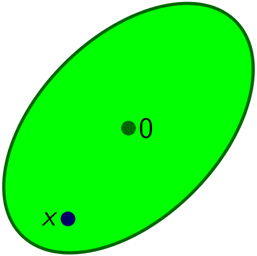

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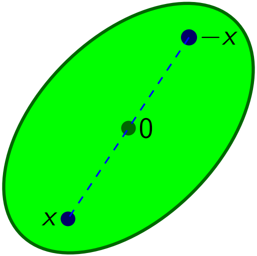

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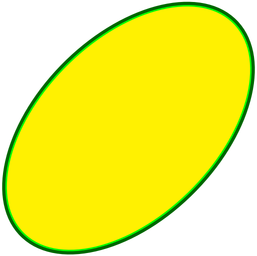

Fact: The image of a convex body $\mathcal{C}$ of $\mathbb{R}^n$ by an invertible linear map $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a convex body $T(\mathcal{C})$ of $\mathbb{R}^n$ with

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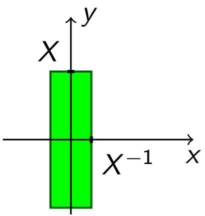

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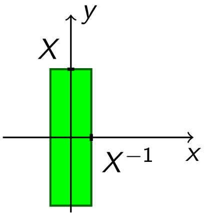

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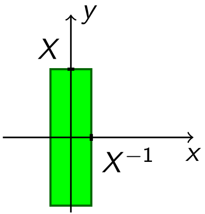

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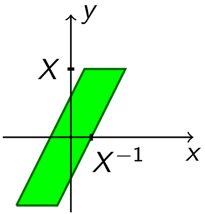

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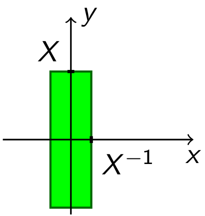

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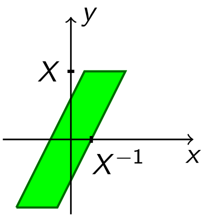

Since $\det(T) = 1$, its volume is also 4.

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$$\Longrightarrow \left| \frac{x}{y} - \xi \right| \leq \frac{X^{-1}}{|y|} \leq \frac{1}{y^2}.$$

$\Longrightarrow$ If $\xi \notin \mathbb{Q}$, there are infinitely many rational numbers $\dfrac{x}{y} \in \mathbb{Q}$ with

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Thue-Siegel-Roth theorem (1909, 1921, 1955)

Let α be an algebraic number of degree $d \geq 3$. For each $\epsilon > 0$, there exists a constant $C > 0$ such that

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Open problem: Can the product $|y(x - y\sqrt[3]{2})|$ be made arbitrarily small for positive integers x, y?

A more general construction

Let $\xi_1, \dots, \xi_n \in \mathbb{R}$. For each $X > 0$, the convex body of $\mathbb{R}^{n+1}$ defined by

$$|x_0 + x_1\xi_1 + \cdots + x_n\xi_n| \leq X^{-n},\ |x_1| \leq X,\ \dots,\ |x_n| \leq X \tag{1}$$

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For each $X > 0$, the equations (1) have a solution in integers $x_0, \ldots, x_n$ not all 0.

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Let $\mathcal{C}$ be a convex body in $\mathbb{R}^n$. For $i = 1, \ldots, n$, the *i-th minimum* of $\mathcal{C}$, denoted $\lambda_i(\mathcal{C})$, is the smallest λ such that $\lambda\mathcal{C}$ contains at least i linearly independent points of $\mathbb{Z}^n$.

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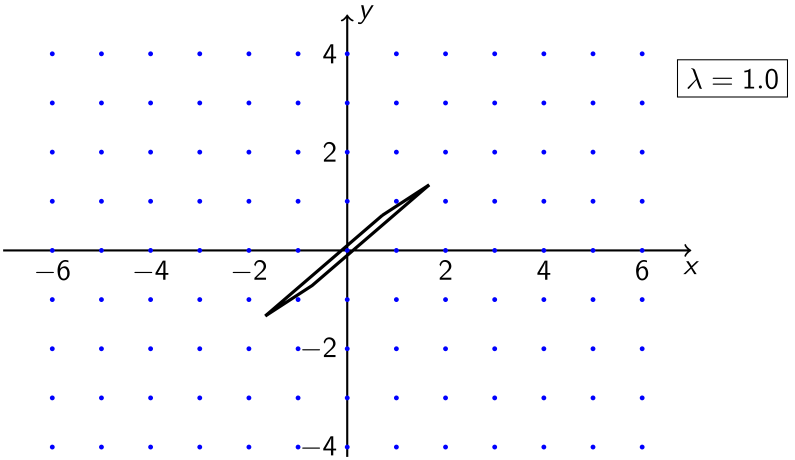

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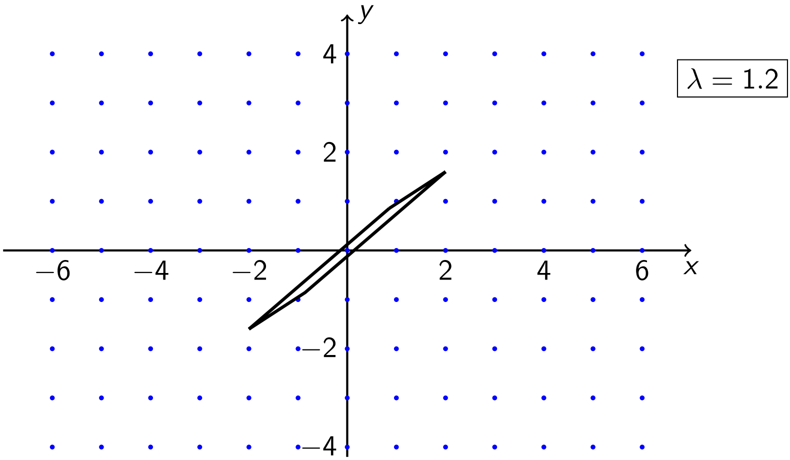

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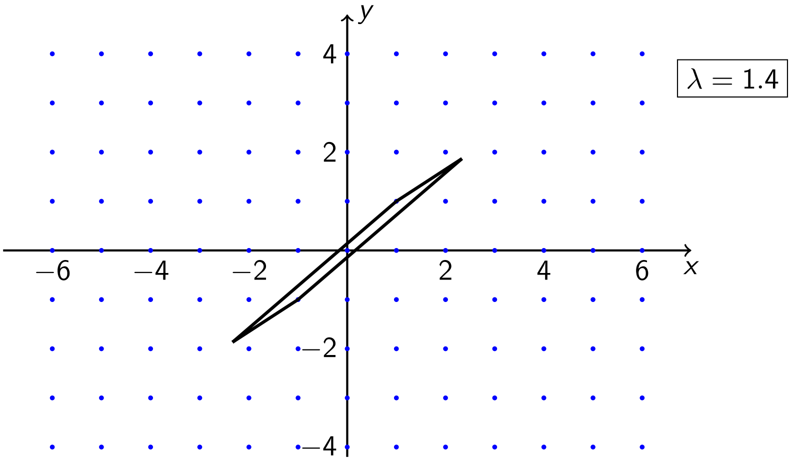

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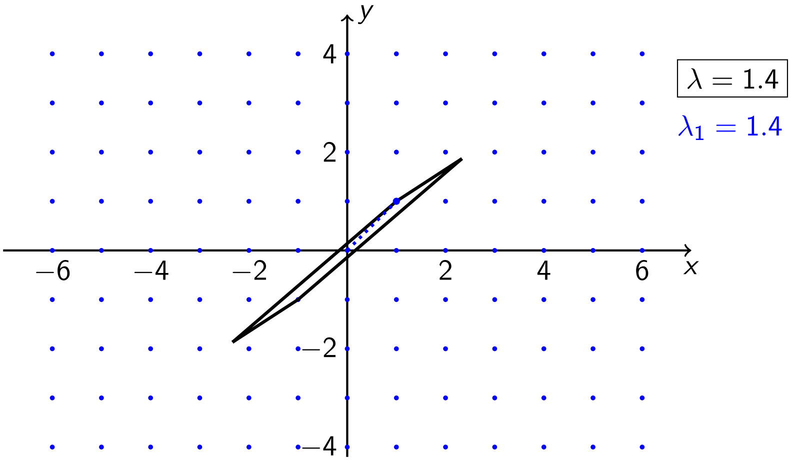

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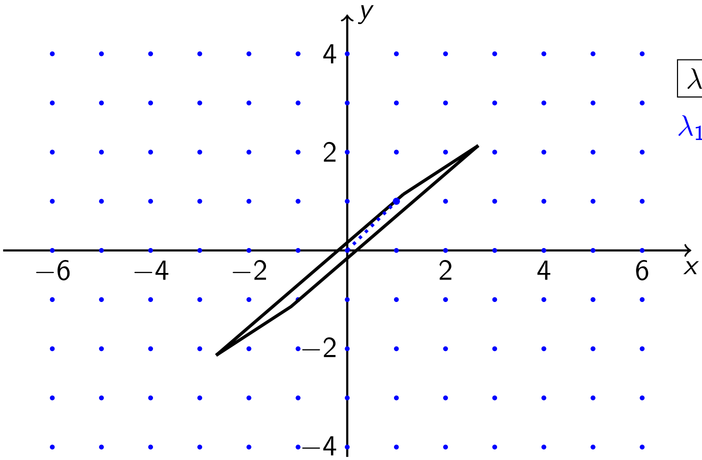

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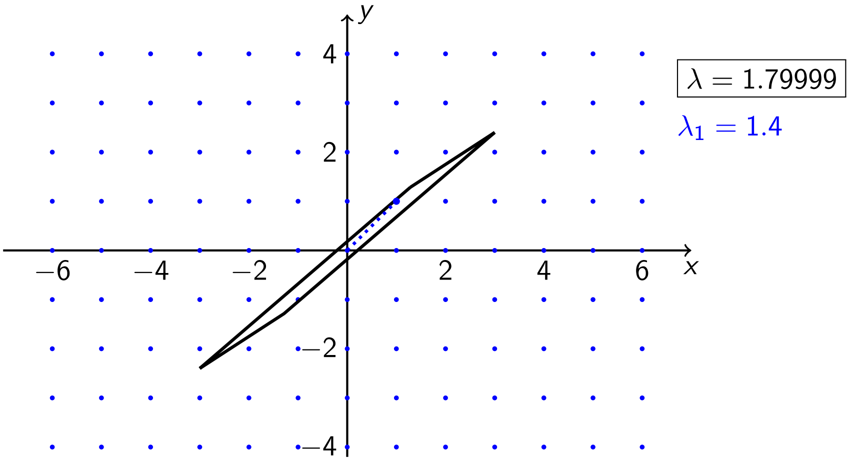

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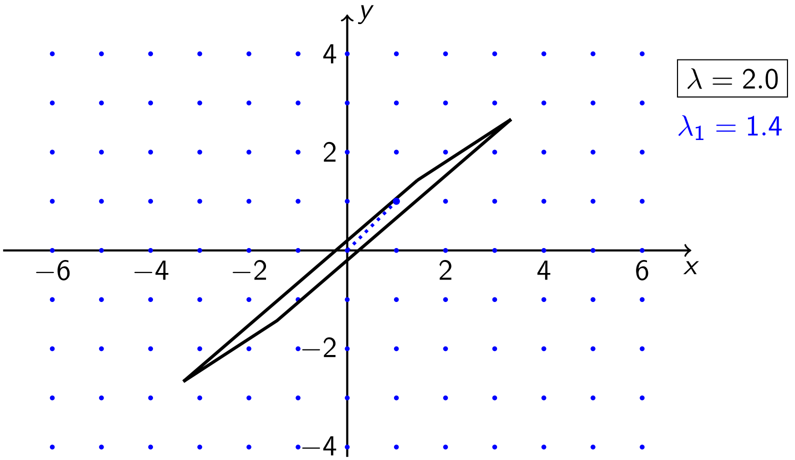

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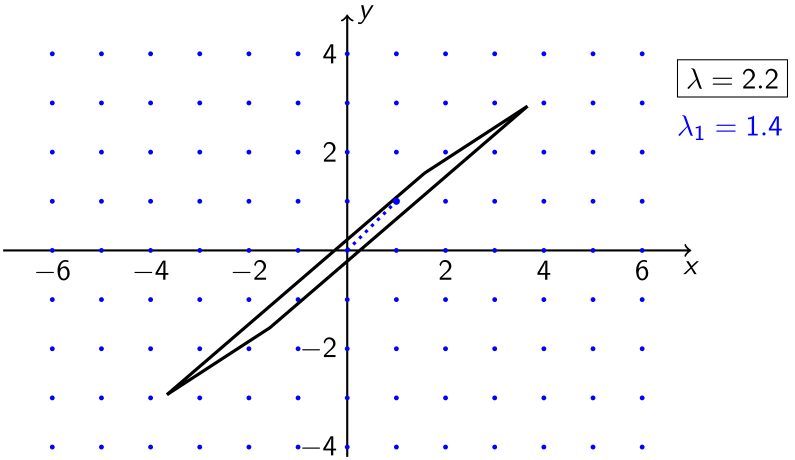

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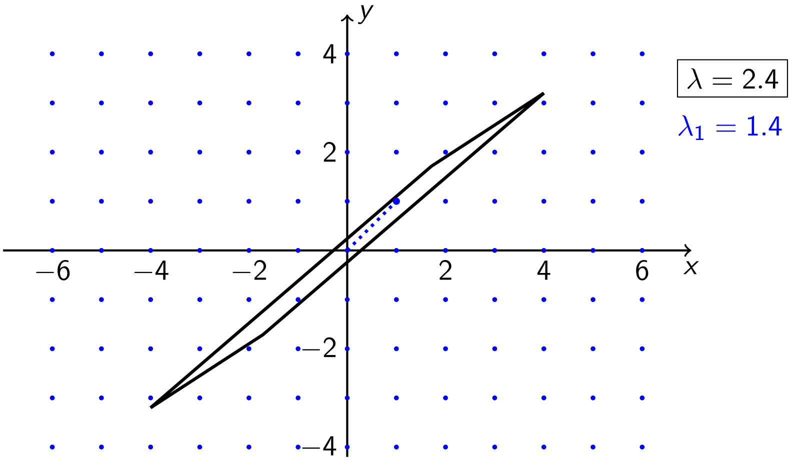

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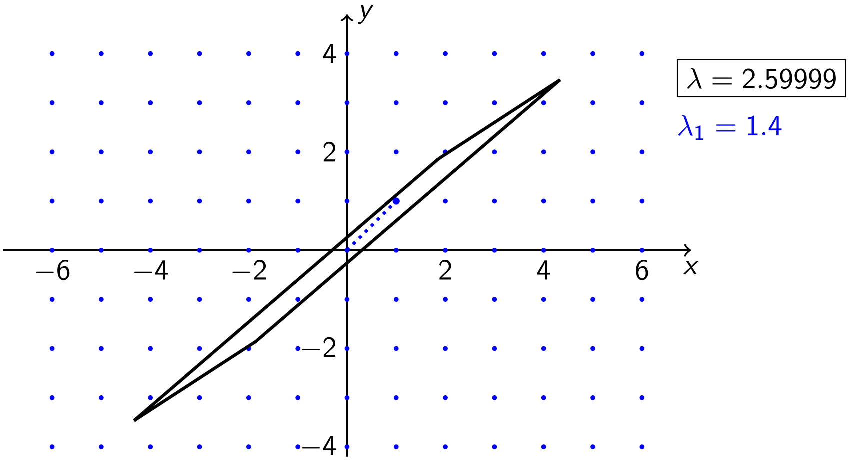

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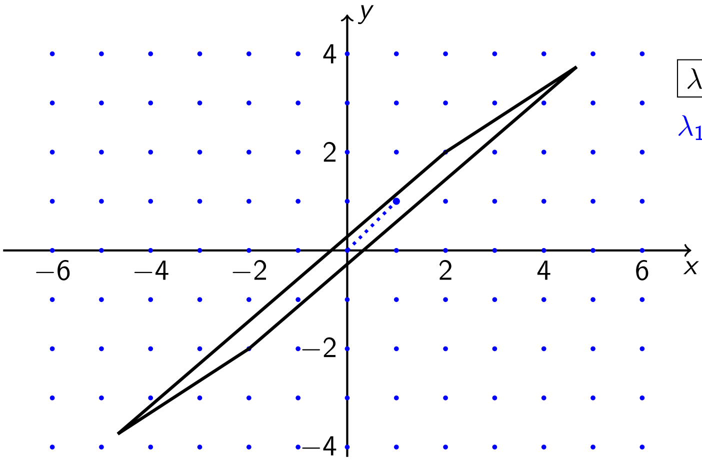

$\boxed{\lambda = 2.79999}$

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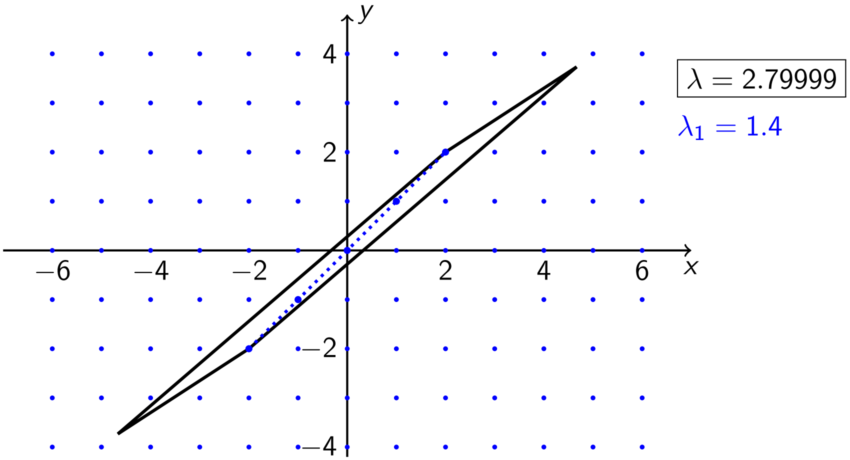

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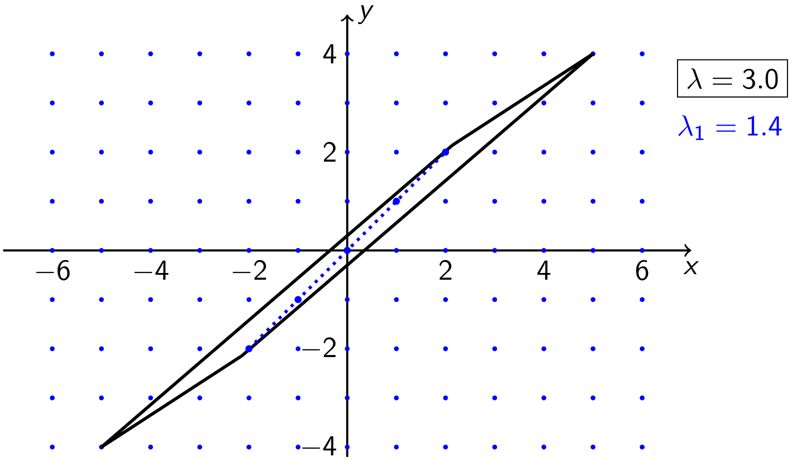

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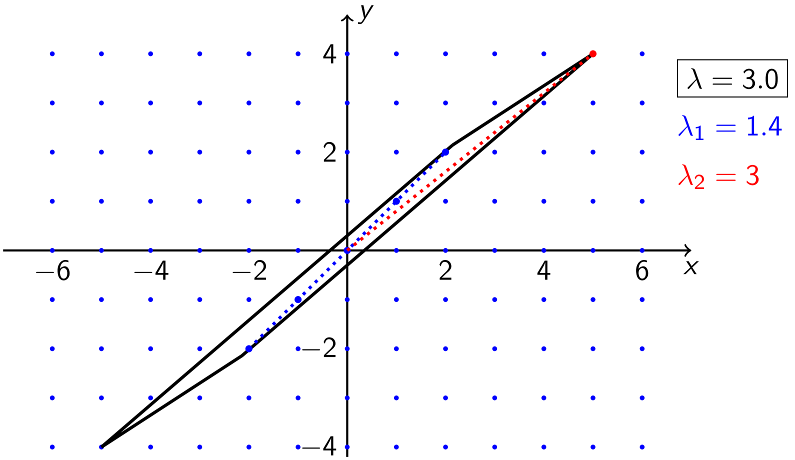

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$$\frac{2^n}{n!} \leq \lambda_1(\mathcal{C}) \cdots \lambda_n(\mathcal{C}) \mathrm{vol}(\mathcal{C}) \leq 2^n.$$

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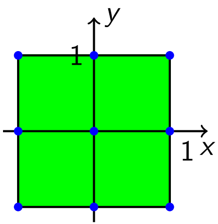

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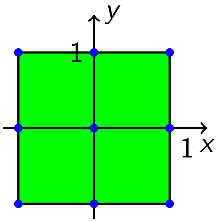

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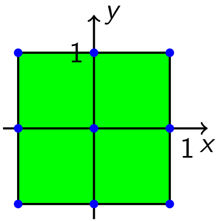

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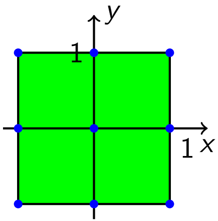

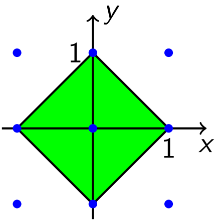

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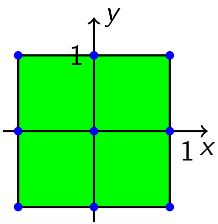

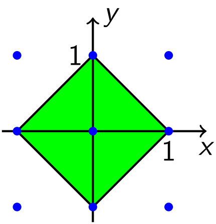

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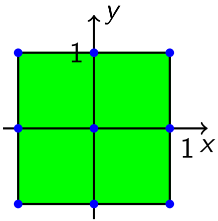

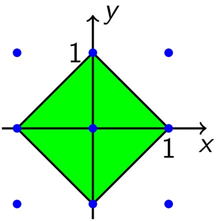

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i.e. $\mathcal{C}$ contains a non-zero point of $\mathbb{Z}^n$.

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Let $\xi_1, \dots, \xi_n \in \mathbb{R}$. For each $X > 0$, let $\mathcal{C}(X)$ denote the convex body of $\mathbb{R}^{n+1}$ defined by

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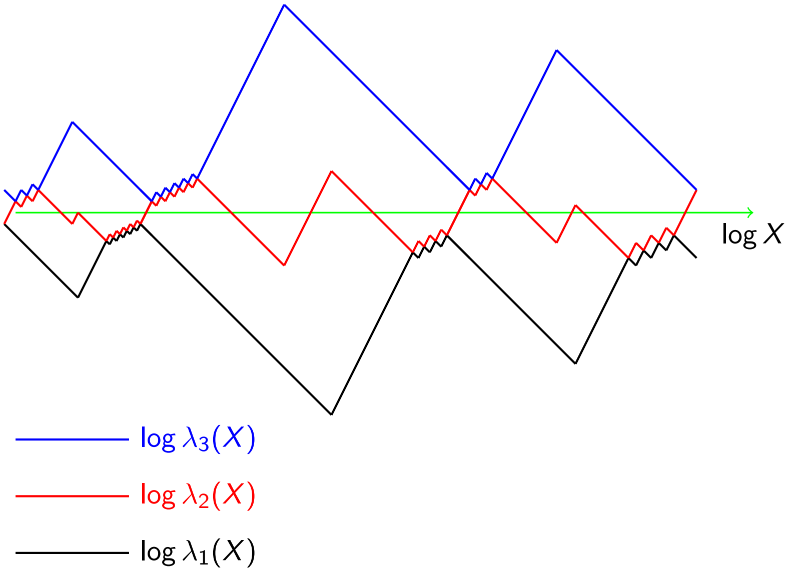
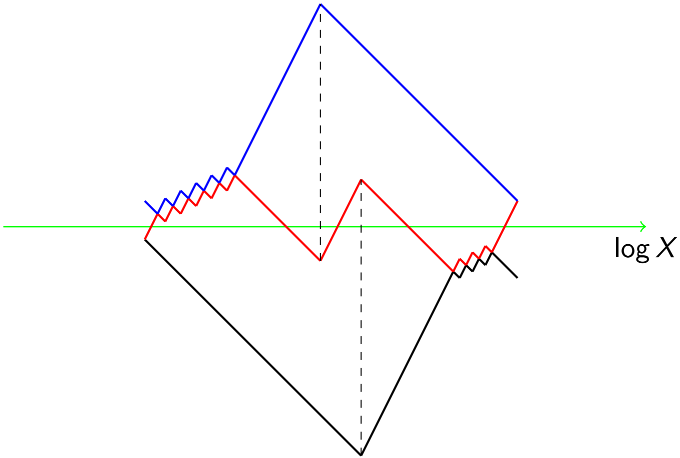
$$\text{Ideally: } \boxed{\sum_{i=1}^{n+1} \log(\lambda_i(X)) = 0}$$

The ideal model of Schmidt and Summerer (2013)



The basic mesh



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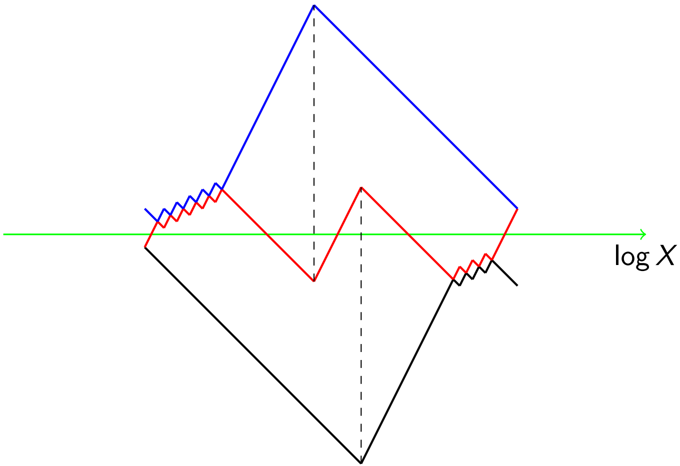

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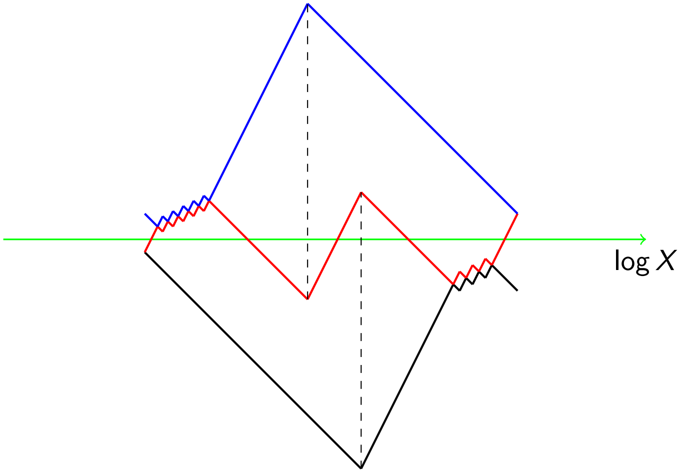

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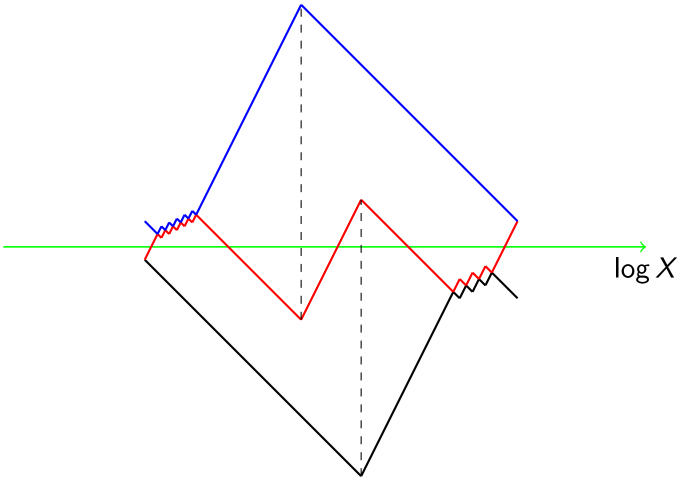

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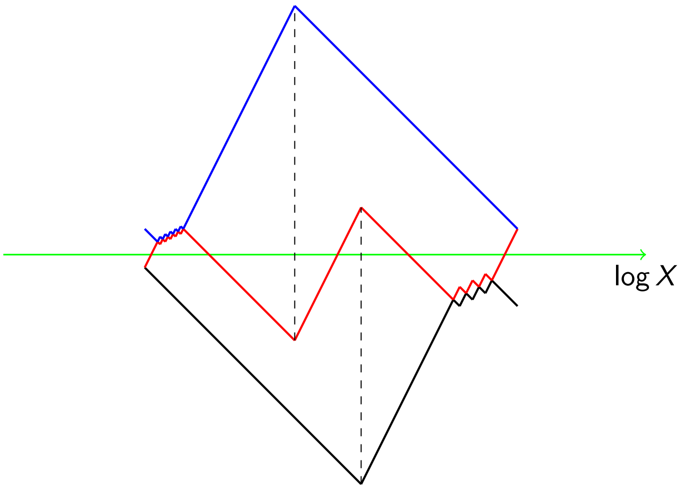

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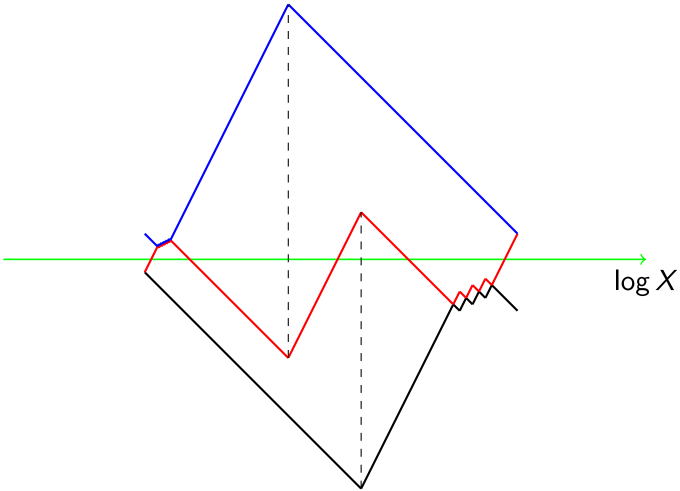

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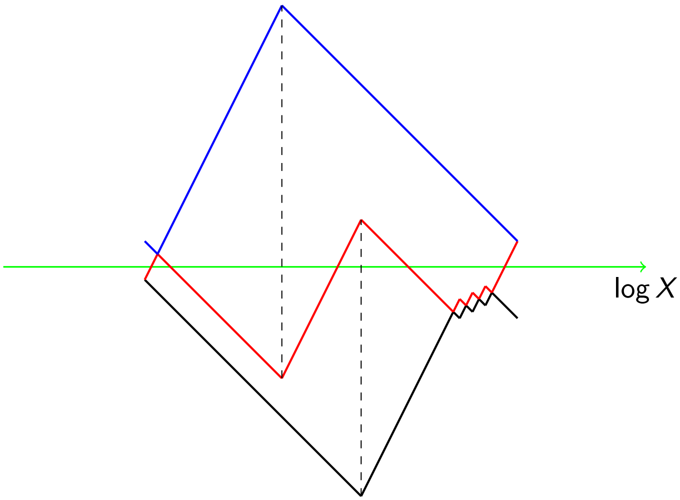

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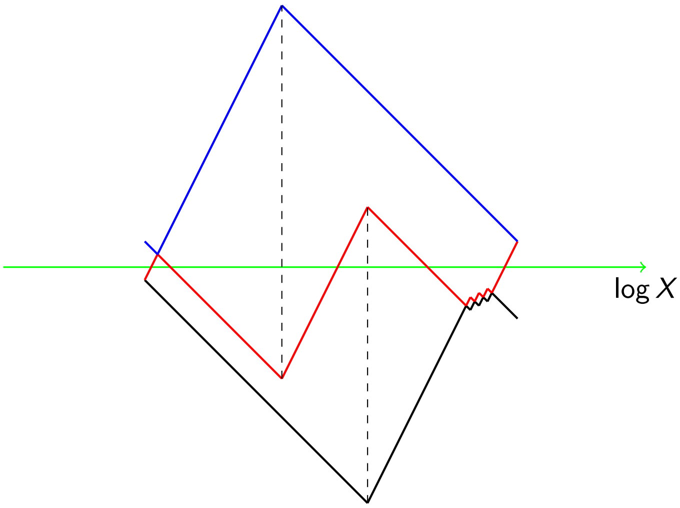

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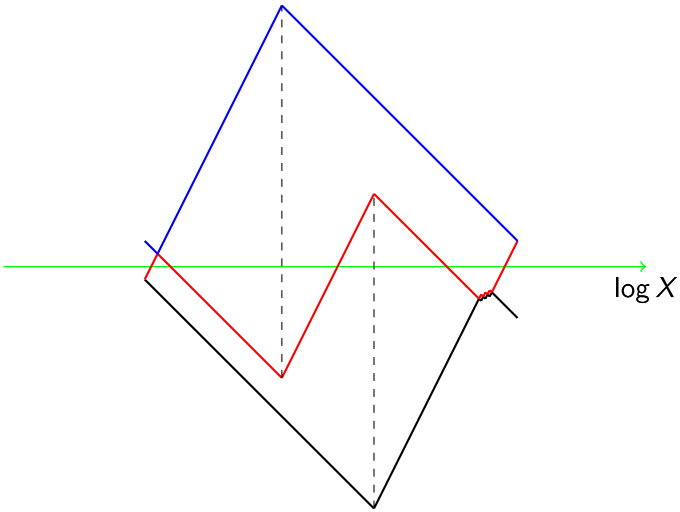

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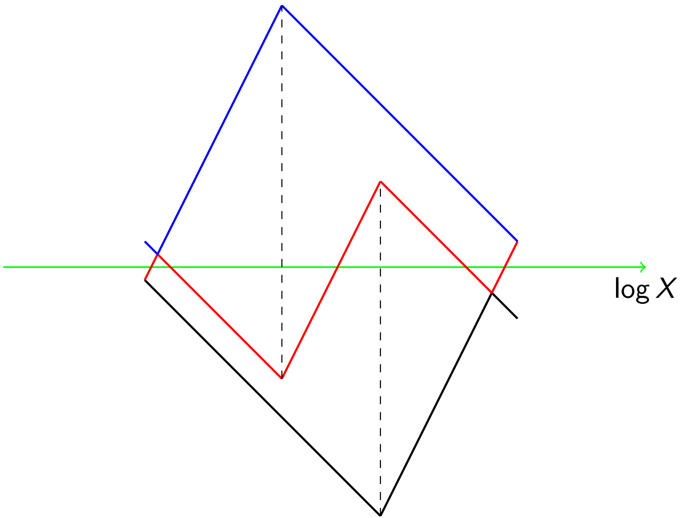

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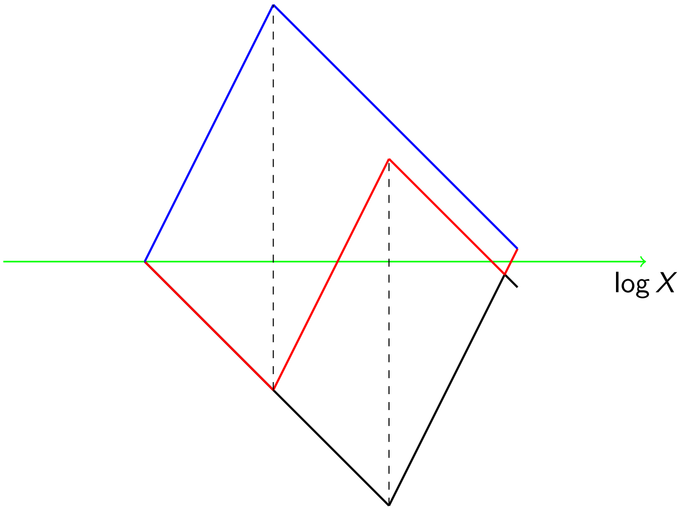

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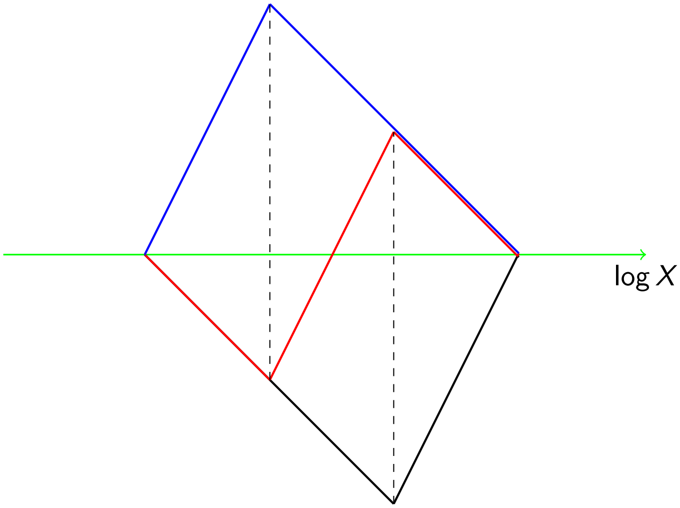

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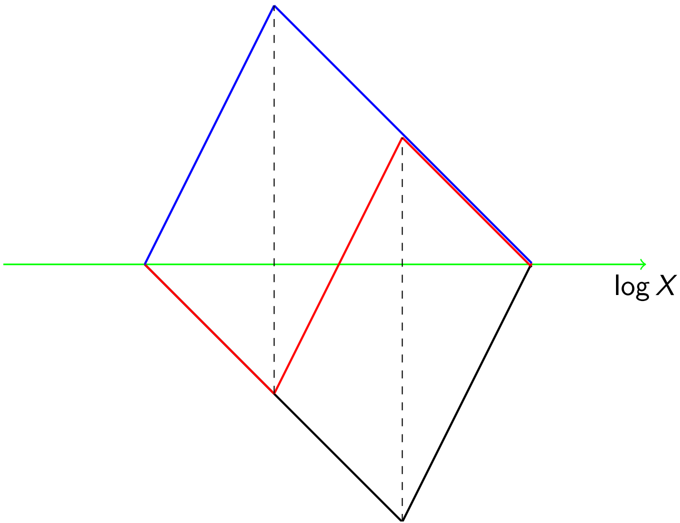

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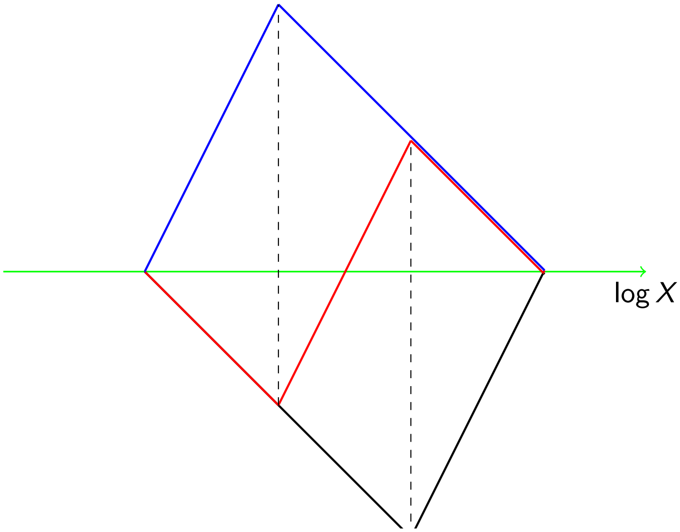

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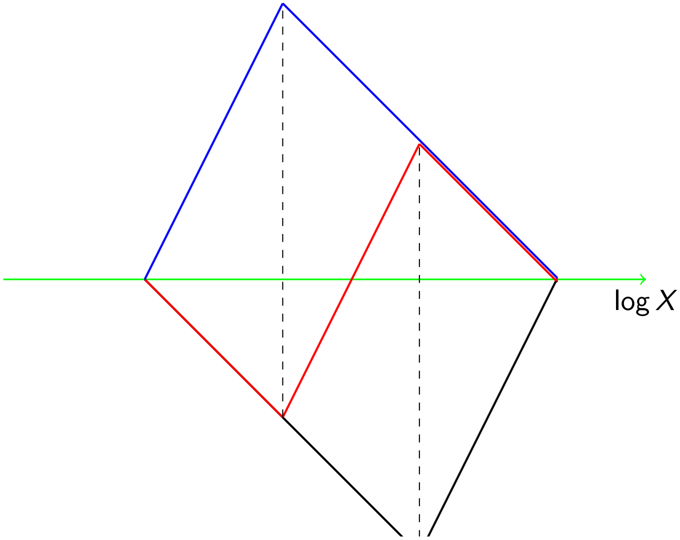

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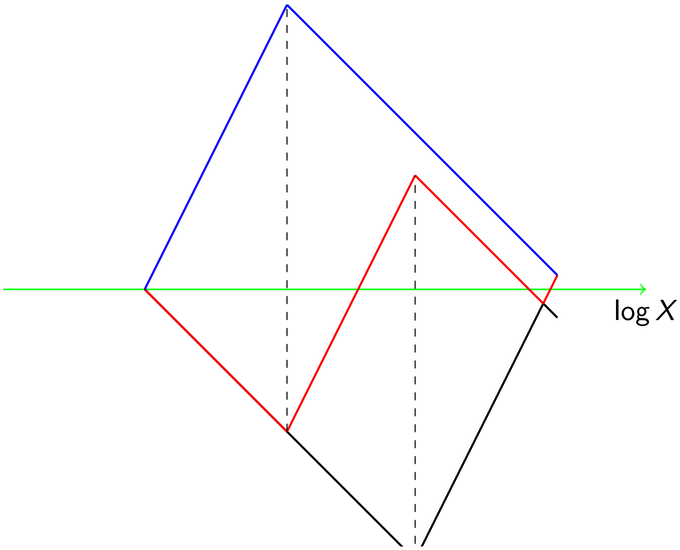

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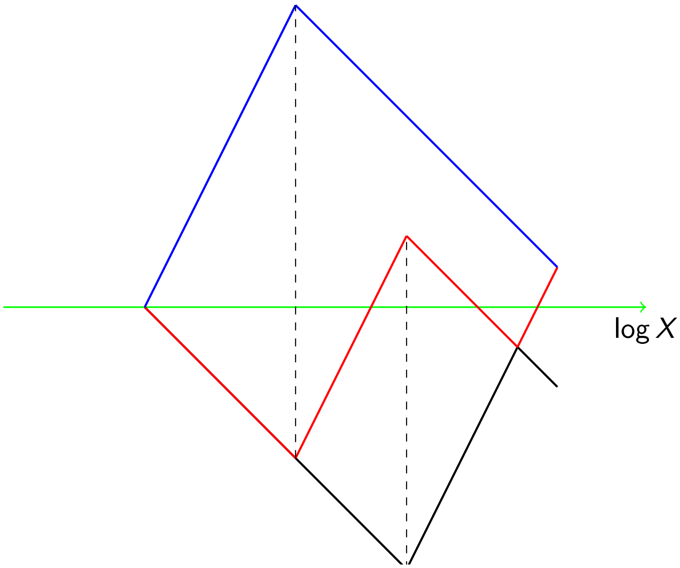

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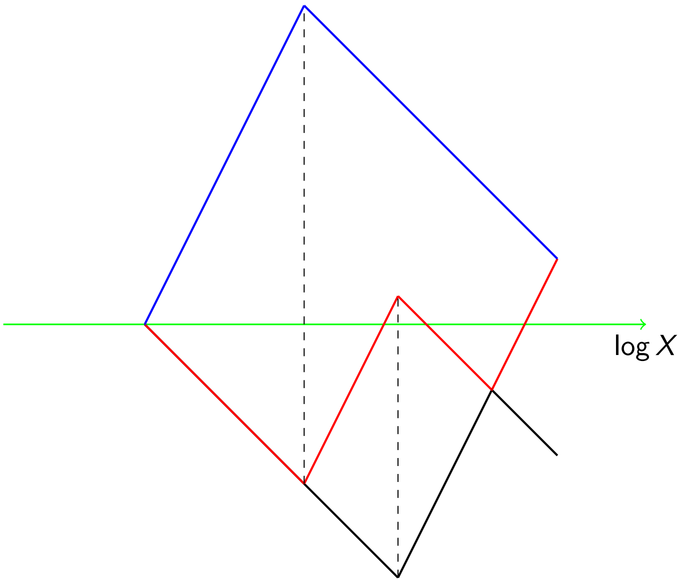

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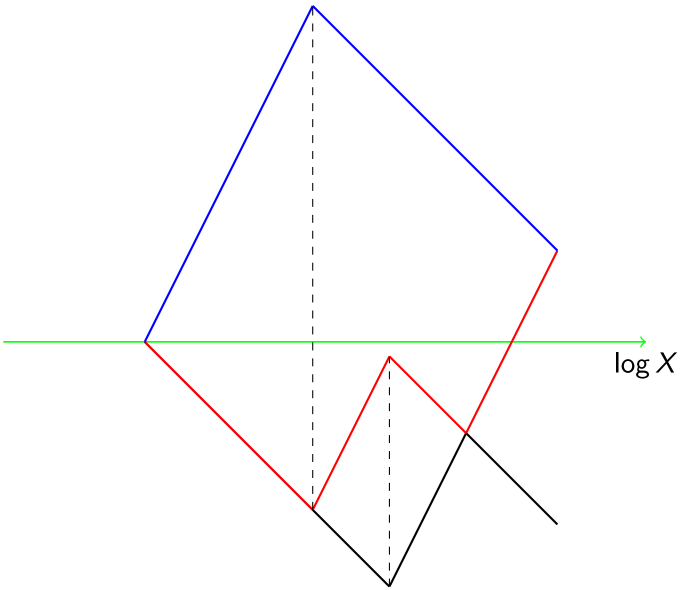

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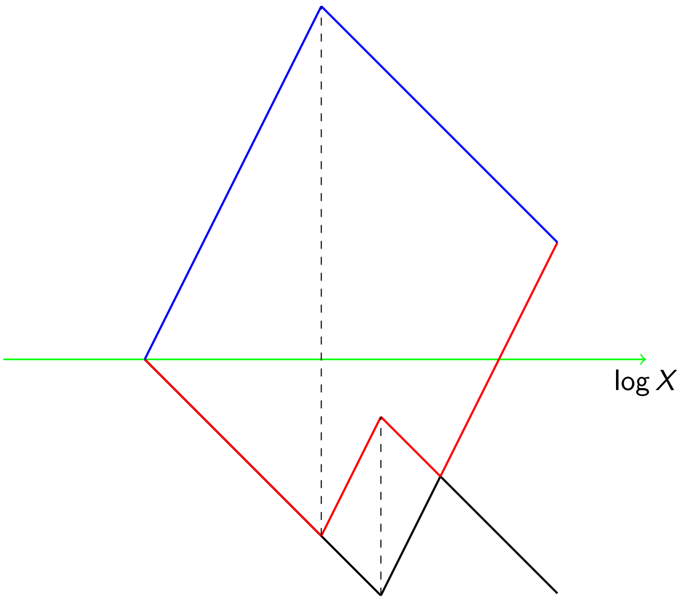

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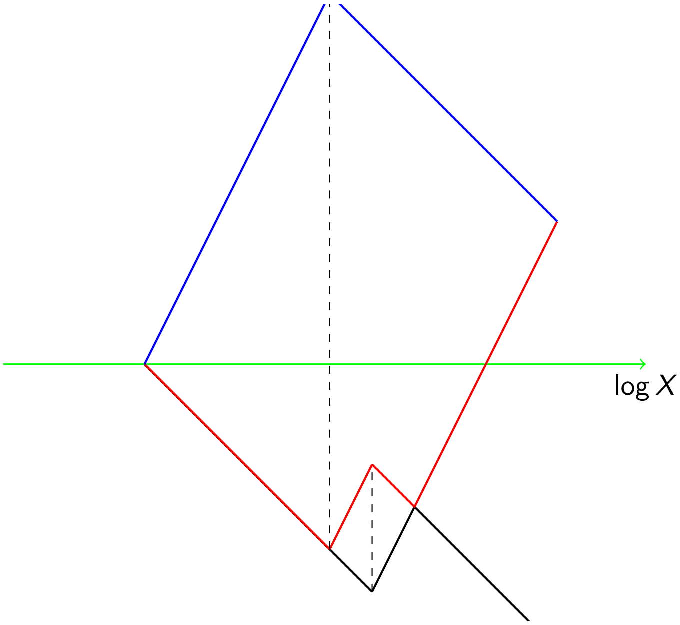

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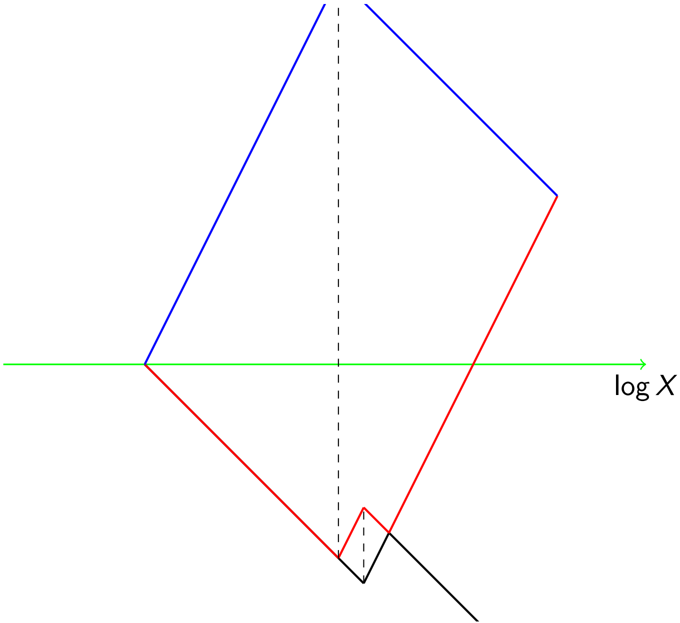

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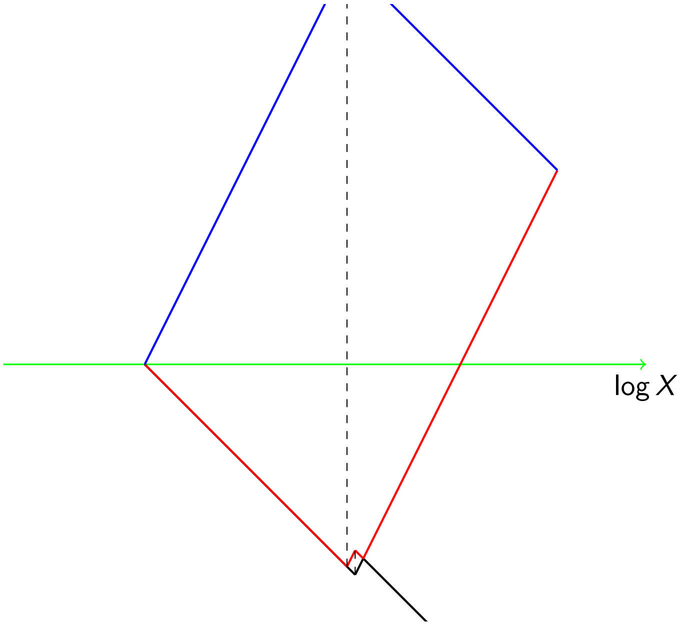

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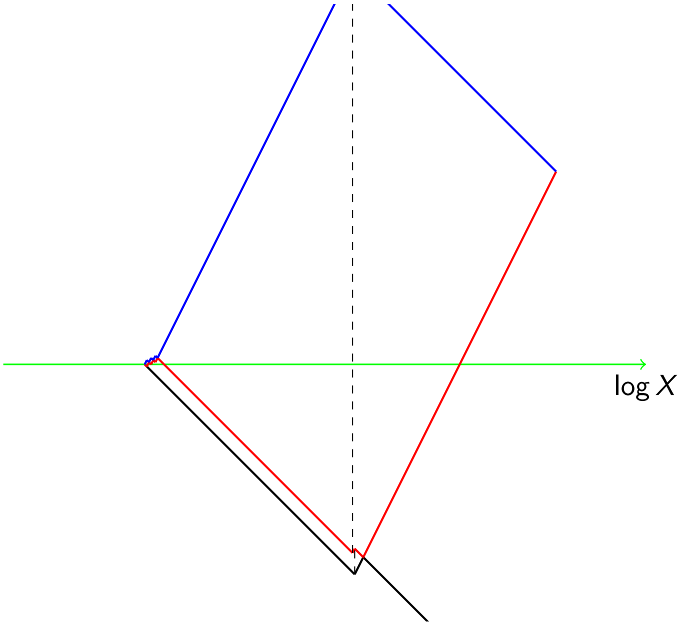

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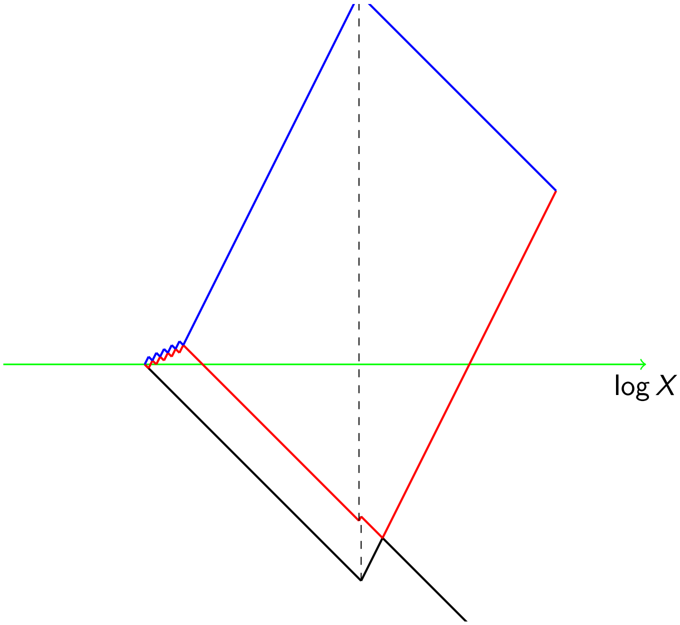

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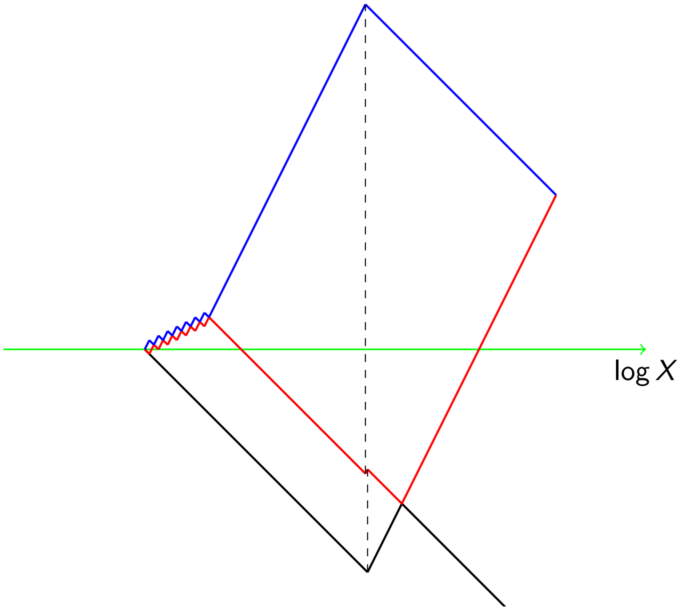
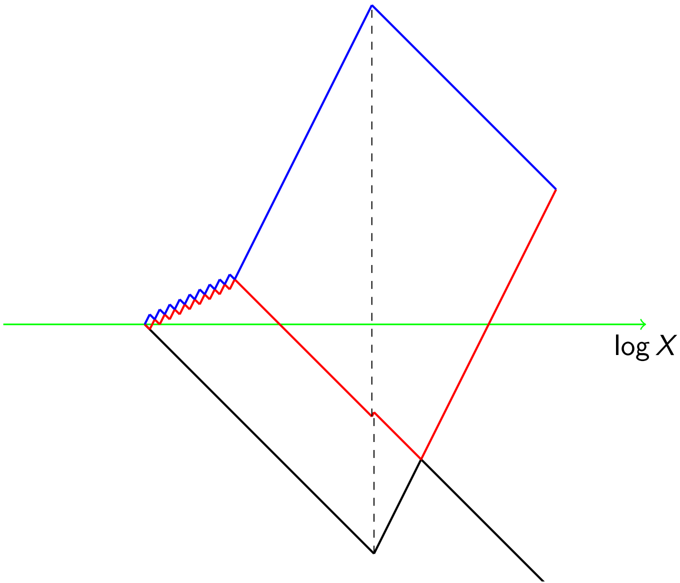
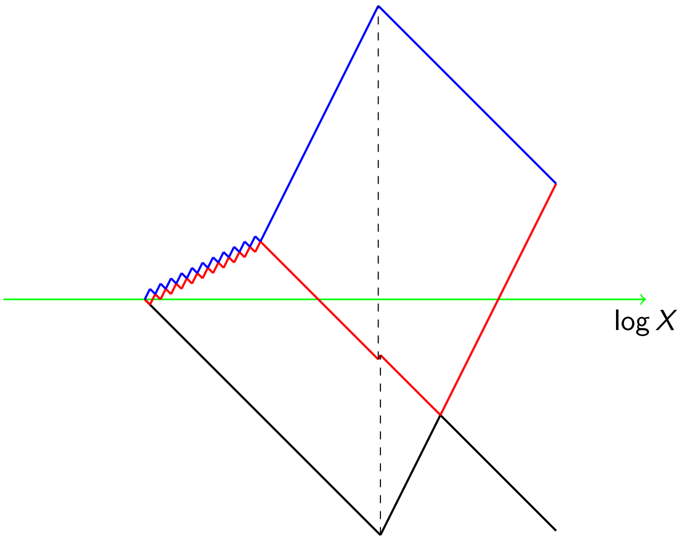


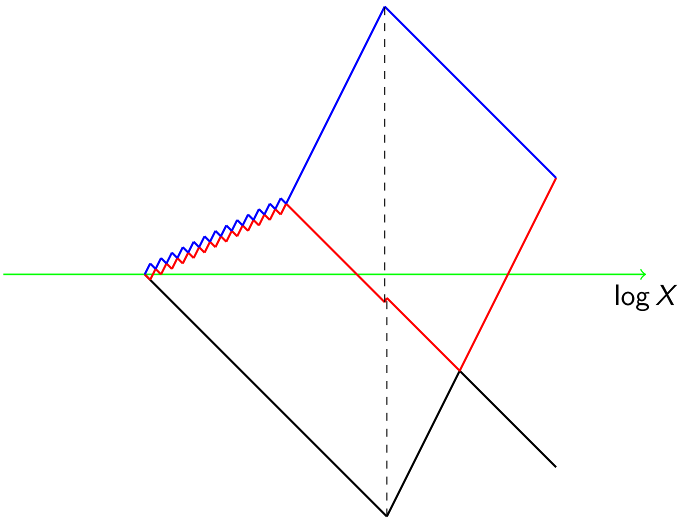
The basic mesh



The basic mesh



The basic mesh



The basic mesh