Parametric geometry of numbers and simultaneous approximation to geometric progressions

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Diophantine Approximation, Fractal Geometry and Related topics

Université Gustave Eiffel

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https://mysite.science.uottawa.ca/droy//talks.html

I. Uniform rational approximation

Let $\mathbf{u}$ be a non-zero point of $\mathbb{R}^{n+1}$ for some integer $n \geq 1$. We define $\widehat{\lambda}(\mathbf{u})$ to be the supremum of the real numbers $\lambda > 0$ for which the inequalities

$$\|\mathbf{x}\| \leq X \quad \text{and} \quad \|\mathbf{x} \wedge \mathbf{u}\| \leq X^{-\lambda}$$

admit a non-zero solution $\mathbf{x} \in \mathbb{Z}^{n+1}$ for each sufficiently large X.

- $\widehat{\lambda}(\mathbf{u}) \geq 1/n$ by a theorem of Dirichlet.
- $\widehat{\lambda}(\mathbf{u}A) = \widehat{\lambda}(\mathbf{u})$ for each $A \in \mathrm{GL}_{n+1}(\mathbb{Q})$.

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For $\xi \in \mathbb{R}$, we set $\widehat{\lambda}_n(\xi) = \widehat{\lambda}(1, \xi, \ldots, \xi^n)$.

- $\widehat{\lambda}_n(\xi) = 1/n$ for almost all $\xi \in \mathbb{R}$ and each $\xi \in \bar{\mathbb{Q}}$ with $[\mathbb{Q}(\xi) : \mathbb{Q}] > n$.
- $\widehat{\lambda}_n(g.\xi) = \widehat{\lambda}(\xi)$ for each $g \in \mathrm{GL}_2(\mathbb{Q})$.

Some estimates

Let $\xi \in \mathbb{R} \setminus \bar{\mathbb{Q}}$. Set $\gamma = (1+\sqrt{5})/2 \cong 1.618$.

1) **Davenport & Schmidt (1969):** $\widehat{\lambda}_n(\xi) \le \begin{cases} 1/\gamma \cong 0.618 & \text{if } n = 2, \\ 1/2 & \text{if } n = 3, \\ 1/\lfloor n/2 \rfloor & \text{if } n \ge 4. \end{cases}$

2) **Laurent (2003):** $\widehat{\lambda}_n(\xi) \le 1/\lceil n/2 \rceil$ if $n \ge 3$.

3) **R. (2003):** $\widehat{\lambda}_2(\xi) = 1/\gamma$ for an infinite countable set of ξ.

4) **R. (2008):** $\widehat{\lambda}_3(\xi) \le \lambda_3 \cong 0.4245$ the positive root of $T^2 - \gamma^3 T + \gamma$.

Goals of the talk:

- similarities between **3)** and **4)**,
- hints for the proof that λ_3 in **4)** can be improved,
- relevance of parametric geometry of numbers.

II. Two families of convex bodies

Let $\mathbf{u} \in \mathbb{R}^n$ with $\mathbb{Q}$-linearly independent coordinates. For each $q \geq 0$, set

$$\mathcal{C}_{\mathbf{u}}(q) = \{\mathbf{x} \in \mathbb{R}^n \,;\, \|\mathbf{x}\| \leq 1 \quad \text{and} \quad |\mathbf{x} \cdot \mathbf{u}| \leq e^{-q}\},$$
$$\mathcal{C}^*_{\mathbf{u}}(q) = \{\mathbf{x} \in \mathbb{R}^n \,;\, \|\mathbf{x}\| \leq 1 \quad \text{and} \quad |\mathbf{x} \wedge \mathbf{u}| \leq e^{-q}\},$$

and, for each $j = 1, \dots, n$, define

$$\begin{aligned} L_{\mathbf{u},j}(q) = {} & \text{smallest } L \geq 0 \text{ such that } e^L \mathcal{C}_{\mathbf{u}}(q) \text{ contains at least} \\ & j \text{ linearly independent points of } \mathbb{Z}^n, \\ L^*_{\mathbf{u},j}(q) = {} & \text{smallest } L \geq 0 \text{ such that } e^L \mathcal{C}^*_{\mathbf{u}}(q) \text{ contains at least} \\ & j \text{ linearly independent points of } \mathbb{Z}^n. \end{aligned}$$

Finally define $\mathbf{L}_{\mathbf{u}} \colon [0,\infty) \to \mathbb{R}^n$ and $\mathbf{L}^*_{\mathbf{u}} \colon [0,\infty) \to \mathbb{R}^n$ by

$$\mathbf{L}_{\mathbf{u}}(q) = (L_{\mathbf{u},1}(q), \dots, L_{\mathbf{u},n}(q)) \quad \text{and} \quad \mathbf{L}^*_{\mathbf{u}}(q) = (L^*_{\mathbf{u},1}(q), \dots, L^*_{\mathbf{u},n}(q)).$$

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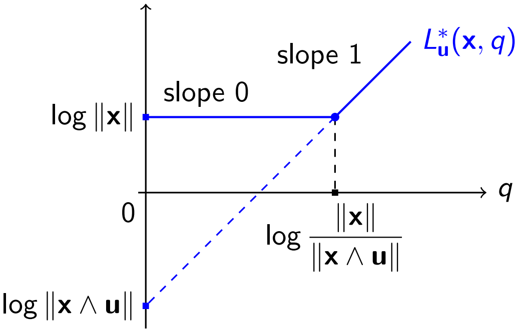
Mahler's duality : $\boxed{L_{\mathbf{u},j}(q) + L^*_{\mathbf{u},n+1-j}(q) = q + \mathcal{O}(1) \ \text{ for } j = 1, \dots, n.}$

The trajectory of a point

The *trajectory* of a non-zero point $\mathbf{x} \in \mathbb{Z}^n$ relative to the family $\mathcal{C}^*_{\mathbf{u}}(q)$ is the map $L^*_{\mathbf{u}}(\mathbf{x}, \cdot) \colon [0, \infty) \to \mathbb{R}$ given by

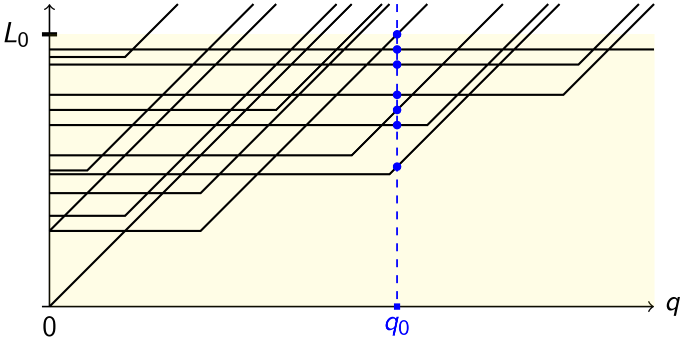
$$\begin{aligned} L^*_{\mathbf{u}}(\mathbf{x}, q) &= \text{smallest } L \text{ such that } \mathbf{x} \in e^L \mathcal{C}^*_{\mathbf{u}}(q) \\ &= \max\{\log \|\mathbf{x}\|, \ q + \log \|\mathbf{x} \wedge \mathbf{u}\|\}. \end{aligned}$$

It is continuous and piecewise linear with slope 0 then 1.



The first minimum

Finitely many non-zero points $\mathbf{x} \in \mathbb{Z}^n$ have their trajectory cross the domain $0 \leq L \leq L_0$: they all have $\log \|\mathbf{x}\| \leq L_0$.



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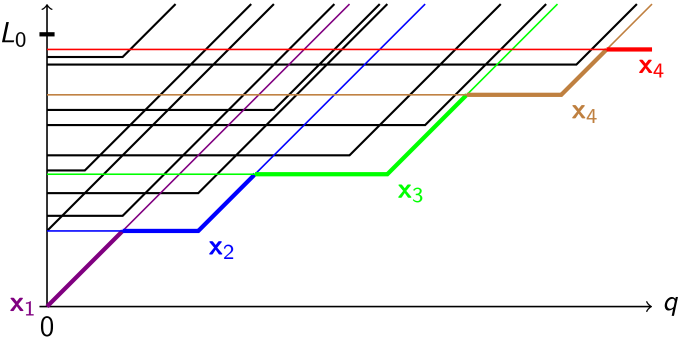
$$L^*_{\mathbf{u},1}(q) = \min\{L^*_{\mathbf{u}}(\mathbf{x}, q)\,;\, \mathbf{x} \in \mathbb{Z}^n \setminus \{0\}\}$$

is a continuous piecewise linear function of $q \ge 0$ with slopes 0 and 1, and it is realized by a sequence $(\mathbf{x}_i)_{i\ge1}$ of integer points called "minimal points".



Link with the exponent $\widehat{\lambda}(\mathbf{u})$

Fix $\lambda > 0$. The following conditions are equivalent:

- There exists a constant $c > 0$ such that the conditions

$$\|\mathbf{x}\| \leq X \quad \text{and} \quad \|\mathbf{x} \wedge \mathbf{u}\| \leq cX^{-\lambda}$$

  admit a non-zero solution $\mathbf{x} \in \mathbb{Z}^n$ for any sufficiently large X.
- We have $\|\mathbf{x}_i \wedge \mathbf{u}\| \ll \|\mathbf{x}_{i+1}\|^{-\lambda}$ for each $i \geq 1$.
- We have $L^*_{\mathbf{u},1}(q) \leq \dfrac{q}{1+\lambda} + \mathcal{O}(1)$ as $q \to \infty$.

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- We have $L^*_{\mathbf{u},1}(q) \le \dfrac{q}{1+\lambda} + \mathcal{O}(1)$ as $q \to \infty$.

Corollary (Schmidt and Summerer (2013))

For any non-zero $\mathbf{u} \in \mathbb{R}^n$, we have

$$\widehat{\lambda}(\mathbf{u}) = \frac{1}{\bar{\varphi}(\mathbf{u})} - 1 \quad \textit{where} \quad \bar{\varphi}(\mathbf{u}) = \limsup_{q\to\infty} \frac{L^*_{\mathbf{u},1}(q)}{q}.$$

III. The n-systems

Let $q_0 \geq 0$. An n-**system** on $[q_0, \infty)$ is a map $\mathbf{P} = (P_1, \ldots, P_n)$ from $[q_0, \infty)$ to $\mathbb{R}^n$ with the following properties.

(S1) Each P_j is continuous and piecewise linear with slopes 0 and 1.

(S2) We have $0 \leq P_1(q) \leq \cdots \leq P_n(q)$ and $P_1(q) + \cdots + P_n(q) = q$ for each $q \geq q_0$.

(S3) For each $j = 1, \ldots, n-1$ and each $q > q_0$ at which $P_1 + \cdots + P_j$ decreases slope from 1 to 0, we have $P_j(q) = P_{j+1}(q)$.

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The **switch points** of such a map $\mathbf{P}$ are q_0 and all points $q > q_0$ at which at least one of the sums $P_1 + \cdots + P_j$ with $1 \leq j < n$ increases slope from 0 to 1.

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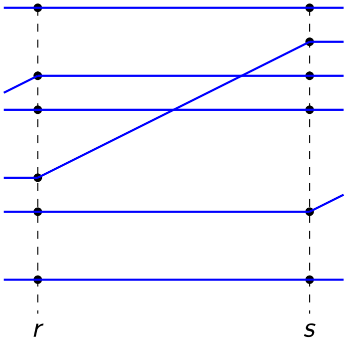
Let $\delta > 0$. We say that $\mathbf{P}$ is **rigid of mesh** $\delta > 0$ if $P_1(q), \dots, P_n(q)$ are distinct positive multiples of δ for each switch point q of $\mathbf{P}$.

The combined graph of a rigid n-system

The combined graph of an n-system $\mathbf{P} = (P_1, \ldots, P_n)$ over an interval is the union of the graphs of $P_1, \ldots, P_n$ over that interval. If $\mathbf{P}$ is rigid and $r < s$ are consecutive switch points of $\mathbf{P}$, then it combined graph has the following form over a neighborhood of $[r, s]$ (here $n = 6$).



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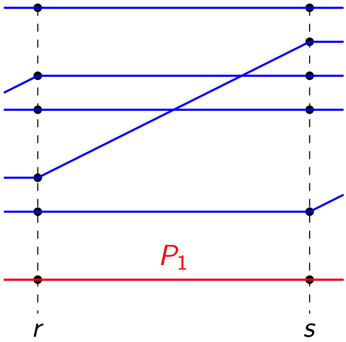

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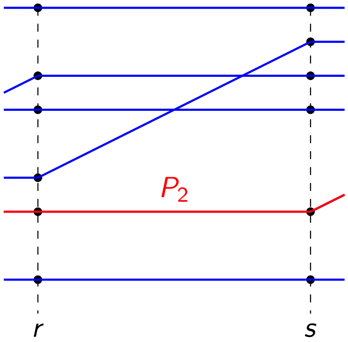

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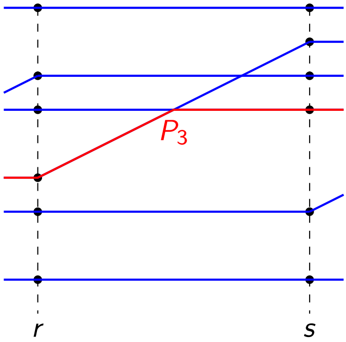

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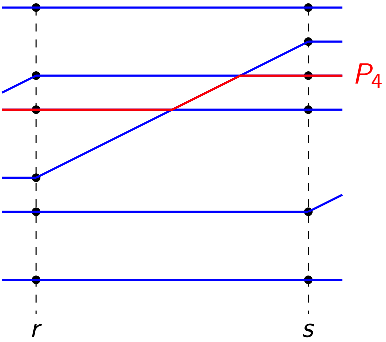

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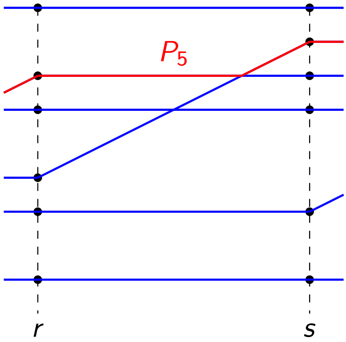

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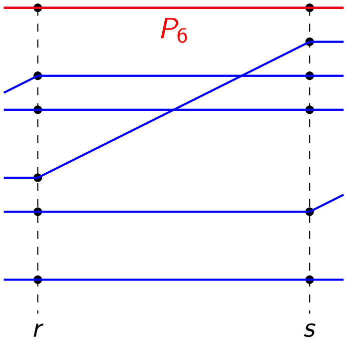

Characterization of the minima up to bounded functions

Theorem (R. 2015)

For each nonzero $\mathbf{u} \in \mathbb{R}^n$ *and each* $\delta > 0$*, there exists a rigid n-system* $\mathbf{P}\colon [q_0, \infty) \to \mathbb{R}^n$ *of mesh* δ *such that* $\mathbf{L_u} - \mathbf{P}$ *is bounded on* $[q_0, \infty)$*.*
Conversely, given any n-system $\mathbf{P}\colon [q_0, \infty) \to \mathbb{R}^n$*, there exists a nonzero* $\mathbf{u} \in \mathbb{R}^n$ *such that* $\mathbf{L_u} - \mathbf{P}$ *is bounded on* $[q_0, \infty)$*.*

- Schmidt and Summerer prove the first assertion with a larger class of functions $\mathbf{P}$ called (n, γ)-systems, where γ is an auxiliary parameter.

Dual n-systems

Let $q_0 \geq 0$. A **dual n-system** on $[q_0, \infty)$ is a map $\mathbf{P}^*\colon [q_0, \infty) \to \mathbb{R}^n$ given by

$$\mathbf{P}^*(q) = (q - P_n(q), \dots, q - P_1(q)) \quad (q \geq q_0)$$

for some n-system $\mathbf{P} = (P_1, \dots, P_n)\colon [q_0, \infty) \to \mathbb{R}^n$.

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Equivalently, this is a map $\mathbf{P}^* = (P_1^*, \ldots, P_n^*)\colon [q_0, \infty) \to \mathbb{R}^n$ with the following properties.

(S1) Each P_j^* is continuous and piecewise linear with slopes 0 and 1.

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(S3) For each $j = 1, \ldots, n-1$ and each $q > q_0$ at which $P_1^* + \cdots + P_j^*$ decreases slope from j to $j-1$, we have $P_j^*(q) = P_{j+1}^*(q)$.

Its **switch points** are q_0 and the points $q > q_0$ at which at least one of the sums $P_1^* + \cdots + P_j^*$ with $1 \leq j < n$ increases slopes from $j-1$ to j.

The combined graph of a dual rigid n-system

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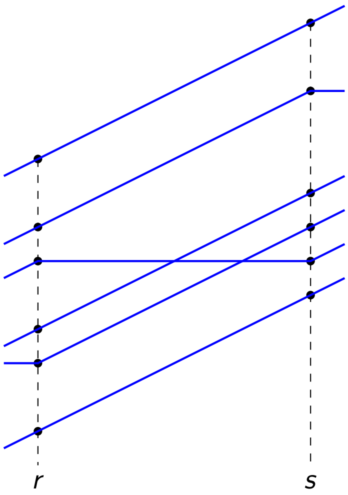

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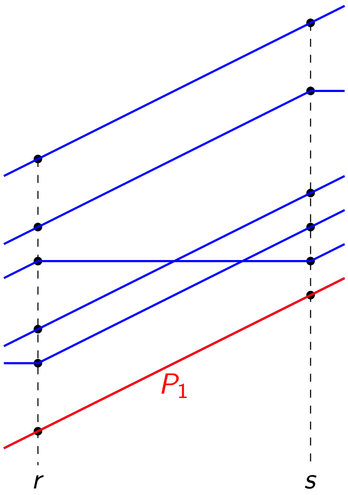

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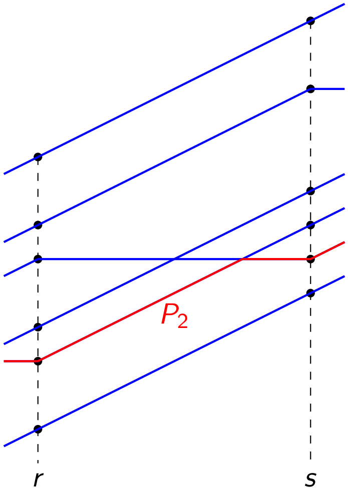

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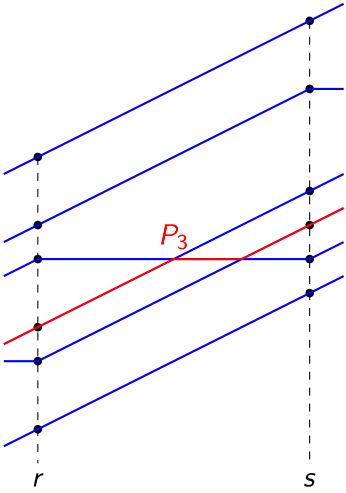

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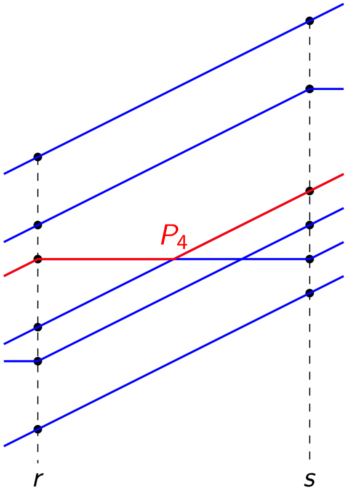

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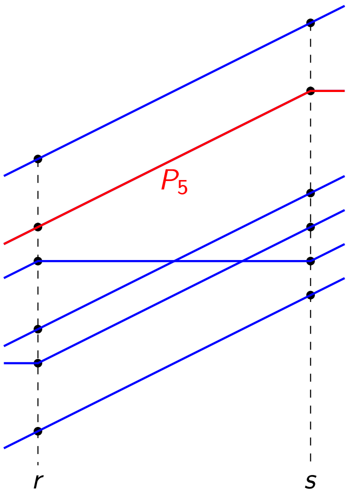

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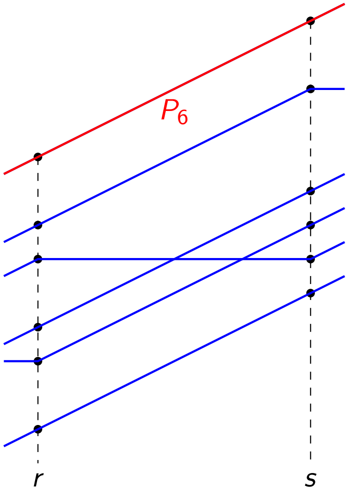
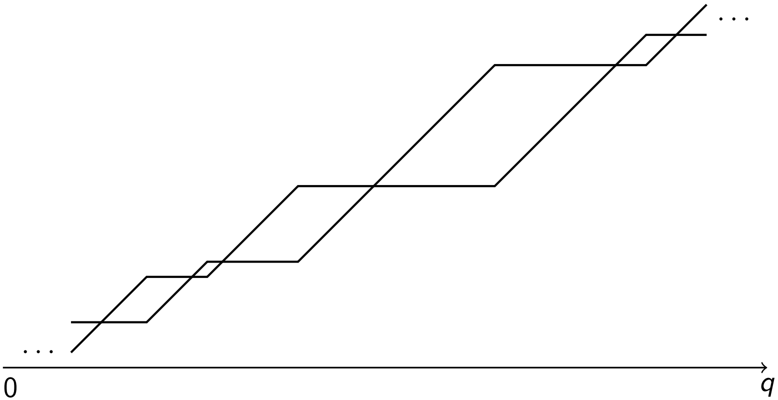


Characterization of the minima up to bounded functions

Corollary

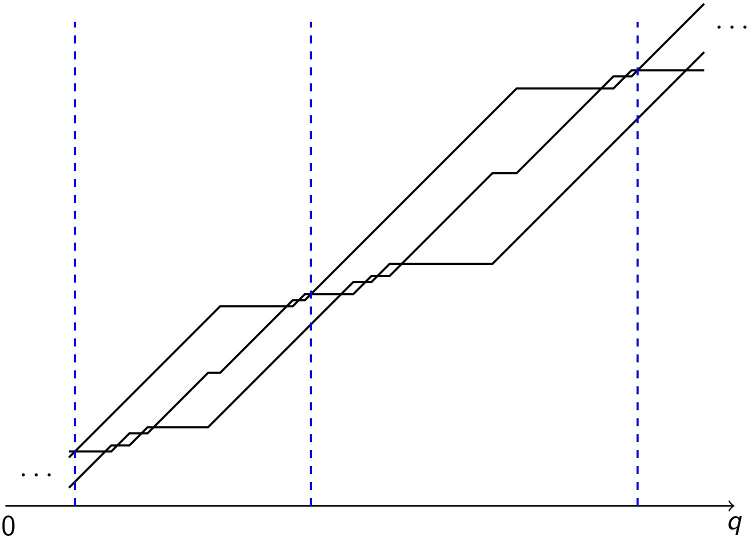
For each nonzero $\mathbf{u} \in \mathbb{R}^n$ *and each* $\delta > 0$*, there exists a dual rigid n-system* $\mathbf{P}^*\colon [q_0, \infty) \to \mathbb{R}^n$ *of mesh* δ *such that* $\mathbf{L}_{\mathbf{u}}^* - \mathbf{P}^*$ *is bounded on* $[q_0, \infty)$*. Conversely, given any dual n-system* $\mathbf{P}^*\colon [q_0, \infty) \to \mathbb{R}^n$*, there exists a nonzero* $\mathbf{u} \in \mathbb{R}^n$ *such that* $\mathbf{L}_{\mathbf{u}}^* - \mathbf{P}^*$ *is bounded on* $[q_0, \infty)$*.*

Combined graph of a dual 2-system

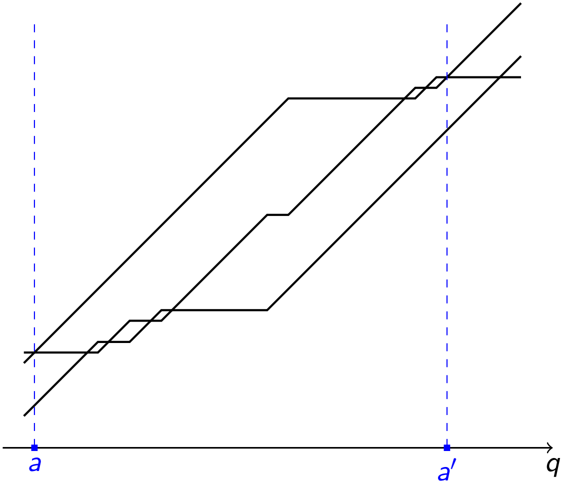


Combined graph of a dual 3-system

There is a repetitive pattern :

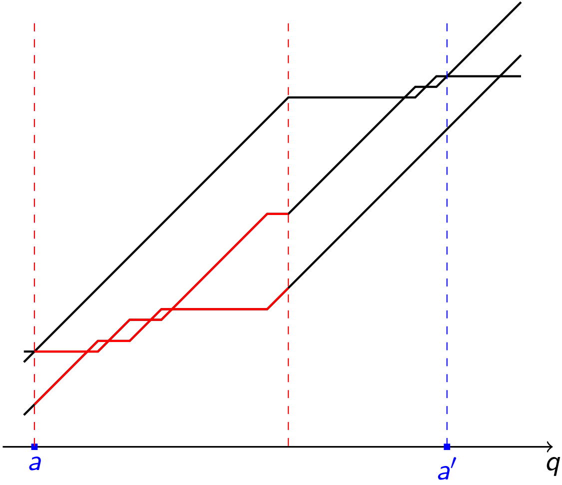


The generic pattern



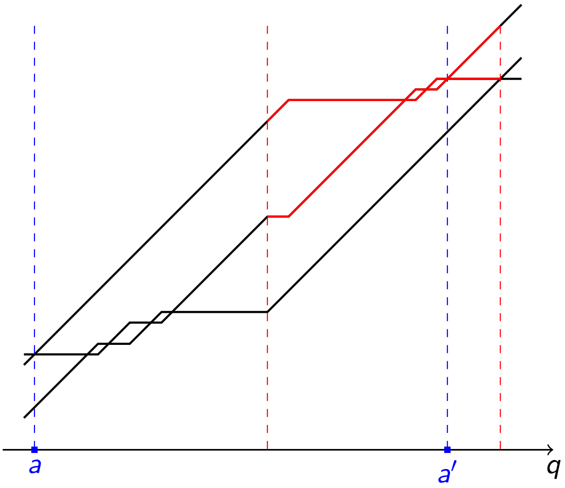
The generic pattern

When P_3^* has slope 1, (P_1^*, P_2^*) behaves like a dual 2-system.



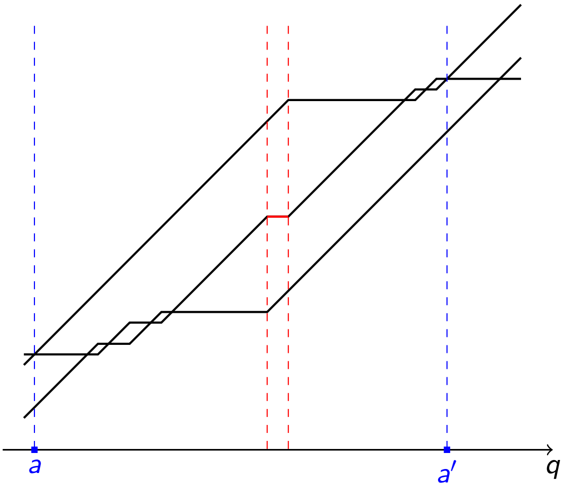
The generic pattern

When P_1^* has slope 1, (P_2^*, P_3^*) behaves like a dual 2-system.

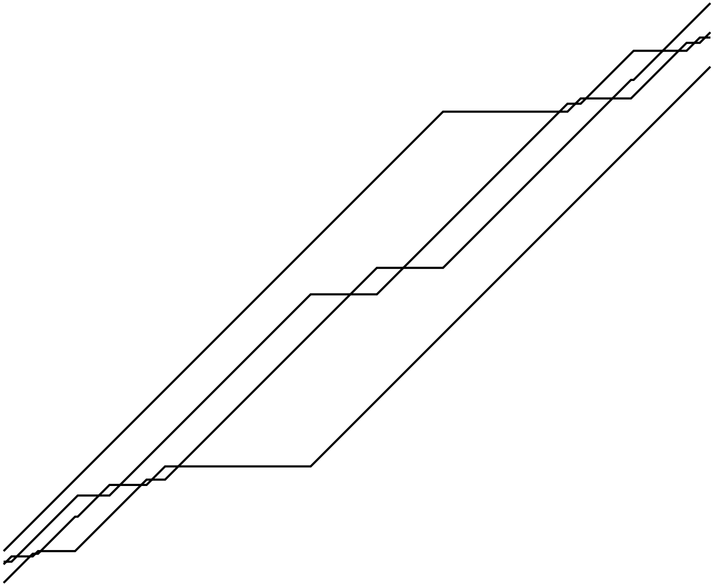


The generic pattern

When P_1^* and P_3^* have slope 1, (P_2^*) behaves like a dual 1-system.

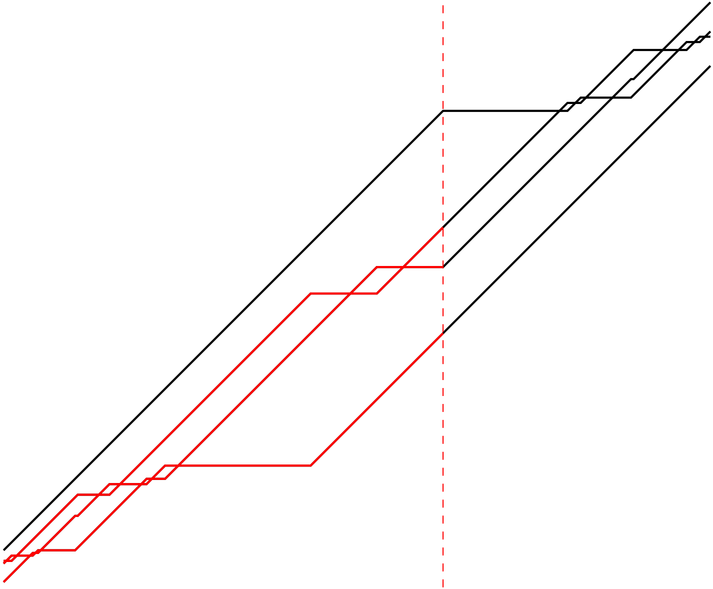


Dual 4-systems



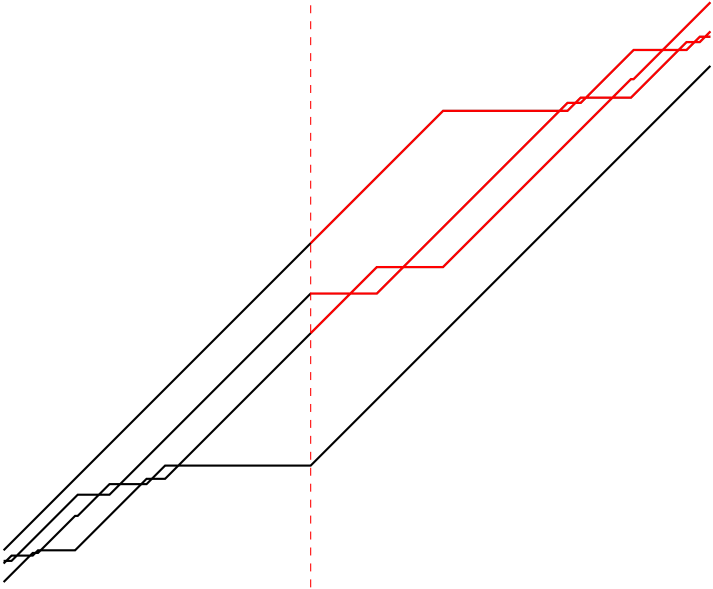
Dual 4-systems

When P_4^* has slope 1, (P_1^*, P_2^*, P_3^*) behaves like a dual 3-system.



Dual 4-systems

When P_1^* has slope 1, (P_2^*, P_3^*, P_4^*) behaves like a dual 3-system.



Dual 4-systems

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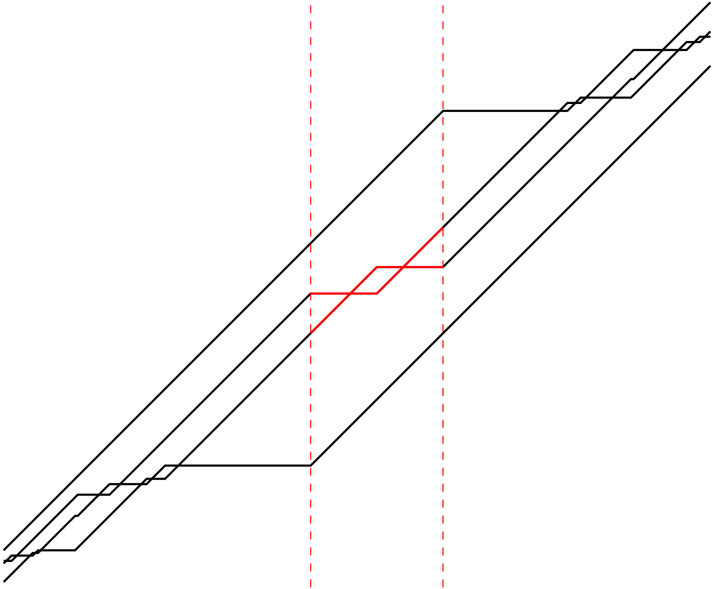
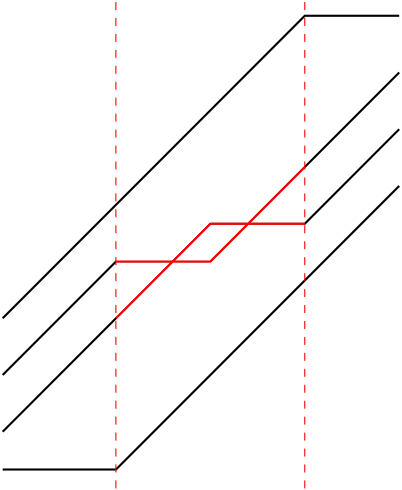


No repetitive pattern for dual 4-systems

Transitions when P_1^* and P_4^* have slope 1 may be qualitatively very different

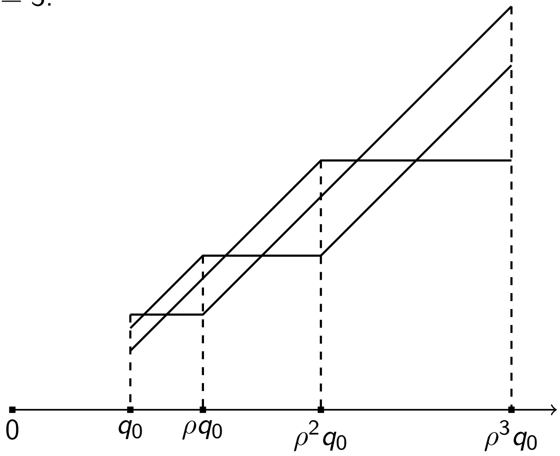


Self-similar dual systems

They are the dual n-systems $\mathbf{P}^*\colon [q_0, \infty) \to \mathbb{R}^n$ which, for some $\rho > 1$, satisfy

$$\mathbf{P}^*(\rho q) = \rho \mathbf{P}^*(q) \text{ for each } q \geq q_0$$

Example for $n = 3$:



IV. The trajectory of a subspace

Let $1 \le k < n$ be integers and let $\mathbf{u} \in \mathbb{R}^n \setminus \{0\}$. Mahler's k-th compound of

$$\mathcal{C}^*_{\mathbf{u}}(q) = \Big\{ \mathbf{x} \in \mathbb{R}^n \,;\, \log \|\mathbf{x}\| \le 1 \text{ and } \log \|\mathbf{x} \wedge \mathbf{u}\| \le -q \Big\}$$

is comparable to

$$(\mathcal{C}^*_{\mathbf{u}})^{(k)}(q) = \Big\{ X \in \textstyle\bigwedge^k \mathbb{R}^n \,;\, \log \|X\| \le -(k-1)q \text{ and } \log \|X \wedge \mathbf{u}\| \le -kq \Big\}.$$

The *trajectory* of a non-zero $X \in \bigwedge^k \mathbb{R}^n$ is

$$L^*_{\mathbf{u}}(X, q) = \max\{\log \|X\| + (k-1)q, \, \log \|X \wedge \mathbf{u}\| + kq\}.$$

The *trajectory* of a k-dimensional subspace V of $\mathbb{R}^n$ **defined over** $\mathbb{Q}$ is

$$L^*_{\mathbf{u}}(V, q) = L(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k, q)$$

where $(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ is any basis of $V \cap \mathbb{Z}^n$.

A glimpse at Mahler's theory

Suppose that I is a sub-interval of $[0,\infty)$ such that

$$L^*_{\mathbf{u},k}(q) < L^*_{\mathbf{u},k+1}(q) \quad \text{for each } q \in I.$$

Then, the subspace V of $\mathbb{R}^n$ generated by the first k minima of $\mathcal{C}^*_{\mathbf{u}}(q)$ in $\mathbb{Z}^n$ is independent of $q \in I$, and we have

$$L^*_{\mathbf{u}}(V,q) \simeq L^*_{\mathbf{u},1}(q) + \cdots + L^*_{\mathbf{u},k}(q) \quad \text{for each } q \in I.$$

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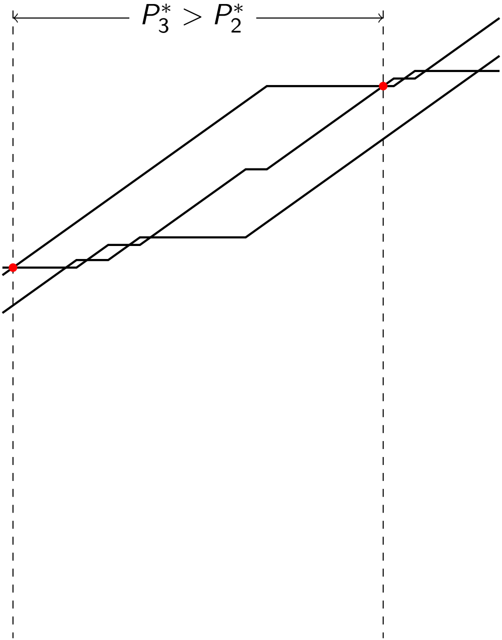
Consequence. Let $\mathbf{P}^* = (P^*_1, \ldots, P^*_n)$ is a dual n-system on $[q_0, \infty)$ for which $c := \|\mathbf{P}^* - \mathbf{L}^*_{\mathbf{u}}\|_\infty < \infty$. Suppose that I is a subinterval of $[q_0, \infty)$ such that

$$P^*_k(q) < P^*_{k+1}(q) - 2c \quad \text{for each } q \in I.$$

for each $q \geq q_0$. Then, the subspace V of $\mathbb{R}^n$ generated by the points $\mathbf{x} \in \mathbb{Z}^n$ with $L^*_{\mathbf{u}}(\mathbf{x}, q) < P^*_{k+1}(q) - c$ for some $q \in I$ has dimension k and

$$L^*_{\mathbf{u}}(V, q) \simeq P^*_1(q) + \cdots + P^*_k(q) \quad \text{for each } q \in I.$$

Illustration for planes in 3-space



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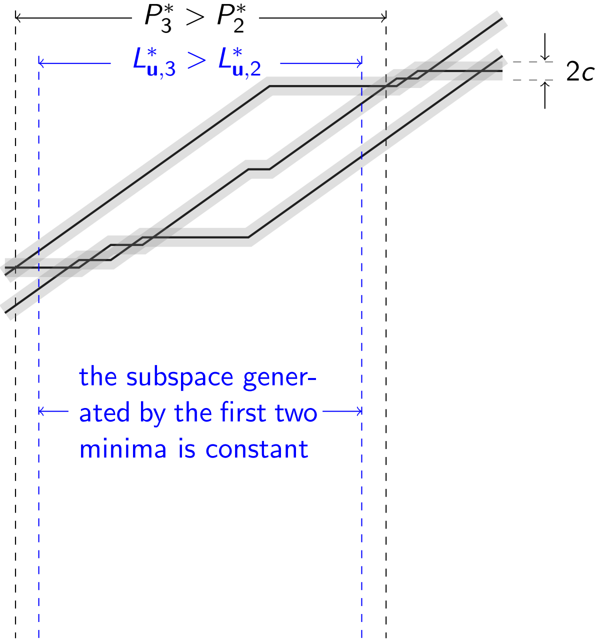

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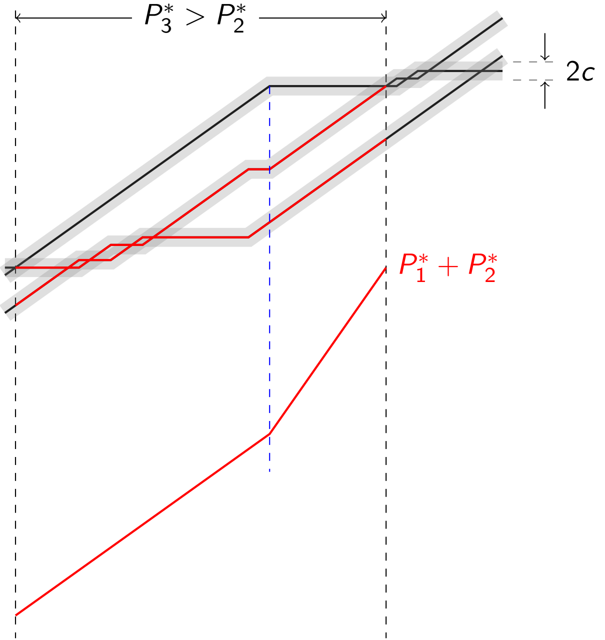

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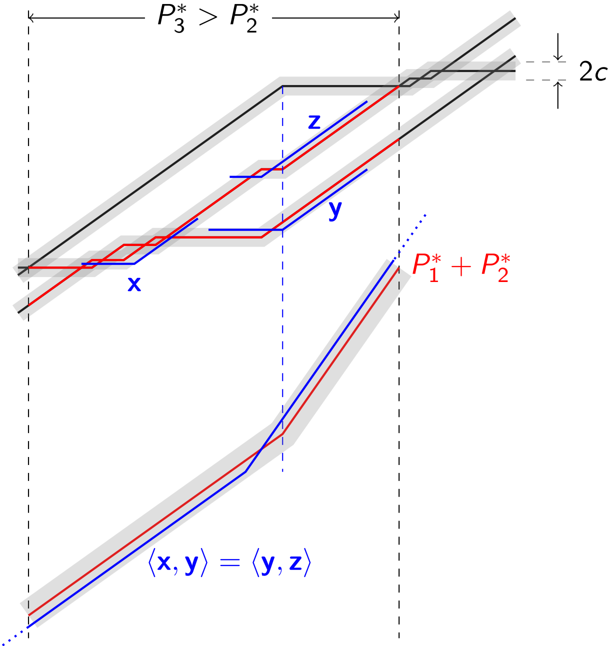


V. Approximation to $(1, \xi, \xi^2)$

Hypothesis

Let $\xi \in \mathbb{R}$ with $[\mathbb{Q}(\xi) : \mathbb{Q}] > 2$. Set $\mathbf{u} = (1, \xi, \xi^2)$ and suppose that there exist $\lambda > 1/2$ and $c > 0$ such that the inequalities

$$\|\mathbf{x}\| \le X \quad \text{and} \quad \|\mathbf{x} \wedge \mathbf{u}\| \le cX^{-\lambda}$$

admit a non-zero solution $\mathbf{x} \in \mathbb{Z}^3$ for each large enough X.

Fix a dual 3-system $\mathbf{P}^* = (P_1^*, P_2^*, P_3^*)$ such that $\mathbf{L}_{\mathbf{u}}^* - \mathbf{P}^*$ is bounded. The last hypothesis becomes

$$P_1^*(q) \le \frac{q}{1+\lambda} + \mathcal{O}(1)$$

as $q \to \infty$.

Exploiting the nature of the point

For each point $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3$, we define

$$\mathbf{x}^- = (x_0, x_1), \quad \mathbf{x}^+ = (x_1, x_2) \quad \text{and} \quad \Delta\mathbf{x} = \mathbf{x}^+ - \xi\mathbf{x}^-.$$

Then, $\|\mathbf{x} \wedge \mathbf{u}\| \asymp \|\Delta\mathbf{x}\|$.

Theorem (Davenport and Schmidt, 1969)

For any minimal point $\mathbf{x} \in \mathbb{Z}^3$ with $\|\mathbf{x}\|$ large enough, we have

$$\det(\mathbf{x}) := \det(\mathbf{x}^-, \mathbf{x}^+) \neq 0.$$

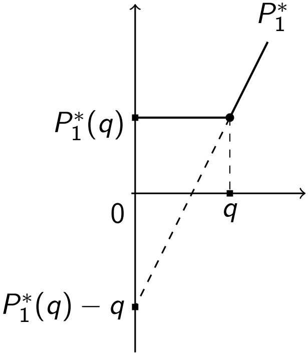
Then, $1 \leq |\det(\mathbf{x}^-, \mathbf{x}^+)| = |\det(\mathbf{x}^-, \Delta\mathbf{x})| \ll \|\mathbf{x}\| \|\Delta\mathbf{x}\|$,

and so, $0 \leq \log\|\mathbf{x}\| + \log\|\mathbf{x} \wedge \mathbf{u}\| + \mathcal{O}(1)$.

Consequence on P_1^*

We have $P_1^*(q) \geq q/2 + \mathcal{O}(1)$ as $q \to \infty$.

Proof. We may assume that P_1^* changes slope from 0 to 1 at q.



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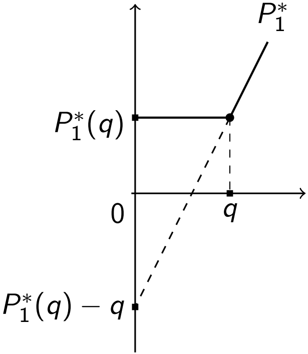
$$P_2^*(q) \geq (P_1^*(q) + P_2^*(q))/2 \geq q/2$$

since $P_1^* + P_2^*$ has slope 1 or 2. So, we may assume that $P_2^*(q) - P_1^*(q)$ is large.



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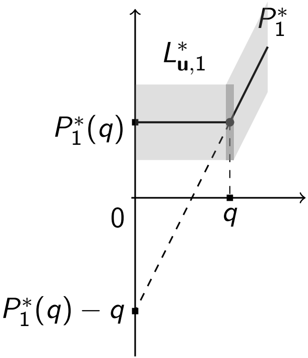
Choose a minimal point $\mathbf{x} \in \mathbb{Z}^3$ such that $L_{\mathbf{u},1}^*(q) = L_{\mathbf{u}}^*(\mathbf{x}, q)$.



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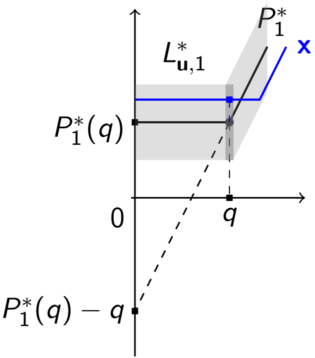

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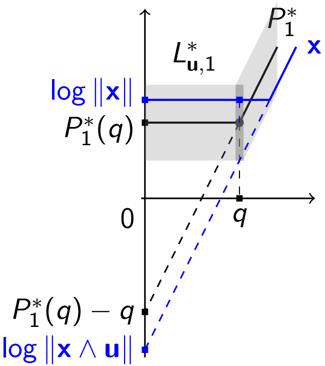
Choose a minimal point $\mathbf{x} \in \mathbb{Z}^3$ such that $L_{\mathbf{u},1}^*(q) = L_{\mathbf{u}}^*(\mathbf{x}, q)$. Then, we have

$$\log \|\mathbf{x}\| \simeq P_1^*(q), \quad \log \|\mathbf{x} \wedge \mathbf{u}\| \simeq P_1^*(q) - q,$$

Thus, $0 \leq \log \|\mathbf{x}\| + \log \|\mathbf{x} \wedge \mathbf{u}\| + \mathcal{O}(1)$
$\leq P_1^*(q) + (P_1^*(q) - q) + \mathcal{O}(1).$



Summary of the constraints

We have

$$[\mathbb{Q}(\xi):\mathbb{Q}] > 2 \iff \mathbf{u} = (1, \xi, \xi^2) \text{ has } \mathbb{Q}\text{-linearly independent coordinates}$$

$$\iff \boxed{P_3^* \text{ changes slope infinitely often}}$$

Moreover

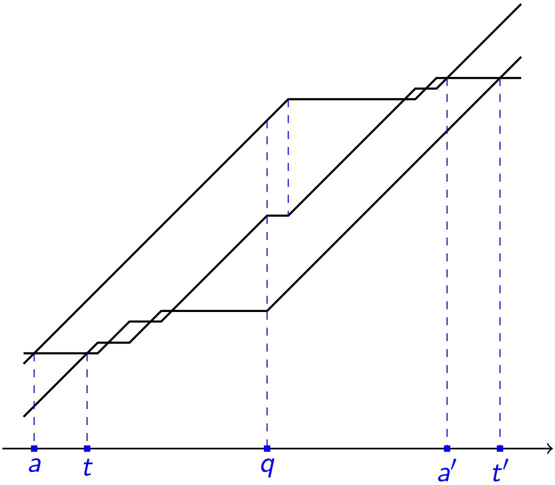
$$\boxed{\frac{q}{2} + \mathcal{O}(1) \leq P_1^*(q) \leq \frac{q}{1+\lambda} + \mathcal{O}(1)}$$

One can show that these conditions imply

Theorem (Davenport and Schmidt, 1969)

$$\lambda \leq 1/\gamma \cong 0.618$$

Application to a generic pattern



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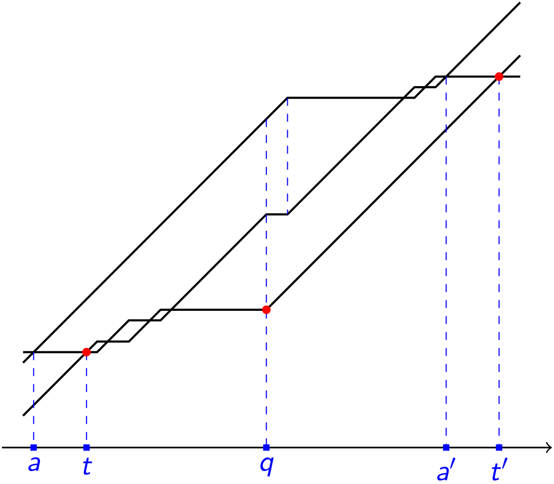

$P_1^*(t) \lesssim t/(1+\lambda)$

$P_1^*(t') \lesssim t'/(1+\lambda)$

$P_1^*(q) \gtrsim q/2$

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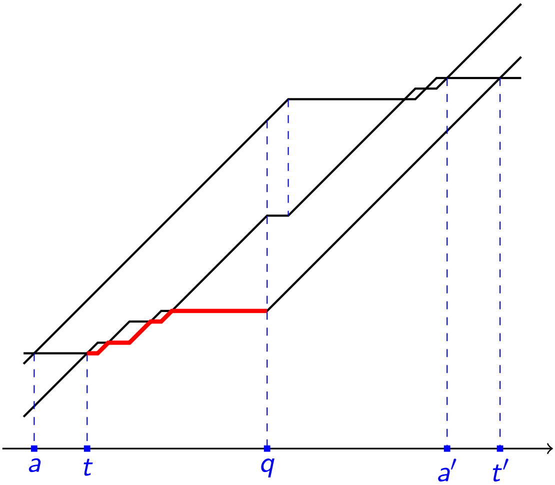

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$P_1^*(q) \geq P_1^*(t)$

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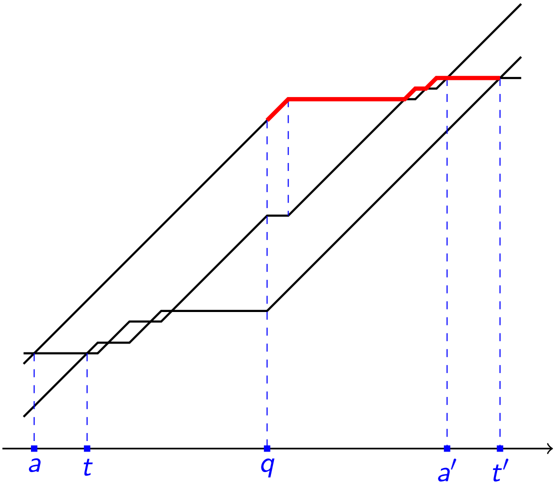

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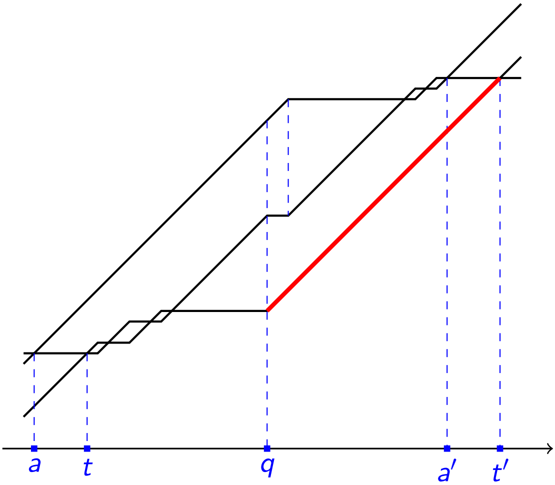
Application to a generic pattern



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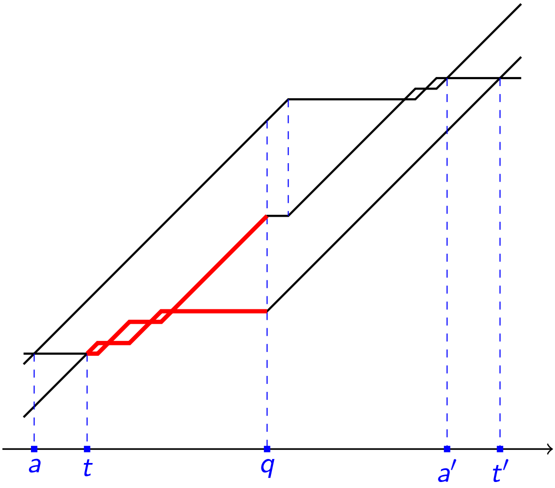
$P_3^*(q) \leq P_1^*(t')$

$t' - P_1^*(t') = q - P_1^*(q)$

Application to a generic pattern



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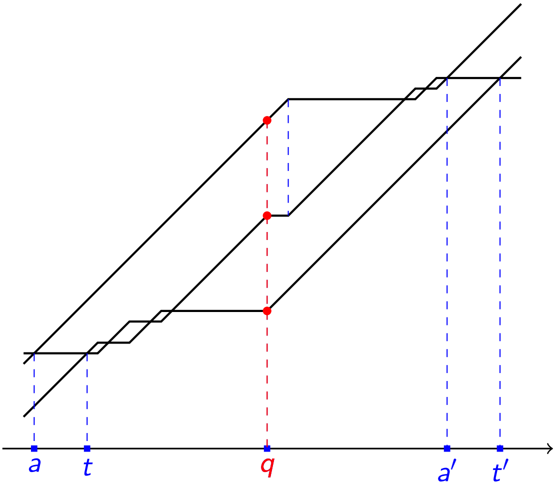
$t' - P_1^*(t') = q - P_1^*(q)$

$2P_1^*(t) - t$
$= P_1^*(q) + P_2^*(q) - q$

Application to a generic pattern



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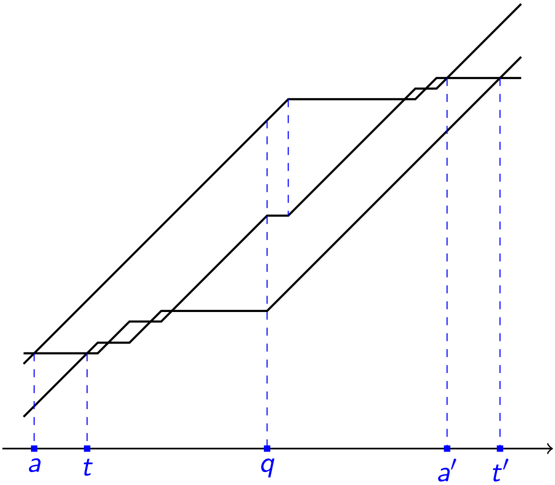
$$t' - P_1^*(t') = q - P_1^*(q)$$

$$2P_1^*(t) - t$$
$$= P_1^*(q) + P_2^*(q) - q$$

$$2q = \sum_{i=1}^{3} P_i^*(q)$$

Application to a generic pattern



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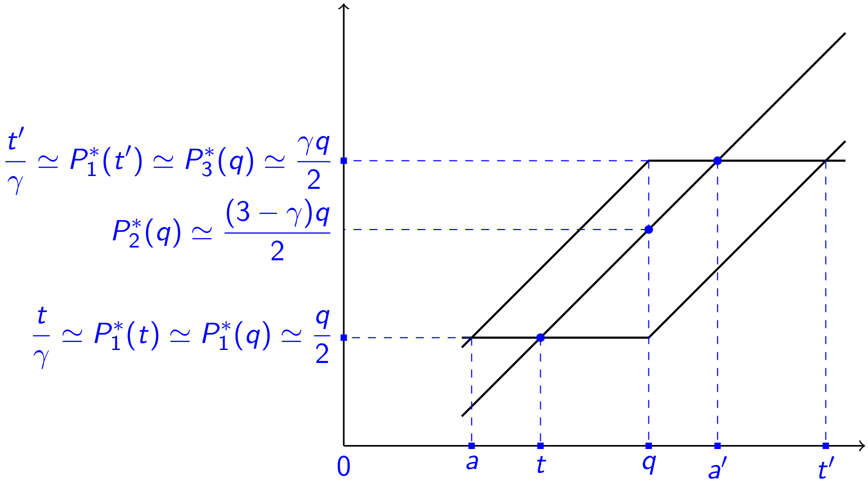
$$2P_1^*(t) - t = P_1^*(q) + P_2^*(q) - q$$

$$2q = \sum_{i=1}^{3} P_i^*(q)$$

Limit case

- Solving the above inequalities yields $\lambda \leq 1/\gamma$.
- If $\lambda = 1/\gamma$, all inequalities are equalities up to a bounded difference:

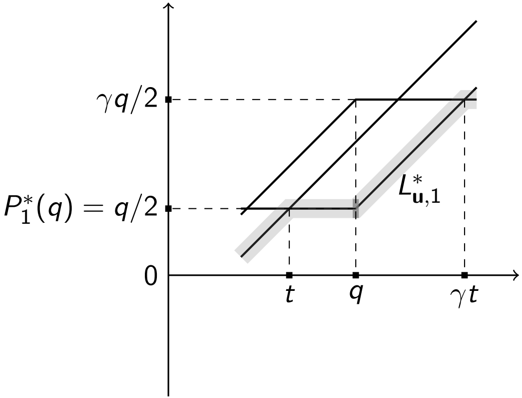


A particular minimal point

As $P_2^*(q) - P_1^*(q) \to \infty$, there is a unique primitive pair $\pm\mathbf{y} \in \mathbb{Z}^3$ with

$$\log \|\mathbf{y}\| \simeq q/2 \quad \text{and} \quad \log \|\mathbf{y} \wedge \mathbf{u}\| \simeq -q/2$$

and thus $|\det(\mathbf{y})| \asymp 1$.



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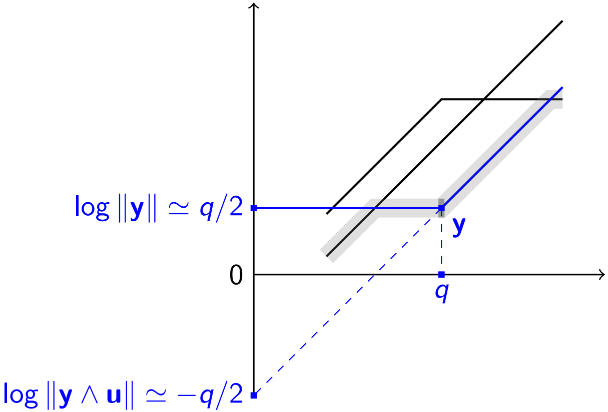
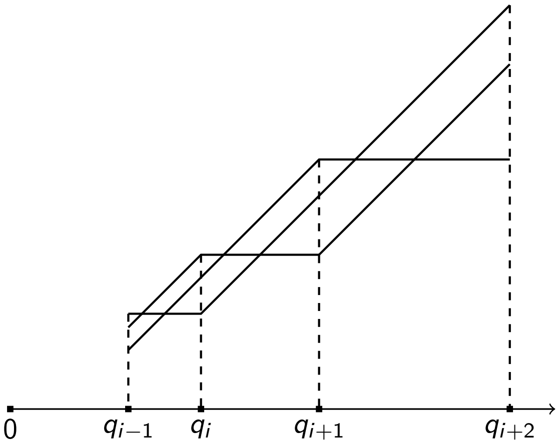


The sequence of these points

We get real numbers $q_i > 0$ in $\mathbb{R}$ and primitive points $\mathbf{y}_i \in \mathbb{Z}^4$ with

$$q_{i+1} \simeq \gamma q_i, \quad \log \|\mathbf{y}_i\| \simeq q_i/2, \quad \log \|\mathbf{y}_i \wedge \mathbf{u}\| \simeq -q_i/2.$$

We may choose $\mathbf{P}^*$ self similar with ratio γ,
so that $q_{i+1} = \gamma q_i$ for each $i \geq 1$.



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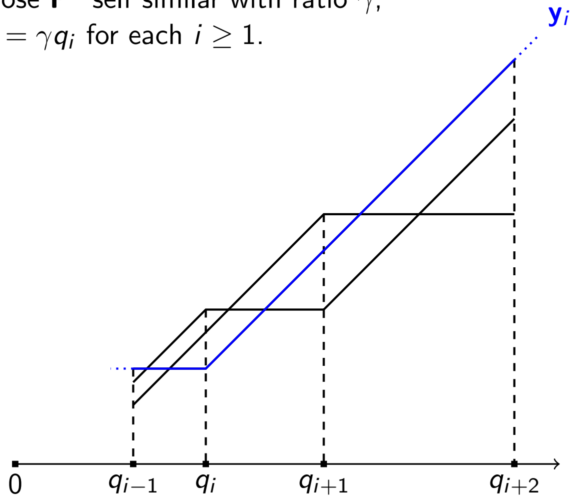

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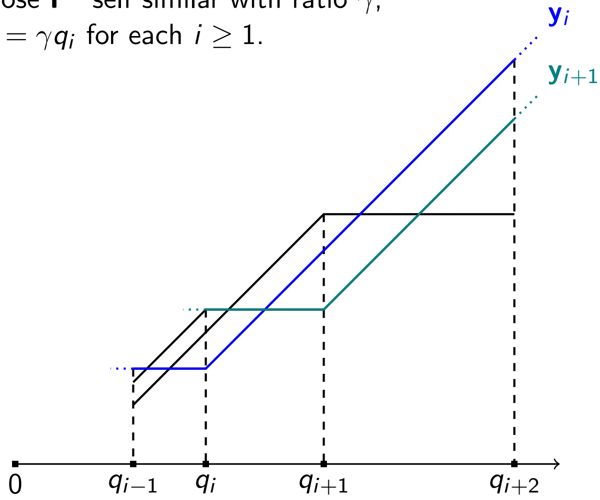

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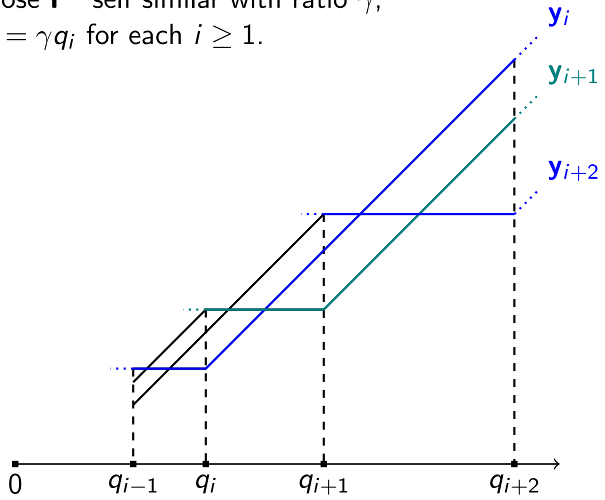
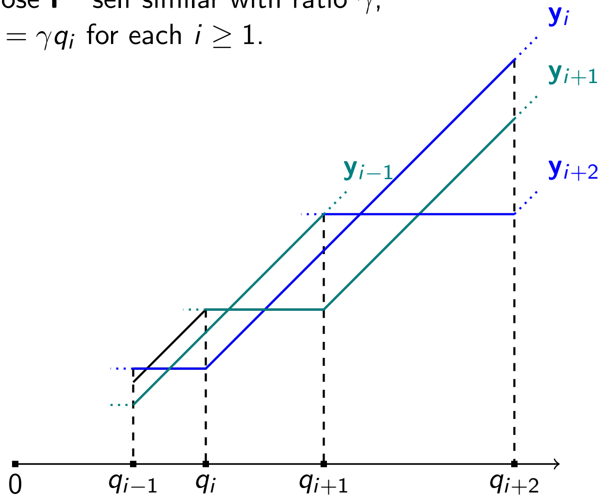


The sequence of these points

We get real numbers $q_i > 0$ in $\mathbb{R}$ and primitive points $\mathbf{y}_i \in \mathbb{Z}^4$ with

$$q_{i+1} \simeq \gamma q_i, \quad \log \|\mathbf{y}_i\| \simeq q_i/2, \quad \log \|\mathbf{y}_i \wedge \mathbf{u}\| \simeq -q_i/2.$$

We may choose $\mathbf{P}^*$ self similar with ratio γ,
so that $q_{i+1} = \gamma q_i$ for each $i \geq 1$.



Linear independence of three consecutive points

Claim. The points $\mathbf{y}_{i-1}, \mathbf{y}_i, \mathbf{y}_{i+1}$ are linearly independent if $i \gg 1$.

Step 1. The trajectory of a non-zero $\mathbf{x} \in \mathbb{Z}^3$ changes slope at

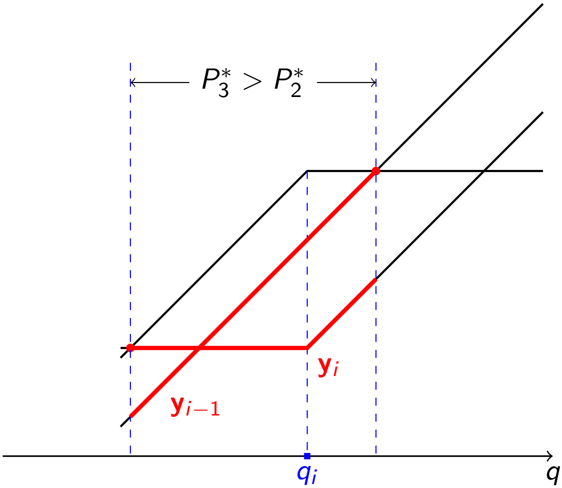
$$q(\mathbf{x}) = \log \frac{\|\mathbf{x}\|}{\|\mathbf{x} \wedge \mathbf{u}\|}.$$

Thus, if $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^3$ are linearly independent, then $q(\mathbf{x}) = q(\mathbf{y})$.

Since $L^*_{\mathbf{u}}(\mathbf{y}_i, q)$ changes slope around q_i and $q_{i+1} - q_i \to \infty$, the points $\mathbf{y}_i$ and $\mathbf{y}_{i+1}$ are linearly independent if $i \gg 1$.

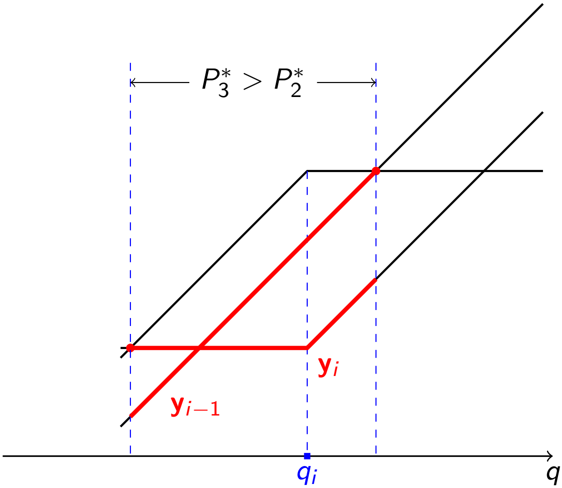
Step 2

The trajectory of $\langle \mathbf{y}_{i-1}, \mathbf{y}_i \rangle_{\mathbb{R}}$ changes slope around q_i.



Step 2

The trajectory of $\langle \mathbf{y}_{i-1}, \mathbf{y}_i \rangle_{\mathbb{R}}$ changes slope around q_i.
$\implies$ That of $\langle \mathbf{y}_i, \mathbf{y}_{i+1} \rangle_{\mathbb{R}}$ changes slope around q_{i+1}.

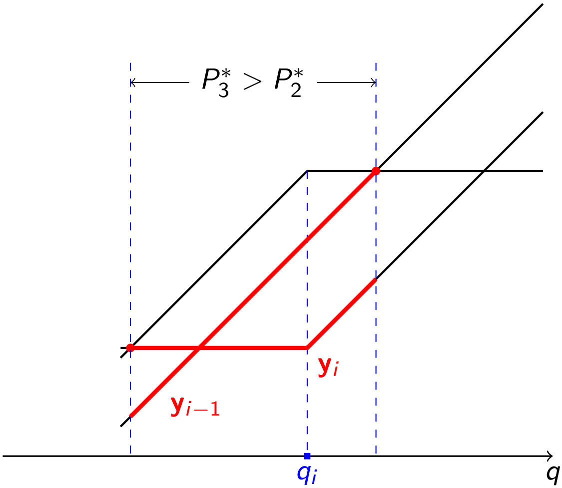


Step 2

The trajectory of $\langle \mathbf{y}_{i-1}, \mathbf{y}_i \rangle_{\mathbb{R}}$ changes slope around q_i.

$\Longrightarrow$ That of $\langle \mathbf{y}_i, \mathbf{y}_{i+1} \rangle_{\mathbb{R}}$ changes slope around q_{i+1}.

$\Longrightarrow$ $\langle \mathbf{y}_{i-1}, \mathbf{y}_i \rangle_{\mathbb{R}} \neq \langle \mathbf{y}_i, \mathbf{y}_{i+1} \rangle_{\mathbb{R}}$ if $i \gg 1$.



Summary

Set $\mathbf{u} = (1, \xi, \xi^2)$ for some $\xi \in \mathbb{R}$ with $[\mathbb{Q}(\xi) : \mathbb{Q}] > 2$. Suppose that, for some $c > 0$,

$$\|\mathbf{x}\| \leq X \quad \text{and} \quad \|\mathbf{x} \wedge \mathbf{u}\| \leq cX^{-1/\gamma}$$

admits a non-zero solution $\mathbf{x} \in \mathbb{Z}^3$ for each large enough X.

Then, there exist an unbounded sequence $(\mathbf{y}_i)_{i \geq 1}$ of primitive points of $\mathbb{Z}^3$ such that, for each large enough i,

- $\|\mathbf{y}_{i+1}\| \asymp \|\mathbf{y}_i\|^\gamma$ and $\|\Delta \mathbf{y}_i\| \asymp \|\mathbf{y}_i \wedge \mathbf{u}\| \asymp \|\mathbf{y}_i\|^{-1}$,
- $|\det(\mathbf{y}_i)| \asymp 1$,
- $\mathbf{y}_{i-1}, \mathbf{y}_i, \mathbf{y}_{i+1}$ are linearly independent.

The polynomial map Ξ

We define a polynomial map $\Xi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\Xi(\mathbf{x}, \mathbf{y}) = (\det(\mathbf{x}^-, \mathbf{y}^+) - \det(\mathbf{x}^+, \mathbf{y}^-))\mathbf{x} - \det(\mathbf{x})\mathbf{y}.$$

where $\det(\mathbf{x}) := \det(\mathbf{x}^-, \mathbf{x}^+) = \begin{vmatrix} x_0 & x_1 \\ x_1 & x_2 \end{vmatrix}$.

Algebraic properties

(i) $\det(\Xi(\mathbf{x}, \mathbf{y})) = \det(\mathbf{x})^2 \det(\mathbf{y})$,

(ii) $\Xi(\mathbf{x}, \Xi(\mathbf{x}, \mathbf{y})) = \det(\mathbf{x})^2 \mathbf{y}$.

Analytic properties

(i) $\|\Xi(\mathbf{x}, \mathbf{y})\| \ll \|\mathbf{x}\|^2 \|\Delta \mathbf{y}\| + \|\mathbf{y}\| \|\Delta \mathbf{y}\|^2$,

(ii) $\|\Delta \Xi(\mathbf{x}, \mathbf{y}))\| \ll \left(\|\mathbf{x}\| \|\Delta \mathbf{y}\| + \|\mathbf{y}\| \|\Delta \mathbf{x}\|\right) \|\Delta \mathbf{x}\|$.

Application

We find

- $\|\Xi(\mathbf{y}_i, \mathbf{y}_{i+1})\| \ll \|\mathbf{y}_{i-2}\| \quad \text{and} \quad \|\Delta\Xi(\mathbf{y}_i, \mathbf{y}_{i+1})\| \ll \|\mathbf{y}_{i-2}\|^{-1}$,

and then

- $|\det(\mathbf{y}_{i-2}, \mathbf{y}_{i-1}, \Xi(\mathbf{y}_i, \mathbf{y}_{i+1}))| \ll \|\mathbf{y}_{i-4}\|^{-1} \to 0$,
- $|\det(\mathbf{y}_{i-3}, \mathbf{y}_{i-2}, \Xi(\mathbf{y}_i, \mathbf{y}_{i+1}))| \ll \|\mathbf{y}_{i-3}\|^{-1} \to 0$.

Thus, for each large enough i,

$$\det(\mathbf{y}_{i-2}, \mathbf{y}_{i-1}, \Xi(\mathbf{y}_i, \mathbf{y}_{i+1})) = 0 \quad \text{and} \quad \det(\mathbf{y}_{i-3}, \mathbf{y}_{i-2}, \Xi(\mathbf{y}_i, \mathbf{y}_{i+1})) = 0,$$

and so $\Xi(\mathbf{y}_i, \mathbf{y}_{i+1}) \propto \mathbf{y}_{i-2}$. As $\Xi(\mathbf{y}_i, \mathbf{y}_{i+1}) \neq 0$, we find

$$\boxed{\Xi(\mathbf{y}_i, \mathbf{y}_{i-2}) \propto \Xi(\mathbf{y}_i, \Xi(\mathbf{y}_i, \mathbf{y}_{i+1})) \propto \mathbf{y}_{i+1},}$$

which determines the primitive point $\mathbf{y}_{i+1}$ as a function of $\mathbf{y}_{i-2}$ and $\mathbf{y}_i$ up to multiplication by ± 1.

Solution to the inverse problem

Choose linearly independent $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \in \mathbb{Z}^3$ with $\det(\mathbf{y}_i) = 1$ for $j = 1, 2, 3$. Then the sequence $(\mathbf{y}_i)_{i \geq 1}$ given recursively by

$$\mathbf{y}_{i+1} = \Xi(\mathbf{y}_i, \mathbf{y}_{i-2}) \text{ for each } i \geq 3$$

belongs to $\mathbb{Z}^3$. For each $i \geq 1$, it has $\det(\mathbf{y}_i) = 1$ and $(\mathbf{y}_i, \mathbf{y}_{i+1}, \mathbf{y}_{i+2})$ is a linearly independent triple.

For an appropriate choice of $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$, the image of $\mathbf{y}_i$ in $\mathbb{P}^2(\mathbb{R})$ converges to the class of $(1, \xi, \xi^2)$ for some $\xi \in \mathbb{R}$ with $[\mathbb{Q}(\xi) : \mathbb{Q}] > 2$ and $\widehat{\lambda}_2(\xi) = 1/\gamma$.

VI. Approximation to $(1, \xi, \xi^2, \xi^3)$

Let $\lambda = \lambda_3 = 0.4245\ldots = $ the positive root of $T^2 - \gamma^3 T + \gamma$.

Hypothesis

Let $\xi \in \mathbb{R}$ with $[\mathbb{Q}(\xi) : \mathbb{Q}] > 3$. Set $\mathbf{u} = (1, \xi, \xi^2, \xi^3)$ and suppose that there exists $c > 0$ such that the inequalities

$$\|\mathbf{x}\| \leq X \quad \text{and} \quad \|\mathbf{x} \wedge \mathbf{u}\| \leq cX^{-\lambda}$$

admit a non-zero solution $\mathbf{x} \in \mathbb{Z}^3$ for each large enough X.

We want to show that this leads to a contradiction. The proof can be adapted to shows that $\widehat{\lambda}_3(\xi) \leq \lambda_3 - \epsilon$ for some small explicit ϵ (not computed).

First main tool : the map C

For each point $\mathbf{x} = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$, we define

$$\mathbf{x}^- = (x_0, x_1, x_2), \quad \mathbf{x}^+ = (x_1, x_2, x_3) \quad \text{and} \quad \Delta\mathbf{x} = \mathbf{x}^+ - \xi\mathbf{x}^-.$$

Then, $\quad \|\Delta\mathbf{x}\| \asymp \|\mathbf{x} \wedge \mathbf{u}\|.$

For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^4$,

- $C(\mathbf{x}, \mathbf{y}) := \big(\det(\mathbf{x}^-, \mathbf{x}^+, \mathbf{y}^-), \det(\mathbf{x}^-, \mathbf{x}^+, \mathbf{y}^+)\big) \in \mathbb{R}^2$ satisfies

$$\begin{aligned} \|C(\mathbf{x}, \mathbf{y})\| &\ll \|\mathbf{x}\|\|\Delta\mathbf{x}\|\|\Delta\mathbf{y}\| + \|\mathbf{y}\|\|\Delta\mathbf{x}\|^2 \\ \|\Delta C(\mathbf{x}, \mathbf{y})\| &\ll \|\mathbf{x}\|\|\Delta\mathbf{x}\|\|\Delta\mathbf{y}\|. \end{aligned}$$

- $\mathbf{w} := C(\mathbf{x}, \mathbf{y})^- \mathbf{z}^+ - C(\mathbf{x}, \mathbf{y})^+ \mathbf{z}^- \in \mathbb{R}^3$ satisfies

$$\begin{aligned} \|\mathbf{w}\| &\ll \|C(\mathbf{x}, \mathbf{y})\|\|\Delta\mathbf{z}\| + \|\mathbf{z}\|\|\Delta C(\mathbf{x}, \mathbf{y})\| \\ \|\Delta\mathbf{w}\| &\ll \|C(\mathbf{x}, \mathbf{y})\|\|\Delta\mathbf{z}\|. \end{aligned}$$

Non-vanishing results

Let $(\mathbf{x}_i)_{i\geq 1}$ denote a sequence of minimal points for ξ in $\mathbb{Z}^4$.

For each sufficiently large i,

- **Davenport and Schmidt 1969 :** $V_i := \langle \mathbf{x}_i^-, \mathbf{x}_i^+ \rangle_{\mathbb{R}} \subseteq \mathbb{R}^3$ has dimension 2 (uses $\lambda > 1/3$),
- **R. 2008 :** $V_i \neq V_{i+1}$ (uses $\lambda > \sqrt{2} - 1 \cong 0.4142$),

thus

$$\boxed{C(\mathbf{x}_i, \mathbf{x}_{i+1}) \neq 0 \quad \text{and} \quad C(\mathbf{x}_{i+1}, \mathbf{x}_i) \neq 0.}$$

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thus

$$\boxed{C(\mathbf{x}_i, \mathbf{x}_{i+1}) \neq 0 \quad \text{and} \quad C(\mathbf{x}_{i+1}, \mathbf{x}_i) \neq 0.}$$

In particular, this gives $1 \leq \|C(\mathbf{x}_i, \mathbf{x}_{i-1})\|$ which yields

$$\|\mathbf{x}_{i+1}\| \ll \|\mathbf{x}_i\|^\theta \quad \text{where} \quad \theta = \frac{1-\lambda}{\lambda}.$$

In terms of a dual 4-system $\mathbf{P}^*$ that approximates $\mathbf{L}_{\mathbf{u}}^*$, we find

$$\boxed{2P_1^*(q) + P_2^*(q) \geq 2q + \mathcal{O}(1).}$$

First reduction

Using the above, we can argue in two ways

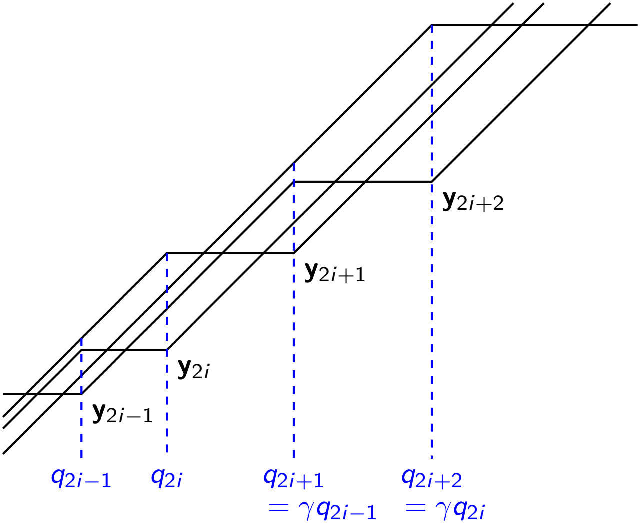
- we can work with minimal points only using Schmidt's height inequalities for subspaces spanned by consecutive minimal points
- or we can use a dual 4-system $\mathbf{P}^*$ with $\|\mathbf{L}_\mathbf{u}^* - \mathbf{P}^*\| < \infty$.

Then, there exist an unbounded sequence $(\mathbf{y}_i)_{i\geq 1}$ of primitive points of $\mathbb{Z}^4$ such that, for each large enough i,

- $|\det(\mathbf{y}_{2i-2}, \mathbf{y}_{2i-1}, \mathbf{y}_{2i}, \mathbf{y}_{2i+1})| \asymp 1$ and $\det(\mathbf{y}_{2i-3}, \mathbf{y}_{2i-2}, \mathbf{y}_{2i-1}, \mathbf{y}_{2i}) = 0$,
- $\|C(\mathbf{y}_{2i}, \mathbf{y}_{2i-1})\| \asymp 1$,
- $\|\mathbf{y}_{2i}\| \asymp \|\mathbf{y}_{2i-1}\|^{\gamma/\theta}$ and $\|\mathbf{y}_{2i+1}\| \asymp \|\mathbf{y}_{2i}\|^{\theta}$,
- $\|\Delta\mathbf{y}_{2i-1}\| \asymp \|\mathbf{y}_{2i}\|^{-\lambda}$ and $\|\Delta\mathbf{y}_{2i}\| \asymp \|\mathbf{y}_{2i+1}\|^{-\lambda}$.

Consequence on $\mathbf{L}^*$

There is a self-similar dual 4-system $\mathbf{P}^*$ with ratio γ such that $\mathbf{L}^* - \mathbf{P}^*$ is bounded. Its combined graph is the following.



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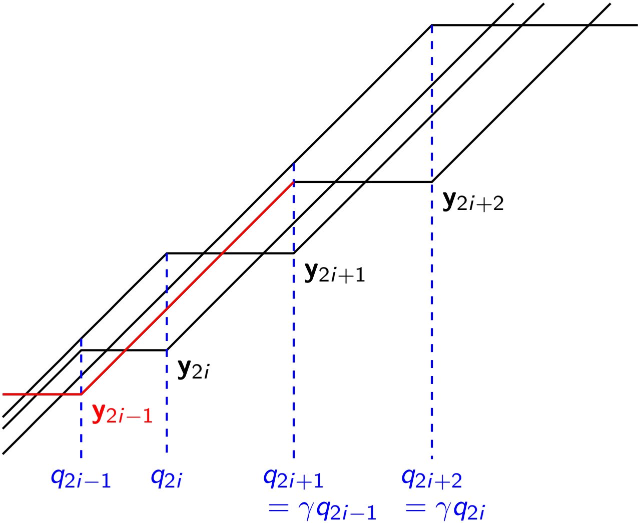

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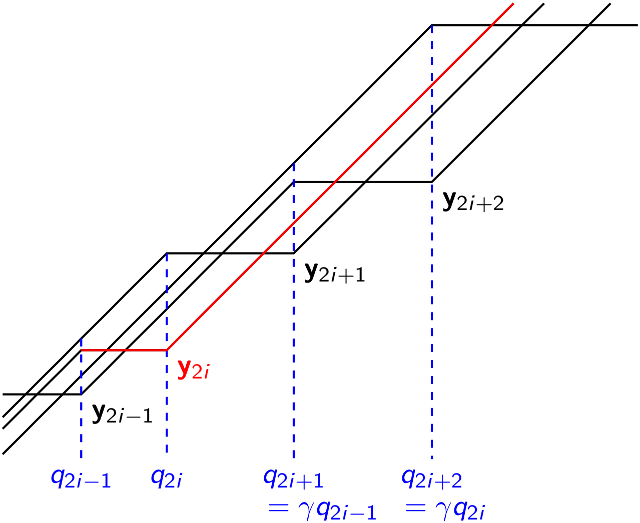

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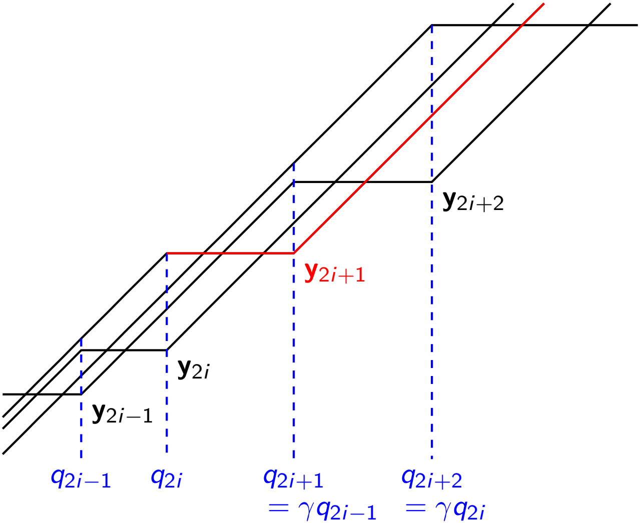

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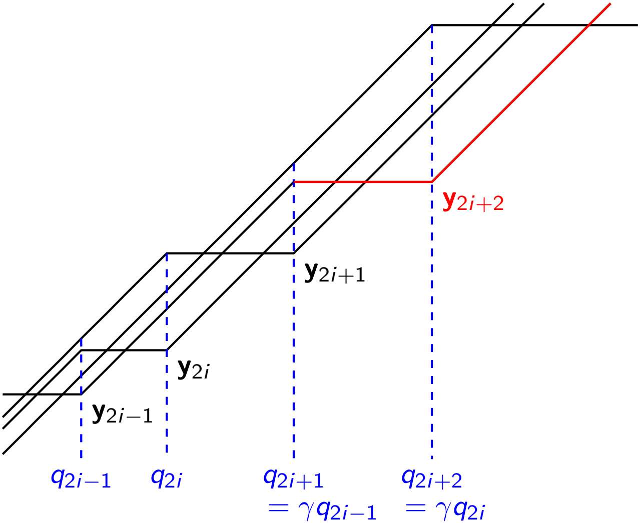

Second main tool : the maps $\Psi_\pm$

For each sign ϵ among $\{-,+\}$, we define $\Psi_\epsilon\colon (\mathbb{R}^4)^3 \to \mathbb{R}^4$ by

$$\Psi_\epsilon(\mathbf{x},\mathbf{y},\mathbf{z}) = C(\mathbf{y},\mathbf{z})^\epsilon \mathbf{x} + E(\mathbf{y},\mathbf{z},\mathbf{x})^\epsilon \mathbf{y} - C(\mathbf{y},\mathbf{x})^\epsilon \mathbf{z}$$

where $E(\mathbf{y},\mathbf{z},\mathbf{x})$ is the unique 3-linear map, symmetric in its first two arguments, such that $E(\mathbf{y},\mathbf{y},\mathbf{x}) = 2C(\mathbf{y},\mathbf{x})$.

General estimates imply that the integer

$$\det(\mathbf{y}_{2i-2},\mathbf{y}_{2i-1},\mathbf{y}_{2i},\Psi_\epsilon(\mathbf{y}_{2i},\mathbf{y}_{2i+1},\mathbf{y}_{2i+2}))$$

vanishes for any sign ϵ if i is large enough. Then algebraic considerations show the existence of non-zero rational numbers c_i and t_i with bounded numerator and denominator such that

1) $C(\mathbf{y}_{2i+1},\mathbf{y}_{2i+2}) = t_i C(\mathbf{y}_{2i},\mathbf{y}_{2i+1})$,
2) $C(\mathbf{y}_{2i+2},\mathbf{y}_{2i+1}) = c_i t_i C(\mathbf{y}_{2i},\mathbf{y}_{2i-1})$,
3) $\det(C(\mathbf{y}_{2i+2},\mathbf{y}_{2i}), C(\mathbf{y}_{2i},\mathbf{y}_{2i-1}) = c_i^2 \det(C(\mathbf{y}_{2i-1},\mathbf{y}_{2i}),\ C(\mathbf{y}_{2i},\mathbf{y}_{2i-1}))$.

Final contradiction

- The condition 2), namely

$$C(\mathbf{y}_{2i+2}, \mathbf{y}_{2i+1}) = c_i t_i C(\mathbf{y}_{2i}, \mathbf{y}_{2i-1}),$$

implies that each $C(\mathbf{y}_{2i}, \mathbf{y}_{2i-1})$ with i large enough is a bounded integer multiple of some fixed primitive integer point of $\mathbb{Z}^2$.

Final contradiction

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- The condition 3), namely

$$\det(C(\mathbf{y}_{2i+2}, \mathbf{y}_{2i}), C(\mathbf{y}_{2i}, \mathbf{y}_{2i-1}) = c_i^2 \det(C(\mathbf{y}_{2i-1}, \mathbf{y}_{2i}),\ C(\mathbf{y}_{2i}, \mathbf{y}_{2i-1})),$$

implies that

$$\|C(\mathbf{y}_{2i-1}, \mathbf{y}_{2i})\| \ll \|C(\mathbf{y}_{2i+2}, \mathbf{y}_{2i})\| \ll \|\mathbf{y}_{2i}\|^{\gamma(1-\lambda\theta\gamma)=0.1113\dots}$$

which is much better than the standard estimate

$$\|C(\mathbf{y}_{2i-1}, \mathbf{y}_{2i})\| \ll \|\mathbf{y}_{2i}\|^{1-2\lambda=0.1509\dots}.$$

With some additional work, this leads to a contradiction.

Similarities with the case $n = 2$

Although the upper bound $\widehat{\lambda}_3(\xi) \leq \lambda_3 = 0.424506\ldots$ can be improved, the analysis of the two cases have similarities.

- Both yield that $\mathbf{L}_{\mathbf{u}}^*$ is approximated by a self-similar dual n-system $\mathbf{P}^*$ with ratio γ, the golden ratio.
- In both cases, we have a subsequence $(\mathbf{y}_i)_{i\geq 1}$ of the sequence of minimal points which realizes the successive minima of $\mathcal{C}_{\mathbf{u}}^*(q)$.
- There are bounded quantities namely $\det(\mathbf{y}_i)$ for $n = 2$, and $C(\mathbf{y}_{2i}, \mathbf{y}_{2-1})$ for $n = 3$.
- There is also a polynomial map $\Xi\colon (\mathbb{R}^4)^3 \to \mathbb{R}^4$ with similar properties, given by

$$\begin{aligned}\Xi(\mathbf{x},\mathbf{y},\mathbf{z}) &= C(\mathbf{z},\mathbf{x})^-\Psi_+(\mathbf{y},\mathbf{x},\mathbf{z}) - C(\mathbf{z},\mathbf{x})^+\Psi_-(\mathbf{y},\mathbf{x},\mathbf{z})\\ &= -\det(E(\mathbf{x},\mathbf{z},\mathbf{y}), C(\mathbf{z},\mathbf{x}))\mathbf{x} - \det(C(\mathbf{x},\mathbf{z}), C(\mathbf{z},\mathbf{x}))\mathbf{y}\\ &\qquad + \det(C(\mathbf{x},\mathbf{y}), C(\mathbf{z},\mathbf{x}))\mathbf{z}.\end{aligned}$$

Properties of Ξ

We can recover $\mathbf{z}$ from $\Xi(\mathbf{x},\mathbf{y},\mathbf{z})$ via the formula

$$\Xi(\mathbf{x},\mathbf{z},\Xi(\mathbf{x},\mathbf{y},\mathbf{z})) = \det(C(\Xi(\mathbf{x},\mathbf{y},\mathbf{z}),\mathbf{x}),\, C(\mathbf{x},\Xi(\mathbf{x},\mathbf{y},\mathbf{z})))\,\mathbf{z}.$$

We also have a factorization for the determinant on the right.

- $C(\Xi(\mathbf{x},\mathbf{y},\mathbf{z}),\mathbf{x}) = \det(C(\mathbf{z},\mathbf{x}),\, C(\mathbf{z},\mathbf{y}))\det(C(\mathbf{x},\mathbf{y}),\, C(\mathbf{x},\mathbf{z}))C(\mathbf{x},\mathbf{z})$,
- $C(\mathbf{x},\Xi(\mathbf{x},\mathbf{y},\mathbf{z})) = \det(C(\mathbf{x},\mathbf{y}),\, C(\mathbf{x},\mathbf{z}))C(\mathbf{z},\mathbf{x})$,

So, $\det(C(\Xi(\mathbf{x},\mathbf{y},\mathbf{z}),\mathbf{x}),\, C(\mathbf{x},\Xi(\mathbf{x},\mathbf{y},\mathbf{z})))$
$= \det(C(\mathbf{z},\mathbf{x}),\, C(\mathbf{z},\mathbf{y}))\det(C(\mathbf{x},\mathbf{y}),\, C(\mathbf{x},\mathbf{z}))^2\det(C(\mathbf{x},\mathbf{z}),\, C(\mathbf{z},\mathbf{x}))$

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Assuming that $\lambda \cong 0.4245$, general estimates imply that

$$\det(\mathbf{y}_{2i-6},\mathbf{y}_{2i-5},\mathbf{y}_{2i-4},\Xi(\mathbf{y}_{2i},\mathbf{y}_{2i+1},\mathbf{y}_{2i+2})) = 0$$

for each large enough i, a polynomial relation of degree 10 in 24 variables.

VII. Relevant dual 4-systems

Suppose that

$$\widehat{\lambda}_3(\xi) > \sqrt{2} - 1 \cong 0.4142$$

for some $\xi \in \mathbb{R}$ with $[\mathbb{Q}(\xi) : \mathbb{Q}] > 3$. We set

$$\mathbf{u} = (1, \xi, \xi^2, \xi^3)$$

and choose a dual 4-system $\mathbf{P}^*$ for which $\mathbf{L}_{\mathbf{u}}^* - \mathbf{P}^*$ is bounded. Then,

$$\lim_{q\to\infty} P_3^*(q) - P_1^*(q) = \infty \quad \text{and} \quad \lim_{q\to\infty} P_4^*(q) - P_2^*(q) = \infty.$$

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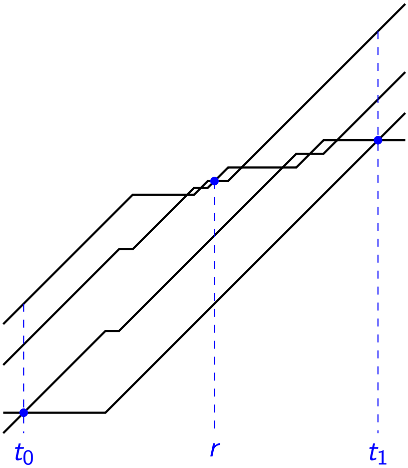
and choose a dual 4-system $\mathbf{P}^*$ for which $\mathbf{L}^*_{\mathbf{u}} - \mathbf{P}^*$ is bounded. Then,

$$\lim_{q\to\infty} P^*_3(q) - P^*_1(q) = \infty \quad \text{and} \quad \lim_{q\to\infty} P^*_4(q) - P^*_2(q) = \infty.$$

Moreover, if $P^*_2(r) = P^*_3(r)$ and $P^*_3(s) = P^*_4(s)$ for some $r < s$, then we have $P^*_1(t) = P^*_2(t)$ for some t with $r < t < s$.

Consequence of the last assertion

Suppose that $t_0 < t_1$ are consecutive points at which P_1^* and P_2^* coincide. Suppose also that there is a point r between t_0 and t_1 where P_3^* and P_4^* coincide. Then the combined graph of $\mathbf{P}^*$ over $[t_0, t_1]$ takes the form



Thank you !

https://tex.stackexchange.com/questions/337529/how-to-randomly-fill-an-area-with-daisies-in-tikz