# An introduction to Parametric Geometry of Numbers

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# Outline

1. Geometry of numbers
2. Motivation: transference inequalities
3. Main result
4. An application

Wolfgang Schmidt

Leonhard Summerer

# 1.1. Geometry of numbers

- A **(Minkowski) convex body** in $\mathbb{R}^n$ is a compact convex neighborhood $\mathcal{C}$ of 0 in $\mathbb{R}^n$ with $\mathcal{C} = -\mathcal{C}$.
- For $j = 1, \ldots, n$, its ***j*-th minimum**, denoted $\lambda_j(\mathcal{C})$ is the smallest $\lambda > 0$ such that $\lambda\mathcal{C}$ contains at least $j$ linearly independent elements of $\mathbb{Z}^n$.

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**Minkowski's second convex body theorem (1889)**

$$\frac{2^n}{n!} \leq \lambda_1(\mathcal{C}) \cdots \lambda_n(\mathcal{C}) \mathrm{vol}(\mathcal{C}) \leq 2^n.$$

If $\mathrm{vol}(\mathcal{C}) \geq 2^n$, then $\lambda_1(\mathcal{C}) \leq 1$ and so $\mathcal{C} \cap \mathbb{Z}^n \neq \{0\}$.

# 1.2. Mahler's dual and compound bodies

**Mahler (1939):** Define the dual of $\mathcal{C}$ by

$$\mathcal{C}^* = \{\mathbf{x} \in \mathbb{R}^n\, ;\, |\mathbf{x} \cdot \mathbf{y}| \leq 1 \text{ for all } \mathbf{y} \in \mathcal{C}\}.$$

Then $\boxed{\lambda_i(\mathcal{C}^*)\lambda_{n+1-i}(\mathcal{C}) \asymp 1}$ $(1 \leq i \leq n)$.

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**Mahler (1955):** For $k = 1, \dots, n$, define the $k$-th compound of $\mathcal{C}$ by

$$\mathcal{C}^{(k)} \;=\; \text{convex hull of } \{\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k \,;\, \mathbf{x}_1, \dots, \mathbf{x}_k \in \mathcal{C}\}$$

in $\bigwedge^k \mathbb{R}^n \simeq \mathbb{R}^N$ where $N = \binom{n}{k}$. Then

- $\lambda_1(\mathcal{C}^{(k)}) \asymp \lambda_1(\mathcal{C}) \cdots \lambda_{k-1}(\mathcal{C})\lambda_k(\mathcal{C})$
- $\lambda_2(\mathcal{C}^{(k)}) \asymp \lambda_1(\mathcal{C}) \cdots \lambda_{k-1}(\mathcal{C})\lambda_{k+1}(\mathcal{C})$

(In general $\lambda_j(\mathcal{C}^{(k)})$ is comparable to the $j$-th smallest product $\lambda_{i_1}(\mathcal{C}) \cdots \lambda_{i_k}(\mathcal{C})$ with $1 \le i_1 < \cdots < i_k \le n$.)

# 1.3. Parallelepipeds

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be a basis of $\mathbb{R}^n$. Then

$$\mathcal{P} : \begin{cases} |\mathbf{u}_1 \cdot \mathbf{x}| \leq 1 \\ \quad \cdots \\ |\mathbf{u}_n \cdot \mathbf{x}| \leq 1 \end{cases} \qquad \text{has} \quad \mathrm{vol}(\mathcal{P}) = 2^n |\det(\mathbf{u}_1, \ldots, \mathbf{u}_n)|.$$

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Its $k$-**th pseudo-compound** $\mathcal{P}^{(k)}$ is the set of all $\boldsymbol{\alpha} \in \bigwedge^k \mathbb{R}^n$ satisfying

$$\big| (\mathbf{u}_{i_1} \wedge \cdots \wedge \mathbf{u}_{i_k}) \cdot \boldsymbol{\alpha} \big| \leq 1$$

for each choice of indices $1 \leq i_1 < \cdots < i_k \leq n$.

W. Schmidt, 1983:

"*The classical theorems of the Geometry of Numbers deal with one convex set and one lattice at a time, while in Diophantine Approximation one deals with parametrized families of lattices [or convex bodies].*"

# 2.1. Simultaneous rational approximation

**Dirichlet (1842).** Let $\mathbf{u} = (1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$. For each integer $Q \geq 1$,

a) there exist $x_1, \dots, x_n \in \mathbb{Z}$ not all zero such that

$$|x_1| \leq Q^{n-1} \quad \text{and} \quad |x_1\xi_2 - x_2|, \dots, |x_1\xi_n - x_n| \leq Q^{-1}\,,$$

b) there exist $y_1, \dots, y_n \in \mathbb{Z}$ not all zero such that

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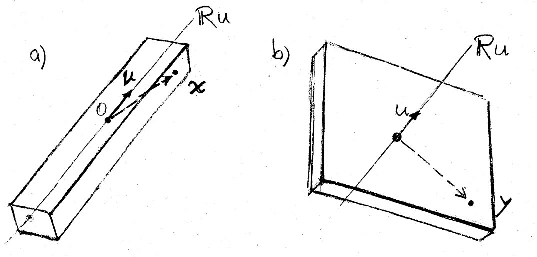
Proof.

Each set of inequalities define a convex body in $\mathbb{R}^n$ of volume $2^n$. □



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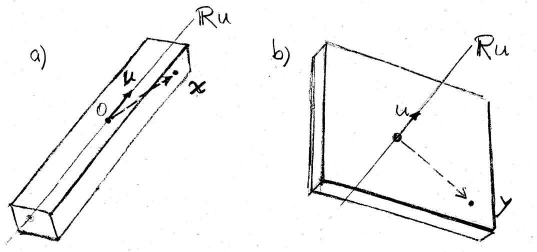

## 2.2. Four exponents of rational approximation

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Dirichlet:
$$\boxed{\lambda(\mathbf{u}) \geq \widehat{\lambda}(\mathbf{u}) \geq \frac{1}{n-1} \quad \text{and} \quad \tau(\mathbf{u}) \geq \widehat{\tau}(\mathbf{u}) \geq n-1}$$

## 2.3. Case of dimension $n = 3$

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**M. Laurent (2009):** The set of all quadruples $(\lambda(\mathbf{u}), \hat{\lambda}(\mathbf{u}), \tau(\mathbf{u}), \hat{\tau}(\mathbf{u}))$ is

$$\left\{ (\lambda, \hat{\lambda}, \tau, \hat{\tau})\,;\, 2 \leq \hat{\tau} \leq \infty,\ \hat{\lambda} = 1 - \frac{1}{\hat{\tau}},\ \frac{\tau(\hat{\tau} - 1)}{\tau + \hat{\tau}} \leq \lambda \leq \frac{\tau - \hat{\tau} + 1}{\hat{\tau}} \right\}$$

# 2.4. Dimension $n \geq 3$

- **Spectrum of** $(\lambda, \tau)$**:**
  - inequalities by **Khintchine (1926, 1928)**;
  - they are best possible: **Jarník (1935, 1936).**

- **Spectrum of** $(\widehat{\lambda}, \widehat{\tau})$**:**
  - inequalities: **Jarník (1938)** for $n = 3$, **German (2012)** for $n > 3$;
  - best possible: **Schmidt and Summerer (2016)**;
  - full spectrum: **Marnat (2018)** and for $n = 3$ **Jarník (1954)**.

- **Spectra of** $(\lambda, \widehat{\lambda})$ **and** $(\tau, \widehat{\tau})$**:**
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# 3.1. Schmidt and Summerer theory (2009, 2013)

Let $\mathbf{u} \in \mathbb{R}^n$ be non-zero. Define

$$\mathcal{C}(Q) = \left\{ \mathbf{x} \in \mathbb{R}^n \; ; \; \|\mathbf{x}\| \leq 1, \; |\mathbf{x} \cdot \mathbf{u}| \leq \frac{1}{Q} \right\} \quad (Q \geq 1),$$

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$$\underline{\varphi}_i(\mathbf{u}) := \liminf_{q \to \infty} \frac{L_i(q)}{q} \quad \text{and} \quad \bar{\varphi}_i(\mathbf{u}) := \limsup_{q \to \infty} \frac{L_i(q)}{q} \quad i = 1, \ldots, n.$$

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- **Problem:** Characterize the family of maps

$$\begin{aligned} \mathbf{L_u} : \quad [0, \infty) \quad &\longrightarrow \quad \mathbb{R}^n \\ q \quad &\longmapsto \quad (L_1(q), \dots, L_n(q)) \end{aligned}$$

... up to bounded functions.

# Formulas for the four exponents

- From the definitions, one gets

$$\tau(\mathbf{u}) = \frac{1}{\underline{\varphi}_1(\mathbf{u})} - 1, \qquad \widehat{\tau}(\mathbf{u}) = \frac{1}{\bar{\varphi}_1(\mathbf{u})} - 1.$$

- Using the duality of Mahler, one gets

$$\lambda(\mathbf{u}) = \frac{\bar{\varphi}_n(\mathbf{u})}{1 - \bar{\varphi}_n(\mathbf{u})}, \qquad \widehat{\lambda}(\mathbf{u}) = \frac{\underline{\varphi}_n(\mathbf{u})}{1 - \underline{\varphi}_n(\mathbf{u})}.$$

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3) $\boxed{L_1, \ldots, L_n \text{ are continuous and monotone increasing}}$

# 3.3. The trajectory of a point

$$\mathcal{C}(Q) = \left\{\mathbf{x} \in \mathbb{R}^n\,;\ \|\mathbf{x}\| \le 1,\ |\mathbf{x}\cdot\mathbf{u}| \le Q^{-1}\right\}$$

For each $\mathbf{x} \in \mathbb{Z}^n \setminus \{0\}$, we define

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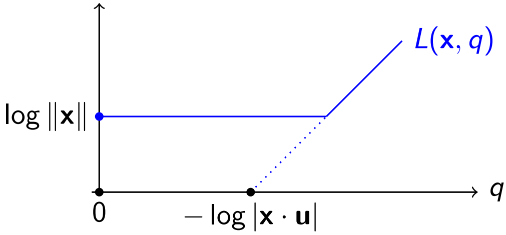

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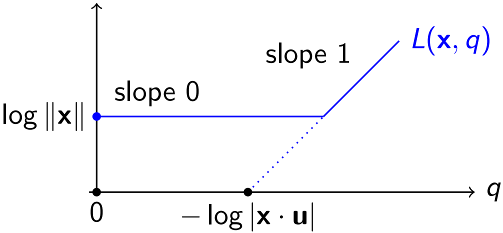
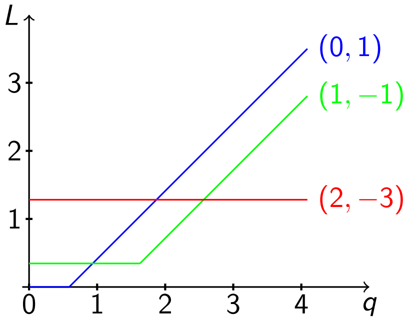


## 3.4. The first minimum

$$L(\mathbf{x}, q) = \max\{\log \|\mathbf{x}\|, q + \log |\mathbf{x} \cdot \mathbf{u}|\}$$

4) $L_1(q) = \min\{L(\mathbf{x}, q)\,;\, \mathbf{x} \in \mathbb{Z}^n \setminus \{0\}\} \quad (q \geq 0)$
is continuous and piecewise linear with slopes 0 and 1.

Example: $n = 2$, $\mathbf{u} = (3, 2)$



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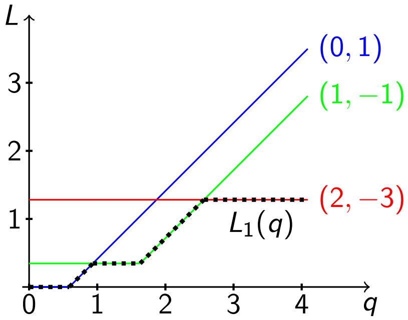

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5) $L_1(q) = L_2(q)$ at each $q$ where $L_1$ changes slope from 1 to 0.

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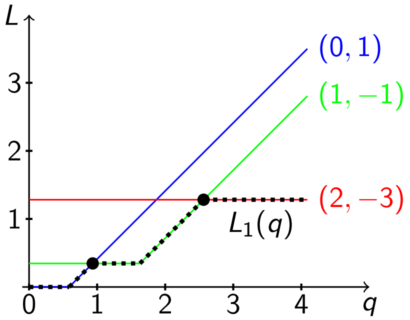

## 3.5. Compound convex bodies (Mahler, 1955)

- Write $\mathbb{R}^n = U \perp \langle \mathbf{u} \rangle_{\mathbb{R}}$ where $U = \mathbf{u}^\perp$,

$$\bigwedge^k \mathbb{R}^n = \bigwedge^k U \perp W^{(k)} \quad \text{where} \quad W^{(k)} = \bigwedge^{k-1} U \wedge \langle \mathbf{u} \rangle_{\mathbb{R}},$$

$$\mathcal{C}^{(k)}(Q) = \left\{ \boldsymbol{\alpha} \in \bigwedge^k \mathbb{R}^n \,;\; \|\boldsymbol{\alpha}\| \le 1,\; \|\operatorname{proj}_{W^{(k)}}(\boldsymbol{\alpha})\| \le Q^{-1} \right\},$$

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$$L_j^{(k)}(q) = \log \lambda_j(\mathcal{C}^{(k)}(e^q)) \quad \left(1 \le j \le \binom{n}{k}\right)$$

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5') At each point $q$ where it changes slope from 1 to 0, we have

$$L_2^{(k)}(q) = L_1^{(k)}(q) \implies \boxed{L_{k+1}(q) = L_k(q) + \mathcal{O}(1)}$$

## 3.6. The theorem of Schmidt and Summerer

For $j = 1, \dots, n$, we define a map $P_j \colon [0, \infty) \to \mathbb{R}$ by

$$P_j(q) := \begin{cases} L_1^{(1)}(q) = L_1(q) & \text{if } j = 1, \\ L_1^{(j)}(q) - L_1^{(j-1)}(q) & \text{if } 2 \le j \le n-1, \\ q - L_1^{(n-1)}(q) & \text{if } j = n. \end{cases}$$

**Theorem (Schmidt-Summerer, 2013)**

*There exists $\gamma > 0$ such that the function $\mathbf{P} = (P_1, \dots, P_n) \colon [0, \infty) \to \mathbb{R}^n$ satisfies*

$$\|\mathbf{P}(q) - \mathbf{L_u}(q)\|_\infty \le \gamma \quad \text{for each } q \ge 0,$$

*and is an $(n, \gamma)$-system according to the following definition.*

# 3.7. $(n, \gamma)$-systems

**Definition.** An $(n, \gamma)$-system is a map $\mathbf{P} = (P_1, \ldots, P_n)\colon [0, \infty) \to \mathbb{R}^n$ which satisfies the following conditions.

(S1) $P_j(q) \le P_{j+1}(q) + \gamma \quad (1 \le j < n,\ 0 \le q)$.

(S2) $P_j(q_1) \le P_j(q_2) + \gamma \quad (1 \le j \le n,\ 0 \le q_1 \le q_2)$.

(S3) For $j = 1, \ldots, n$, the function $M_j := P_1 + \cdots + P_j\colon [0, \infty) \to \mathbb{R}$ is continuous and piecewise linear with slopes 0 and 1.

(S4) $M_n(q) = q \quad (0 \le q)$.

(S5) If, for $j \in \{1, \ldots, n-1\}$, the function $M_j$ changes slope from 1 to 0 at a point $q > 0$, then $P_{j+1}(q) \le P_j(q) + \gamma$.

## 3.7. $(n,\gamma)$-systems

**Definition.** An $(n,\gamma)$-system is a map $\mathbf{P} = (P_1,\dots,P_n)\colon [0,\infty) \to \mathbb{R}^n$ which satisfies the following conditions.

(S1) $P_j(q) \le P_{j+1}(q) + \gamma \quad (1 \le j < n,\ 0 \le q)$.

(S2) $P_j(q_1) \le P_j(q_2) + \gamma \quad (1 \le j \le n,\ 0 \le q_1 \le q_2)$.

(S3) For $j = 1,\dots,n$, the function $M_j := P_1 + \cdots + P_j\colon [0,\infty) \to \mathbb{R}$ is continuous and piecewise linear with slopes 0 and 1.

(S4) $M_n(q) = q \quad (0 \le q)$.

(S5) If, for $j \in \{1,\dots,n-1\}$, the function $M_j$ changes slope from 1 to 0 at a point $q > 0$, then $P_{j+1}(q) \le P_j(q) + \gamma$.

(S3) $\implies$ $P_1,\dots,P_n$ are continuous and piecewise linear with slopes $-1$, 0 and 1

## 3.8. Ideal case

**Definition.** An $(n,0)$-system is a map $\mathbf{P} = (P_1, \ldots, P_n)\colon [q_0, \infty) \to \mathbb{R}^n$ which satisfies the following conditions.

(1) $P_j(q) \leq P_{j+1}(q) + 0 \quad (1 \leq j < n,\ q_0 \leq q)$.

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(3) For $j = 1, \ldots, n$, the function $M_j := P_1 + \cdots + P_j\colon [q_0, \infty) \to \mathbb{R}$ is continuous and piecewise linear with slopes 0 and 1.

(4) $M_n(q) = q \quad (q_0 \leq q)$.

(5) If, for $j \in \{1, \ldots, n-1\}$, the function $M_j$ changes slope from 1 to 0 at a point $q > q_0$, then $P_{j+1}(q) \leq P_j(q) + 0$.

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**Definition.** An *$n$-system* is a map $\mathbf{P} = (P_1, \ldots, P_n) \colon [q_0, \infty) \to \mathbb{R}^n$ which satisfies the following conditions.

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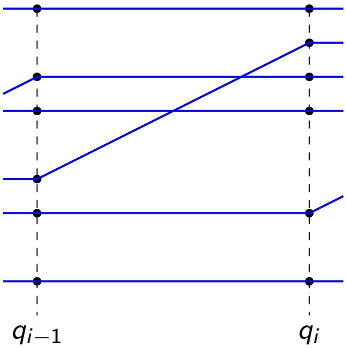
Let $\delta > 0$. We that $\mathbf{P}$ is a *rigid system with mesh $\delta$* if, for $q = q_0$ and for each $q > q_0$ at which some $M_j$ changes slope from 0 to 1, the coordinates of $\mathbf{P}(q)$ form a **strictly increasing** sequence of positive multiples of $\delta$.

## 3.9. The combined graph of a rigid system

Let $\mathbf{P} = (P_1, \ldots, P_n) \colon [q_0, \infty) \to \mathbb{R}^n$ be a rigid system with mesh $\delta$.

Denote by $q_1 < q_2 < \ldots$ the points where at least one of the sums $M_j = P_1 + \cdots + P_j$ changes slope from 0 to 1.

The combined graph of $P_1, \ldots, P_n$ has the following shape:



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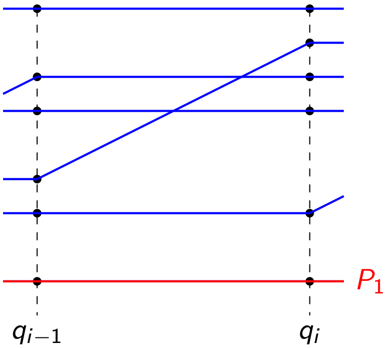

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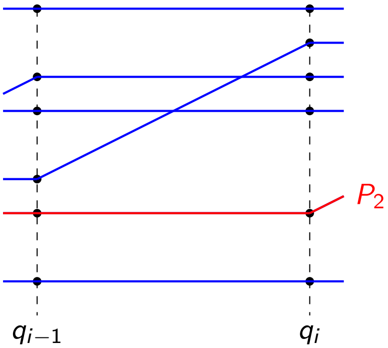

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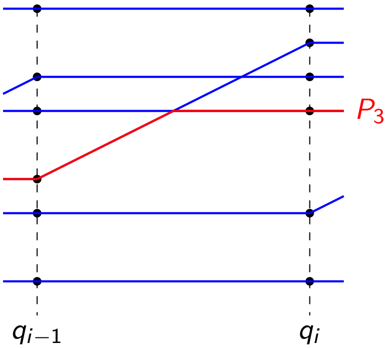

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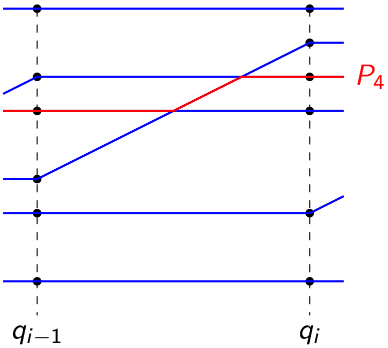

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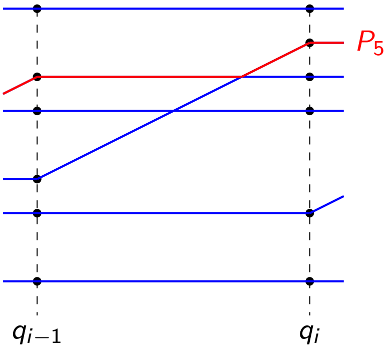

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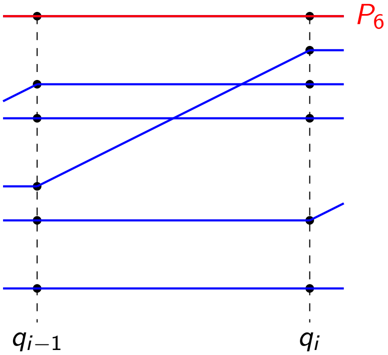

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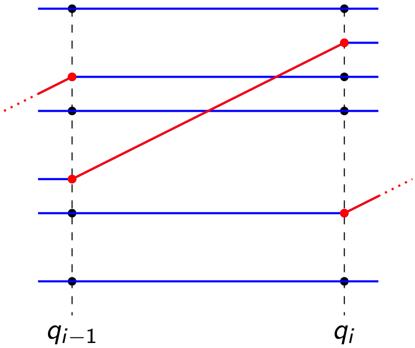

# 3.10. Characterization up to bounded maps

**Theorem (R. 2015)**

*For each nonzero* $\mathbf{u} \in \mathbb{R}^n$ *and each* $\delta > 0$*, there exists a rigid n-system* $\mathbf{P}$ *of mesh* $\delta$ *such that* $\mathbf{L_u} - \mathbf{P}$ *is a bounded function on* $[0, \infty)$*. Conversely, given any n-system* $\mathbf{P}$*, there exists a nonzero* $\mathbf{u} \in \mathbb{R}^n$ *such that* $\mathbf{L_u} - \mathbf{P}$ *is bounded.*

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- non-zero points $\mathbf{u} \in \mathbb{R}^n$
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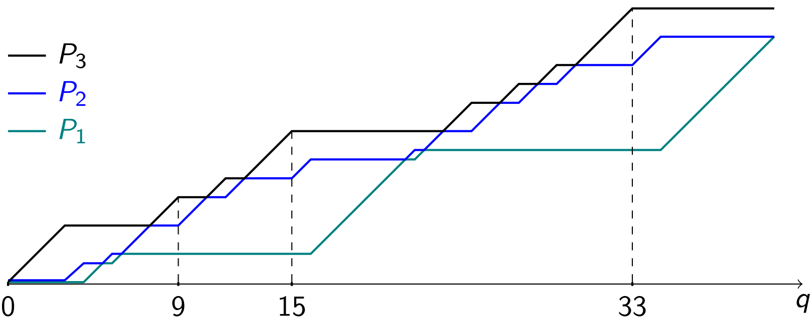
## 3.11. Interpretation as a game (Luca Ghidelli)

We can view an $n$-system as giving the positions of $n$ players $P_1, \ldots, P_n$ moving on a line, as a function of the time $q$, according to the following rules.

- At time $q = 0$, they all stand at position 0.
- They always remain in the same order ($P_1$ cannot overpass $P_2$, nor $P_2$ can overpass $P_3$, etc).
- At any time, only the player who has the ball can move and he moves at constant speed 1.
- The player who holds the ball can only pass it to a player that is behind him or next to him.

# Combined graph of the 3-system in the animation



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Set $\varphi_i(q) = \dfrac{P_i(q)}{q} \quad (1 \leq i \leq n,\ q > 0)$. Then

$$0 \leq \varphi_1(q) \leq \cdots \leq \varphi_n(q) \quad \text{and} \quad \varphi_1(q) + \cdots + \varphi_n(q) = 1,$$

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Moreover $0 \le \underline{\varphi}_1(\mathbf{u}) \le \dfrac{1}{n}$, thus $\boxed{n - 1 \le \tau(\mathbf{u}) \le \infty}$.

# The above inequalities describe the spectrum of $(\lambda, \tau)$

Let $a, b \in \mathbb{R}$ with $0 \leq a \leq \frac{1}{n}$ and $(n-1)a + b \leq 1 \leq a + (n-1)b$.

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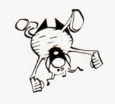

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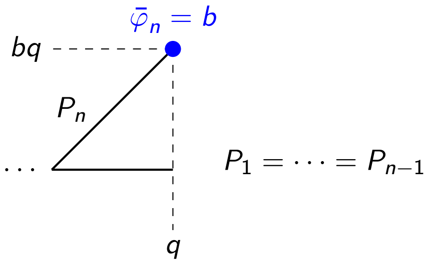


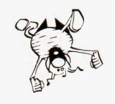

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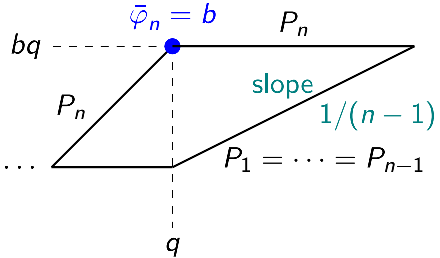


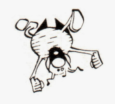

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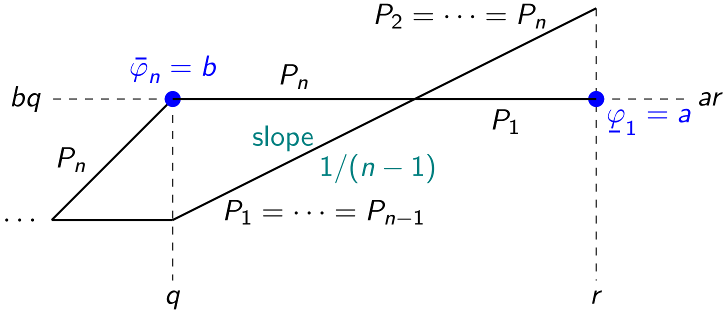


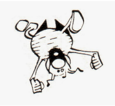

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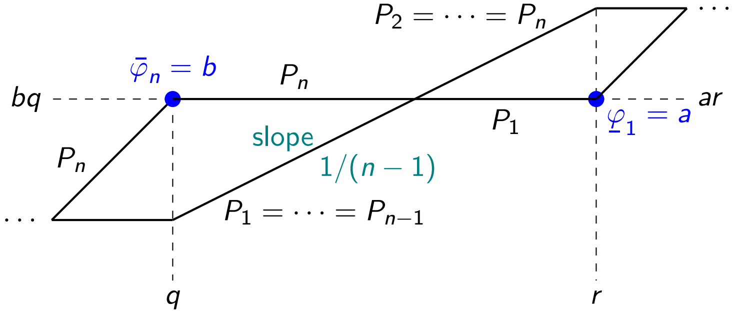


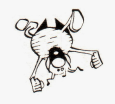

Thank you!