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Philippon's Criterion for Algebraic Independence
Lectures 3 and 4

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A typical transcendence argument starts with the construction of a sequence of auxiliary polynomials taking small values at many points of a finitely generated subgroup Γ of a commutative algebraic group G. These values belong to some finitely generated extension $K = \mathbb{Q}(\theta_1, \dots, \theta_m)$ of $\mathbb{Q}$, assuming that the group G itself is defined over K. If these values are ordinary integers and if we know that their absolute values are < 1, then they all vanish and we can apply a zero estimate to conclude. More generally, if these values are algebraic numbers and if we have sufficiently good upper bounds for their degree over $\mathbb{Q}$ as well as for their (naive) height (the largest absolute of the coefficients of their minimal polynomial over $\mathbb{Z}$), then we can instead apply Liouville's inequality and hopefully conclude that these values are zero as well. This is the situation when the field K is algebraic over $\mathbb{Q}$. When it has transcendence degree one over $\mathbb{Q}$, a substitute for Liouville's inequality is given by Gel'fond's criterion. When the transcendence degree is higher, a substitute is Philippon's criterion. The purpose of these two lectures is to present a proof of the latter.

The main tool for the proof is the use of (Cayley-) Chow forms which are generalizations of the Sylvester resultant of two polynomials. Basically, we will need to factor these forms and to recursively specialize their arguments while keeping track of several norms attached to them. In Philippon's paper [11], all norms are multiplicative. This makes factoring easy but complicates the specialization arguments. Here we use the approach of [8]. The norms that we consider instead are attached to convex bodies in such a way that specialization becomes simple. These norms being quasi-multiplicative, we do not really lose much neither when factoring the forms. This approach was used in [8] to provide a version of Philippon's criterion with "multiplicities", using slightly more general convex bodies than those which we will consider here.

In the next section, we study norms attached to convex bodies in rings of polynomials. In Section 2, we define Chow forms and present some of their properties. Then, in Section 3, we use these forms to define heights of projective algebraic sets defined over $\mathbb{Q}$ relative to convex bodies. In section 4, we look at specific convex bodies adapted to the context of Philippon's criterion and provide estimates for the corresponding norms. In Sections 5 and 6, we look at the behavior of these norms under specialization, an operation which geometrically corresponds to taking intersection of algebraic sets with hypersurfaces. We conclude in Section 7 with the statement and proof of Philippon's criterion.

1. Norms of polynomials with respect to convex bodies

In this section, we define the notion of a convex body $\mathcal{C}$ of $\mathbb{C}^n$ and, for each polynomial $F \in \mathbb{C}[U_1, \dots, U_n]$ in n-variables, we define the *norm* of F with respect to $\mathcal{C}$ to be the maximum of the absolute values of F on $\mathcal{C}$. Following the presentation in [8], we show that this norm is essentially multiplicative up to constant factors depending only on the degree of the product and the number of variables n. To be more precise we work in fact with multi-homogeneous polynomials and corresponding cartesian products of convex bodies. This is motivated by the applications in Section 3 when we define heights for $\mathbb{Q}$-subvarieties of $\mathbb{P}_m(\mathbb{C})$. The notion of norm with respect to a convex body is reminiscent of the twisted height of J. L. Thunder [15] and the work of S. Zhang [17].

1.1. **A generalization of John's theorem.** Fix a positive integer n.

Definition 1. A *convex body* of $\mathbb{C}^n$ is a compact neighborhood $\mathcal{C}$ of the origin in $\mathbb{C}^n$ with the property that

$$\lambda \mathbf{x} + \mu \mathbf{y} \in \mathcal{C}$$

for any choice of points $\mathbf{x}$, $\mathbf{y}$ of $\mathcal{C}$ and any choice of elements λ, μ of $\mathbb{C}$ with $|\lambda| + |\mu| \leq 1$.

If, in this definition, we replace $\mathbb{C}$ by $\mathbb{R}$, we find the usual notion of convex body of $\mathbb{R}^n$, in the sense of Minkowski. In the present context, if $L_1, \dots, L_k$ are linear forms from $\mathbb{C}^n$ to $\mathbb{C}$ with the origin as their only common zero, and if $\epsilon_1, \dots, \epsilon_k$ are positive real numbers, then the set of points $\mathbf{x} \in \mathbb{C}^n$ which satisfy the conditions $|L_i(\mathbf{x})| \leq \epsilon_i$ for $i = 1, \dots, k$ is a convex body of $\mathbb{C}^n$. This is the type of convex body that we shall encounter later in the applications.

For any bounded subset S of $\mathbb{C}^n$, there is a smallest convex body of $\mathbb{C}^n$ containing S. We call it the *symmetric convex hull* of S. It is the topological closure of the set of all linear combinations $\lambda_1 \mathbf{x}_1 + \cdots + \lambda_s \mathbf{x}_s$ with $\mathbf{x}_1, \dots, \mathbf{x}_s \in S$ and $\lambda_1, \dots, \lambda_s \in \mathbb{C}$ satisfying $\sum |\lambda_i| \leq 1$.

We also endow $\mathbb{C}^n$ with the maximum norm $\|(x_1, \dots, x_n)\| = \max\{|x_1|, \dots, |x_n|\}$ and denote by

$$\mathcal{B} = \left\{ \mathbf{x} \in \mathbb{C}^n \, ; \, \|\mathbf{x}\| \leq 1 \right\}$$

the *unit ball* of $\mathbb{C}^n$ relative to this norm. It is again a convex body.

John's theorem states that, for any convex body $\mathcal{C}$ of $\mathbb{R}^n$ there is an ellipsoid E of $\mathbb{R}^n$ such that $E \subseteq \mathcal{C} \subseteq \sqrt{n} E$ (see [6] or [14, Chap. IV, Thm. 2A]). However an ellipsoid E of $\mathbb{R}^n$ is simply the image of the unit Euclidean ball of $\mathbb{R}^n$ by a linear map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$. Here, we prove an analog result in $\mathbb{C}^n$ using the unit ball $\mathcal{B}$ of $\mathbb{C}^n$ with respect to the maximum norm instead of the unit Euclidean ball. The statement below remains true and the proof is the same if we replace everywhere $\mathbb{C}$ by $\mathbb{R}$.

Proposition 1.1. *Let $\mathcal{C}$ be a convex body of $\mathbb{C}^n$. Then, there is a linear map $\varphi\colon \mathbb{C}^n \to \mathbb{C}^n$ such that*

$$\varphi(\mathcal{B}) \subseteq \mathcal{C} \subseteq n\,\varphi(\mathcal{B}). \tag{1}$$

Proof. Denote by $\mathcal{E}$ the set of all linear maps $\varphi\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $\varphi(\mathcal{B}) \subseteq \mathcal{C}$. Since $\mathcal{C}$ is a compact neighborhood of zero, the set $\mathcal{E}$ is not empty and compact. Therefore, it contains an element φ for which $|\det(\varphi)|$ is maximal. We will show that such a choice of φ satisfies the condition (1).

Indeed, assume on the contrary that $\mathcal{C} \not\subseteq n\varphi(\mathcal{B}) = \varphi(n\mathcal{B})$. Then there exists $\underline{\xi} \in \mathbb{C}^n$ with $\varphi(\underline{\xi}) \in \mathcal{C}$ and $\|\underline{\xi}\| > n$. Since $\varphi(\mathcal{B}) \subseteq \mathcal{C}$, this implies that $\varphi(\mathcal{C}') \subseteq \mathcal{C}$, where $\mathcal{C}'$ denotes the symmetric convex hull of $\mathcal{B} \cup \{\underline{\xi}\}$.

Now, fix an index i with $|\xi_i| = \|\underline{\xi}\|$, and denote by $\{e_1, \dots, e_n\}$ the canonical basis of $\mathbb{C}^n$. Upon multiplying $\underline{\xi}$ by an appropriate element of $\mathbb{C}$ of absolute value 1, we may assume without loss of generality that ξ_i is real and positive, and thus that $\xi_i > n$. For any real number t with $0 \le t \le 1$, define ψ_t to be the $\mathbb{C}$-linear endomorphism of $\mathbb{C}^n$ which satisfies

$$\psi_t(e_i) = te_i + (1-t)\underline{\xi} \quad \text{and} \quad \psi_t(e_j) = te_j \quad \text{for } j \neq i.$$

Then, the map ψ_t satisfies $\psi_t(\mathcal{B}) \subseteq \mathcal{C}'$, and so $\varphi \circ \psi_t$ belongs to $\mathcal{E}$. Moreover, we find

$$\det(\psi_t) = v(t) \quad \text{where } v(t) = \big(t + (1-t)\xi_i\big)\, t^{n-1} \in \mathbb{R}.$$

Since $\xi_i > n$, the function $v(t)$ achieves its maximum on the interval $[0,1]$ at the point $t_0 = (1-1/n)/(1-1/\xi_i)$ and is strictly monotone decreasing in the interval $[t_0, 1]$. Define $\psi = \psi_{t_0}$. Since $t_0 < 1$, we find that $|\det(\psi)| = v(t_0) > v(1) = 1$. Then, the composite $\varphi \circ \psi$ is an element of $\mathcal{E}$ with $|\det(\varphi \circ \psi)| > |\det(\varphi)|$, against the choice of φ. □

1.2. **Norm attached to a convex body.** Given a convex body $\mathcal{C}$ of $\mathbb{C}^n$ and a polynomial $F \in \mathbb{C}[U_1, \dots, U_n]$, we define the norm of F relative to $\mathcal{C}$ by

$$\|F\|_{\mathcal{C}} = \sup\{|F(\mathbf{x})|\,;\, \mathbf{x} \in \mathcal{C}\}.$$

A more standard norm on $\mathbb{C}[U_1, \dots, U_n]$ denoted $\|\ \|$ is given, for $F = \sum_{\underline{\tau}} a_{\underline{\tau}} U_1^{\tau_1} \cdots U_n^{\tau_n}$, by the maximum of the absolute values of its coefficients:

$$\|F\| = \max_{\underline{\tau}} |a_{\underline{\tau}}|\,.$$

The next lemma compares these norms when $\mathcal{C}$ is the unit ball $\mathcal{B}$ of $\mathbb{C}^n$ for the maximum norm. We leave it as an exercise.

Lemma 1.2. *Let $\mathcal{B}$ be the unit ball of $\mathbb{C}^n$ for the maximum norm, let $F \in \mathbb{C}[U_1, \dots, U_n]$ be a polynomial and let N be the number of non-zero coefficients of F. Then, we have:*

$$\|F\| \le \|F\|_{\mathcal{B}} \le N\|F\|.$$

Our goal is to show that the norm attached to a convex body is essentially multiplicative. We start by the case of the unit ball of $\mathbb{C}^n$, and proceed by comparison with the *Mahler's measure*. We recall that the Mahler's measure of a non-zero polynomial $F \in \mathbb{C}[U_1, \dots, U_n]$ is given by

$$M(F) = \exp\left\{\int_0^1 \cdots \int_0^1 \log|F(\exp(2\pi \mathrm{i} t_1), \cdots, \exp(2\pi \mathrm{i} t_n))| \, \mathrm{d}t_1 \cdots \mathrm{d}t_n\right\}$$

(see [9]). For $F = 0$, we put $M(F) = 0$. This function is multiplicative in the sense that, for any polynomials $F, G \in \mathbb{C}[U_1, \dots, U_n]$, we have

$$M(FG) = M(F)M(G).$$

For the applications to Chow forms, it is crucial to have good inequalities of comparison between $\|F\|_{\mathcal{B}}$ and $M(F)$, especially for the dependence in the number of variables. In [11, Lemma 1.13], P. Philippon proves such precise inequalities of comparison between $\|F\|$ and $M(F)$. We will admit the following result whose proof is similar (see [8, Lemma 3.5]).

Lemma 1.3. *Let $\mathcal{B}$ be the unit ball of $\mathbb{C}^n$ for the maximum norm. Let $n_1, \dots, n_k$ be positive integers with sum n, and let $U_{j,i}$, $(1 \le j \le k,\ 1 \le i \le n_j)$, be indeterminates. Put $\mathbf{U}_j = (U_{j,1}, \dots, U_{j,n_j})$ for $j = 1, \dots, k$. If $F \in \mathbb{C}[\mathbf{U}_1, \dots, \mathbf{U}_k]$ is a multi-homogeneous polynomial of multi-degree $(d_1, \dots, d_k)$ in the sets of variables $\mathbf{U}_1, \dots, \mathbf{U}_k$, then we have:*

$$M(F) \le \|F\|_{\mathcal{B}} \le n_1^{d_1} \cdots n_k^{d_k} M(F).$$

In the above statement, the assumption of multi-homogeneity on F simply means that each monomial in F has degree d_j in the set of variables $\mathbf{U}_j$, for $j = 1, \dots, k$. We deduce:

Lemma 1.4. *Let $\mathcal{B}$, $n_1, \dots, n_k$, $\mathbf{U}_1, \dots, \mathbf{U}_k$ be as in Lemma 1.3. If $F_1, \dots, F_s \in \mathbb{C}[\mathbf{U}_1, \dots, \mathbf{U}_k]$ are multi-homogeneous polynomials in the sets of variables $\mathbf{U}_1, \dots, \mathbf{U}_k$ and if their product $F = F_1 \cdots F_s$ has multi-degree $(d_1, \dots, d_k)$, then we have:*

$$\|F\|_{\mathcal{B}} \le \prod_{i=1}^{s} \|F_i\|_{\mathcal{B}} \le n_1^{d_1} \cdots n_k^{d_k} \|F\|_{\mathcal{B}}.$$

Proof. For $i = 1, \dots, s$, denote by $(d_{i,1}, \dots, d_{i,k})$ the multi-degree of F_i. Using Lemma 1.3 and the multiplicativity of the Mahler's measure, we find

$$\prod_{i=1}^{s} \|F_i\|_{\mathcal{B}} \le \prod_{i=1}^{s} \left(n_1^{d_{i,1}} \cdots n_k^{d_{i,k}} M(F_i)\right) = n_1^{d_1} \cdots n_k^{d_k} M(F) \le n_1^{d_1} \cdots n_k^{d_k} \|F\|_{\mathcal{B}}.$$

The other inequality is clear. □

Combining Lemma 1.4 with Proposition 1.1, we find:

Proposition 1.5. *Let $\mathcal{B}$, $n_1, \dots, n_k$, $\mathbf{U}_1, \dots, \mathbf{U}_k$ be as in Lemma 1.3, and let $\mathcal{C}$ be a convex body of $\mathbb{C}^n = \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_k}$ in the form of a Cartesian product $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_k$ where $\mathcal{C}_j$ is a convex body of $\mathbb{C}^{n_j}$ for $j = 1, \dots, k$. If $F_1, \dots, F_s \in \mathbb{C}[\mathbf{U}_1, \dots, \mathbf{U}_k]$ are multi-homogeneous polynomials in the sets of variables $\mathbf{U}_1, \dots, \mathbf{U}_k$ and if their product $F = F_1 \cdots F_s$ has multi-degree $(d_1, \dots, d_k)$, then we have:*

$$\|F\|_{\mathcal{C}} \le \prod_{i=1}^{s} \|F_i\|_{\mathcal{C}} \le \left(n_1^{d_1} \cdots n_k^{d_k}\right)^2 \|F\|_{\mathcal{C}}.$$

Proof. The lower bound $\prod_{i=1}^{s} \|F_i\|_{\mathcal{C}} \ge \|F\|_{\mathcal{C}}$ is clear and is valid for any convex body $\mathcal{C}$ of $\mathbb{C}^n$. To prove the upper bound, denote by $\mathcal{B}_j$ the unit ball of $\mathbb{C}^{n_j}$ relative to the maximum norm, for $j = 1, \dots, k$. By Proposition 1.1, there is a linear map $\varphi_j \colon \mathbb{C}^{n_j} \to \mathbb{C}^{n_j}$ such that

$$\varphi_j(\mathcal{B}_j) \subseteq \mathcal{C}_j \subseteq \varphi_j(n_j \mathcal{B}_j).$$

For $i = 1, \dots, s$, define $G_i = F_i \circ (\varphi_1, \dots, \varphi_k)$. Put also $\mathcal{B} = \mathcal{B}_1 \times \cdots \times \mathcal{B}_k$ and $G = G_1 \cdots G_s$. Then, $\mathcal{B}$ is the unit ball of $\mathbb{C}^n$ for the maximum norm and we have $G = F \circ (\varphi_1, \dots, \varphi_k)$. Using the above inclusions together with Lemma 1.4, we find:

$$\begin{aligned}\prod_{i=1}^{s} \|F_i\|_{\mathcal{C}} \le \prod_{i=1}^{s} \|G_i\|_{(n_1\mathcal{B}_1)\times\cdots\times(n_k\mathcal{B}_k)} &= n_1^{d_1} \cdots n_k^{d_k} \prod_{i=1}^{s} \|G_i\|_{\mathcal{B}} \\ &\le \left(n_1^{d_1} \cdots n_k^{d_k}\right)^2 \|G\|_{\mathcal{B}} \le \left(n_1^{d_1} \cdots n_k^{d_k}\right)^2 \|F\|_{\mathcal{C}}.\end{aligned}$$

□

2. Chow forms

From now on, we fix a positive integer m and denote by $\mathbb{C}[\mathbf{X}]$ the ring $\mathbb{C}[X_0, X_1, \dots, X_m]$ of polynomials in the variables $\mathbf{X} = (X_0, X_1, \dots, X_m)$. We also denote by $\mathbb{P}_m(\mathbb{C})$ the projective m-space over $\mathbb{C}$. Its elements are the equivalence classes of non-zero points of $\mathbb{C}^{m+1}$ under the relation $\underline{\alpha} \sim \underline{\alpha}'$ if $\underline{\alpha}' = \lambda \underline{\alpha}$ for some $\lambda \in \mathbb{C}^\times$. We denote by $(\alpha_0 : \alpha_1 : \cdots : \alpha_m)$ the equivalence class of a non-zero point $\underline{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_m)$ in $\mathbb{C}^{m+1}$ and say that $\underline{\alpha}$ is a *set of homogeneous coordinates* of that point.

2.1. Geometric preliminaries.

General references for this section are [4, Chap. 1, §2 and §7] and [18, Chap. VII].

If a homogeneous polynomial P of $\mathbb{C}[\mathbf{X}]$ vanishes at some non-zero point $\underline{\alpha}$ in $\mathbb{C}^{m+1}$, then it vanishes at each point of the equivalence class of $\underline{\alpha}$. So it is natural to define the set of zeros $Z(P)$ of P in $\mathbb{P}_m(\mathbb{C})$ to be the set of equivalence classes of zeros of P in $\mathbb{C}^{m+1} \setminus \{0\}$.

More generally, a (closed) algebraic subset of $\mathbb{P}_m(\mathbb{C})$ is the set of common zeros $Z(\mathcal{F})$ of a family of homogeneous polynomials $\mathcal{F}$ of $\mathbb{C}[\mathbf{X}]$. Here, we consider only algebraic subsets W

of $\mathbb{P}_n(\mathbb{C})$ that are *defined over* $\mathbb{Q}$ meaning by this that they can be expressed as the set of common zeros of homogeneous polynomials of $\mathbb{Q}[\mathbf{X}]$. When such a set cannot be written as the union of two proper algebraic subsets defined over $\mathbb{Q}$, we say that it is *irreducible over* $\mathbb{Q}$ and we call it a $\mathbb{Q}$-*subvariety* of $\mathbb{P}_m(\mathbb{C})$.

Given an algebraic subset W of $\mathbb{P}_m(\mathbb{C})$ defined over $\mathbb{Q}$, we denote by $I(W)$ the ideal of $\mathbb{Q}[\mathbf{X}]$ generated by all homogeneous polynomials of that ring which vanish at each point of W. This ideal is *radical* in the sense that, if it contains some power of a polynomial, then it contains this polynomial as well. It can be shown that the assignment $W \mapsto I(W)$ establishes a bijection between the set of algebraic subsets of $\mathbb{P}_m(\mathbb{C})$ defined over $\mathbb{Q}$ and the set of homogeneous radical ideals of $\mathbb{Q}[\mathbf{X}]$. Under this bijection, $\mathbb{Q}$-subvarieties of $\mathbb{P}_m(\mathbb{C})$ correspond to homogeneous prime ideals of $\mathbb{Q}[\mathbf{X}]$. It can be shown that any algebraic subset W of $\mathbb{P}_m(\mathbb{C})$ defined over $\mathbb{Q}$ can be written as a finite union of $\mathbb{Q}$-subvarieties of $\mathbb{P}_m(\mathbb{Q})$

$$W = W_1 \cup \cdots \cup W_s$$

with $W_i \not\subseteq W_j$ when $i \neq j$. These $\mathbb{Q}$-subvarieties $W_1, \dots, W_s$ are uniquely determined by W (up to permutation) and are called the $\mathbb{Q}$-*irreducible components* of W.

For each integer $D \geq 0$, we denote by $\mathbb{C}[\mathbf{X}]_D$ (resp. $\mathbb{Q}[\mathbf{X}]_D$) the subspace of $\mathbb{C}[\mathbf{X}]$ (resp. $\mathbb{Q}[\mathbf{X}]$) which consists of all homogeneous polynomials of degree D in that ring (with the convention that 0 is homogeneous of each degree). If W is a non-empty algebraic subset of $\mathbb{P}_m(\mathbb{C})$ defined over $\mathbb{Q}$, we define its *Hilbert function* by

$$H(W;\, D) = \dim_{\mathbb{Q}} \mathbb{Q}[\mathbf{X}]_D / (I(W) \cap \mathbb{Q}[\mathbf{X}]_D)$$

for each $D \in \mathbb{N}$. It can be shown that it coincides with a polynomial in D for each sufficiently large value of D. The degree of this polynomial is an integer t with $0 \leq t \leq m$ called the *dimension* of W and denoted $\dim(W)$. Moreover, the product by $t!$ of the leading coefficient of that polynomial is a positive integer called the *degree* of W, and denoted $\deg(W)$. It can be shown that $\dim(W)$ is the largest dimension of the $\mathbb{Q}$-irreducible components of W and that $\deg(W)$ is the sum of the degrees of the $\mathbb{Q}$-irreducible components of W with dimension $\dim(W)$. We say that W is *equidimensional* if all its $\mathbb{Q}$-irreducible components have the same dimension. We will frequently use the fact that, if W is a $\mathbb{Q}$-subvariety of $\mathbb{P}_m(\mathbb{C})$ of dimension $t \geq 1$ and if $P \in \mathbb{Q}[\mathbf{X}]$ is an homogeneous polynomial which does not belong to $I(W)$, then the intersection $W \cap Z(P)$ of W with the set of zeros of P is equidimensional of dimension $t - 1$.

2.2. **Chow forms.** We will admit the following fact.

Theorem 2.1. *Let W be a $\mathbb{Q}$-subvariety of $\mathbb{P}_m(\mathbb{C})$ of dimension $t \geq 0$. For any choice of integers $D_0, \dots, D_t \in \mathbb{N}$, there exists, up to multiplication by ± 1, a unique polynomial*

$$F \in \mathbb{Z}[\mathbf{U}^{(0)}, \dots, \mathbf{U}^{(t)}] \quad \textit{where} \quad \mathbf{U}^{(k)} = (U_{\mathbf{j}}^{(k)}\,;\, \mathbf{j} \in \mathbb{N}^{m+1},\ |\mathbf{j}| = D_k) \quad \textit{for } k = 0, \dots, t,$$

which satisfies the following properties:

1) *F is irreducible (in particular, the set of its coefficients is relatively prime),*

2) *for any choice of $\mathbf{u}^{(k)} = (u_{\mathbf{j}}^{(k)}\,;\, \mathbf{j} \in \mathbb{N}^{m+1},\ |\mathbf{j}| = D_k) \in \mathbb{C}^{\binom{D_k+m}{m}}$ $(k = 0, \ldots, t)$, we have $F(\mathbf{u}^{(0)}, \ldots, \mathbf{u}^{(t)}) = 0 \iff \sum_{\mathbf{j}} u_{\mathbf{j}}^{(0)} \mathbf{X}^{\mathbf{j}}, \ldots, \sum_{\mathbf{j}} u_{\mathbf{j}}^{(t)} \mathbf{X}^{\mathbf{j}}$ have a common zero on W,*

3) *F is homogeneous of degree $D_0 \cdots \widehat{D_k} \cdots D_t \deg(W)$ in the set of variables $\mathbf{U}^{(k)}$ for each $k = 0, \ldots, t$.*

We say that such a polynomial F is a *Cayley-Chow form* or simply a *Chow form* of W in degree $(D_0, \ldots, D_t)$.

For example, when $m = 1$ and $W = \mathbb{P}_1(\mathbb{C})$, the form F is the Sylvester resultant of two homogeneous polynomials. When $W = \mathbb{P}_m(\mathbb{C})$ and $D_0 = \cdots = D_m = 1$, then F is simply the determinant of $m + 1$ linear forms.

For the proof, Propositions 1.3 (ii), 1.4 and 1.5 (iii) of Philippon's paper [11] show the existence and uniqueness of a polynomial F satisfying 1) and 2). Lemma 1.8 of [11] together with Remark 1) on page 15 of [11] show that it satisfies 3). For the more general case where W is a subvariety of a product of projective spaces, see the paper of G. Rémond [13]. For the special case where $W = \mathbb{P}_m(\mathbb{C})$, a very nice exposition using homological methods is given by M. Chardin in [2]. For the special case $D_0 = \cdots = D_t = 1$, see Yu. Nesterenko's paper [10, §1] or the classical reference [5, Chap. X].

In the sequel, we view a Chow form in degree $(D_0, \ldots, D_t)$ as a polynomial map

$$F\colon \mathbb{C}[\mathbf{X}]_{D_0} \times \cdots \times \mathbb{C}[\mathbf{X}]_{D_t} \longrightarrow \mathbb{C}$$

whose set of zeros are the $(t+1)$-tuples of polynomials $(P_0, \ldots, P_t)$ which have a common zero on W, upon identifying each P_k with its set of coefficients.

Before we continue with the exposition of algebraic properties of the Chow forms, this is probably the right place to stop and explain why we are interested in these forms. To this end, fix a $\mathbb{Q}$-subvariety W of dimension $t \geq 0$, a point $\theta \in W$ with projective coordinates $\underline{\theta}$, positive integers $D_0, \ldots, D_t$, and a Chow form F of W in degree $(D_0, \ldots, D_t)$. Suppose that there exist homogeneous polynomials $P_0, \ldots, P_t$ in $\mathbb{Z}[\mathbf{X}]$ of respective degrees $D_0, \ldots, D_t$ with no common zeros on W, then $F(P_0, \ldots, P_t)$ is a non-zero integer and so $|F(P_0, \ldots, P_t)| \geq 1$. Now suppose that $P_0, \ldots, P_t$ all have small absolute values at the point $\underline{\theta}$. Then a small perturbation in their coefficients will give rise to homogeneous polynomials $\tilde{P}_0, \ldots, \tilde{P}_t$ of the same respective degrees (but with complex coefficients) which all vanish at θ. Then we have $F(\tilde{P}_0, \ldots, \tilde{P}_t) = 0$. However, the value $F(\tilde{P}_0, \ldots, \tilde{P}_t)$ should be close to $F(P_0, \ldots, P_t)$. This is impossible if $|P_0(\underline{\theta})|, \dots, |P_t(\underline{\theta})|$ are smaller than a certain positive number depending only on W, $D_0, \dots, D_t$, $\|P_0\|, \dots, \|P_t\|$ and $\|\underline{\theta}\|$. Our first goal will be to make this statement explicit in the case where $D_0 = \cdots = D_t$. This will be achieved with Corollary 4.5 at the end of Section 4.

2.3. **Properties of Chow forms.** The following three lemmas will be useful.

Lemma 2.2. *Let W be a $\mathbb{Q}$-subvariety of $\mathbb{P}_m(\mathbb{C})$ of dimension $t \geq 0$, and let F be a Chow form of W in degree $(D_0, \dots, D_t)$.*

(i) *Let σ be a permutation of $\{0, \dots, t\}$ and let F_σ be a Chow form of W in degree $(D_{\sigma(0)}, \dots, D_{\sigma(t)})$. Then,*

$$F_\sigma(P_{\sigma(0)}, \dots, P_{\sigma(t)}) = \pm F(P_0, \dots, P_t) \tag{2}$$

for any choice of polynomials $P_j \in \mathbb{C}[\mathbf{X}]_{D_j}$ for $j = 0, \dots, t$.

(ii) *Let I be a finite set of indices and let $\big(D^{(i)}\big)_{i \in I}$ be a sequence of positive integers indexed by I such that $D_t = \sum_{i \in I} D^{(i)}$. For each $i \in I$, let F_i be a Chow form of W in degree $(D_0, \dots, D_{t-1}, D^{(i)})$. Then,*

$$F\Big(P_0, \dots, P_{t-1}, \prod_{i \in I} Q_i\Big) = \pm \prod_{i \in I} F_i(P_0, \dots, P_{t-1}, Q_i) \tag{3}$$

for any choice of polynomials $Q_i \in \mathbb{C}[\mathbf{X}]_{D^{(i)}}$, $i \in I$, and $P_j \in \mathbb{C}[\mathbf{X}]_{D_j}$, $j = 0, \dots, t-1$.

Proof. To prove (i), view both sides of (2) as defining functions on $\prod_{j=0}^{t} \mathbb{C}[\mathbf{X}]_{D_j}$. Since these polynomial maps come from irreducible polynomials over $\mathbb{Z}$ and have the same zeros, they differ by a factor ± 1.

The proof of (ii) is more delicate. Here, we view both sides of (3) as defining functions on the product $\Big(\prod_{j=0}^{t-1} \mathbb{C}[\mathbf{X}]_{D_j}\Big) \times \Big(\prod_{i \in I} \mathbb{C}[\mathbf{X}]_{D^{(i)}}\Big)$. These polynomial maps come from polynomials over $\mathbb{Z}$ and have the same zeros because

$$\begin{aligned} F\Big(P_0, \dots, P_{t-1}, \prod_{i \in I} Q_i\Big) = 0 &\iff W \cap Z(P_0, \dots, P_{t-1}, \prod_{i \in I} Q_i) \neq \emptyset \\ &\iff W \cap Z(P_0, \dots, P_{t-1}, Q_i) \neq \emptyset \quad \text{for some } i \in I \\ &\iff F_i(P_0, \dots, P_{t-1}, Q_i) = 0 \quad \text{for some } i \in I \\ &\iff \prod_{i \in I} F_i(P_0, \dots, P_{t-1}, Q_i) = 0 \end{aligned}$$

So, they have the same irreducible factors over $\mathbb{Z}$. Moreover, all factors on the right hand side of (3) are distinct and irreducible over $\mathbb{Z}$. So the right hand side divides the left hand side. Finally, both sides are homogeneous of the same degree in the coefficients of P_j for

$j = 0, \dots, t-1$ as well as in the coefficients of Q_i for each $i \in I$, so we have

$$F\Big(P_0, \dots, P_{t-1}, \prod_{i \in I} Q_i\Big) = a \prod_{i \in I} F_i(P_0, \dots, P_{t-1}, Q_i)$$

for some integer a. Showing that $a = \pm 1$ requires local analysis at each prime number p. We will omit this. □

Lemma 2.3. *Let W be a $\mathbb{Q}$-subvariety of $\mathbb{P}_m(\mathbb{C})$ of dimension $t \geq 0$, let F be a Chow form of W in degree $(D_0, \dots, D_t)$ and let $Q \in \mathbb{Z}[\mathbf{X}]_{D_t}$. If $t = 0$, then $F(Q)$ is an integer, and this integer is non-zero if and only if $Q \notin I(W)$. Assume now that $t \geq 1$. Then the polynomial map F' from $\mathbb{C}[\mathbf{X}]_{D_0} \times \cdots \times \mathbb{C}[\mathbf{X}]_{D_{t-1}}$ to $\mathbb{C}$ given by*

$$F'(P_0, \dots, P_{t-1}) = F(P_0, \dots, P_{t-1}, Q)$$

is non-zero if and only if $Q \notin I(W)$. In that case, let $W_1', \dots, W_s'$ denote the $\mathbb{Q}$-irreducible components of $W \cap Z(Q)$ (all have dimension $t-1$) and, for each $j = 1, \dots, s$, let F_j' be a Chow form of W_j' in degree $(D_0, \dots, D_{t-1})$. Then there exist positive integers $e_1, \dots, e_s$ and a non-zero integer b such that

$$F'(P_0, \dots, P_{t-1}) = \pm\, b^{D_0 \cdots D_{t-1}}\, (F_1')^{e_1} \cdots (F_s')^{e_s} \tag{4}$$

for any choice of polynomials $P_j \in \mathbb{C}[\mathbf{X}]_{D_j}$ $(j = 0, \dots, t-1)$. Moreover, the integers b and $e_1, \dots, e_s$ are independent of $D_0, \dots, D_{t-1}$ and satisfy

$$\sum_{k=1}^{s} e_k \deg(W_k') = D_t \deg(W). \tag{5}$$

It can be shown that $e_1, \dots, e_s$ are the *intersection multiplicities* of W and of the divisor attached to Q in the sense of intersection theory (see the appendix of [8]).

Proof. In the case $t = 0$, we have

$$F(Q) \neq 0 \iff W \not\subseteq Z(Q) \iff Q \notin I(W).$$

Assume from now on that $t \geq 1$. If $Q \in I(W)$, then for any choice of polynomials $P_j \in \mathbb{C}[\mathbf{X}]_{D_j}$ $(j = 0, \dots, t-1)$ we find

$$W \cap Z(P_0, \dots, P_{t-1}, Q) = W \cap Z(P_0, \dots, P_{t-1}) \neq \emptyset,$$

and so $F'(P_0, \dots, P_{t-1}) = F(P_0, \dots, P_{t-1}, Q) = 0$, meaning that F' is the constant zero. Assume now that $Q \notin I(W)$. For polynomials $P_0, \dots, P_{t-1}$ as above, we have

$$\begin{aligned} F'(P_0, \dots, P_{t-1}) = 0 &\iff W \cap Z(P_0, \dots, P_{t-1}, Q) \neq \emptyset \\ &\iff W_j' \cap Z(P_0, \dots, P_{t-1}) \neq \emptyset \quad \text{for some } j \in \{1, \dots, s\} \\ &\iff F_j'(P_0, \dots, P_{t-1}) = 0 \quad \text{for some } j \in \{1, \dots, s\}. \end{aligned}$$

Since the polynomials $F'_1 \dots, F'_s$ are irreducible over $\mathbb{Z}$, while F' has integer coefficients, this implies that F' factors as a product

$$F' = a(F'_1)^{e_1} \cdots (F'_s)^{e_s} \tag{6}$$

for some non-zero integer a and some positive integers $e_1, \dots, e_s$. Comparing the degree of both sides in their first polynomial argument, we find the relation (5).

To complete the proof, choose linear forms $L_0, \dots, L_{t-1} \in \mathbb{C}[\mathbf{X}]_1$ and evaluate both sides of (6) at the points $L_0^{D_0}, \dots, L_{t-1}^{D_{t-1}}$. This gives

$$F\big(L_0^{D_0}, \dots, L_{t-1}^{D_{t-1}}, Q\big) = a \prod_{k=1}^{s} F'_k\big(L_0^{D_0}, \dots, L_{t-1}^{D_{t-1}}\big)^{e_k}$$

On the other hand, let E be a Chow form of W in degree $(1, \dots, 1, D_t)$ and for $j = 1, \dots, s$, let E'_j be a Chow form of W'_j in degree $(1, \dots, 1)$. Lemma 2.2 shows that, upon putting $N = D_0 \cdots D_{t-1}$, we have

$$\begin{aligned} F\big(L_0^{D_0}, \dots, L_{t-1}^{D_{t-1}}, Q\big) &= \pm\, E(L_0, \dots, L_{t-1}, Q)^N, \\ F'_j\big(L_0^{D_0}, \dots, L_{t-1}^{D_{t-1}}\big) &= \pm\, E'_j(L_0, \dots, L_{t-1})^N \quad (j = 1, \dots, s) \end{aligned}$$

with signs that are independent of $L_0, \dots, L_{t-1}$. Substituting these expressions into the previous equality, we deduce from the unique factorization property of polynomials that

$$E(L_0, \dots, L_{t-1}, Q) = b \prod_{k=1}^{s} E'_k(L_0, \dots, L_{t-1})^{e_k}$$

where b is an integer with $b^N = \pm a$. Thus, $e_1, \dots, e_s$ are independent of $D_0, \dots, D_{t-1}$ and a has the requested form for some non-zero integer b which is also independent of $D_0, \dots, D_{t-1}$. □

For any integer $D \geq 0$, and any point $\underline{\alpha} \in \mathbb{C}^{m+1}$, we denote by $\mathcal{L}_{\underline{\alpha}}$ the linear map from $\mathbb{C}[\mathbf{X}]_D$ to $\mathbb{C}$ which sends a polynomial $P \in \mathbb{C}[\mathbf{X}]_D$ to its value $P(\underline{\alpha})$ at $\underline{\alpha}$. With this notation, we are ready to prove:

Lemma 2.4. *Let W and F be as in the previous lemmas. Choose $Q_j \in \mathbb{Z}[\mathbf{X}]_{D_j}$ for $j = 1, \dots, t$, and put $N_0 = D_1 \cdots D_t \deg(W)$. Then, there exist non-zero points $\underline{\alpha}_1, \dots, \underline{\alpha}_{N_0}$ of $\mathbb{C}^{m+1}$ which are independent of D_0 and represent (not necessarily distinct) elements of W, and there exists $\xi \in \mathbb{C}$ such that*

$$F(P, Q_1, \dots, Q_t) = \xi \prod_{k=1}^{N_0} P(\underline{\alpha}_k),$$

for any $P \in \mathbb{C}[\mathbf{X}]_{D_0}$.

Proof. Denote by G the polynomial map from $\mathbb{C}[\mathbf{X}]_{D_0}$ to $\mathbb{C}$ given by $G(P) = F(P, Q_1, \dots, Q_t)$ for each $P \in \mathbb{C}[\mathbf{X}]_{D_0}$. Lemma 2.3 shows, by induction on t, that G is identically zero unless $W \cap Z(Q_1, \dots, Q_t)$ has dimension 0, a condition that is independent of D_0. Moreover, in the latter case, if $W'_1, \dots, W'_s$ denote the $\mathbb{Q}$-irreducible components of $W \cap Z(Q_1, \dots, Q_t)$, then G factors as a product

$$G = a \prod_{k=1}^{s} G_k^{e_k}$$

where a is a non-zero integer, where $e_1, \dots, e_s$ are positive integers that are independent of D_0, and where G_k is a Chow form of W'_k in degree D_0 for each $k = 1, \dots, s$. Furthermore, we have $\sum_{k=1}^{s} e_k \deg(W'_k) = N_0$. Since $\mathbb{Q}$ has characteristic zero, each set W'_k consists of $\deg(W'_k)$ distinct points which are conjugate over $\mathbb{Q}$. Put $n_k = \deg(W'_k)$ and choose representatives $\underline{\alpha}_{k,1}, \dots, \underline{\alpha}_{k,n_k}$ for these points in $\mathbb{C}^{m+1} \setminus \{0\}$. Since G_k is homogeneous of degree n_k and since its set of zeros in $\mathbb{C}[\mathbf{X}]_{D_0}$ is the union of the sets of zeros of the maps $\mathcal{L}_{\underline{\alpha}_{k,j}} : L[\mathbf{X}]_{D_0} \to L$ for $j = 1, \dots, n_k$, we get $G_k = \xi_k \prod_{j=1}^{n_k} \mathcal{L}_{\underline{\alpha}_{k,j}}$ for some $\xi_k \in \mathbb{C}^\times$. Then, we conclude that $G = \xi \prod_{k=1}^{s} \prod_{j=1}^{n_k} \mathcal{L}_{\underline{\alpha}_{k,j}}$ with $\xi = a \prod_{k=1}^{s} \xi_k$. If, on the contrary, G is zero, we write $G = \xi \prod_{k=1}^{N_0} \mathcal{L}_{\underline{\alpha}_k}$ with $\xi = 0$ and any choice of points $\underline{\alpha}_1, \dots, \underline{\alpha}_{N_0} \in \mathbb{C}^{m+1} \setminus \{0\}$ representing elements of W. □

3. Heights of algebraic sets

For each integer $D \geq 1$, we identify $\mathbb{C}[\mathbf{X}]_D$ with $\mathbb{C}^{\binom{D+m}{m}}$ by mapping a polynomial to the set of its coefficients in some order. So, we can talk about a convex body of $\mathbb{C}[\mathbf{X}]_D$. Given integers $D_0, \dots, D_t \geq 0$, we define a *convex body in degree* $(D_0, \dots, D_t)$ to be a convex body for $\mathbb{C}[\mathbf{X}]_{D_0} \times \cdots \times \mathbb{C}[\mathbf{X}]_{D_t}$ which has the form of a Cartesian product $\mathcal{C} = \mathcal{C}_0 \times \cdots \times \mathcal{C}_t$ where $\mathcal{C}_j$ is a convex body for $\mathbb{C}[\mathbf{X}]_{D_j}$, $j = 0, \dots, t$.

Fix such a convex body $\mathcal{C}$, and let W be a $\mathbb{Q}$-subvariety of $\mathbb{P}_m(\mathbb{C})$ of dimension t. We define the *height* of W relative to $\mathcal{C}$ by the formula

$$h_{\mathcal{C}}(W) = \log \|F\|_{\mathcal{C}},$$

where F denotes a Chow form of W in degree $(D_0, \dots, D_t)$. We also define the *normalized height* of W with respect to $\mathcal{C}$ by

$$\tilde{h}_{\mathcal{C}}(W) = (D_0 \cdots D_t \deg(W))^{-1} h_{\mathcal{C}}(W),$$

Applying Lemma 2.2 (i), we find:

Lemma 3.1. *Let $\mathcal{C}_0, \dots, \mathcal{C}_t$ be as above, and let W be a $\mathbb{Q}$-subvariety of $\mathbb{P}_m(\mathbb{C})$ of dimension $t \geq 0$. For any permutation σ of $\{0, \dots, t\}$, we have*

$$\tilde{h}_{\mathcal{C}_{\sigma(0)} \times \cdots \times \mathcal{C}_{\sigma(t)}}(W) = \tilde{h}_{\mathcal{C}_0 \times \cdots \times \mathcal{C}_t}(W).$$

We now define a notion of product of convex bodies and show that height $\tilde{h}_{\mathcal{C}_0\times\cdots\times\mathcal{C}_t}(W)$ is quasi-linear with respect to factorizations of each of the convex bodies $\mathcal{C}_0,\dots,\mathcal{C}_t$.

Definition 2. Let $(D^{(i)})_{i\in I}$ be non-negative integers indexed by a finite set I, and let D denote their sum. Moreover, let $\mathcal{C}^{(i)}$ be a convex body of $\mathbb{C}[\mathbf{X}]_{D^{(i)}}$ for each $i\in I$. We define the *product* $\prod_{i\in I}\mathcal{C}^{(i)}$ of these convex bodies as the smallest convex body of $\mathbb{C}[\mathbf{X}]_D$ containing the products $\prod_{i\in I}Q^{(i)}$ with $Q^{(i)}\in\mathcal{C}^{(i)}$ for each $i\in I$.

Recall that, for any point $\underline{\alpha}\in\mathbb{C}^{m+1}$, we denote by $\mathcal{L}_{\underline{\alpha}}\colon\mathbb{C}[\mathbf{X}]_D\to\mathbb{C}$ the linear map of evaluation at $\underline{\alpha}$. According to the definitions from §1.2, we have

$$\|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}}=\sup_{Q\in\mathcal{C}}|\mathcal{L}_{\underline{\alpha}}(Q)|=\sup_{Q\in\mathcal{C}}|Q(\underline{\alpha})|.$$

We can now state.

Lemma 3.2. *Let $\underline{\alpha}\in\mathbb{C}^{m+1}$ be any point. With the notations of the above definition, we have*

$$\|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}}=\prod_{i\in I}\|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}^{(i)}}.$$

Proof. Any element of $\mathcal{C}$ can be approximated arbitrarily well by a linear combination $Q=\sum_{j=1}^{s}\lambda_j\prod_{i\in I}Q_j^{(i)}$ where $Q_j^{(i)}\in\mathcal{C}^{(i)}$ for each $i\in I$ and $j=1,\dots,s$, and where the coefficients $\lambda_1,\dots,\lambda_s\in\mathbb{C}$ satisfy $\sum_i|\lambda_i|\le 1$. Since any such linear combination satisfies $|Q(\underline{\alpha})|\le\prod_{i\in I}\|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}^{(i)}}$, we deduce by continuity that $\|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}}\le\prod_{i\in I}\|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}^{(i)}}$. To establish the reverse inequality, we choose $Q=\prod_{i\in I}Q^{(i)}$ with $Q^{(i)}\in\mathcal{C}^{(i)}$, and take the supremum of both sides of the equality $|Q(\underline{\alpha})|=\prod_{i\in I}|Q^{(i)}(\underline{\alpha})|$ over the set of all such products Q. □

The main result of this section is the following.

Proposition 3.3. *Let $\mathcal{C}=\mathcal{C}_0\times\cdots\times\mathcal{C}_t$ be a convex body in degree $(D_0,\dots,D_t)$ and let s be an integer with $0\le s\le t$. Suppose that D_s is written as a finite sum of positive integers $D_s=\sum_{i\in I}D^{(i)}$ and that we have a corresponding decomposition of $\mathcal{C}_s$ into a product*

$$\mathcal{C}_s=\prod_{i\in I}\mathcal{C}^{(i)},$$

where, for each $i\in I$, $\mathcal{C}^{(i)}$ is a convex body of $\mathbb{C}[\mathbf{X}]_{D^{(i)}}$. For each $i\in I$, denote by $\mathcal{E}^{(i)}$ the convex body which, as a Cartesian product, has the same factors as $\mathcal{C}$ except that the s-th factor $\mathcal{C}_s$ is replaced by $\mathcal{C}^{(i)}$. Then, for any $\mathbb{Q}$-subvariety W of $\mathbb{P}_m(\mathbb{C})$ of dimension t, we have

$$-2t\eta N\ \le\ h_{\mathcal{C}}(W)-\sum_{i\in I}h_{\mathcal{E}^{(i)}}(W)\ \le\ 2\eta N,$$

where $\eta=\log(m+1)$ and $N=D_0\cdots D_t\deg(W)$.

Proof. Since, by Lemma 3.1 (i), the height of W relative to a Cartesian product of convex bodies does not change under a permutation of its factors, we may assume, without loss of generality, that $s = t$.

Let F be a Chow form of W in degree $(D_0, \dots, D_t)$ and, for $i \in I$, let F_i be a Chow form of W in degree $(D_0, \dots, D_{t-1}, D^{(i)})$. Lemma 2.2 (iii) shows that

$$F\Big(P_0, \dots, P_{t-1}, \prod_{i\in I} Q_i\Big) = \pm \prod_{i\in I} F_i(P_0, \dots, P_{t-1}, Q_i) \tag{7}$$

for any choice of polynomials $Q_i \in \mathbb{C}[\mathbf{X}]_{D^{(i)}}$ for $i \in I$ and $P_j \in \mathbb{C}[\mathbf{X}]_{D_j}$ for $j = 0, \dots, t-1$. Define $N_0 = N/D_0, \dots, N_t = N/D_t$.

For the upper bound, fix a choice of polynomials $P_j \in \mathcal{C}_j$ for $j = 0, \dots, t-1$. Define polynomial maps $G\colon \mathbb{C}[\mathbf{X}]_{D_t} \to \mathbb{C}$ and $G_i\colon \mathbb{C}[\mathbf{X}]_{D^{(i)}} \to \mathbb{C}$ for $i \in I$ by putting

$$G(Q) = F(P_0, \dots, P_{t-1}, Q) \quad \text{and} \quad G_i(Q_i) = F_i(P_0, \dots, P_{t-1}, Q_i)$$

for any $Q \in \mathbb{C}[\mathbf{X}]_{D_t}$ and any $Q_i \in \mathbb{C}[\mathbf{X}]_{D^{(i)}}$ with $i \in I$. Lemma 2.4 shows that there exist non-zero elements $\underline{\alpha}_1, \dots, \underline{\alpha}_{N_t}$ of $\mathbb{C}^{m+1}$ and constants $\xi \in \mathbb{C}$ and $\xi_i \in \mathbb{C}$ for $i \in I$ such that

$$G(Q) = \xi \prod_{k=1}^{N_t} Q(\underline{\alpha}_k) \quad \text{and} \quad G_i(Q_i) = \xi_i \prod_{k=1}^{N_t} Q_i(\underline{\alpha}_k)$$

for any choice of polynomials Q and Q_i with $i \in I$, as above. By virtue of (7), we have $G(Q) = \pm \prod_{i\in I} G_i(Q_i)$ whenever $Q = \prod_{i\in I} Q_i$ and therefore $\xi = \pm \prod_{i\in I} \xi_i$. Applying Proposition 1.5 to the above factorizations of the maps G and G_i into products of linear forms, we find

$$\|G\|_{\mathcal{C}_t} \le |\xi| \prod_{k=1}^{N_t} \|\mathcal{L}_{\underline{\alpha}_k}\|_{\mathcal{C}_t} \quad \text{and} \quad |\xi_i| \prod_{k=1}^{N_t} \|\mathcal{L}_{\underline{\alpha}_k}\|_{\mathcal{C}^{(i)}} \le \binom{D^{(i)}+m}{m}^{2N_t} \|G_i\|_{\mathcal{C}^{(i)}} \le (m+1)^{2N_t D^{(i)}} \|G_i\|_{\mathcal{C}^{(i)}}.$$

On the other hand, Lemma 3.2 shows that, for any $\underline{\alpha} \in \mathbb{C}^{m+1}$, we have $\|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}_t} = \prod_{i\in I} \|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}^{(i)}}$. Combining this with the previous inequalities and using the relation $\xi = \pm \prod_{i\in I} \xi_i$, we get

$$\|G\|_{\mathcal{C}_t} \le (m+1)^{2N_t D_t} \prod_{i\in I} \|G_i\|_{\mathcal{C}^{(i)}} = (m+1)^{2N} \prod_{i\in I} \|G_i\|_{\mathcal{C}^{(i)}}.$$

Taking the supremum of both sides of this inequality over all choices of $P_0, \dots, P_{t-1}$, we deduce that

$$\|F\|_{\mathcal{C}} \le (m+1)^{2N} \prod_{i\in I} \|F_i\|_{\mathcal{E}^{(i)}}.$$

Then, taking logarithms, we get

$$h_{\mathcal{C}}(W) \le \sum_{i\in I} h_{\mathcal{E}^{(i)}}(W) + 2N \log(m+1).$$

For the lower bound, fix a choice of polynomials $Q_i \in \mathcal{C}^{(i)}$ for $i \in I$. Put $Q = \prod_{i\in I} Q_i$ and define polynomial maps E and E_i for $i \in I$ on the product space $\prod_{j=0}^{t-1} \mathbb{C}[\mathbf{X}]_{D_j}$ by putting

$$E(P_0,\dots,P_{t-1}) = F(P_0,\dots,P_{t-1},Q) \quad \text{and} \quad E_i(P_0,\dots,P_{t-1}) = F_i(P_0,\dots,P_{t-1},Q_i)$$

for any choice of $P_j \in \mathbb{C}[\mathbf{X}]_{D_j}$ for $j = 0,\dots,t-1$. By virtue of (7), we have $E = \pm\prod_{i\in I} E_i$. Since E is multi-homogeneous of multi-degree $(N_0,\dots,N_{t-1})$, Proposition 1.5 gives

$$\begin{aligned}\prod_{i\in I} \|E_i\|_{\mathcal{C}_0\times\cdots\times\mathcal{C}_{t-1}} &\le \left(\prod_{j=0}^{t-1}\binom{D_j+m}{m}^{2N_j}\right)\|E\|_{\mathcal{C}_0\times\cdots\times\mathcal{C}_{t-1}}\\ &\le \left(\prod_{j=0}^{t-1}(m+1)^{2N_jD_j}\right)\|E\|_{\mathcal{C}_0\times\cdots\times\mathcal{C}_{t-1}}\\ &= (m+1)^{2tN}\|E\|_{\mathcal{C}_0\times\cdots\times\mathcal{C}_{t-1}}.\end{aligned}$$

Taking the supremum of both sides over all choices of polynomials $Q_i \in \mathcal{C}^{(i)}$ with $i \in I$ and using the fact that $Q = \prod_{i\in I} Q_i$ then belongs to $\mathcal{C}_t$, we get

$$\prod_{i\in I}\|F_i\|_{\mathcal{E}^{(i)}} \le (m+1)^{2tN}\|F\|_{\mathcal{C}}$$

and, by taking logarithms,

$$\sum_{i\in I} h_{\mathcal{E}^{(i)}}(W) \le h_{\mathcal{C}}(W) + 2tN\log(m+1).$$

□

Corollary 3.4. *Let $\mathcal{C}_0\times\cdots\times\mathcal{C}_t$ be a convex body in degree $(D_0,\dots,D_t)$. Suppose that, for each $s = 0,\dots,t$, the integer D_s is written as a finite sum of positive integers $D_s = \sum_{i\in I_s} D_s^{(i)}$ and that we have a corresponding decomposition of $\mathcal{C}_s$ into a product*

$$\mathcal{C}_s = \prod_{i\in I_s}\mathcal{C}_s^{(i)},$$

where, for each $i \in I_s$, $\mathcal{C}_s^{(i)}$ is a convex body of $\mathbb{C}[\mathbf{X}]_{D_s^{(i)}}$. Then, for any $\mathbb{Q}$-subvariety W of $\mathbb{P}_m(\mathbb{C})$ of dimension t, we have

$$-2t(t+1)\eta N \;\le\; h_{\mathcal{C}_0\times\cdots\times\mathcal{C}_t}(W) - \sum_{i_0\in I_0}\cdots\sum_{i_t\in I_t} h_{\mathcal{C}_0^{(i_0)}\times\cdots\times\mathcal{C}_t^{(i_t)}}(W) \;\le\; 2(t+1)\eta N,$$

where $\eta = \log(m+1)$ and $N = D_0\cdots D_t\deg(W)$.

4. An application

Throughout this section, we fix a point θ in $\mathbb{P}^m(\mathbb{C})$ and a set of projective coordinates $\underline{\theta} = (\theta_0,\dots,\theta_m)$ in $\mathbb{C}^{m+1}$ for that point. We also fix a positive integer D and a positive real number V. We attach to these parameters two convex bodies $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{C}[\mathbf{X}]_D$ and two convex bodies $\mathcal{D}$ and $\mathcal{E}$ of $\mathbb{C}[\mathbf{X}]_1$ by putting

$$\begin{aligned}
\mathcal{B} &= \big\{ P \in \mathbb{C}[\mathbf{X}]_D \ ; \ \|P\| \le 1 \big\},\\
\mathcal{C} &= \Big\{ P \in \mathcal{B} \ ; \ |P(\underline{\theta})| \le \exp(-V)\, \|\underline{\theta}\|^D \Big\},\\
\mathcal{D} &= \big\{ L \in \mathbb{C}[\mathbf{X}]_1 \ ; \ \|L\| \le 1 \big\},\\
\mathcal{E} &= \Big\{ L \in \mathcal{D} \ ; \ |L(\underline{\theta})| \le \exp(-V)\, \|\underline{\theta}\| \Big\}.
\end{aligned}$$

Note that $\mathcal{C}$ and $\mathcal{D}$ depend only on θ and not on the particular choice of projective coordinates $\underline{\theta}$, so that we may always assume that $\|\underline{\theta}\| = 1$. We first provide factorizations for $\mathcal{B}$ and $\mathcal{C}$ (compare with Lemma 5.1 of [8]).

Lemma 4.1. *We have the inclusions*

$$\begin{aligned}
(m+1)^{-D}\, \mathcal{D}^D \ &\subseteq \mathcal{B} \ \subseteq \ (m+1)^D \mathcal{D}^D\,,\\
(m+1)^{-D}\, \mathcal{D}^{D-1}\mathcal{E} \ &\subseteq \mathcal{C} \ \subseteq \ (2(m+1))^D\, \mathcal{D}^{D-1}\mathcal{E}\,.
\end{aligned}$$

Proof. The inclusions relative to $\mathcal{B}$ express standard properties of the maximum norm for polynomials. So, we concentrate on the inclusions relative to $\mathcal{C}$. Without loss of generality, we may assume that $|\theta_0| = \|\underline{\theta}\| = 1$.

To prove the left inclusion, we note that, for any choice of linear forms $M_1 \in \mathcal{E}$ and $M_2, \dots, M_D \in \mathcal{D}$, the product $P = M_1 \cdots M_D$ satisfies both $\|P\| \le (m+1)^D$ and $|P(\underline{\theta})| \le (m+1)^D e^{-V}$. Hence, P belongs to $(m+1)^D\mathcal{C}$. This shows that $\mathcal{D}^{D-1}\mathcal{E} \subseteq (m+1)^D\mathcal{C}$.

For the right inclusion, we first observe that, for each $\underline{\nu} = (\nu_0, \dots, \nu_m) \in \mathbb{N}^{m+1}$ with $|\underline{\nu}| = D$, we have

$$(8) \qquad \theta_0^D \mathbf{X}^{\underline{\nu}} - \underline{\theta}^{\underline{\nu}} X_0^D = \sum_{k=1}^{m} \left(\prod_{i=0}^{k-1} (\theta_0 X_i)^{\nu_i} \right) \left(\prod_{i=k+1}^{m} (\theta_i X_0)^{\nu_i} \right) \Big((\theta_0 X_k)^{\nu_k} - (\theta_k X_0)^{\nu_k} \Big)$$

and that, for each $k = 1, \dots, m$ such that $\nu_k \ge 1$, we also have

$$(9) \qquad (\theta_0 X_k)^{\nu_k} - (\theta_k X_0)^{\nu_k} = (\theta_0 X_k - \theta_k X_0) \sum_{\ell=0}^{\nu_k - 1} (\theta_0 X_k)^{\ell}\, (\theta_k X_0)^{\nu_k - \ell - 1} \in \nu_k \mathcal{D}^{D-1}\mathcal{E},$$

thus $\theta_0^D \mathbf{X}^{\underline{\nu}} - \underline{\theta}^{\underline{\nu}} X_0^D \in D\mathcal{D}^{D-1}\mathcal{E}$. So, if $P(\mathbf{X}) = \sum_{|\underline{\nu}|=D} p_{\underline{\nu}} \mathbf{X}^{\underline{\nu}}$ is any polynomial of $\mathcal{C}$, we find that

$$(10) \qquad P(\mathbf{X}) = \theta_0^{-D} P(\underline{\theta}) X_0^D + \theta_0^{-D} \sum_{|\underline{\nu}|=D} p_{\underline{\nu}} (\theta_0^D \mathbf{X}^{\underline{\nu}} - \underline{\theta}^{\underline{\nu}} X_0^D) \in \left(1 + D\binom{D+m}{m} \right) \mathcal{D}^{D-1}\mathcal{E},$$

upon noting furthermore that $\theta_0^{-D} P(\underline{\theta}) X_0 \in \mathcal{E}$ and that all coefficients $p_{\underline{\nu}}$ of P have absolute value at most 1. This gives proves the second inclusion for $\mathcal{C}$ since

$$1 + D\binom{D+m}{m} \le (D+1)\binom{D+m}{m} \le 2^D (m+1)^D.$$

□

By combining the above factorizations with Corollary 3.4, we obtain the following upper bounds for the heights of a $\mathbb{Q}$-subvariety W of $\mathbb{P}_m(\mathbb{C})$ relative to $\mathcal{B}$ and $\mathcal{C}$ in terms of its heights relative to $\mathcal{D}$ and $\mathcal{E}$, where, for simplicity, we simply write $h_{\mathcal{B}}(W)$ to mean $h_{\mathcal{B}\times\cdots\times\mathcal{B}}(W)$ and similarly for $\tilde{h}_{\mathcal{B}}(W)$, $h_{\mathcal{C}}(W)$, ... (compare with [11, Prop. 2.8] and [8, Prop. 5.3]).

Proposition 4.2. *Let W be a $\mathbb{Q}$-subvariety of $\mathbb{P}_m(\mathbb{C})$ of dimension $t \geq 0$. We have the upper bounds*

$$\tilde{h}_{\mathcal{B}}(W) \leq \tilde{h}_{\mathcal{D}}(W) + a \quad \text{and} \quad \tilde{h}_{\mathcal{C}}(W) \leq \tilde{h}_{\mathcal{D}}(W) + D^{-(t+1)}\big(\tilde{h}_{\mathcal{E}}(W) - \tilde{h}_{\mathcal{D}}(W)\big) + b$$

with $a = 3(t+1)\eta$ and $b = 4(t+1)\eta$, where $\eta = \log(m+1)$.

Proof. We simply establish the upper bound concerning $\tilde{h}_{\mathcal{C}}(W)$ leaving the one for $\tilde{h}_{\mathcal{B}}(W)$ as an exercise. By Lemma 4.1, we have $\mathcal{C} \subseteq (2(m+1))^D\mathcal{C}'$ where $\mathcal{C}' = \mathcal{D}^{D-1}\mathcal{E}$. Since a Chow form for W in degree $(D,\dots,D)$ is multi-homogeneous of degree $D^t\deg(W)$ in each of its $t+1$ polynomial arguments, it follows easily from the definition of the height that

$$h_{\mathcal{C}}(W) \leq h_{\mathcal{C}'}(W) + (t+1)D^t\deg(W)\log\big((2(m+1))^D\big),$$

and so

$$\tilde{h}_{\mathcal{C}}(W) \;\leq\; \tilde{h}_{\mathcal{C}'}(W) + (t+1)(\eta + \log 2) \;\leq\; \tilde{h}_{\mathcal{C}'}(W) + 2(t+1)\eta.$$

For each subset J of $\{0,\dots,t\}$, define a convex body $\mathcal{E}^{(J)} = \mathcal{E}_0^{(J)} \times \cdots \times \mathcal{E}_t^{(J)}$ in degree $(1,\dots,1)$ by putting $\mathcal{E}_j^{(J)} = \mathcal{D}$ if $j \in J$ and $\mathcal{E}_j^{(J)} = \mathcal{E}$ otherwise. By Corollary 3.4, we have

$$h_{\mathcal{C}'}(W) \leq \sum_{J\subseteq\{0,\dots,t\}} (D-1)^{\mathrm{Card}(J)} h_{\mathcal{E}^{(J)}}(W) + 2(t+1)\eta D^{t+1}\deg(W).$$

For a non-empty subset $J \subseteq \{0,\dots,t\}$, we use the upper bound $h_{\mathcal{E}^{(J)}}(Z) \leq h_{\mathcal{D}}(Z)$ which comes from the inclusion $\mathcal{E}^{(J)} \subseteq \mathcal{D}\times\cdots\times\mathcal{D}$. Since

$$\sum_{\emptyset\neq J\subseteq\{0,\dots,t\}} (D-1)^{\mathrm{Card}(J)} = D^{t+1} - 1,$$

this gives

$$h_{\mathcal{C}'}(W) \leq D^{t+1}h_{\mathcal{D}}(W) + \Big(h_{\mathcal{E}}(W) - h_{\mathcal{D}}(W)\Big) + 2(t+1)\eta D^{t+1}\deg(W)\,.$$

The upper bound for $\tilde{h}_{\mathcal{C}}(W)$ follows by dividing both sides of this inequality by $D^{t+1}\deg(W)$, and by combining the result with the upper bound for $\tilde{h}_{\mathcal{C}}(W)$ in terms of $\tilde{h}_{\mathcal{C}'}(W)$. □

The next proposition provides an estimate for $h_{\mathcal{E}}(W)$ when $\theta \in W$ (see also [8, Prop. 5.5]).

Proposition 4.3. *Let W be a $\mathbb{Q}$-subvariety of $\mathbb{P}_m(\mathbb{C})$ containing the point θ and let $t = \dim(W)$. Then we have:*

$$h_{\mathcal{E}}(W) - h_{\mathcal{D}}(W) \leq -V + \log\big(e(t+1)\deg(W)\big).$$

Proof. Put $N = (t+1)\deg(W)$, and let F denote the Chow form of W in degree $(1,\dots,1)$. Since

$$h_{\mathcal{E}}(W) - h_{\mathcal{D}}(W) = \log \frac{\|F\|_{\mathcal{E}}}{\|F\|_{\mathcal{D}}},$$

we simply need to show

$$|F(L_0,\dots,L_t)| \le \|F\|_{\mathcal{D}}\, e^{-V+1}N, \tag{11}$$

for any choice of linear forms $L_0,\dots,L_t \in \mathcal{E}$.

To this end, choose an index ℓ with $|\theta_\ell| = \|\underline{\theta}\|$ and define

$$M_j(\mathbf{X}) = \frac{L_j(\underline{\theta})}{\theta_\ell} X_\ell \qquad \text{for} \qquad 0 \le j \le t,$$

so that all linear forms $L_j - M_j$ vanish at the point $\underline{\theta}$. Since $\theta \in W$, we get

$$F(L_0 - M_0, \dots, L_t - M_t) = 0.$$

Let R be any real number > 1. We define rational functions φ and f on $\mathbb{C}$ by

$$\varphi(z) = \frac{z-1}{1-R^{-2}z} \quad \text{and} \quad f(z) = F\big(L_0 + \varphi(z)M_0, \dots, L_t + \varphi(z)M_t\big).$$

Since $f(0) = 0$, the Schwarz' lemma provides the upper bound

$$|F(L_0,\dots,L_t)| = |f(1)| \le \frac{|f|_R}{R},$$

where $|f|_R$ denotes the maximum of $|f(z)|$ on the disk $|z| \le R$. Note that, with the same notation, the Blaschke function $\varphi(z)$ satisfies $|\varphi|_R = R$. Moreover, by definition of $\mathcal{E}$, we have $\|M_j\| \le e^{-V}$ for each $j = 0,\dots,t$. Since the polynomial F is homogeneous of degree N in the whole set of coefficients of $L_0,\dots,L_t$, we obtain

$$|f|_R \le \big(1 + Re^{-V}\big)^N \|F\|_{\mathcal{D}}.$$

If $e^V \le N$, the inequality (11) is obvious since $\mathcal{E} \subseteq \mathcal{D}$. Otherwise, we may choose $R = e^V/N > 1$ and then (11) follows from the estimates

$$|f(1)| \le |f|_R \le Ne^{-V}(1+1/N)^N \|F\|_{\mathcal{D}} \le Ne^{-V+1}\|F\|_{\mathcal{D}}.$$

□

Combining the last two propositions, we finally obtain the following statement where the *height* $h(W)$ of a $\mathbb{Q}$-subvariety W of $\mathbb{P}_m(\mathbb{C})$ is defined as

$$h(W) = \log \|F\|$$

where F stands for a Chow form of W in degree $(1,\dots,1)$.

Corollary 4.4. *Let W and t be as in Proposition 4.3. Then, we have:*

$$h_{\mathcal{C}}(W) \le -V + D^{t+1}h(W) + 5(t+2)\eta D^{t+1}\deg(W).$$

Proof. By Proposition 4.2, we have

$$h_{\mathcal{C}}(W) \le D^{t+1}h_{\mathcal{D}}(W) + \big(h_{\mathcal{E}}(W) - h_{\mathcal{D}}(W)\big) + 4(t+1)\eta D^{t+1}\deg(W)$$

while Proposition 4.3 gives

$$h_{\mathcal{E}}(W) - h_{\mathcal{D}}(W) \le -V + \log\big(e(t+1)\deg(W)\big) \le -V + 3\eta\deg(W).$$

Finally, applying Lemma 1.2 to a Chow form F of W in degree $(1,\dots,1)$, we find

$$h_{\mathcal{D}}(W) \le h(W) + (t+1)\eta\deg(W).$$

The conclusion follows. □

Corollary 4.5. *Let W and t be as in Proposition 4.3 and let T be a positive real number. Suppose that*

$$V > D^{t+1}h(W) + (t+1)D^tT\deg(W) + 5(t+2)\eta D^{t+1}\deg(W).$$

Then any sequence of homogeneous polynomials $P_0,\dots,P_t \in \mathbb{Z}[\mathbf{X}]_D$ with $\|P_j\| \le e^T$ and $|P_j(\underline{\theta})| \le e^{-V}\|P_j\|\,\|\underline{\theta}\|^D$ for $j=0,\dots,t$ have a common zero on W.

Proof. The conditions on $P_0,\dots,P_t$ imply that $e^{-T}P_j \in \mathcal{C}$ for $j=0,\dots,t$. Let F be a Chow form of W in degree $(D,\dots,D)$, and write $N = D^t\deg(W)$. Since F is homogeneous of degree N in each of its $t+1$ polynomial arguments, we find

$$|F(P_0,\dots,P_t)| = \exp((t+1)NT)|F(e^{-T}P_0,\dots,e^{-T}P_t)| \le \exp((t+1)NT)\,\|F\|_{\mathcal{C}}.$$

On the other hand, Corollary 4.4 together with the hypothesis on V gives

$$\|F\|_{\mathcal{C}} = \exp(h_{\mathcal{C}}(W)) < \exp(-(t+1)NT),$$

and so the preceding inequality leads to $|F(P_0,\dots,P_t)| < 1$. Since $F(P_0,\dots,P_t)$ is an integer, we conclude that this integer is zero and so $P_0,\dots,P_t$ have a common zero on W. □

5. Height of a section by an hypersurface

As the proof of the above Corollary 4.5 shows, the notion of height with respect to convex bodies is particularly well suited for specialization of polynomial arguments. The next proposition formalizes this idea in a general setting (compare with [11, Lemma 2.2]).

Proposition 5.1. *Let W be a $\mathbb{Q}$-subvariety of $\mathbb{P}_m(\mathbb{C})$ of dimension $t \ge 0$, let $\mathcal{C}_0\times\cdots\times\mathcal{C}_t$ be a convex body in degree $(D_0,\dots,D_t) \in \mathbb{N}^{t+1}$, and let $P \in \mathbb{Z}[\mathbf{X}]_{D_t}$. Assume that P does not vanish identically on W, and choose $\lambda \in \mathbb{C}$ such that $P \in \lambda\,\mathcal{C}_t$. If $t \ge 1$, then there exists a $\mathbb{Q}$-irreducible component W' of $W \cap Z(P)$ of dimension $t-1$ with*

$$\tilde{h}_{\mathcal{C}_0\times\cdots\times\mathcal{C}_{t-1}}(W') \le \tilde{h}_{\mathcal{C}_0\times\cdots\times\mathcal{C}_t}(W) + \frac{\log|\lambda|}{D_t} + 2t\eta,$$

where $\eta = \log(m+1)$. If $t=0$, the same inequality holds provided that the left hand side is replaced by 0.

Proof. Let F be a Chow form of W in degree $(D_0,\dots,D_t)$. Write $N_t := D_0\cdots D_{t-1}\deg(W)$ and consider the polynomial map F' from $\mathbb{C}[\mathbf{X}]_{D_0}\times\cdots\times\mathbb{C}[\mathbf{X}]_{D_{t-1}}$ to $\mathbb{C}$ given by

$$F'(P_0,\dots,P_{t-1}) := F(P_0,\dots,P_{t-1},P)$$

for any choice of $P_j\in\mathbb{C}[\mathbf{X}]_{D_j}$ for $j=0,\dots,t-1$. Since F is homogeneous of degree N_t on the factor $\mathbb{C}[\mathbf{X}]_{D_t}$ and since $P\in\lambda\,\mathcal{C}_t$, we have the upper bound

$$\|F'\|_{\mathcal{C}_0\times\cdots\times\mathcal{C}_{t-1}}\le|\lambda|^{N_t}\|F\|_{\mathcal{C}},$$

and thus,

$$(D_0\cdots D_t\deg(W))^{-1}\log\|F'\|_{\mathcal{C}_0\times\cdots\times\mathcal{C}_{t-1}}\le\tilde{h}_{\mathcal{C}_0\times\cdots\times\mathcal{C}_t}(W)+D_t^{-1}\log|\lambda|. \tag{12}$$

If $t=0$, the map F' is a constant non-zero integer and so the left hand side of this inequality is ≥ 0. Otherwise, Lemma 2.3 shows that F' factors as a product $a(F_1')^{e_1}\cdots(F_s')^{e_s}$ where $F_1',\dots,F_s'$ are the Chow forms in degree $(D_0,\dots,D_{t-1})$ of the $\mathbb{Q}$-irreducible components $W_1',\dots,W_s'$ of $W\cap Z(P)$, where a is a non-zero integer, and where $e_1,\dots,e_s$ are positive integers with $\sum_{k=1}^{s}e_s\deg(W_k')=D_t\deg(W)$. Applying Proposition 1.5, we then find

$$\begin{aligned}\sum_{k=1}^{s}e_k\log\|F_k'\|_{\mathcal{C}_0\times\cdots\times\mathcal{C}_{t-1}}&\le\log\|F'\|_{\mathcal{C}_0\times\cdots\times\mathcal{C}_{t-1}}+2D_0\cdots D_t\deg(W)\sum_{j=1}^{t-1}\frac{1}{D_j}\log\binom{D_j+m}{m}\\&\le\log\|F'\|_{\mathcal{C}_0\times\cdots\times\mathcal{C}_{t-1}}+2t\eta D_0\cdots D_t\deg(W).\end{aligned}$$

and so

$$\sum_{k=1}^{s}\lambda_k\tilde{h}_{\mathcal{C}_0\times\cdots\times\mathcal{C}_{t-1}}(W_k')\le(D_0\cdots D_t\deg(W))^{-1}\log\|F'\|_{\mathcal{C}_0\times\cdots\times\mathcal{C}_{t-1}}+2t\eta,$$

where $\lambda_k=e_k\deg(W_k')/(D_t\deg(W))$ for $k=1,\dots,s$. Since $\lambda_1,\dots,\lambda_s$ are positive real numbers with sum 1, it follows that there exists at least one $\mathbb{Q}$-irreducible component W' of $W\cap Z(P)$ such that

$$\tilde{h}_{\mathcal{C}_0\times\cdots\times\mathcal{C}_{t-1}}(W')\le(D_0\cdots D_t\deg(W))^{-1}\log\|F'\|_{\mathcal{C}_0\times\cdots\times\mathcal{C}_{t-1}}+2t\eta.$$

The conclusion follows by combining this inequality with (12). □

6. Philippon's metric Bézout's theorem

We define the *distance* between two points θ and α of $\mathbb{P}^m(\mathbb{C})$ by

$$\operatorname{dist}(\theta,\,\alpha)=\max_{0\le i,j\le m}\frac{|\theta_i\alpha_j-\theta_j\alpha_i|}{\|\underline{\theta}\|\,\|\underline{\alpha}\|}$$

where $\underline{\theta}=(\theta_0,\dots,\theta_m)$ and $\underline{\alpha}=(\alpha_0,\dots,\alpha_m)$ denote respectively sets of projective coordinates for θ and α in $\mathbb{C}^{m+1}$ (the result is independent of such choices). We also define the *distance* between a point θ and a subset E of $\mathbb{P}^m(\mathbb{C})$ as the infimum of the distance between θ and a point of E.

Throughout this section, we fix a point $\theta \in \mathbb{P}_m(\mathbb{C})$ and a set of projective coordinates $\underline{\theta} \in \mathbb{C}^{m+1}$ for that point. We also fix a positive integer D and a positive real number V. As in Section 4, we define a convex body $\mathcal{C}$ of $\mathbb{C}[\mathbf{X}]_D$ by

$$\mathcal{C} \;=\; \left\{ Q \in \mathbb{C}[\mathbf{X}]_D \,;\, \|Q\| \le 1,\; |Q(\underline{\theta})| \le e^{-V}\, \|\underline{\theta}\|^D \right\}$$

and, for any $\mathbb{Q}$-subvariety W of $\mathbb{P}_m(\mathbb{C})$ of dimension $t \ge 0$, we define $h_{\mathcal{C}}(W)$ to be $h_{\mathcal{C}\times\cdots\times\mathcal{C}}(W)$ where $\mathcal{C}\times\cdots\times\mathcal{C}$ stands for the product of $t+1$ copies of $\mathcal{C}$. We first establish the following special case of [8, Lemma 5.2].

Lemma 6.1. *Let $\alpha \in \mathbb{P}^m(\mathbb{C})$ and $\rho_\alpha = \mathrm{dist}(\theta, \alpha)$. Then, for any set of projective coordinates $\underline{\alpha} \in \mathbb{C}^{m+1}$ of α with $\|\underline{\alpha}\| = 1$, we have*

$$\max\left\{e^{-V},\, \rho_\alpha\right\} \le \|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}} \le (2(m+1))^D \max\left\{e^{-V},\, \rho_\alpha\right\}$$

where $\mathcal{L}_{\underline{\alpha}} \colon \mathbb{C}[\mathbf{X}]_D \to \mathbb{C}$ stands for the evaluation at the point $\underline{\alpha}$.

Proof. We may assume without loss of generality that $|\theta_0| = \|\underline{\theta}\| = 1$. Select projective coordinates $\underline{\alpha} = (\alpha_0, \dots, \alpha_m)$ of the point α with norm $\|\underline{\alpha}\| = 1$. Let i and j be indices for which $\rho_\alpha = |\theta_j\alpha_i - \theta_i\alpha_j|$ and let k be an index with $|\alpha_k| = 1$. Since the convex body $\mathcal{C}$ contains both $e^{-V}X_k^D$ and $(\theta_j X_i - \theta_i X_j)X_k^{D-1}$, we find the lower bound

$$\|\mathcal{L}_\alpha\|_{\mathcal{C}} = \sup_{P\in\mathcal{C}} |P(\underline{\alpha})| \ge \max\left\{e^{-V},\, \rho_\alpha\right\}.$$

For the other inequality, we go back to the proof of Lemma 4.1. For any $\underline{\nu} \in \mathbb{N}^{m+1}$ with $|\underline{\nu}| = D$, the formulas (8) and (9) imply that

$$\left|\theta_0^D \underline{\alpha}^{\underline{\nu}} - \underline{\theta}^{\underline{\nu}} \alpha_0^D\right| \le D\rho_\alpha.$$

So, for any polynomial $P(\mathbf{X}) = \sum_{|\underline{\nu}|=D} p_{\underline{\nu}} \mathbf{X}^{\underline{\nu}}$ in $\mathcal{C}$, the formula (10) leads to

$$\begin{aligned} |P(\underline{\alpha})| &\le |P(\underline{\theta})| + \sum_{|\underline{\nu}|=D} |p_{\underline{\nu}}| D\rho_\alpha \\ &\le \left(1 + D\binom{D+m}{m}\right) \max\{e^{-V},\, \rho_\alpha\} \le (2(D+1))^D \max\{e^{-V},\, \rho_\alpha\}, \end{aligned}$$

and so the last estimate on the right is an upper bound for $\|\mathcal{L}_\alpha\|_{\mathcal{C}}$. □

The next proposition is the key to the proof of Philippon's criterion (see the next section). It is essentially Proposition 2.5 of [11] (see also Lemma 4 of [?]). Its proof exploits the factorization property of the Chow forms stated in Lemma 2.4.

Proposition 6.2. *Let W be a $\mathbb{Q}$-subvariety of $\mathbb{P}_m(\mathbb{C})$ of dimension t, and let ρ denote the distance between the point θ and W. Suppose that there exists a homogeneous polynomial*

$P' \in \mathbb{Z}[\mathbf{X}]$ of degree $D' \le D$ which does not vanish identically on W and choose $\lambda \ge 1$ such that

$$\frac{|P'(\underline{\theta})|}{\|P'\|\,\|\underline{\theta}\|^{D'}} \le \lambda \max\{\rho,\, e^{-V}\}.$$

If $t \ge 1$, then there exists a $\mathbb{Q}$-subvariety W' of $\mathbb{P}_m(\mathbb{C})$ contained in W of dimension $t-1$ such that

$$\tilde{h}_{\mathcal{C}}(W') \le \tilde{h}_{\mathcal{C}}(W) + \frac{\log(\lambda\,\|P'\|)}{D} + 2(t+2)\eta,$$

where $\eta = \log(m+1)$. If $t = 0$, the same inequality holds provided that the left hand side is replaced by 0.

Proof. Without loss of generality, we may assume that $\|\underline{\theta}\| = 1$. Since W is non-empty, there exists an integer ℓ such that X_ℓ does not vanish identically on W. Then, the product $P = X_\ell^{D-D'} P'$ is a homogeneous polynomial of degree D which does not not vanish identically on W (the ideal of W in $\mathbb{Q}[\mathbf{X}]$ is prime).

Let F be a Chow form of W in degree $(D, \dots, D)$. Write $N = D^t \deg(W)$ and consider the polynomial map F' from $\big(\mathbb{C}[\mathbf{X}]_D\big)^t$ to $\mathbb{C}$ given by

$$F'(P_0, \dots, P_{t-1}) := F(P_0, \dots, P_{t-1}, P)$$

for any choice of $P_0, \dots, P_{t-1} \in \mathbb{C}[\mathbf{X}]_D$. If $t \ge 1$, then arguing as in the end of the proof of Proposition 5.1 shows the existence of a $\mathbb{Q}$-irreducible component W' of $W \cap Z(P)$ such that

$$\tilde{h}_{\mathcal{C}}(W') \le (D^{t+1} \deg(W))^{-1} \log \|F'\|_{\mathcal{C}} + 2t\eta.$$

If $t = 0$, the map F' is a constant non-zero integer and so the same inequality holds when the left hand side is replaced by ≥ 0. So, it remains to show that

$$(D^{t+1} \deg(W))^{-1} \log \|F'\|_{\mathcal{C}} \le \tilde{h}_{\mathcal{C}}(W) + \frac{\log(\lambda\,\|P'\|)}{D} + 4\eta. \tag{13}$$

To this end, choose polynomials $Q_0, \dots, Q_{t-1} \in \mathcal{C}$ such that

$$\|F'\|_{\mathcal{C}} \;=\; |F'(Q_0, \dots, Q_{t-1})|,$$

and consider the polynomial map $E \colon \mathbb{C}[\mathbf{X}]_D \to \mathbb{C}$ given by

$$E(Q) = F(Q_0, \dots, Q_{t-1}, Q)$$

for any $Q \in \mathbb{C}[\mathbf{X}]_D$. By construction, we have

$$\|F'\|_{\mathcal{C}} = |E(P)| \quad \text{and} \quad \|E\|_{\mathcal{C}} \le \|F\|_{\mathcal{C}} = \exp(h_{\mathcal{C}}(W)). \tag{14}$$

Our goal is therefore to compare $|E(P)|$ and $\|E\|_{\mathcal{C}}$. For this we apply Lemma 2.4. It shows the existence of $\xi \in \mathbb{C}$ and elements $\underline{\alpha}_1, \dots, \underline{\alpha}_N$ of $\mathbb{C}^{m+1}$ of norm 1 representing points $\alpha_1, \dots, \alpha_N$ of W such that

$$E(Q) = \xi \prod_{k=1}^{N} Q(\underline{\alpha}_k)$$

for any $Q \in \mathbb{C}[\mathbf{X}]_D$. Applying Proposition 3.7 to this factorization of E gives

$$|\xi| \prod_{k=1}^{N} \|\mathcal{L}_{\underline{\alpha}_k}\|_{\mathcal{C}} \leq \binom{D+m}{m}^{2N} \|E\|_{\mathcal{C}} \leq (m+1)^{2DN} \|E\|_{\mathcal{C}}, \tag{15}$$

where $\mathcal{L}_{\underline{\alpha}_k} : \mathbb{C}[\mathbf{X}]_D \to \mathbb{C}$ denotes as usual the evaluation at $\underline{\alpha}_k$. Moreover, for each k, Lemma 6.1 gives

$$\|\mathcal{L}_{\underline{\alpha}_k}\|_{\mathcal{C}} \geq \max\{\mathrm{dist}(\theta, \alpha_k),\, e^{-V}\} \geq \max\{\rho,\, e^{-V}\} \tag{16}$$

where the last estimate uses the fact that $\alpha_k \in W$. Now, choose an index j such that $|\theta_j| = \|\underline{\theta}\| = 1$. The polynomial

$$\frac{1}{3\,\lambda\,\|P\|}\left(P(\mathbf{X}) - \frac{P(\underline{\theta})}{\theta_j^D} X_j^D\right)$$

belongs to $\mathcal{C}$ because it vanishes at the point $\underline{\theta}$ and its norm is at most

$$\frac{\|P\| + |P(\underline{\theta})|}{3\lambda\,\|P\|} \leq \frac{\|P'\| + |P'(\underline{\theta})|}{3\lambda\,\|P'\|} \leq \frac{1 + \lambda\,\max\{\rho,\, e^{-V}\}}{3\lambda} \leq 1,$$

using $\|P\| = \|P'\|$ and $|P(\underline{\theta})| = |\theta_\ell|^{D-D'}|P'(\underline{\theta})| \leq |P'(\underline{\theta})|$. Therefore, for each k, we have

$$\begin{aligned}
|P(\underline{\alpha}_k)| &\leq |P(\underline{\theta})| + 3\lambda\,\|P\|\,\|\mathcal{L}_{\underline{\alpha}_k}\|_{\mathcal{C}} \\
&\leq |P'(\underline{\theta})| + 3\lambda\,\|P'\|\,\|\mathcal{L}_{\underline{\alpha}_k}\|_{\mathcal{C}} \\
&\leq \lambda\,\|P'\|\max\{\rho,\, e^{-V}\} + 3\lambda\,\|P'\|\,\|\mathcal{L}_{\underline{\alpha}_k}\|_{\mathcal{C}} \\
&\leq 4\lambda\,\|P'\|\,\|\mathcal{L}_{\underline{\alpha}_k}\|_{\mathcal{C}}
\end{aligned}$$

where the last inequality uses (16). Combining this with (15), we conclude that

$$|E(P)| = |\xi| \prod_{k=1}^{N} |P(\underline{\alpha}_k)| \leq (4\lambda\,\|P'\|)^N |\xi| \prod_{k=1}^{N} \|\mathcal{L}_{\underline{\alpha}_k}\|_{\mathcal{C}} \leq (4\lambda\,\|P'\|)^N (m+1)^{2DN} \|E\|_{\mathcal{C}}.$$

By (14) this means that

$$\log \|F'\|_{\mathcal{C}} \leq h_{\mathcal{C}}(W) + N \log(4\lambda\,\|P'\|) + 2DN\eta,$$

and (13) follows upon dividing both sides of this inequality by $D^{t+1}\deg(W) = DN$ and noting that $\log(4) \leq 2\eta$. □

Corollary 6.3. *Let W and t be as in the statement of Proposition 6.2. Denote by ρ the distance between θ and W and assume that $\rho > 0$. Then, we have*

$$\tilde{h}_{\mathcal{C}}(W) \geq \frac{t+1}{D} \log(\rho) - (t+1)(t+2)\eta.$$

Proof. Suppose that W_s is a $\mathbb{Q}$-subvariety of $\mathbb{P}_m(\mathbb{C})$ of dimension s, with $0 \le s \le t$, contained in W. Select a coordinate index k such that W_s is not contained in the hyperplane $X_k = 0$. Then the polynomial $P' = X_k$ satisfies the hypotheses of Proposition 6.2 with $D' = 1$ and $\lambda = 1/\rho$. If $s \ge 1$, this proposition ensures the existence of a $\mathbb{Q}$-subvariety W_{s-1} of $\mathbb{P}_m(\mathbb{C})$ of dimension $s-1$, contained in W_s with

$$\tilde{h}_{\mathcal{C}}(W_{s-1}) \le \tilde{h}_{\mathcal{C}}(W_s) - \frac{\log \rho}{D} + 2(s+2)\eta\,.$$

If $s = 0$, the same inequality holds with the left hand side replaced by 0. Starting with $W_t = W$, this process generates recursively a sequence a sequence of $\mathbb{Q}$-subvarieties $W_t \supset \cdots \supset W_1 \supset W_0$. The conclusion follows by combining the inequalities that we get at each step. □

7. Philippon's criterion for algebraic independence

We now state and prove Philippon's criterion (Theorem 2.11 of [11]). For the notion of distance in $\mathbb{P}_m(\mathbb{C})$, the reader is referred to the preceding section §6.

Theorem 7.1. *Let $\underline{\theta} = (1, \theta_1, \dots, \theta_m) \in \mathbb{C}^{m+1}$, let θ denote the corresponding point of $\mathbb{P}_m(\mathbb{C})$, and let k be an integer with $0 \le k \le m$. Moreover, let $(D_n)_{n\ge1}$ be a non-decreasing sequence of positive integers, and let $(T_n)_{n\ge1}$ and $(V_n)_{n\ge1}$ be non-decreasing sequences of positive real numbers such that*

$$\limsup_{n\to\infty} \frac{V_n}{(D_n + T_n)D_n^k} = \infty.$$

Suppose also that for each $n \ge 2$ there exists a non-empty family $\mathcal{F}_n$ consisting of homogeneous polynomials in $\mathbb{Z}[\mathbf{X}]$ which satisfy the following two properties.

(i) *For every $P \in \mathcal{F}_n$, we have*

$$\deg(P) = D_n, \quad h(P) \le T_n \quad \textit{and} \quad |P(\underline{\theta})| \le e^{-V_n}\, \|P\|\, \|\underline{\theta}\|^{D_n}.$$

(ii) *The polynomials of $\mathcal{F}_n$ have no common zero α in $\mathbb{P}^m(\mathbb{C})$ with*

$$\mathrm{dist}(\theta, \alpha) \le \exp\left(-V_{n-1}\right).$$

Then, we have $k < m$ and the transcendence degree over $\mathbb{Q}$ of the field $\mathbb{Q}(\theta_1, \dots, \theta_m)$ is $\ge k+1$.

Notes. For $k = 1$, the condition (ii) can be strengthened by asking simply that θ is not a zero of $\mathcal{F}_n$. This type of result goes back to Gel'fond [3] with more recent improvements by W. D. Brownawell [1] and M. Waldschmidt [16]. It was conjectured for a long time that the same would be true for $k \ge 2$ but this is false as P. Philippon showed by expanding a construction of Khintchine in the Appendix to [11]. To reconcile both results, Philippon's original criterion asks that the polynomials of $\mathcal{F}_n$ have at most finitely many common zeros at distance at most $\exp(-V_{n-1})$ from θ. For lack of time, we do not consider this situation here. Another feature of the original Philippon's criterion is that it allows one to deal with smaller free regions than the one imposed in (ii). However, the exercise stated after the proof provides a more efficient strategy to deal with such constraints, using the above theorem as it stands.

Proof. Let W denote the smallest algebraic subset of $\mathbb{P}_m(\mathbb{C})$ defined over $\mathbb{Q}$ and containing the point θ. Then, W is irreducible over $\mathbb{Q}$ of dimension t equal to the transcendence degree over $\mathbb{Q}$ of the field $\mathbb{Q}(\theta_1, \dots, \theta_m)$. We proceed by contradiction assuming, contrary to the conclusion of the Theorem, that $k \geq t$. This means that we have

$$\lim_{n\to\infty} \frac{V_n}{(D_n + T_n)D_n^t} = \infty. \tag{17}$$

Fix a large positive integer n and define

$$\mathcal{C} = \left\{Q \in \mathbb{C}[\mathbf{X}]_{D_n}\,;\; \|Q\| \leq 1,\ |Q(\underline{\theta})| \leq e^{-V_n}\,\|\underline{\theta}\|^{D_n}\right\}.$$

By Corollary 4.4, we have

$$h_{\mathcal{C}}(W) \leq -V_n + D_n^{t+1} h(W) + 5(t+2)\eta D_n^{t+1} \deg(W),$$

and so, in view of (17), we obtain

$$\tilde{h}_{\mathcal{C}}(W) \leq -\frac{V_n}{2D_n^{t+1}\deg(W)}$$

if n is sufficiently large.

We claim that for each $s = 0, \dots, t$, there exists a $\mathbb{Q}$-subvariety W_s of $\mathbb{P}_m(\mathbb{C})$ contained in W of dimension s such that

$$\tilde{h}_{\mathcal{C}}(W_s) \leq -\frac{V_n}{2^{t-s+1}D_n^{t+1}\deg(W)} \tag{18}$$

provided that n is large enough.

For $s = t$, this condition is fulfilled with $W_t = W$. Now, assume that W_s has been constructed for some s with $0 \leq s \leq t$, and denote by ρ the distance between θ and W_s. We consider two cases.

a) Suppose first that $\rho < \exp(-V_n)$. Then, since the family $\mathcal{F}_n$ has no common zero α with $\mathrm{dist}(\theta, \alpha) \leq \exp(-V_{n-1})$ and since $V_n \geq V_{n-1}$, the algebraic set W_s is not contained in $Z(\mathcal{F}_n)$ and so it is not contained in $Z(P)$ for some $P \in \mathcal{F}_n$. Then, Proposition 6.2 applies (with W, t, P', D', D and λ replaced respectively by W_s, s, P, D_n, D_n and 1). If $s \geq 1$, it shows the existence of a $\mathbb{Q}$-subvariety W_{s-1} of $\mathbb{P}_m(\mathbb{C})$ contained in W_s of dimension $s-1$ such that

$$\tilde{h}_{\mathcal{C}}(W_{s-1}) \leq \tilde{h}_{\mathcal{C}}(W_s) + \frac{T_n}{D_n} + 2(t+2)\eta,$$

since $\log\|P\| \le T_n$. Then, the induction hypothesis (18) together with (17) shows that W_{s-1} satisfies the required estimate if n is sufficiently large. If $s = 0$, the above inequality holds with the left hand side replaced by 0, but this is incompatible with (18) if n is sufficiently large. So, if $s = 0$ and n is sufficiently large, we must have $\rho \ge \exp(-V_n)$.

b) Suppose now that $\rho \ge \exp(-V_n)$. Corollary 6.3 together with the hypothesis (18) shows that ρ can be made arbitrarily small by choosing n sufficiently large. Thus, if n is large enough, there exists an integer ℓ with $2 \le \ell \le n$ such that

$$\exp(-V_\ell) \le \rho < \exp(-V_{\ell-1}).$$

Arguing as in the previous case, this means that W_s is not contained in $Z(P)$ for some $P \in \mathcal{F}_\ell$. Such a polynomial P is homogeneous of degree $D_\ell \le D_n$. It also satisfies $\log\|P\| \le T_\ell \le T_n$ and

$$\frac{|P(\underline{\theta})|}{\|P\|\,\|\underline{\theta}\|^{D_\ell}} \le e^{-V_\ell} \le \rho.$$

So, Proposition 6.2 applies (with W, t, P', D', D and λ replaced respectively by W_s, s, P, D_ℓ, D_n and 1). If $s \ge 1$, it shows the existence of a $\mathbb{Q}$-subvariety W_{s-1} of $\mathbb{P}_m(\mathbb{C})$ contained in W_s of dimension $s-1$ such that

$$\tilde{h}_{\mathcal{C}}(W_{s-1}) \le \tilde{h}_{\mathcal{C}}(W_s) + \frac{T_n}{D_n} + 2(t+2)\eta,$$

and so W_{s-1} satisfies the required estimate if n is sufficiently large. If $s = 0$, the above inequality holds with the left hand side replaced by 0, but this is impossible if n is sufficiently large.

This proves our claim by constructing recursively $W_t, \dots, W_0$ whenever the initial choice of n is made sufficiently large. However, it leads to a contradiction when we reach W_0. So, we must indeed have $t < k$. □

Exercise. Let $(E_n)_{n\ge1}$ be a non-decreasing sequence of positive integers. Suppose that all the hypotheses of Theorem 7.1 are satisfied except that, instead of condition (ii), we make the weaker hypothesis that, for each sufficiently large n, the polynomials of $\mathcal{F}_n$ have no common zero α in $\mathbb{P}^m(\mathbb{C})$ with

$$\mathrm{dist}(\theta, \alpha) \le \exp(-E_{n-1}V_{n-1}).$$

Show that the conclusion of Theorem 7.1 still holds provided that

$$\lim_{n\to\infty} \frac{V_n}{(D_n + T_n)D_n^k E_n^k} = \infty.$$

Hint. Apply Theorem 7.1 to the sequences $(D'_n)_{n\ge1}$, $(T'_n)_{n\ge1}$ and $(V'_n)_{n\ge1}$ defined, for each $n \ge 1$, by

$$D'_n = 2E_nD_n, \quad T'_n = 2E_n(T_n + D_n\log(m+1)) \quad \text{and} \quad V'_n = E_nV_n,$$

and to the sets of polynomials given by $\mathcal{F}'_n = \left\{P^{2E_n}\,;\, P \in \mathcal{F}_n\right\}$ for each $n \ge 2$.

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