Simultaneous rational approximation to successive powers of a real number (Part II)

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Webinar on Diophantine approximation and homogeneous dynamics

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Introduction

Fix $\xi \in \mathbb{R} \setminus \overline{\mathbb{Q}}$. Recall that for each $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}$

$$L_\xi(\mathbf{x}) = \max\{|x_0\xi - x_1|, \dots, |x_0\xi^n - x_n|\}.$$

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$\hat{\lambda}_n(\xi)$ is the supremum of the reals numbers $\lambda \geq 0$ such that

$$\|\mathbf{x}\| \leq X \quad \text{and} \quad L_\xi(\mathbf{x}) \leq X^{-\lambda}$$

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Theorem (P.-Roy, 2021)

$\hat{\lambda}_n(\xi) \leq \dfrac{1}{n/2 + a\sqrt{n} + 1/3}$ where $a = (1 - \log 2)/2 \cong 0.153$ for $n \geq 2$.

Truncated points

Given $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ and $\ell \in \{0, 1, \dots, n\}$, define

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Suppose that $\hat{\lambda}_n(\xi) > 1/(n-\ell+1)$ for some integer ℓ with $0 \leq \ell \leq n/2$. Then $\dim \mathcal{U}^{\ell}(\mathbf{x}_i) = \ell + 1$ for each sufficiently large i .

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By the proposition $\hat{\lambda}_n(\xi) \leq \max\{1/(n-\ell+1), 1/\ell\}$, so $\hat{\lambda}_n(\xi) \leq 1/\lfloor n/2 \rfloor$ (Davenport-Schmidt, 1969).

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Goal: increase the dimension by considering $\mathcal{U}^\ell(\mathbf{x}_i, \dots, \mathbf{x}_j)$, get an estimate for its height, and thus a smaller upper bound for $\hat{\lambda}_n(\xi)$.

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remark. For $\ell = \lfloor n/2 \rfloor$, the space $\mathcal{U}^\ell(\mathbf{x}_i) \subseteq \mathbb{R}^{n+1-\ell}$ is the whole space (n even) or an hyperplane (n odd).

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- How to estimate $H(\mathcal{U}^\ell(\mathbf{x}_i, \dots, \mathbf{x}_j))$?
- How to control the size of the points $\mathbf{x}_i, \dots, \mathbf{x}_{j+1}$?

Definitions

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Main goal

Study of the spaces $\mathcal{U}^\ell(A_j(i))$ (Dimension? Height?)

Properties $\mathcal{P}(j,\ell)$

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- If $\hat{\lambda}_n(\xi)>1/(n-\ell+1)$, then $\mathcal{P}(0,\ell)$ holds (BS, 2021).
- $\mathcal{P}(j,\ell)\Rightarrow\mathcal{P}(j+1,\ell-1)$ (properties of $\ell\mapsto\dim\mathcal{U}^\ell(A)$).

First general height estimates

Proposition D

Suppose that $\mathcal{P}(j,\ell)$ holds (with $j,\ell \in \{0,\dots,n\}$). Then

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- For $j=0$, we get $H(\mathcal{U}^\ell(\mathbf{x}_i)) \ll Y_{-1}(i)^{1-\ell\lambda} = X_i^{1-\ell\lambda}$.
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Sketch of the proof (for $j=3$). Write

$$\begin{aligned} A_1(i) &= \langle \mathbf{x}_i, \mathbf{x}_{i+1}\rangle = \langle \mathbf{x}_i,\dots,\mathbf{x}_q\rangle,\\ A_2(i) &= \langle \mathbf{x}_i,\dots,\mathbf{x}_q,\mathbf{x}_{q+1}\rangle = \langle \mathbf{x}_i,\dots,\mathbf{x}_r\rangle,\\ A_3(i) &= \langle \mathbf{x}_i,\dots,\mathbf{x}_r,\mathbf{x}_{r+1}\rangle. \end{aligned}$$

First general height estimates

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Note that $(Y_0(i), Y_1(i), Y_2(i)) = (X_{i+1}, X_{q+1}, X_{r+1})$.

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Since $\mathbf{x}_{r+1} \notin \langle \mathbf{x}_i, \dots, \mathbf{x}_r \rangle$, we have $\dim\langle \mathbf{x}_q, \mathbf{x}_{q+1}, \dots, \mathbf{x}_{r+1} \rangle \geq 3$.

Proof of the proposition (end)

By $\mathcal{P}(3,\ell)$, we have $\dim \mathcal{U}^{\ell}(\mathbf{x}_{r+1}) = \ell + 1$, as well as

- $\dim \mathcal{U}^{\ell}(\mathbf{x}_r, \mathbf{x}_{r+1}) \geq \ell + 2$,
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- one point $\mathbf{y}$ among $\mathbf{x}_r^{(0,\ell)}, \dots, \mathbf{x}_r^{(\ell,\ell)} \Rightarrow L_\xi(\mathbf{y}) \leq L_r$,
- one point $\mathbf{z}$ among the points $\mathbf{x}_k^{(0,\ell)}, \dots, \mathbf{x}_k^{(\ell,\ell)}$ $(q \leq k \leq r)$, $\Rightarrow L_\xi(\mathbf{z}) \leq L_q$,
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$$\begin{aligned} \Rightarrow H(\mathcal{U}^\ell(A_j(i))) &\ll X_{r+1} L_{r+1}^\ell L_r L_q L_i \ll X_{r+1}^{1-\ell\lambda} X_{r+1}^{-\lambda} X_{q+1}^{-\lambda} X_{i+1}^{-\lambda} \\ &\ll Y_2(i)^{1-\ell\lambda} Y_2(i)^{-\lambda} Y_1(i)^{-\lambda} Y_0(i)^{-\lambda}. \end{aligned}$$

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Problem 2. How to ensure that $\mathcal{P}(j,\ell)$ holds (for a large j)?

Growth's estimates

Proposition E

Suppose that $\mathcal{P}(j,\ell)$ holds for some integers $1 \leq j \leq \ell < n$. Then, for each $i \geq 0$, we have

$$Y_j(i)^{(\ell-j+1)\lambda} \ll Y_{j-1}(i)\Big(\prod_{m=1}^{j-1} Y_m(i)^{-2\lambda}\Big) Y_0(i)^{-\lambda}.$$

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Corollary

Suppose that $\mathcal{P}(j,\ell)$ holds for some integers $1 \leq j \leq \ell < n$, and that $\theta^{j-1}+\theta^j \geq 1$, where $\theta = \ell\lambda/(1-\lambda)$. Then we have $Y_m(i)^\theta \ll Y_{m-1}(i)$ for each $i \geq 0$ and each $m = 0, 1, \ldots, j$.

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Ingredients for Proposition E: an alternative height estimate.

Ideas of the proof of Proposition E

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with q and r maximal.

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We construct a basis of $\mathcal{U}^\ell(A_j(i))$ by starting with the points $\mathbf{x}_r^{(0,\ell)}, \dots, \mathbf{x}_r^{(\ell,\ell)}$ and by completing as previously.

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Problem. Factor X_r potentially way bigger than $Y_{j-1}(i)$. How to get rid of it?

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Solution. Schmidt's inequality $H(U+V)H(U\cap V) \ll H(U)H(V)$.

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hence $1 \ll H(U) Y_j(i)^{-m\lambda} \cdots$.

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We conclude with the estimate $H(A_j(i)) \ll Y_{j-1}(i)(Y_{j-1}(i) \cdots Y_0(i))^{-\lambda}$ combined with $1 \ll H(U) Y_j(i)^{-m\lambda} \cdots$.

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Proposition F (Corollary of Proposition C)

Let j,ℓ with $2\ell + j \leq n$. Suppose that $\mathcal{P}(j,\ell-1)$ holds but not $\mathcal{P}(j,\ell)$. Then, there are infinitely many i such that

$$1 \ll H(\mathcal{U}^{\ell-1}(A_j((i))) L_{i-1}^m \qquad (1)$$

where $m = n - j - 2\ell + 2$.

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Let j, ℓ with $2\ell + j \leq n$. Suppose that $\mathcal{P}(j, \ell - 1)$ holds but not $\mathcal{P}(j, \ell)$. Then, there are infinitely many i such that

$$1 \ll H(\mathcal{U}^{\ell-1}(A_j((i)))L_{i-1}^{m} \tag{1}$$

where $m = n - j - 2\ell + 2$.

Since $\mathcal{P}(j, \ell - 1)$, we can use our estimate $H(\mathcal{U}^{\ell-1}(A_j(i))$ as well as $Y_k(i)^{\theta} \ll Y_{k-1}(i)$ for $k = 0, \ldots, j - 1$.

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Hence $\hat{\lambda}_n(\xi)\le 1/(\ell+\theta+\cdots+\theta^{j-1}+m\theta^j)$.

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So $\hat{\lambda}_n(\xi) \le \max\{\rho_n, \rho'_n\}$. By optimizing the choice of ℓ and j, we get

$$\hat{\lambda}_n(\xi) \le \frac{1}{n/2 + a\sqrt{n} + 1/3}.$$

Thank you.

Ideas of the proof of Proposition F

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$$H(\mathcal{U}^{\ell-1}(A_i(j))) \asymp H(V)^m,$$

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Raising to the power m, we get $1 \ll H(\mathcal{U}^{\ell-1}(A_i(j)))L_{i-1}^m$.