On the topology of Diophantine approximation Spectra

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Classical setting

Let $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n \setminus \{0\}$.

Define $\omega_{n-1}(\mathbf{u})$ = supremum of all $\omega \geq 0$ for which the inequalities

$$\|\mathbf{x}\| = \max\{|x_1|, \ldots, |x_n|\} \leq Q\,, \quad |\mathbf{x} \cdot \mathbf{u}| = |x_1 u_1 + \cdots + x_n u_n| \leq Q^{-\omega}$$

have a solution $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n \setminus \{0\}$ for **arbitrarily large** Q's.

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Hermite (1873), Mahler (1932): $\omega_{n-1}(1, e, \dots, e^{n-1}) = n - 1$, so $e \notin \bar{\mathbb{Q}}$.

There are many other exponents of Diophantine approximation

$$\omega_i(\mathbf{u}),\ \hat{\omega}_i(\mathbf{u})\ (1 \leq i \leq n-1),\ \text{etc.}\dots$$

The spectrum of a family of exponents $\tau_1, \ldots, \tau_m$ is the set

$$\{(\tau_1(\mathbf{u}), \ldots, \tau_m(\mathbf{u}))\,;\, \mathbf{u} \in \mathbb{R}^n \text{ has } \mathbb{Q}\text{-linearly independent coordinates}\}$$

A new tool: Parametric Geometry of numbers

- Schmidt 1983
- Schmidt and Summerer 2009, 2013
- Roy 2015

Reformulation in the new language

Consider the one-parameter family of Minkowski convex bodies

$$\mathcal{C}_{\mathbf{u}}(q) = \left\{\mathbf{x} \in \mathbb{R}^n \; ; \; \|\mathbf{x}\| \le 1, \; |\mathbf{x} \cdot \mathbf{u}| \le e^{-q}\right\} \quad (q \ge 0).$$

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For $i = 1, \dots, n$, define

$$\begin{aligned} L_{\mathbf{u},i}(q) = {} & \text{the smallest } \lambda \text{ such that } e^{\lambda}\mathcal{C}_{\mathbf{u}}(q) \text{ contains at least} \\ & i \text{ linearly independent elements of } \mathbb{Z}^n \end{aligned}$$

and form the map

$$\begin{aligned} \mathbf{L}_{\mathbf{u}} \colon [0,\infty) & \longrightarrow \mathbb{R}^n \\ q & \longmapsto (L_{\mathbf{u},1}(q), \dots, L_{\mathbf{u},n}(q)) \end{aligned}$$

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Classical exponents of approximation can be computed from $\mathbf{L}_{\mathbf{u}}$:

$$\omega_{n-1}(\mathbf{u}) = \frac{1}{\underline{\varphi}_1(\mathbf{u})} - 1 \quad \text{where} \quad \underline{\varphi}_i(\mathbf{u}) := \liminf_{q \to \infty} \frac{L_{\mathbf{u},i}(q)}{q}$$

$$\hat{\omega}_{n-1}(\mathbf{u}) = \frac{1}{\bar{\varphi}_1(\mathbf{u})} - 1 \quad \text{where} \quad \bar{\varphi}_i(\mathbf{u}) := \limsup_{q \to \infty} \frac{L_{\mathbf{u},i}(q)}{q}$$

Some known spectra

- $(\underline{\varphi}_1, \bar{\varphi}_n)$: Khintchine (1926,1928), Jarník (1935, 1936)
- $(\bar{\varphi}_1, \underline{\varphi}_n)$: Jarník (1938), German (2012),
  Schmidt and Summerer ($\geq$ 2016), Marnat ($\geq$ 2016)
- $(\underline{\varphi}_1, \bar{\varphi}_1, \underline{\varphi}_3, \bar{\varphi}_3)$ for $n = 3$: Laurent (2009)
- $(\underline{\psi}_1, \ldots, \underline{\psi}_{n-1})$: Schmidt (1967), Laurent (2009), Roy (2016)

where

$$\underline{\psi}_i(\mathbf{u}) = \liminf_{q\to\infty} q^{-1}(L_{\mathbf{u},1}(q) + \cdots + L_{\mathbf{u},i}(q)) \quad (1 \leq i \leq n-1).$$

n-systems

Definition (Schmidt and Summerer 2013)

An *n-system* is a map $\begin{array}{rcl} \mathbf{P}\colon [q_0,\infty) & \longrightarrow & \mathbb{R}^n \\ q & \longmapsto & (P_1(q),\dots,P_n(q)) \end{array}$ such that

- $0 \le P_1(q) \le \cdots \le P_n(q)$ and $P_1(q)+\cdots+P_n(q) = q$ $(q \ge q_0)$,
- each P_i is continuous and piecewise linear with slopes 0 and 1,
- for each $q > q_0$, there is exactly one pair of indices k and ℓ for which
$$P_k'(q^-) = 1 \quad \text{and} \quad P_\ell'(q^+) = 1,$$
- whenever $\ell > k$, we have $P_k(q) = \cdots = P_\ell(q)$.

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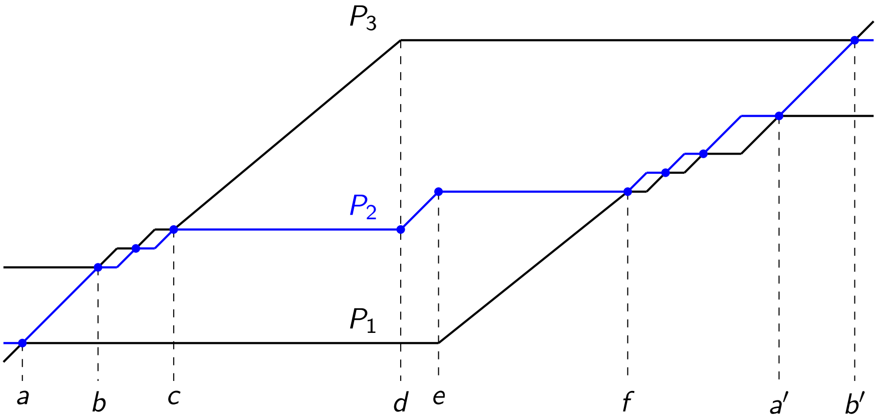
Theorem (Roy 2015)

For each $\mathbf{u} \in \mathbb{R}^n \setminus \{0\}$, there is an n-system $\mathbf{P}$ such that $\mathbf{P} - \mathbf{L}_\mathbf{u}$ is bounded, and conversely.

Basic pattern for the combined graph of a 3-system



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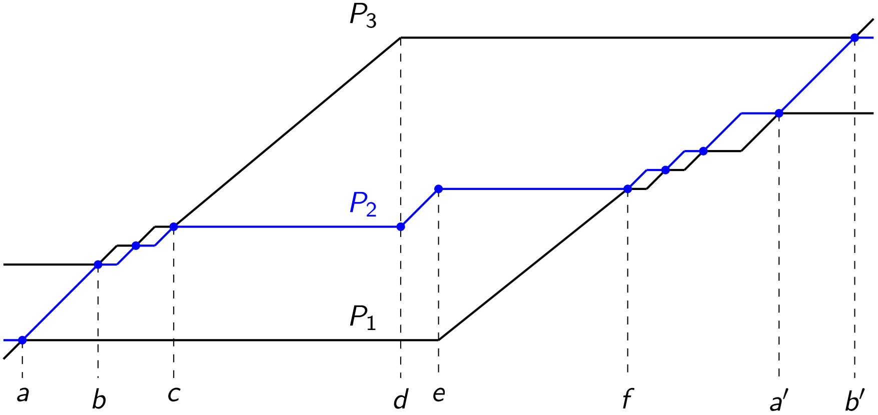


The knowledge of P_2 determines the whole map $\mathbf{P} : [q_0, \infty) \to \mathbb{R}^3$.

A general notion of spectrum

Suppose that $\mathbf{P} - \mathbf{L}_\mathbf{u}$ is bounded for some $\mathbf{u} \in \mathbb{R}^n \setminus \{0\}$. Then

- $\underline{\varphi}_i(\mathbf{u}) = \liminf_{q\to\infty} q^{-1} P_i(q) \quad (1 \le i \le n)$
- the coordinates of $\mathbf{u}$ are linearly independent over $\mathbb{Q}$ iff $\lim_{q\to\infty} P_1(q) = \infty$: such a system is said to be *proper*.

Definition

Let $\begin{array}{rccc} T: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ & \mathbf{x} & \longmapsto & (T_1(\mathbf{x}), \dots, T_m(\mathbf{x})) \end{array}$ be a linear map.

For each n-system $\mathbf{P}$ define

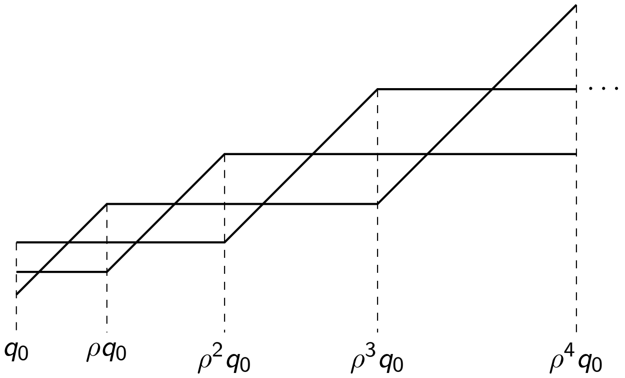
$$\mu_T(\mathbf{P}) = \left(\liminf_{q\to\infty} T_1(q^{-1}\mathbf{P}(q)), \dots, \liminf_{q\to\infty} T_m(q^{-1}\mathbf{P}(q)) \right).$$

The spectrum of T is $\{\mu_T(\mathbf{P})\,;\, \mathbf{P} \text{ is a proper } n\text{-system}\}$

Self-similar systems

We say that an n-system $\mathbf{P}\colon [q_0, \infty) \to \mathbb{R}^n$ is self-similar if there exists $\rho > 1$ such that $\mathbf{P}(\rho q) = \rho \mathbf{P}(q)$ for each $q \geq q_0$.

Example (n=3):



Main result

Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then

- The set
$$\{\mu_T(\mathbf{P})\,;\ \mathbf{P} \text{ is a proper self-similar } n\text{-system}\}$$
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- The spectrum of T is a compact connected subset of $\mathbb{R}^m$.
- If $n = 3$, then the spectrum of T is a **semi-algebraic subset** of $\mathbb{R}^m$. Moreover, it is closed under componentwise minimum: if $(x_1, \dots, x_m)$ and $(y_1, \dots, y_m)$ belong to the spectrum of T, then
$$(\min(x_1, y_1), \dots, \min(x_m, y_m))$$
also belong to that spectrum.

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Application: Computation of the spectrum of $(\underline{\varphi}_1, \bar{\varphi}_1, \underline{\varphi}_2, \bar{\varphi}_2, \underline{\varphi}_3, \bar{\varphi}_3)$ when $n = 3$.

Open problem: Is the spectrum always semi-algebraic?