Rational approximation and extension of scalars

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joint work with Anthony Poëls

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Mathematical Congress of the Americas

Special session "Number Theory in the Americas",

Universidad de Buenos Aires,

July 14, 15 and 21, 2021

1.1. Simultaneous rational approximation

Dirichlet (1842). Let $\boldsymbol{\xi} = (1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$. For each $Q \geq 1$, there exist $x_1, \ldots, x_n \in \mathbb{Z}$ not all zero such that

$$|x_1| \leq Q \quad \text{and} \quad |x_1\xi_2 - x_2|, \ldots, |x_1\xi_n - x_n| \leq Q^{-1/(n-1)} .$$

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Proof

- These inequalities define a (Minkowski) convex body in $\mathbb{R}^n$
  $=$ a compact convex neigborhood $\mathcal{C}$ of 0 in $\mathbb{R}^n$ with $\mathcal{C} = -\mathcal{C}$.
- It has volume 2^n.
- By Minkowski's first convex body theorem (1889), such a convex body $\mathcal{C}$ contains a non-zero integer point.

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- It has volume 2^n.
- By Minkowski's first convex body theorem (1889), such a convex body $\mathcal{C}$ contains a non-zero integer point.

- Stronger estimates if $1, \xi_2, \dots, \xi_n$ are linearly dependent over $\mathbb{Q}$.
- Can we do better otherwise?

1.2. Two exponents of approximation

For $\boldsymbol{\xi} = (1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$, define

- $\lambda(\boldsymbol{\xi})$ = supremum of all $\lambda \geq 0$ such that
$$|x_1| \leq Q \quad \text{and} \quad |x_1\xi_2 - x_2|, \ldots, |x_1\xi_n - x_n| \leq Q^{-\lambda}$$
has a nonzero solution $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ for arbitrarily large Q.
- $\widehat{\lambda}(\boldsymbol{\xi})$ = same but for all sufficiently large values of Q.

- Dirichlet: $\lambda(\boldsymbol{\xi}) \geq \widehat{\lambda}(\boldsymbol{\xi}) \geq 1/(n-1)$
- Metrical result: equalities $\lambda(\boldsymbol{\xi}) = \widehat{\lambda}(\boldsymbol{\xi}) = 1/(n-1)$ for almost all $\boldsymbol{\xi} \in \mathbb{R}^n$ with respect to Lebesgue measure.
- Schmidt subspace theorem: equalities hold as well if $\boldsymbol{\xi}$ has $\mathbb{Q}$-linearly independent **algebraic** coordinates.

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For $\boldsymbol{\xi} \in \mathbb{R}^n \setminus \{0\}$, define

- $\lambda(\boldsymbol{\xi})$ = supremum of all $\lambda \geq 0$ such that

  $$\|\mathbf{x}\| \leq Q \quad \text{and} \quad \|\mathbf{x} \wedge \boldsymbol{\xi}\| \leq Q^{-\lambda}$$

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- Schmidt subspace theorem: equalities hold as well if $\boldsymbol{\xi}$ has $\mathbb{Q}$-linearly independent **algebraic** coordinates.
- $\dfrac{\|\mathbf{x} \wedge \boldsymbol{\xi}\|}{\|\boldsymbol{\xi}\|}$ = distance from $\mathbf{x}$ to $\mathbb{R}\boldsymbol{\xi}$ for the Euclidean norm.

1.3. Very singular points

Definition

A point $\boldsymbol{\xi} \in \mathbb{R}^n$ is *very singular* if its coordinates are linearly independent over $\mathbb{Q}$ and if $\widehat{\lambda}(\boldsymbol{\xi}) > 1/(n-1)$.

Theorem (Davenport and Schmidt 1969, R. 2003)

$$\widehat{\lambda}(1, \xi, \xi^2) \le 1/\gamma \cong 0.618 \quad \textit{for all} \quad \xi \in \mathbb{R} \setminus \bar{\mathbb{Q}}$$

where $\gamma = (1+\sqrt{5})/2$, *with equality for an infinite countable set of* ξ.

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- Plenty of very singular points of the form $(1, \xi, \xi^2)$: Bugeaud-Laurent, Fischler, Poëls, ...
- No singular point $(1, \xi)$ in $\mathbb{R}^2$: $\widehat{\lambda}(1, \xi) = 1$ if $\xi \in \mathbb{R} \setminus \mathbb{Q}$
- No known singular points $(1, \xi, \xi^2, \dots, \xi^n)$ with $n \ge 3$: problem strongly connected with approximation to real numbers by algebraic numbers of degree $\le n$, by algebraic integers $\le n + 1$, etc

1.4. Algebraic curves with very singular points

All known algebraic curves $Z \subset \mathbb{P}^{n-1}(\mathbb{R})$ defined over $\mathbb{Q}$ with very singular points are plane conics, i.e. the zero set of a homogenous quadratic polynomial $q(t_0, t_1, t_2) \in \mathbb{Q}[t_0, t_1, t_2]$ in $\mathbb{P}^2(\mathbb{R})$. For all of them,

$$\widehat{\lambda}(Z) := \sup\left\{\widehat{\lambda}(\boldsymbol{\xi})\,;\, \boldsymbol{\xi} \in Z \text{ has } \mathbb{Q}\text{-l.i. coordinates }\right\} = 1/\gamma \cong 0.618$$

Examples:

- $q = t_0 t_2 - t_1^2$: $\sup\{\,\widehat{\lambda}(1, \xi, \xi^2)\,;\, \xi \in \mathbb{R} \setminus \bar{\mathbb{Q}}\,\} = 1/\gamma$.
- $q = t_1^2 - 2t_0^2$: $\sup\{\,\widehat{\lambda}(1, \sqrt{2}, \xi)\,;\, \xi \in \mathbb{R} \setminus \bar{\mathbb{Q}}\,\} = 1/\gamma$.

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Problem

Find algebraic curves Z defined over $\mathbb{Q}$ of degree > 2 containing very singular points.

Remark: In general, [Poëls–R. 2020] computes $\widehat{\lambda}(Z)$ for any quadratic hypersurface defined over $\mathbb{Q}$ in $\mathbb{P}^n(\mathbb{R})$.

2.1. Approximation over K

Let K be a number field of degree d. Define

- $M(K)$ = the set of all places of K,
- K_v = the completion of K at $v \in M(K)$,
- $d_v = [K_v : \mathbb{Q}_v]$, the local degree of K at v,
- $\|\mathbf{x}\|_v$ = the maximum norm of $\mathbf{x} \in K_v^n$.

We assume that K admits a real place w (so $d_w = 1$) and we fix a non-zero point

$$\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in K_w^n = \mathbb{R}^n\,.$$

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$$\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in K_w^n = \mathbb{R}^n .$$

For each non-zero point $\mathbf{x} = (x_1, \ldots, x_n) \in K^n$, we set

$$H(\mathbf{x}) = \prod_{v \in M(K)} \|\mathbf{x}\|_v^{d_v/d} \qquad \text{(absolute Weil height of } \mathbf{x}\text{)},$$

$$D_{\boldsymbol{\xi}}(\mathbf{x}) = \|\mathbf{x} \wedge \boldsymbol{\xi}\|_w^{d_w/d} \prod_{v \neq w} \|\mathbf{x}\|_v^{d_v/d},$$

which depend only on the class of $\mathbf{x}$ in $\mathbb{P}^n(K)$.

2.2. Exponents of approximation over K

For each non-zero $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in K_w^n = \mathbb{R}^n$, we define

- $\widehat{\lambda}_{K,w}(\boldsymbol{\xi})$ = supremum of all $\lambda \geq 0$ such that
$$H(\mathbf{x}) \leq Q \quad \text{and} \quad D_{\boldsymbol{\xi}}(\mathbf{x}) \leq Q^{-\lambda}$$
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Alternative definition

- $\widehat{\lambda}_{K,w}(\boldsymbol{\xi})$ = supremum of all $\lambda \geq 0$ such that

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  has a nonzero solution $\mathbf{x} \in K^n$ for all sufficiently large Q.

- $\lambda_{K,w}(\boldsymbol{\xi})$ = same but for arbitrarily large values of Q.
- Alternative definition is amenable to adelic geometry of numbers.
- For $K = \mathbb{Q} \subset \mathbb{R}$, we recover the usual exponents.

2.3. A result of P. Bel

It is easy to show that: $\widehat{\lambda}_{K,w}(1,\xi) = 1$ for each $\xi \in \mathbb{R} \setminus K$.

A much deeper result is :

Theorem (Bel, 2013)

$$\sup\left\{\widehat{\lambda}_{K,w}(1,\xi,\xi^2)\,;\,\xi \in \mathbb{R} \setminus \bar{\mathbb{Q}}\right\} = \frac{1}{\gamma} \cong 0.618$$

2.3. Extension of scalars

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ be a basis of K over $\mathbb{Q}$. For a given non-zero $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in K_w^n = \mathbb{R}^n$, we construct

$$\boldsymbol{\xi} \otimes \boldsymbol{\alpha} = (\alpha_1\xi_1, \ldots, \alpha_d\xi_1, \ldots, \alpha_1\xi_n, \ldots, \alpha_d\xi_n) \in \mathbb{R}^{dn}.$$

Theorem A

$$d\left(1 + \frac{1}{\widehat{\lambda}_{K,w}(\boldsymbol{\xi})}\right) = 1 + \frac{1}{\widehat{\lambda}(\boldsymbol{\xi} \otimes \boldsymbol{\alpha})}, \quad d\left(1 + \frac{1}{\lambda_{K,w}(\boldsymbol{\xi})}\right) = 1 + \frac{1}{\lambda(\boldsymbol{\xi} \otimes \boldsymbol{\alpha})}.$$

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Corollary

(i) $\widehat{\lambda}(\boldsymbol{\alpha}, \xi\boldsymbol{\alpha}) = \dfrac{1}{2d-1}$ *for each* $\xi \in \mathbb{R} \setminus \bar{\mathbb{Q}}$.

(ii) $\sup\left\{\widehat{\lambda}(\boldsymbol{\alpha}, \xi\boldsymbol{\alpha}, \xi^2\boldsymbol{\alpha})\,;\, \xi \in \mathbb{R} \setminus \bar{\mathbb{Q}}\right\} = \dfrac{1}{\gamma^2 d - 1}$.

In particular, there are very singular points of the form $(\boldsymbol{\alpha}, \xi\boldsymbol{\alpha}, \xi^2\boldsymbol{\alpha})$.

3.1. Dual exponents of approximation (over $\mathbb{Q}$)

For each $\boldsymbol{\xi} \in \mathbb{R}^n \setminus \{0\}$, define

- $\widehat{\omega}(\boldsymbol{\xi}) =$ supremum of all $\omega \geq 0$ such that
$$\|\mathbf{y}\| \leq Q \quad \text{and} \quad |\mathbf{y} \cdot \boldsymbol{\xi}| \leq Q^{-\omega}$$
has a nonzero solution $\mathbf{y} \in \mathbb{Z}^n$ for all sufficiently large Q.
- $\omega(\boldsymbol{\xi}) =$ same but for arbitrarily large values of Q.

- Dirichlet: $\omega(\boldsymbol{\xi}) \geq \widehat{\omega}(\boldsymbol{\xi}) \geq n - 1$
- $\dfrac{|\mathbf{y} \cdot \boldsymbol{\xi}|}{\|\boldsymbol{\xi}\|} =$ distance from $\mathbf{y}$ to $(\mathbb{R}\boldsymbol{\xi})^{\perp}$ for the Euclidean norm.

3.2. Transference inequalities over K

For each $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$ with linearly independent coordinates over $\mathbb{Q}$, we have

- **Khintchine (1926/28) :**
$$\frac{\omega(\boldsymbol{\xi})}{(n-1)\omega(\boldsymbol{\xi})+n} \le \lambda(\boldsymbol{\xi}) \le \frac{\omega(\boldsymbol{\xi})-(n-1)}{n}\,.$$

- **German (2012) :**
$$\frac{\widehat{\omega}(\boldsymbol{\xi})-1}{(n-1)\widehat{\omega}(\boldsymbol{\xi})} \le \widehat{\lambda}(\boldsymbol{\xi}) \le \frac{\widehat{\omega}(\boldsymbol{\xi})-(n-1)}{\widehat{\omega}(\boldsymbol{\xi})}\,.$$

- **Jarník (1938) :** if $n=2$ (i.e. $\boldsymbol{\xi} \in \mathbb{R}^3$), then $\widehat{\lambda}(\boldsymbol{\xi}) = 1 - \dfrac{1}{\widehat{\omega}(\boldsymbol{\xi})}$, i.e.
$$\left(\widehat{\omega}(\boldsymbol{\xi})-1\right)\left(\frac{1}{\widehat{\lambda}(\boldsymbol{\xi})}-1\right) = 1.$$

3.3. Dual exponents over K

For non-zero $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in K_w^n = \mathbb{R}^n$, we define

- $\widehat{\omega}_{K,w}(\boldsymbol{\xi})$ = supremum of all $\omega \geq 0$ such that

$$H(\mathbf{x}) \leq Q \quad \text{and} \quad \left(\frac{|\mathbf{x} \cdot \boldsymbol{\xi}|_w}{\|\mathbf{x}\|_w} \right)^{1/d} H(\mathbf{x}) \leq Q^{-\omega}$$

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has a nonzero solution $\mathbf{x} \in K^n$ for all sufficiently large Q.

- $\omega_{K,w}(\boldsymbol{\xi})$ = same but for arbitrarily large values of Q.

3.4. Extension of scalars (bis)

Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$ be a basis of K over $\mathbb{Q}$. Theorem A admits the following dual statement.

Theorem B

For each non-zero $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in K_w^n = \mathbb{R}^n$, we have

$$d\big(1 + \widehat{\omega}_{K,w}(\boldsymbol{\xi})\big) = 1 + \widehat{\omega}(\boldsymbol{\xi} \otimes \boldsymbol{\alpha}), \quad d\big(1 + \omega_{K,w}(\boldsymbol{\xi})\big) = 1 + \omega(\boldsymbol{\xi} \otimes \boldsymbol{\alpha})\,.$$

Theorem C

- *All transference inequalities mentioned earlier remain true over K.*
- *In particular, if $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) \in K_w^3 = \mathbb{R}^3$ has linearly independent coordinates over K, then*

$$\Big(\widehat{\omega}_{K,w}(\boldsymbol{\xi}) - 1\Big)\left(\frac{1}{\widehat{\lambda}_{K,w}(\boldsymbol{\xi})} - 1\right) = 1$$

3.5. Corollaries

Corollary

If $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) \in K_w^3 = \mathbb{R}^3$ *has linearly independent coordinates over* K*, then*

$$\left(\widehat{\omega}(\boldsymbol{\xi} \otimes \boldsymbol{\alpha}) - (2d-1)\right)\left(\frac{1}{\widehat{\lambda}(\boldsymbol{\xi} \otimes \boldsymbol{\alpha})} - (2d-1)\right) = d^2 .$$

Corollary

$$\sup\left\{\widehat{\omega}(\boldsymbol{\alpha}, \boldsymbol{\xi}\boldsymbol{\alpha}, \xi^2\boldsymbol{\alpha})\,;\, \xi \in \mathbb{R} \setminus \bar{\mathbb{Q}}\right\} = d(\gamma^2+1) - 1.$$

4.1. Geometry of numbers

Let $\mathcal{C}$ be a *(Minkowski) convex body* in $\mathbb{R}^n$.

For $j = 1, \ldots, n$, the j-th minimum of $\mathcal{C}$, denoted $\lambda_j(\mathcal{C})$ is the smallest $\lambda > 0$ such that $\lambda\mathcal{C}$ contains at least j linearly independent elements of $\mathbb{Z}^n$.

$$\lambda_1(\mathcal{C}) \leq \cdots \leq \lambda_n(\mathcal{C})$$

Main results:

- Minkowski's second convex body theorem (1889):

$$\frac{2^n}{n!} \leq \lambda_1(\mathcal{C}) \cdots \lambda_n(\mathcal{C}) \mathrm{vol}(\mathcal{C}) \leq 2^n.$$

- Mahler's theory of polar (dual) convex bodies.
- Mahler's theory of compound bodies.

4.2. Parametric geometry of numbers

For a given non-zero point $\boldsymbol{\xi} \in \mathbb{R}^n$, define

- $\mathcal{C}(Q) = \mathcal{C}_{\boldsymbol{\xi}}(Q) = \left\{ \mathbf{x} \in \mathbb{R}^n \; ; \; \|\mathbf{x}\| \le 1, \; |\mathbf{x} \cdot \boldsymbol{\xi}| \le \dfrac{1}{Q} \right\} \quad (Q \ge 1),$
- $L_i(q) = \log \lambda_i(\mathcal{C}_{\boldsymbol{\xi}}(e^q)) \quad (q \ge 0, \; 1 \le i \le n)$
- $$\begin{array}{rccc} \mathbf{L}_{\boldsymbol{\xi}}: & [0,\infty) & \longrightarrow & \mathbb{R}^n \\ & q & \longmapsto & (L_1(q), \dots, L_n(q)) \end{array}$$

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Theorem (Schmidt and Summerer (2009, 2013))

There exists a constant $\gamma = \gamma(n) > 0$ and an "(n,γ)-system"

$$\begin{array}{rccc} \mathbf{P}: & [0,\infty) & \to & \mathbb{R}^n \\ & q & \mapsto & (P_1(q), \dots, P_n(q)) \end{array}$$

such that $\mathbf{L}_{\boldsymbol{\xi}} - \mathbf{P}$ is bounded. For such $\mathbf{P}$, we have

$$\limsup_{q\to\infty} \frac{P_1(q)}{q} = \limsup_{q\to\infty} \frac{L_{\boldsymbol{\xi},1}(q)}{q} = \frac{1}{\widehat{\omega}(\boldsymbol{\xi})+1}, \quad \dots etc$$

4.3. A simplification and a converse

- The $(n, 0)$-systems, also called n-systems, are piecewise linear functions characterized by simple combinatorial conditions.
- The proper n-systems $\mathbf{P} = (P_1, \dots, P_n)$ are those for which P_1 is unbounded.

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Theorem (R. 2015)

- *The set of maps $\mathbf{L}_{\boldsymbol{\xi}}$ where $\boldsymbol{\xi}$ runs through the set of all non-zero points $\boldsymbol{\xi}$ of $\mathbb{R}^n$ coincides with the set of n-systems modulo the additive group of bounded functions.*
- *The set of maps $\mathbf{L}_{\boldsymbol{\xi}}$ where $\boldsymbol{\xi}$ has $\mathbb{Q}$-linearly independent coordinates coincides with the set of proper n-systems modulo bounded functions.*

4.4. Parametric geometry of numbers over K

Let $K_{\mathbb{A}}$ denote the adele ring of K. For given $\boldsymbol{\xi} \in K_w^n = \mathbb{R}^n$, and each $Q \geq 1$, define

- $\mathcal{C}_{\boldsymbol{\xi}}(Q) = \left\{(\mathbf{x}_v) \in K_{\mathbb{A}}^n \;;\; |\mathbf{x}_w \cdot \boldsymbol{\xi}| \leq Q^{-1} \text{ and } \|\mathbf{x}\|_v \leq 1 \text{ if } v \neq w\right\}$,
- $L_i(q) = \log \lambda_i(\mathcal{C}_{\boldsymbol{\xi}}(e^q)) \quad (q \geq 0,\ 1 \leq i \leq n)$
- $$\begin{array}{rccc} \mathbf{L}_{\boldsymbol{\xi}}: & [0,\infty) & \longrightarrow & \mathbb{R}^n \\ & q & \longmapsto & (L_1(q), \dots, L_n(q)) \end{array}$$

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Lemma

- $\displaystyle \limsup_{q\to\infty} \frac{L_{\boldsymbol{\xi},1}(q)}{q} = \frac{1}{d(\widehat{\omega}_{K,w}(\boldsymbol{\xi})+1)}$,
- $\displaystyle \liminf_{q\to\infty} \frac{L_{\boldsymbol{\xi},1}(q)}{q} = \frac{1}{d(\omega_{K,w}(\boldsymbol{\xi})+1)}$, $\ldots etc$

4.5. Main result

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ be a basis of K over $\mathbb{Q}$. For a given $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in K_w^n = \mathbb{R}^n$, we obtain two families of adelic convex bodies

$$\mathcal{C}_{\boldsymbol{\xi}}(Q) \subset K_{\mathbb{A}}^n \quad \text{and} \quad \mathcal{C}_{\boldsymbol{\alpha}\otimes\boldsymbol{\xi}}(Q) \subset \mathbb{Q}_{\mathbb{A}}^{dn}.$$

Theorem

For each $i = 1, \ldots, n$, each $j = 1, \ldots, d$ and each $Q \geq 1$, we have

$$\lambda_{d(i-1)+j}\big(\mathcal{C}_{\boldsymbol{\xi}\otimes\boldsymbol{\alpha}}(Q)\big) \asymp \lambda_i\big(\mathcal{C}_{\boldsymbol{\xi}}(Q)\big)$$

with implied constant that depend only on K, n and $\boldsymbol{\xi}$.

Proof uses Jeff Thunder's principle to derive the adelic Minkowski theorem of McFeat and Bombieri–Vaaler over K from the standard one over $\mathbb{Q}$.

$$\boxed{\text{Theorem} \Rightarrow \text{Theorems A, B and C.}}$$

¡Muchísimas gracias por su atención!

Thank you!

Merci!

Description of n-systems as games (Luca Ghidelli)

We can view an n-system as giving the positions of n players $P_1, \ldots, P_n$ moving on a line, as a function of the time q, according to the following rules.

- At time $q = 0$, they all stand at position 0.
- They always remain in the same order (P_1 cannot overpass P_2, nor P_2 can overpass P_3, etc).
- At any time, only the player who has the ball can move and he moves at constant speed 1.
- The player who holds the ball can only pass it to a player that is behind him or next to him.

Combined graph of the 3-system in the animation