Recent progress on Diophantine approximation in small degree

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Diophantine approximation and analytic number theory : a tribute to Cam Stewart
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Let

$$\gamma := \frac{1+\sqrt{5}}{2} = 1 + \frac{1}{\gamma} = 1.618\ldots$$

Theorem (Davenport and Schmidt, 1969)

Let $\xi \in \mathbb{R}$ s. t. $1, \xi, \xi^2$ are $\mathbb{Q}$-linearly independent. There exists a constant $c = c(\xi) > 0$ s. t.

$$|x_0| \le X, \quad |x_0\xi - x_1| \le cX^{-1/\gamma}, \quad |x_0\xi^2 - x_2| \le cX^{-1/\gamma}$$

has no non-zero solution $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3$ for arbitrarily large values of X.

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By Jarník's transference principle, this is equivalent to

Corollary

There exists a constant $c = c(\xi) > 0$ s. t.

$$|x_0 + x_1\xi + x_2\xi^2| \le cX^{-\gamma^2}, \quad |x_1| \le X, \quad |x_2| \le X$$

has no non-zero solution $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3$ for arbitrarily large values of X.

Corollary (D.& S., 1969)

There are infinitely many algebraic integers α *of degree* ≤ 3 *over* $\mathbb{Q}$ *s. t.* $|\xi - \alpha| \ll H(\alpha)^{-\gamma^2}$.

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Corollary (Bugeaud & Laurent, 2005)

Let $\theta \in \mathbb{R}$. There are ∞-many $Q \in \mathbb{Z}[T]_{\leq 2}$ s. t.

$$|\theta - Q(\xi)| \ll \|Q\|^{-\gamma}.$$

Theorem (2003)

There exists $\xi \in \mathbb{R}$ with $1, \xi, \xi^2$ $\mathbb{Q}$-lin. indep. s. t., for an appropriate constant $c = c(\xi) > 0$, the system

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- In the p-adic case:

  Zelo, Bel: the exponent is optimal

  Is there an analog of the notion of extremal number?

Example

- $\xi = [0, 1, 2, 1, 1, 2, 1, 1, 2, 2, \dots]$ is extremal.

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- if ξ is extremal, then $\dfrac{a\xi + b}{c\xi + d}$ is also extremal for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$.

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Definition

Identify any $\mathbf{x} = (x_0, x_1, x_2)$ with $\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix}$, and set

$$\det(\mathbf{x}) = x_0 x_2 - x_1^2.$$

Theorem (R. 2004)

A real number ξ is extremal

$\Longleftrightarrow$ *there exists a sequence of primitive points $(\mathbf{x}_i)_{i\geq 1}$ in $\mathbb{Z}^3$ and a matrix $M \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ with ${}^tM \neq \pm M$ such that, upon putting $X_i = \|\mathbf{x}_i\|$, we have*

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1) $\mathbf{x}_{i+2} \propto \mathbf{x}_{i+1} M_{i+1} \mathbf{x}_i$ *where* $M_i = \begin{cases} M & \text{if } i \text{ is even,} \\ {}^tM & \text{else,} \end{cases}$

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2) $X_{i+1} \asymp X_i^{\gamma}$ *and* $\lim_{i\to\infty} X_i = \infty$,
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4) $1 \leq |\det(\mathbf{x}_i)| \ll 1$.

An example for W. Schmidt's talk

- Consider the related parametric family of convex bodies

$$K(q) : |x_0| \leq e^{2q}, \quad |x_0\xi - x_1| \leq e^{-q}, \quad |x_0\xi^2 - x_2| \leq e^{-q}.$$

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- For each $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3 \setminus \{0\}$, we have

$$\begin{aligned} \lambda_{\mathbf{x}}(q) &:= \inf\{\lambda > 0\,;\, \mathbf{x} \in \lambda K(q)\} \\ &= \max\{e^{-2q}|x_0|, e^{q}|x_0\xi - x_1|, e^{q}|x_0\xi^2 - x_2|\} \end{aligned}$$

and so

$$L_{\mathbf{x}_i}(q) := \log \lambda_{\mathbf{x}_i}(q) = \max\{-2q + \log X_i, q - \log X_i\} + \mathcal{O}(1).$$

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- Moreover, if $\frac{2}{3}\log X_i \le q < \frac{2}{3}\log X_{i+1}$, then

$$\lambda_{\mathbf{x}_{i-1}}(q) \asymp \frac{e^q}{X_{i-1}}, \quad \lambda_{\mathbf{x}_i}(q) \asymp \frac{e^q}{X_i}, \quad \lambda_{\mathbf{x}_{i+1}}(q) \asymp \frac{X_{i+1}}{e^{2q}}$$

with product $\asymp 1$. So e^q/X_{i-1}, e^q/X_i and X_{i+1}/e^{2q} are bounded multiples of the successive minima of $K(q)$.

Theorem (2004)

Let $\xi \in \mathbb{R}$ *be extremal. There exists* $t \geq 0$ *such that*

- $\left|\xi - \dfrac{p}{q}\right| \gg q^{-2}(1 + \log q)^{-t}$ *for all* $\dfrac{p}{q} \in \mathbb{Q}$, $q \geq 1$.

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- $\left|\xi - \dfrac{p}{q}\right| \gg q^{-2}(1+\log q)^{-t}$ *for all* $\dfrac{p}{q} \in \mathbb{Q}$, $q \geq 1$.
- *There is a sequence of quadratic numbers* $(\alpha_i)_{i\geq 1}$ *in* $\mathbb{R}$ *such that*

$$|\xi - \alpha_i| \ll H(\alpha_i)^{-2\gamma^2} \quad \text{and} \quad H(\alpha_{i+1}) \asymp H(\alpha_i)^{\gamma}.$$

Any other quadratic α *has* $|\xi - \alpha| \gg H(\alpha)^{-4}$.

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Theorem (Adamczewski & Bugeaud (to appear))

For any integer $d \geq 1$ *and any algebraic number* α *of degree* $\leq d$, *we have*

$$|\xi - \alpha| \geq H(\alpha)^{-exp\{c(log 3d)^2(l \log\log 3d)^2\}}$$

Markoff extremal numbers

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Theorem

Let ξ *be extremal and let* $(\alpha_i)_{i\geq 1}$ *its associated sequence of best quadratic approximations.*

- $\xi' := \lim_{k\to\infty} \alpha_{2k}$ *and* $\xi'' := \lim_{k\to\infty} \alpha_{2k+1}$ *are extremal numbers in the* $GL_2(\mathbb{Q})$*-equivalence class of* ξ*.*

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- $\nu(\xi) = \nu(\xi') = \nu(\xi'')$
- $\nu(\xi) = 1/3 \quad \Longleftrightarrow \quad \nu(\alpha_i) > 1/3$ *for each sufficiently large i.*

Fix an extremal number ξ with conjugates $\xi \pm 3$.

Theorem

Let $d \in \{3, 4, 5\}$ and let $R \in \mathbb{Z}[T]$ be polynomial of degree d. For any $Q \in \mathbb{Z}[T]_{\leq 2}$, we have $|R(\xi) - Q(\xi)| \gg \|Q\|^{-\gamma}$.

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Corollary

For any root α of a polynomial of the form $T^d + aT^2 + bT + c \in \mathbb{Z}[T]$, we have $|\xi - \alpha| \gg H(\alpha)^{-\gamma-1}$.

Approximation to Markoff extremal numbers (with D. Zelo)

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Theorem

There are infinitely many α which are roots of a polynomial of the form $2T^6 + aT^2 + bT + c \in \mathbb{Z}[T]$ and satisfy

$$|\xi - \alpha| \ll H(\alpha)^{-\gamma-1}(\log\log H(\alpha))^{-1}.$$

On a question of Cam Stewart (with S. Lozier)

Theorem

Let $\xi \in \mathbb{R}$ such that $\underline{\xi} = (1, \xi, \xi^3)$ is $\mathbb{Q}$-linearly independent. Then, we have

$$\hat{\omega}_{sim}(\underline{\xi}) \leq 0.7115\ldots$$