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Summer School in Analytic Number Theory and Diophantine Approximation.

Small points on varieties of algebraic tori

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1 Introduction : from torsion to small points

The former Manin-Mumford conjecture predicts that the set of torsion points of a curve of genus ≥ 2 embedded in its jacobian is finite. More generally, let $\mathbb{G}$ be a semi-abelian variety and V an irreducible[1] algebraic subvariety of $\mathbb{G}$, defined over some algebraically closed field K. We say that V is a *torsion variety* if V is a translate of a proper subtorus by a torsion point of $\mathbb{G}$. We also denote by V_{tors} the set of torsion points of $\mathbb{G}$ lying on V. Then we have the following generalization of the Manin-Mumford conjecture.

Theorem 1.1

i) If V is not a torsion variety, then the set V_{tors} of torsion points of $\mathbb{G}$ lying on V is not Zariski dense.

ii) The Zariski closure of V_{tors} is a finite union of torsion varieties.

The two assertions are clearly equivalent. Theorem 1.1 was proved by Raynaud ([Ray 1983]) when $\mathbb{G}$ is an abelian variety, by Laurent ([Lau 1984]) if $\mathbb{G} = \mathbb{G}_m^n$, and finally by Hindry ([Hin 1988]) in the general situation.

We assume from now on that all varieties are algebraic and defined over $\overline{\mathbb{Q}}$. Bogomolov ([Bog 1981]) gave the following generalization of the former Manin-Mumford conjecture. Let $\mathcal{C}$ be a curve of genus ≥ 2 embedded in its jacobian. Then $\mathcal{C}(\overline{\mathbb{Q}})$ is discrete for the metric induced by the Néron-Tate height. In other words, Bogomolov conjectures that the set of points of "sufficiently small" height on $\mathcal{C}$ is finite, while the former Manin-Mumford conjecture makes a similar assertion on the set of torsion points (which are precisely the points of zero height).

[1]By irreduciblle we mean geometrically irreducible

More generally, let $\mathbb{G}$ be a semi-abelian variety and let $\hat{h}$ be a normalized height on $\mathbb{G}(\overline{\mathbb{Q}})$. Hence, $\hat{h}$ is the Neron-Tate height if $\mathbb{G}$ is abelian, and it is the Weil height if $\mathbb{G} = \mathbb{G}_m^n \hookrightarrow \mathbb{P}_n$. In particular $\hat{h}$ is a non-negative function on $\mathbb{G}$ and $\hat{h}(P) = 0$ if and only if P is a torsion point. Given an algebraic subvariety of $\mathbb{G}$, we denote by V^* the complement in V of the Zariski closure of the set of torsion points of V. Therefore, by theorem 1.1, $V \backslash V^* = \overline{V_{\text{tors}}}$ is a finite union of torsion varieties.

Theorem 1.2 *Let V be an irreducible subvariety of a semi-abelian variety $\mathbb{G}$. Then:*

i) If V is not a torsion variety, then there exists $\theta > 0$ such that the set $V(\theta) = \{P \in V \text{ s.t. } \hat{h}(P) \leq \theta\}$ is not Zariski dense in V.

ii) V^* is discrete for the metric induced by $\hat{h}$, i.e.

$$\inf\{\hat{h}(P) \text{ s.t. } P \in V^*\} > 0.$$

It is easy to see that the two assertions are equivalent. In this formulation, theorem 1.2 was proved for $\mathbb{G} = \mathbb{G}_m^n$ by Zhang (see [Zha 1995]). In the abelian case, Ullmo (see [Ull 1998]) proved Bogomolov's original formulation for curves ($\dim(V) = 1$); immediately after Zhang (see [Zha 1998]) proved theorem 1.2. The semi-abelian case was solved by David and Philippon (see [Dav-Phi 2000]).

In these lessons we describe some quantitative versions of theorem 1.2 for a torus $\mathbb{G} = \mathbb{G}_m^n$ and we sketch proofs of theorems which prove these conjectures "up to an ε".

2 Algebraic numbers.

2.1 Height of algebraic numbers

Let $\alpha \in \overline{\mathbb{Q}}$ and let K be *any* number field containing α. We denote by $\mathcal{M}_K$ the set of places of K. For $v \in K$, let K_v be the completion of K at v and let $|\cdot|_v$ be the (normalized) absolute value of the place v. Hence

$$|\alpha|_v = |\sigma\alpha|,$$

if v is an archimedean place associated to the embedding $\sigma \colon K \hookrightarrow \overline{\mathbb{Q}}$. If v is a non archimedean place associated with the prime ideal $\wp$ over the rational prime, we have

$$|\alpha|_v = p^{-\lambda/e},$$

where e is the ramification index of $\wp$ and λ is the exponent of $\wp$ in the factorization of the ideal (α) in the ring of integers of K. This standard normalization agrees with the product formula

$$\prod_{v \in \mathcal{M}_K} |\alpha|_v^{[K_v:\mathbb{Q}_v]} = 1$$

which holds for any $\alpha \in K^*$. For further references we recall that for any rational place w (thus $w = \infty$ or $w =$ a prime number)

$$\sum_{v|w} [K_v : \mathbb{Q}_v] = [K : \mathbb{Q}].$$

For a proof of these assertions and for more details on absolute values on number fields, see [Wal 2000] and [Ave 2008].

We define the Weil height of α by

$$h(\alpha) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in \mathcal{M}_K} [K_v : \mathbb{Q}_v] \log \max\{|\alpha|_v, 1\}.$$

It is easy to see that this definition does not depend on the field K containing α; hence, it defines a function $h\colon \overline{\mathbb{Q}} \to \mathbb{R}^+$.

The Weil height of an algebraic number is related to the Mahler measure of a polynomial. Let $P \in \mathbb{C}[x]$ be non-zero; then its Mahler measure is

$$M(P) = \exp \int_0^1 \log \left|P\left(e^{2\pi i t}\right)\right| dt.$$

We also agree that $M(0) = 0$. The Mahler measure are some nice properties. It is a multiplicative function and it is invariant by the morphism $P(x) \to P(x^l)$ ($l \in \mathbb{N}$). Let $\alpha_1, \dots, \alpha_d$ be the roots of P and let P_d be its leading coefficient. By Jensen's formula we easily see that

$$M(P) = |P_d| \prod_{j=1}^{d} \max\{|\alpha_j|, 1\}. \tag{2.1.1}$$

Let K be a number field and let $f \in K[\mathbf{x}]$. We define:

$$\hat{h}(f) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in \mathcal{M}_K} [K_v : \mathbb{Q}_v] \log M_v(f),$$

where $M_v(f)$ is the maximum of the v-adic absolute values of the coefficients of f if v is non archimedean, and $M_v(f)$ is the Mahler measure of σf if v is an archimedean place associated with the embedding $\sigma\colon K \hookrightarrow \overline{\mathbb{Q}}$. As for the Weil height, this definition does not depend on the field K containing the coefficients of f. Moreover, by the product formula, $\hat{h}(\lambda f) = \hat{h}(f)$ for any $\lambda \in K^*$. We also remark that $\hat{h}$ is an additive function. Indeed, $M_v(*)$ is a multiplicative function at least for $v \mid \infty$. By a simple exercise this property still hold for $v \nmid \infty$. By the above properties and by (2.1.1), $\hat{h}(f)$ is the sum of the Weil's height of its roots. As a special case

$$h(\alpha) = \frac{\log M(f)}{[\mathbb{Q}(\alpha) : \mathbb{Q}]}. \tag{2.1.2}$$

where $f \in \mathbb{Z}[x]$ is the minimal polynomial of α over $\mathbb{Z}$ (*i.e.* f is irreducible in $\mathbb{Z}[x]$, $f(\alpha) = 0$ and its leading coefficient is positive).

Let $\|P\|_1$ be the sum of the absolute values of the coefficients of $P \in \mathbb{C}[x]$ (the "length" of P). Since the maximum of $|P|$ on the unit disk is bounded by $\|P\|_1$, we have $M(P) \leq \|P\|_1$. Moreover,

$$\|P\|_1 \leq 2^{\deg(P)} M(P). \tag{2.1.3}$$

This follows from (2.1.1) and from the usual formulas for the coefficients of a polynomial as symmetric functions of its roots. Inequality (2.1.3) implies a theorem of Northcott: the set of algebraic numbers of bounded height and degree is finite. Indeed if $h(\alpha) \leq B$, by the above inequality the coefficients of the minimal polynomial of α are bounded by $2^{[\mathbb{Q}(\alpha):\mathbb{Q}]}B$. Thus the minimal polynomials of the algebraic numbers of bounded height and degree belong to a finite set.

We now state some other important properties of the height. Let α, $\beta \in \overline{\mathbb{Q}}^*$. Then $h(\alpha\beta) \leq h(\alpha) + h(\beta)$. This follows from the inequality $\max\{xy, 1\} \leq \max\{x, 1\} \max\{y, 1\}$ (for x, $y > 0$) applied at each place. Moreover, if β is a root of unity, $h(\alpha\beta) = h(\alpha)$. Indeed roots of unity have absolute value 1 at each place. Let $\alpha \in \overline{\mathbb{Q}}$ and $n \in \mathbb{Z}$. Then $h(\alpha^n) = |n| h(\alpha)$. If $n \geq 0$ this is obvious from the definition. If $n < 0$ this follows from the fact that $h(\alpha^{-1}) = h(\alpha)$, by the product formula.

2.2 Lehmer's problem and Dobrowolski's theorem.

The last property implies that $h(\alpha) = 0$ if and only if α is a root of unity. This is a theorem of Kronecker, and it is precisely the simplest case of Zhang's theorem on the Bogomolov toric conjecture. The problem of finding sharp lower bounds for the height of a non-zero algebraic number α which is not a root of unity is a famous problem of Lehmer. Let $f \in \mathbb{Z}[x]$ be a nonconstant irreducible polynomial. Assume $f \neq \pm x$ and that $\pm f$ is not a cyclotomic polynomial. Lehmer (see [Leh 1933]) asks whether there exists an absolute constant $C > 1$ such that $M(f) \geq C$. An equivalent formulation in term of the height is the following. Let α be a non-zero algebraic number of degree d which is not a root of unity. Then *Lehmer's conjecture* state as follow: there exists an absolute constant $c > 0$ such that

$$h(\alpha) \geq \frac{c}{d}.$$

This should be the best possible lower bound for the height (without any further assumption on α), since $h(2^{1/d}) = (\log 2)/d$. The best known result in the direction of Lehmer's conjecture is Dobrowolski's lower bound

$$h(\alpha) \geq \frac{c}{d}\left(\frac{\log d}{\log\log d}\right)^{-3}$$

which holds for any α of degree $d \geq 2$ as in Lehmer's conjecture. Here c is an absolute constant. In the original statement ([Dob 1979]) $c = 1/1200$; later Voutier ([Vou 1996]) shows that one can take $c = 1/4$.

2.3 A simple proof of a weaker result

In this section we prove a very simple lower bound for the height, once again due to Dobrowolski [Do 1978].

Theorem 2.1 *Let α be a non-zero algebraic number of degree d. Assume that α is not a root of unity. Then, there exists an absolute constant $c > 0$ such that*

$$h(\alpha) \geq \frac{c}{d^3}\ .$$

Proof. We can assume that α is an algebraic integer, otherwise $h(\alpha) \geq (\log 2)/d$. Let $\alpha_1, \ldots, \alpha_d$ be the algebraic conjugates of α. For $n \in \mathbb{N}$ let $S_n = \alpha_1^n + \cdots + \alpha_d^n \in \mathbb{Z}$. Then, for any prime number p,

$$S_{np} \equiv S_n^p \equiv S_n \bmod p.$$

We choose a prime number p such that $2ed < p \leq 4ed$. Let H be the maximum of the absolute values of the algebraic conjugates of α. Assume by contradiction

$$H \leq 1 + \frac{1}{4ed^2}.$$

Then, for $n = 1, 2, \ldots, d$,

$$|S_n - S_{np}| \le 2dH^{np} \le 2d\left(1 + \frac{1}{4ed^2}\right)^{4ed^2} \le 2de < p\,.$$

Since $p \mid (S_{np} - S_n)$ we have $S_{np} = S_n$ for $n = 1, \ldots, d$. By Newton's formulas, the symmetric functions on d variable take the same values on $\alpha_1, \ldots, \alpha_d$ and on $\alpha_1^p, \ldots, \alpha_d^p$. Thus α^p is a conjugate of α. Since $h(\alpha^p) = ph(\alpha)$, this implies $h(\alpha) = 0$. This is a contradiction. Thus $H > 1 + \frac{1}{4ed^2}$ and

$$h(\alpha) \ge \frac{\log H}{d} \ge \frac{c}{d^3}\,.$$

2.4 Lower bound for the height in abelian extensions

In some special cases not only Lehmer's conjecture is true, but it can also be sharpened. Assume for instance that L is a totally real number field or a CM field (a totally complex quadratic extension of a totally real number field). Then, as a special case of a more general result, Schinzel proved that

$$h(\alpha) \ge \frac{1}{2}\log\frac{1+\sqrt{5}}{2} = 0.2406...$$

if $\alpha \in L^*$ *and* $|\alpha| \ne 1$. In particular, by Kronecker's theorem, this inequality holds if α is an algebraic integer different from zero and from the roots of unity. Although this additional assumption makes no harm if one is interested in proving Lehmer's problem, it may happen that non-integers in CM fields have Weil's height smaller than $\frac{1}{2}\log\frac{1+\sqrt{5}}{2}$. For instance, the roots of the irreducible polynomial $2x^4 - 3x^2 + 2 \in \mathbb{Z}[x]$ belong to an abelian extension and have absolute value 1, hence their height is $\frac{\log 2}{4} = 0.1732....$

When the extension $L/\mathbb{Q}$ is an immaginary Galois extension, L is CM if and only if the complex conjugation lies in the center of the Galois group. Assume further that $L/\mathbb{Q}$ is abelian. In this section we prove the following somewhat weaker form of the main result of [Amo-Dvo 2000].

Theorem 2.2 *Let $K/\mathbb{Q}$ be an abelian extension and let $\alpha \in K^*$, α not a root of unity. Then*

$$h(\alpha) \ge \frac{\log(5/2)}{10}.$$

A little more complicated proof give the better bound

$$h(\alpha) \geq \frac{\log 5}{12} = 0.1341...$$

For a natural number $m \geq 3$ we denote by ζ_m a primitive m-root of unity and we let $K_m = \mathbb{Q}(\zeta_m)$ be the m-th cyclotomic field. We need two lemmas.

Lemma 2.3 *Let p be a rational prime. Then there exists $\sigma = \sigma_p \in \mathrm{Gal}(K_m/\mathbb{Q})$ with the following two properties.*

i) If $p \nmid m$, then

$$p \mid (\gamma^p - \sigma\gamma)$$

for any integer $\gamma \in K_m$.

ii) If $p \mid m$, then

$$p \mid (\gamma^p - \sigma\gamma^p)$$

for any integer $\gamma \in K_m$. Moreover, if $\sigma\gamma^p = \gamma^p$ for some $\gamma \in K_m$, then there exists a root of unity $\zeta \in K_m$ such that $\zeta\gamma$ is contained in a proper cyclotomic subextension of K_m.

Proof. Assume first that $p \nmid m$. Let $\sigma \in \mathrm{Gal}(K_m/\mathbb{Q})$ be the Frobenius automorphism defined by $\sigma\zeta_m = \zeta_m^p$. For any integer $\gamma \in K_m$ we have $\gamma = f(\zeta_m)$ for some $f \in \mathbb{Z}[x]$; hence

$$\gamma^p \equiv f(\zeta_m^p) \equiv f(\sigma\zeta_m) \equiv \sigma\gamma \,(\mathrm{mod}\, p).$$

Assume now that $p|m$. The Galois group $\mathrm{Gal}(K_m/K_{m/p})$ is cyclic of order $k = p$ or $k = p - 1$ depending on whether $p^2|m$ or not. Let σ be one of its generators; hence $\sigma\zeta_m = \zeta_p\zeta_m$ for some primitive p-root of unity ζ_p. For any integer $\gamma = f(\zeta_m) \in \mathbb{Z}[\zeta_m]$, we have

$$\gamma^p \equiv f(\zeta_m^p) \equiv f(\sigma\zeta_m^p) \equiv \sigma\gamma^p \,(\mathrm{mod}\, p).$$

Suppose finally that $\sigma\gamma^p = \gamma^p$: then $\sigma\gamma = \zeta_p^u\gamma$ for some integer u. It follows that $\sigma(\gamma/\zeta_m^u) = \gamma/\zeta_m^u$, hence γ/ζ_m^u belongs to the fixed field $K_{m/p}$, as desired.

□

Lemma 2.4 *Let K be a number field and let v be a non-archimedean place of K. Then, for any $\alpha \in K^*$ there exists an algebraic integer β such that $\beta\alpha$ is also integer and*

$$|\beta|_v = \max\{1, |\alpha|_v\}^{-1}.$$

Proof. Let Σ_0 be the set of non-archimedean places w of K such that $\max\{1, |\alpha|_w\} > 1$. Let $\Sigma = \Sigma_0 \cup \{v\}$ and choose an arbitrary non-archimedean place w_0. By the "strong approximation theorem" (see [Cas-Fro 1957], Chapter II, § 15, page 67), there exists $\beta \in K$ such that $|\beta - \alpha^{-1}|_w < \max\{1, |\alpha|\}^{-1}$ for any $w \in \Sigma$ and $|\beta|_w \leq 1$ if $w \notin \Sigma \cup \{w_0\}$. Using the ultrametric inequality, we deduce that

$$|\beta|_w = \max\{1, |\alpha|_w\}^{-1}$$

for any $w \in \Sigma$ and $|\beta|_w \leq 1$ if $w \notin \Sigma \cup \{w_0\}$. Therefore, $|\beta|_w \leq 1$ and $|\alpha\beta|_w \leq 1$ for any finite place w (and so β and $\alpha\beta$ are both integers) and $|\beta|_v = \max\{1, |\alpha|_v\}^{-1}$, since $v \in \Sigma$.

□

Proof of theorem 2.2. Let $p \geq 3$ be a prime number and let $\alpha \in K_m^*$, α not a root of unity. We show that

$$h(\alpha) \geq \frac{\log(p/2)}{2p} .$$

Choosing $p = 5$, this gives, *via* Kronecker-Weber's theorem, the lower bound

$$h(\alpha) \geq \frac{\log(5/2)}{10}$$

for the height of a non-zero algebraic number α (α not a root of unity) lying in an abelian extension.

Let $K = K_m$ and let $\sigma = \sigma_p$ the homomorphism given by lemma 2.3. Assume first that $p \nmid m$. Let v be a place of K dividing p (thus $|p|_v = 1/p$). By lemma 2.4, there exists an algebraic integer $\beta = \beta_v \in K$ such that $\alpha\beta$ is integer and

$$|\beta|_v = \max\{1, |\alpha|_v\}^{-1}.$$

Then

$$|(\alpha\beta)^p - \sigma(\alpha\beta)|_v \leq p^{-1} \quad \text{and} \quad |\beta^p - \sigma\beta|_v \leq p^{-1}.$$

Using the ultrametric inequality, we deduce that

$$\begin{aligned}|\alpha^p - \sigma\alpha|_v &= |\beta|_v^{-p} |(\alpha\beta)^p - \sigma(\alpha\beta) + (\sigma\beta - \beta^p)\sigma\alpha|_v \\ &\le |\beta|_v^{-p} \max\big(|(\alpha\beta)^p - \sigma(\alpha\beta)|_v, |\beta^p - \sigma\beta|_v |\sigma\alpha|_v\big) \\ &\le p^{-1} \max(1, |\alpha|_v)^p \max(1, |\sigma\alpha|_v).\end{aligned}$$

Suppose now that v is a finite place not dividing p. Then we have

$$|\alpha^p - \sigma(\alpha)|_v \le \max(1, |\alpha|_v)^p \max(1, |\sigma(\alpha)|_v) \ .$$

Finally, if $v|\infty$,

$$|\alpha^p - \sigma(\alpha)|_v \le 2 \max(1, |\alpha|_v)^p \max(1, |\sigma(\alpha)|_v) \ .$$

Moreover $\alpha^p \neq \sigma\alpha$, since α is not a root of unity. We now apply the product formula to $\gamma = \alpha^p - \sigma\alpha$, using

$$\sum_{v|p} [K_v : \mathbb{Q}_v] = \sum_{v|\infty} [K_v : \mathbb{Q}_v] = [K : \mathbb{Q}].$$

We get

$$\begin{aligned} 0 &= \sum_{\substack{v\nmid\infty \\ v\nmid p}} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log|\gamma|_v + \sum_{v|p} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log|\gamma|_v + \sum_{v|\infty} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log|\gamma|_v \\ &\le \sum_{v} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} (p \log^+ |\alpha|_v + \log^+ |\sigma\alpha|_v) - \sum_{v|p} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log p \\ &\qquad + \sum_{v|\infty} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log 2 \\ &= p h(\alpha) + h(\sigma\alpha) - \log p + \log 2 \\ &= (p+1) h(\alpha) - \log(p/2).\end{aligned}$$

Therefore,

$$h(\alpha) \ge \frac{\log(p/2)}{p+1} \ge \frac{\log(p/2)}{2p}.$$

Assume now that $p \mid m$. Let v be a place of K dividing p and let $\beta = \beta_v \in K$ as in the first part of the proof. Then

$$|(\alpha\beta)^p - \sigma(\alpha\beta)^p|_v \le p^{-1} \quad \text{and} \quad |\beta^p - \sigma\beta^p|_v \le p^{-1}.$$

Using the ultrametric inequality, we find

$$\begin{aligned}|\alpha^p - \sigma\alpha^p|_v &= |\beta|_v^{-p} |(\alpha\beta)^p - \sigma(\alpha\beta)^p + (\sigma\beta^p - \beta^p)\sigma\alpha^p|_v \\ &\leq p^{-1} \max(1, |\alpha|_v)^p \max(1, |\sigma\alpha|_v)^p.\end{aligned}$$

Moreover, we can assume $\alpha^p \neq \sigma\alpha^p$. Otherwise, by lemma 2.3, there exists a root of unity $\zeta \in K$ such that $\zeta\alpha$ is contained in a proper cyclotomic subextension of K; hence $h(\alpha) = h(\zeta\alpha)$ and, by induction, $h(\zeta\alpha) \geq \frac{\log(p/2)}{2p}$. Applying the product formula to $\gamma = \alpha^p - \sigma\alpha^p$ as in the first part of the proof, we get

$$0 \leq ph(\alpha) + ph(\sigma\alpha) - \log p + \log 2 = 2ph(\alpha) - \log(p/2).$$

Again

$$h(\alpha) \geq \frac{\log(p/2)}{2p}.$$

□

2.5 Sketch of the proof of Dobrowolski's theorem

In this section we give a sketch of the proof of the main result of Dobrowolski:

Theorem 2.5 *For any algebraic number $\alpha \in \overline{\mathbb{Q}}^*$ of degree $d \geq 2$ which is not a root of unity we have*

$$h(\alpha) \geq \frac{c}{d} \left(\frac{\log d}{\log \log d} \right)^{-3}$$

for some absolute constant $c > 0$.

Sketch of the proof. As usual, we can assume that α is an algebraic integer. Let f be its minimal polynomial over $\mathbb{Z}$ and let p be a prime number. Then, by Fermat's little theorem,

$$f(x)^p \equiv f(x^p) \bmod p\mathbb{Z}[x].$$

Thus

$$|f(\alpha^p)|_v \leq p^{-1}$$

for any $v \mid p$. Let now $F \in \mathbb{Z}[x]$ be a polynomial of degree L vanishing on α with multiplicity $\geq T$ for some *parameters* L and T with $L \geq dT$. Then

$$|F(\alpha^p)|_v \leq p^{-T}$$

for any $v \mid p$. Moreover $|F(\alpha^p)|_v \leq 1$ for $v \nmid \infty$ and

$$|F(\alpha^p)|_v \leq \|F\|_1 \max(1, |\alpha|_v)^{pL}$$

if $v \mid \infty$. Assume

$$F(\alpha^p) \neq 0. \tag{2.5.1}$$

Then, by product formula,

$$0 = -T \log p + \log |F|_1 + pLh(\alpha)\ . \tag{2.5.2}$$

This gives

$$h(\alpha) \geq \frac{T \log p - \log \|F\|_1}{pL}.$$

We choose $L = d$, $T = 1$ and $F = f$. The non vanishing condition (2.5.1) is satisfied. Indeed, if α is not a root of unity, then α^p is not a conjugate of α, as we have remarked before. Thus we get

$$h(\alpha) \geq \frac{\log p - \log \|f\|_1}{pd}\ .$$

Unfortunately, $\log \|f\|_1$ can be as large as a power of d, even if the height of α is very small (see [Amo 1995]). Thus, to get a positive lower bound, we must choose p exponential in d^c and the argument ends with a poor lower bound of the shape $h(\alpha) \geq e^{-d^c}$.

The use of Siegel's Lemma ([Bom-Vaa 1983]), a classical tool in diophantine approximation, improves enormously the quality of this bound. Using this lemma we find a non-zero polynomial $F \in \mathbb{Z}[x]$ of degree $\leq L$ vanishing on α with multiplicity $\geq T$ as required and such that

$$\log \|F\|_\infty \leq \frac{dT}{L + 1 - dT}(T \log(L + 1) + Lh(\alpha)). \tag{2.5.3}$$

Here $\|F\|_\infty$ denotes the maximum of the absolute values of the coefficients of F.

The proof now follow the scheme of a classical transcendence proof: construction of an auxiliary function, extrapolation, zero's lemma. During the proof we assume that the height of α is pathologically small and at the end we get a contradiction.

We use in a non standard way the symbols $\approx$, $\ll$ and $\gg$. We write $A \approx B$ if and only if $c_1 B < A < c_2 B$ with c_1, $c_2 > 0$. The constants c_1,c_2 are eventually assumed to be sufficiently large (or small) in such a way that the forthcoming assumptions are verified. Similarly, $A \ll B$ (or $B \gg A$) if and only if $A \leq cB$ where $c > 0$ has the same meaning as before.

- *Choice of the auxiliary function*

  Since $\log \|F\|_1 \leq (L+1) \log \|F\|_\infty$, by (2.5.3) we have

$$\log \|F\|_1 \leq \log(L+1) + \frac{dT}{L+1-dT}(T \log(L+1) + Lh(\alpha)) \, .$$

This inequality cannot give nothing better than $\log \|F\|_1 \ll \log(L+1)$. Therefore, it is reasonable to choose L and T in such a way that

$$\frac{dT^2}{L+1-dT} \approx 1,$$

say $L = dT^2$, and to assume $Lh(\alpha) \leq T \log(L+1)$. Assume further that $\log \log T \ll \log d$. This implies $\log(L+1) \approx \log d$. Thus if $h(\alpha) \ll \log d/dT$, the lenght of the auxiliary polynomial satisfies $\log \|F\|_1 \ll \log d$.

- *Extrapolation*

  We fix a third parameter N. We assume that our primes p satisfy $N/2 \leq p \leq N$. As for T, we suppose that $\log N \ll \log \log d$. We want to show that F vanishes on α^p for all p as before. Assume that for some p we have $F(\alpha^p) \neq 0$. Then, by (2.5.2),

$$0 \ll -T \log \log d + \log d + NT^2 dh(\alpha). \tag{2.5.4}$$

We choose $T \log \log d \approx \log d$ and we assume $NT^2 dh(\alpha) \leq \log d$. Thus

$$T \approx \frac{\log d}{\log \log d}$$

and

$$h(\alpha) \ll \frac{(\log \log d)^2}{N d \log d}. \tag{2.5.5}$$

Note that 2.5.5 implies the previous assumption $h(\alpha) \ll \log d/dT$. Then, choosing in an appropriate way the implicit constants in the parameters, equation (2.5.4) cannot hold. This force F to vanish on α^p for all $N/2 \leq p \leq N$ as required.

- *Zero's lemma and conclusion.*

  Since α is not a root of unity, α^{p_1} and α^{p_2} are not conjugates for primes $p_1 \neq p_2$. Assume

$$[\mathbb{Q}(\alpha^n) : \mathbb{Q}] = d \tag{2.5.6}$$

for all integer n. Let

$$\Sigma = \{\sigma(\alpha^p), \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),\ p \text{ prime},\ N/2 \le p \le N\}\ .$$

Then, by the Prime Number Theorem,

$$\#\Sigma = \sum_{N/2 \le p \le N} d \gg \frac{dN}{\log N}.$$

We choose $\frac{dN}{\log N} \approx L$, *i.e.*

$$N \approx \frac{(\log d)^2}{\log\log d}\ .$$

Then, choosing in an appropriate way the implicit constants in the parameters,

$$L < \#\Sigma \le \deg F \le L\ .$$

This contradiction shows that, at least if α satisfies the additional hypothesis (2.5.6), the inequality (2.5.5) cannot hold. Thus

$$h(\alpha) \gg \frac{\log\log d}{Nd\log d} \gg \frac{1}{d}\left(\frac{\log d}{\log\log d}\right)^{-3}$$

as required.

Thus, Dobrowolski's theorem is proved under the additional assumption (2.5.6). Assume that for some $n > 1$ (2.5.6) does not hold. We follow an argument of [Rausch 1985]. We have $k = [\mathbb{Q}(\alpha) : \mathbb{Q}(\alpha^n)] > 1$. Let β be the norm of α from $\mathbb{Q}(\alpha)$ to $\mathbb{Q}(\alpha^n)$. Then $\beta = \zeta\alpha^k$ for some root of unity ζ and $h(\beta) = h(\alpha^k) = kh(\alpha)$. We remark that

$$\varepsilon(d) = \left(\frac{\log\log 16d}{\log 16d}\right)^3$$

is a decreasing function. Since $[\mathbb{Q}(\beta) : \mathbb{Q}] < d$, by induction we get

$$dh(\alpha) = [\mathbb{Q}(\alpha^n) : \mathbb{Q}]h(\beta) \ge [\mathbb{Q}(\beta) : \mathbb{Q}]h(\beta) \gg \varepsilon([\mathbb{Q}(\beta) : \mathbb{Q}]) \ge \varepsilon(d).$$

$\square$

2.6 Relative results

We can "mix" the lower bound in abelian extensions (theorem 2.2) with Dobrowolski's result. Let K be a fixed number field and let L/K be an abelian extension. In ([Amo-Zan 2000]), we prove that for $\alpha \in L^*$ not a root of unity

$$h(\alpha) \geq \frac{c(K)}{D}\left(\frac{\log\log 5D}{\log 2D}\right)^{13},$$

where $D = [L(\alpha) : L]$ and where $c(K) > 0$. In ([Amo-Del 2007]) we refine the error term in this inequality and we compute a lower bound for $c(K)$. As the proof of the original paper suggested, this lower bound depends on the degree *and* on the discriminant of K.

We go back to lower bounds for the height on an abelian extension L of a number field K. As a very special case of the result of [Amo-Zan 2000], the height in L^*, outside roots of unity, is bounded from below by a positive function depending only on K. The following question arises. Is it true that we can choose a function depending only on the degree $[K : \mathbb{Q}]$? In [Amo-Zan 2008] we give a positive answer to this problem. Let L/K be as before. Then for any $\alpha \in \overline{\mathbb{Q}}^*$ which is not a root of unity we have

$$h(\alpha) > 3^{-d^2-2d-6}$$

where $d = [K : \mathbb{Q}]$. This result has some amusing consequence. For instance, let L be a dihedral extension of the rational field of degree, say, $2n$. Then L is an abelian extension of its quadratic subfield K fixed by the normal cyclic group of order n. Thus for any $\alpha \in L^*$ which is not a root of unity we have

$$h(\alpha) \geq 3^{-14}.$$

3 Normalized height on subvarieties of $\mathbb{G}_{\mathrm{m}}^n$.

3.1 Some geometry

We consider a torus $\mathbb{G}_m^n$ and we fix the "standard embedding" $\iota\colon \mathbb{G}_m^n \hookrightarrow \mathbb{P}_n$,

$$\iota(x_1, \ldots, x_n) = (1 : x_1 : \cdots : x_n).$$

By subvariety of $\mathbb{G}_m^n$ we mean an algebraic subvariety V defined over some number field K. The degree of V is the degree of its Zariski closure in $\mathbb{P}_n$. We shall say that V is irreducible if its Zariski closure is geometrically irreducible. Similarly, we say that V is irreducible over K if its Zariski closure is irreducible over K.

Note that any m-tuple of vectors $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_m \in \mathbb{Z}^n$ defines a regular map $\varphi\colon \mathbb{G}_{\mathrm{m}}^n \to \mathbb{G}_{\mathrm{m}}^m$ by $\varphi(\mathbf{x}) := (\mathbf{x}^{\boldsymbol{\lambda}_1}, \dots, \mathbf{x}^{\boldsymbol{\lambda}_m})$. This map is plainly an algebraic group homomorphism, called *monoidal*. When $m = n$, the homomorphism φ is invertible if and only if $\det(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_m) = \pm 1$; in this case it is called a *monoidal automorphism* of $\mathbb{G}_{\mathrm{m}}^n$. We shall often use a special monoidal morphism. Let $l \in \mathbb{N}$. We denote by $[l]\colon \mathbb{G}_{\mathrm{m}}^n \to \mathbb{G}_{\mathrm{m}}^n$ the "multiplication" by $[l]$, *i.e.* the morhism $\mathbf{x} \mapsto \mathbf{x}^l = (x_1^l, \dots, x_n^l)$. Thus the kernel $\mathrm{Ker}[l]$ is the set of l-torsion points. It is a subgroup $\cong (\mathbb{Z}/l\mathbb{Z})^n$.

By algebraic subgroup of $\mathbb{G}_m^n$ we mean a closed algebraic subvariety stable under the group operations. An irreducible algebraic subgroup is called a torus. Any algebraic subgroup is a finite disjoint union of translates of a torus. Given an algebraic subgroup H we denote by H^0 its connected component containing $\mathbf{1}$. Let $\Lambda \subseteq \mathbb{Z}^n$ be a subgroup. Then

$$H_\Lambda = \{\mathbf{x} \in \mathbb{G}_{\mathrm{m}}^n,\ \forall \boldsymbol{\lambda} \in \Lambda,\ \mathbf{x}^{\boldsymbol{\lambda}} = \mathbf{1}\}$$

is an algebraic group. Moreover, one can see that $\Lambda \mapsto H_\Lambda$ is a bijection between subgroups of $\mathbb{Z}^n$ and algebraic subgroups of $\mathbb{G}_{\mathrm{m}}^n$. All these statements are proved in [Bom-Gub 2006].

Let V be an irreducible subvariety of $\mathbb{G}_{\mathrm{m}}^n$. We define its stabilizer as $\mathrm{Stab}(V) = \{\boldsymbol{\alpha} \in \mathbb{G}_m^n \text{ s.t. } \boldsymbol{\alpha} V = V\}$. Thus

$$\mathrm{Stab}(V) = \bigcap_{\mathbf{x} \in V} \mathbf{x}^{-1} V.$$

This shows that $\mathrm{Stab}(V)$ is an algebraic subgroup of dimension $\leq \dim(V)$. We remark that we have equality of the dimensions if and only if V is a translate of a torus. Moreover

$$\deg(\mathrm{Stab}(V)) \leq \deg(V)^{\dim(V)+1}. \tag{3.1.1}$$

Indeed let s be the dimension of $\mathrm{Stab}(V)$ and let d be the dimension of V. Then, by standard algebraic geometry, V is a component of an intersection of $n-d$ hypersurfaces Z_j of degree $\leq \deg(V)$. By definition of $\mathrm{Stab}(V)$, there exist $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{d-s} \in V$ and distinct indexes $1 \leq j_1, \dots, j_{d-s} \leq n-d$ such that the connected components of $\mathrm{Stab}(V)$ are components of

$$\mathbf{x}_0^{-1} V \bigcap \left(\bigcap_{1 \leq i \leq d-s} \mathbf{x}_i^{-1} Z_{j_i} \right),$$

By Bézout theorem,

$$\deg(\mathrm{Stab}(V)) \leq \deg(V)^{d-s+1} \leq \deg(V)^{d+1},$$

as required.

Let l be a positive integer. We are interested in relations between the degree of V and the degrees of $[l]^{-1}V = \{\boldsymbol{\alpha} \in \mathbb{G}_m^n \text{ s.t. } \boldsymbol{\alpha}^l \in V\}$ and of $[l]V = \{\boldsymbol{\alpha}^l \text{ s.t. } \boldsymbol{\alpha} \in V\}$. For $[l]^{-1}V$ we have

$$\deg([l]^{-1}V) = l^{\operatorname{codim}(V)} \deg(V).$$

For a hypersurface, this statement is clear. Indeed let f be an equation of V. Then $f(\mathbf{x}^l)$ is an equation of $[l]^{-1}V$. We consider the general case. Let d be the dimension of V and let $W_1, \ldots, W_d$ be generic hypersurfaces of degree $D_1, \ldots, D_d$ such that $X = V \cap W_1 \cap \cdots \cap W_d$ is a finite set of $\deg(V)D_1 \cdots D_d$ points. Then $[l]^{-1}X = [l]^{-1}V \cap [l]^{-1}W_1 \cap \cdots \cap [l]^{-1}W_d$ is a set of cardinality $l^n|X|$. On the other hand, for what we have seen for hypersurfaces, this set has cardinality $\deg([l]^{-1}V)l^d D_1 \cdots D_d$. Thus $\deg([l]^{-1}V) = l^{n-d}\deg(V)$ as required.

For the degree of $[l]V$ we have

$$\deg([l]V) = \frac{l^{\dim(V)}\deg(V)}{|\mathrm{Ker}[l] \cap \mathrm{Stab}(V)|}. \tag{3.1.2}$$

This equality follows from the previous one. Indeed $[l]^{-1}[l]V = \mathrm{Ker}[l]V$ and $\mathrm{Ker}[l]V$ is a union of

$$\frac{l^n}{|\mathrm{Ker}[l] \cap \mathrm{Stab}(V)|}$$

distinct components. Thus

$$l^{\operatorname{codim}(V)}\deg([l]V) = \deg([l]^{-1}[l]V) = \frac{l^n \deg([l]V)}{|\mathrm{Ker}[l] \cap \mathrm{Stab}(V)|}.$$

We conclude this section with an additional remark. We have

$$|\mathrm{Ker}([l]) \cap \mathrm{Stab}(V)| = l^{\dim \mathrm{Stab}(V)}|\mathrm{Ker}[l] \cap (\mathrm{Stab}(V)/\mathrm{Stab}(V)^0)|.$$

Let l be an integer coprime with $[\mathrm{Stab}(V) : \mathrm{Stab}(V)^0]$. Then $\ker[l]V$ is a union of $l^{\operatorname{codim}(\mathrm{Stab}\,V)}$ distinct components (which are translates of V by l-torsion points).

3.2 Points

Let $\boldsymbol{\alpha} = (\alpha_0 : \cdots : \alpha_n) \in \mathbb{P}_n(K)$ and let K be any number field containing $\alpha_0, \ldots, \alpha_n$. We define the Weil height of $\boldsymbol{\alpha}$ by:

$$h(\boldsymbol{\alpha}) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in \mathcal{M}_K} [K_v : \mathbb{Q}_v] \log \max\{|\alpha_0|_v, \ldots, |\alpha_n|_v\}.$$

As for the height of algebraic number, this definition does not depend on the number field K; moreover it does not depend on the projective coordinates of $\boldsymbol{\alpha}$ (by the product formula).

This provides a height function $\hat{h}(x_1, \ldots, x_n) = h(1 : x_1 : \cdots : x_n)$ on $\mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$. The following properties hold:

i) the function $\hat{h}$ is a positive function on $\mathbb{G}_m^n(\overline{\mathbb{Q}})$, vanishing only on its torsion points;

ii) $\hat{h}(\boldsymbol{\alpha\beta}) \leq \hat{h}(\boldsymbol{\alpha}) + \hat{h}(\boldsymbol{\beta})$. Moreover, if $\boldsymbol{\zeta}$ is a torsion point, $\hat{h}(\boldsymbol{\zeta\alpha}) = \hat{h}(\boldsymbol{\alpha})$. If $n \in \mathbb{N}$ then $\hat{h}(\boldsymbol{\alpha}^n) = n\hat{h}(\boldsymbol{\alpha})$;

iii) a subset of $\mathbb{G}_m^n(\overline{\mathbb{Q}})$ of bounded height and bounded degree is finite.

The proofs are similar to those in dimension 1.

3.3 Hypersurfaces

We have a "natural" definition of height on hypersurfaces rising from an extension of the Mahler measure to polynomials in several variables. Let $P \in \mathbb{C}[x_1, \ldots, x_n]$; we define its Mahler measure as:

$$M(P) = \exp \int_0^1 \cdots \int_0^1 \log \left| f\left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\right) \right| dt_1 \ldots dt_n$$

and we make the convention $M(0) = 0$. As in dimension 1, the Mahler measure is a multiplicative function and it is invariant by $P(\mathbf{x}) \mapsto P(\mathbf{x}^{\boldsymbol{\lambda}})$ for any $\boldsymbol{\lambda} \in \mathbb{N}^n$. Let K be a number field and let $f \in K[\mathbf{x}]$ be a polynomial. We define, as we do in section 2.1,

$$\hat{h}(f) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in \mathcal{M}_K} [K_v : \mathbb{Q}_v] \log M_v(f),$$

where $M_v(f)$ is the maximum of the v-adic absolute values of the coefficients of f if v is non archimedean, and $M_v(f)$ is the Mahler measure of σf if v

is an archimedean place associated with the embedding $\sigma\colon K \hookrightarrow \overline{\mathbb{Q}}$. As in section 2.1, this definition does not depend on the field K containing the coefficients of f and $\hat{h}$ defines a positive and additive function on $\overline{\mathbb{Q}}[\mathbf{x}]$. Let

$$V = \{\boldsymbol{\alpha} \in \mathbb{G}_m^n \text{ s.t. } f(\boldsymbol{\alpha}) = 0\}$$

be a hypersurface in $\mathbb{G}_m^n$ defined by some square-free polynomial $f \in K[\mathbf{x}]$. We define the normalized height of V as

$$\hat{h}(V) = \hat{h}(f).$$

This definition does not depend on the equation we choose for V. We also remark that the height of V is invariant under inverse image by monoidal morphisms: let $\varphi\colon \mathbb{G}_\mathrm{m}^n \to \mathbb{G}_\mathrm{m}^m$ be a monoidal morphism, then $\hat{h}(\varphi^{-1}(V)) = \hat{h}(V)$.

Following Schinzel, we say that an irreducible $f \in \mathbb{Z}[\mathbf{x}]$ is an *extended cyclotomic polynomial* if there exist a cyclotomic polynomial ϕ and $\boldsymbol{\lambda}$, $\boldsymbol{\mu} \in \mathbb{Z}^n$ such that

$$f(\mathbf{x}) = \pm \mathbf{x}^{\boldsymbol{\lambda}} \phi(\mathbf{x}^{\boldsymbol{\mu}}) \ .$$

In other words, an irreducible polynomial $f \in \mathbb{Z}[\mathbf{x}]$ is extended cyclotomic if and only if the hypersurface $\{f = 0\}$ in $\mathbb{G}_\mathrm{m}^n$ is an union of torsion varieties. In this context, Zhang's theorem on the toric Bogomolov conjecture can be paraphrased as follows. Let $f \in \mathbb{Z}[\mathbf{x}]$ be irreducible. Then $M(f) = 1$ if and only if $f = \pm x_j$ or if f is an extended cyclotomic polynomial. This result was proved earlier by [Boy 1980], [Law 1977] and [Smy 1982] independently.

The normalized height of an irreducible hypersurface has a nice behaviour under the action of pull back and pull out by multiplication by $[l]$. Indeed

$$\hat{h}([l]^{-1}V) = \hat{h}(V)$$

and

$$\hat{h}([l]V) = \frac{l^n \hat{h}(V)}{|\mathrm{Ker}[l] \cap \mathrm{Stab}(V)|}.$$

The first equality is a special case of the invariance of $\hat{h}(V)$ under inverse image by monoidal morphisms. The second equality follows from the first one and from the additivity of $\hat{h}$, exactly as the corresponding formulas for the degree.

The normalized height of a hypersurface can be computed as a limit. Let $f \in \mathbb{C}[\mathbf{x}]$. From (2.1.3) we deduce by induction on n (see [Mig 1992] for details)

$$\|f\|_1 \leq 2^{d_1 + \cdots + d_n} M(f),$$

where $d_1, \dots, d_n$ are the partial degrees of f. Let $\|\cdot\|$ be any norm on $\mathbb{C}[\mathbf{x}]$ such that

$$\log \|f\| = \log \|f\|_1 + O(\deg f) \tag{3.3.1}$$

We define a height on hypersurfaces of $\mathbb{G}_m^n$ by choosing the norm $\|\cdot\|$ at the archimedean places. Let as before

$$V = \{\boldsymbol{\alpha} \in \mathbb{G}_m^n \text{ s.t. } f(\boldsymbol{\alpha}) = 0\}$$

be a hypersurface in $\mathbb{G}_m^n$ defined by some square-free polynomial $f \in K[\mathbf{x}]$. Let define

$$h(V) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in \mathcal{M}_K} [K_v : \mathbb{Q}_v] \log H_v(f),$$

where $H_v(f) = M_v(f)$ if v is non archimedean, and $H_v(f) = \|\sigma f\|$ if v is an archimedean place associated with the embedding $\sigma\colon K \hookrightarrow \overline{\mathbb{Q}}$. Then,

$$\hat{h}(V) = h(V) + O(\deg(V)). \tag{3.3.2}$$

Let l be a positive integer. Using the relations between degrees and heights of V and $[l]V$ we see that

$$\hat{h}(V) = \frac{\hat{h}([l]V)\deg(V)}{l \deg([l]V)}.$$

Thus, replacing in (3.3.2) V by $[l]V$,

$$\hat{h}(V) = \frac{h([l]V)\deg(V)}{l\deg([l]V)} + O(l^{-1}\deg(V)).$$

This shows

$$\lim_{l \mapsto \infty} \frac{h([l]V)\deg(V)}{l\deg([l]V)} = \hat{h}(V).$$

3.4 Subvarieties of arbitrary dimension

The last remark suggests a "simple" definition of normalized height on subvarieties of $\mathbb{G}_m^n$, alternative to the one commonly used in Arakelov theory. We start by choosing a height on subvarieties. Let V be a d dimensional irreducible subvariety and let F be the Chow form of its Zariski closure in $\mathbb{P}_n$. The Chow form is an irreducible multihomogeneous polynomial $F(u_0^1, \dots, u_n^1, \dots, u_0^{d-1}, \dots, u_n^{d-1})$ vanishing precisely if the intersection of V with the hyperplanes of coordinates $\mathbf{u}^1, \dots, \mathbf{u}^{d-1}$ is non empty. We define a height $h(V)$ as the height of the hypersurface in $\mathbb{G}_m^{(d-1)n}$ defined by

$\{F = 0\}$, where one choose any reasonable norm at the archimedean places[2]. David and Philippon (see [Dav-Phi 1999]) prove that the limit

$$\hat{h}(V) = \lim_{l \to +\infty} \frac{h([l]V)\deg(V)}{l \deg([l]V)}$$

exists. We can see (compute the Chow form) that this definition of normalized height specializes to the previous ones if V is a point or if V is a hypersurface (see [Dav-Phi 1999]). Moreover:

i) the function $\hat{h}(\cdot)$ is non-negative;

ii) for every $l \in \mathbb{N}$ we have

$$\hat{h}([l]^{-1}V) = l^{\operatorname{codim}(V)-1}\hat{h}(V)$$

and

$$\hat{h}([l]V) = \frac{l^{\dim(V)+1}\hat{h}(V)}{|\operatorname{Ker}[l] \cap \operatorname{Stab}(V)|}.$$

iii) for every torsion point $\boldsymbol{\zeta}$ we have $\hat{h}(\boldsymbol{\zeta}V) = \hat{h}(V)$.

For details on this construction of the normalized height on tori, see [Pon 2008].

4 The zero and small height problems for tori.

4.1 Conjectures and results.

Using property iii) and ii) of the normalized height, we see that a torsion variety $V = \boldsymbol{\zeta}H$ has height zero. Indeed, if $\boldsymbol{\zeta}$ is a torsion point and H is a subtorus, then $\hat{h}(\boldsymbol{\zeta}H) = \hat{h}(H)$ and $\hat{h}(H) = \hat{h}([l]H) = l\hat{h}(H)$ for any $l \in \mathbb{N}$ (since $H = [l]H$ and $|\operatorname{Ker}[l] \cap H| = l^{\dim(H)}$).

Are torsion varieties the only varieties of zero height? The answer is positive; more precisely, this question is equivalent to the multiplicative analogue of the former Bogomolov's conjecture. To see this, let us define the essential minimum $\hat{\mu}^{\mathsf{ess}}(V)$ of an irreducible subvariety V as the infimum of the set of $\theta > 0$ such that

$$V(\theta) = \{P \in V \text{ s.t. } \hat{h}(P) \leq \theta\}$$

[2] *i.e.* a norm satisfying (3.3.1).

is Zariski dense in V. Theorem 1.2 asserts that $\hat{\mu}^{\mathsf{ess}}(V) = 0$ if and only if V is torsion. By a special case of an inequality of Zhang (see [Zha 1995], theorem 5.2.), we have, for an irreducible V,

$$\hat{\mu}^{\mathsf{ess}}(V) \leq \frac{\hat{h}(V)}{\deg(V)} \leq (\dim(V)+1)\hat{\mu}^{\mathsf{ess}}(V). \tag{4.1.1}$$

This inequality shows that $\hat{h}(V) = 0$ if and only if $\hat{\mu}^{\mathsf{ess}}(V) = 0$. The problem of finding sharp lower bounds for $\hat{\mu}^{\mathsf{ess}}(V)$ for non-torsion subvarieties of $\mathbb{G}_m^n$ is a generalization of Lehmer's problem. Lower bounds for the essential minimum of a non-torsion subvariety will depend on some geometric invariants of V, for instance its degree. Moreover, if we do not make any further geometric assumption on the variety, such a bound must also depend on its field of definition ("arithmetic case"). Indeed, let H be a proper subtorus of $\mathbb{G}_m^n$ and let $\boldsymbol{\alpha}_n$ be a sequence of non-torsion points whose height tends to zero (for instance, $\boldsymbol{\alpha}_n = (2^{1/n}, \ldots, 2^{1/n})$). Then, the varieties $V_n = H\boldsymbol{\alpha}_n$ have fixed degree $\deg(H)$ and essential minimum $\hat{\mu}^{\mathsf{ess}}(V_n) \leq \hat{h}(\boldsymbol{\alpha}_n) \to 0$. In spite of that, if we further assume that V is not a translate of a proper subtorus (even by a point of infinite order), then Bombieri and Zannier ([Bom-Zan 1995]) proved that the essential minimum of V can be bounded from below only in terms of the degree of V ("geometric case").

Let V be a subvariety of $\mathbb{G}_m^n$ and let K be a subfield of $\overline{\mathbb{Q}}$. We define the "absolute obstruction index" $\omega(V)$ of V as the minimum of $\deg(Z)$ where Z is a hypersurface containing V. Similarly, we define the "rational obstruction index" $\omega_{\mathbb{Q}}(V)$ as the minimum of $\deg(Z)$ where Z is a hypersurface defined over $\mathbb{Q}$ containing V.

For instance, let α be an algebraic number of degree d. Then $\omega_{\mathbb{Q}}(\alpha) = d$. More generally, let $\boldsymbol{\alpha} \in \mathbb{G}_m^n(\overline{\mathbb{Q}})$. Then, by standard linear algebra,

$$\omega_{\mathbb{Q}}(\boldsymbol{\alpha}) \leq n[\mathbb{Q}(\boldsymbol{\alpha}) : \mathbb{Q}]^{1/n} \tag{4.1.2}$$

Even more generally, let V be a subvariety of $\mathbb{G}_n^m$. Then, if V is irreducible,

$$\omega(V) \leq n \deg(V)^{1/\mathrm{codim}(V)}.$$

Similarly, if V is defined and irreducible over the rational field, $\omega_{\mathbb{Q}}(V) \leq n \deg(V)^{1/\mathrm{codim}(V)}$. Both inegalities are special cases of a result of Chardin ([Cha 1988]).

It turns out that $\omega_{\mathbb{Q}}(V)$, and not the degree of V, is the right invariant to formulate the sharpest conjectures on $\hat{\mu}^{\mathsf{ess}}(V)$ in the "arithmetic case".

Similarly, $\omega(V)$ is the right invariant in the "geometric case". Although, in order to get statements depending on ω we need to assume, in the geometric case, not only that V is not a translate but also that V is not contained in any proper translate. Indeed, consider a curve $\mathcal{C} \subseteq \mathbb{G}_{\mathrm{m}}^{n-1}$. Let $\mathcal{C}' = \mathcal{C} \times \{1\} \subseteq \mathbb{G}_{\mathrm{m}}^n$ and choose, for $l \in \mathbb{N}$, an irreducible component V_l of $[l]^{-1}\mathcal{C}'$. Then $\hat{\mu}^{\mathsf{ess}}(V_l) \mapsto 0$, while $\omega(V_l) = 1$ since V_l is contained in the hypersurface $x_n = 1$. We shall say that an irreducible variety V is a "transverse" if it is not contained in any proper translate. Similarly, in the arithmetic case we need to assume that V is not contained in any proper torsion variety. Such a V will be called a "weak-transverse" variety. Let $\boldsymbol{\alpha} \in \mathbb{G}_{\mathrm{m}}(\mathbb{Q})$. We remark that the 0-dimensional variety $\{\boldsymbol{\alpha}\}$ is weak-transverse if and only if $\alpha_1, \dots, \alpha_n$ are multiplicatively dependent.

In [Amo-Dav 1999] we propose the following conjecture, which generalizes Lehmer's one:

Conjecture 4.1 *Let V be a weak-transverse subvariety of $\mathbb{G}_m^n$. Then, there exists a constant $c(n)$ such that*

$$\hat{\mu}^{\mathsf{ess}}(V) \geq \frac{c(n)}{\omega_{\mathbb{Q}}(V)}.$$

In [Amo-Dav 1999] (case $\dim V = 0$), [Amo-Dav 2000] (case $\mathrm{codim} V = 1$) and [Amo-Dav 2001] (general case) the following analogue of Dobrowolski theorem on $\mathbb{G}_m^n$ is proved:

Theorem 4.2 *Let V be a tranverse subvariety of $\mathbb{G}_m^n$ of codimension k. Let assume that V is not contained in any torsion variety. Then there exists two positive constants $c(n)$ and $\kappa(k) = (k+1)(k+1)!^k - k$ such that*

$$\hat{\mu}^{\mathsf{ess}}(V) \geq \frac{c(n)}{\omega(V)} \left(\log 3\omega_{\mathbb{Q}}(V)\right)^{-\kappa(k)}.$$

This theorem sometimes produces lower bounds for the height of algebraic numbers which are even stronger than what is expected by Lehmer's conjecture. Let $\alpha_1, \dots, \alpha_n$ multiplicatively independent algebraic numbers of height $\leq h$, lying in a number field of degree d. Then $\hat{\mu}^{\mathsf{ess}}(\boldsymbol{\alpha}) \leq h$ and, by (4.1.2),

$$\omega_{\mathbb{Q}}(\boldsymbol{\alpha}) \leq nd^{1/n}.$$

Thus, by theorem 4.2,

$$h \geq \frac{c(n)}{d^{1/n}} \left(\log 3d\right)^{-\kappa(n)}.$$

for some $c(n) > 0$.

Assuming that the subvariety V is tranverse, we now look for lower bounds for $\hat{\mu}^{\mathsf{ess}}(V)$ which do not depend on the field of definition of V (geometric case). In ([Amo-Dav 2003]) we formulate the following conjecture.

Conjecture 4.3 *Let V be a transverse subvariety of $\mathbb{G}_m^n$. Then, there exists a positive constant $c(n)$ such that*

$$\hat{\mu}^{\mathsf{ess}}(V) \geq \frac{c(n)}{\omega_{\overline{\mathbb{Q}}}(V)}.$$

In the same paper the following analogue of theorem 4.2 is proved :

Theorem 4.4 *Let V be a transverse subvariety of $\mathbb{G}_m^n$ of codimension k. Then there exist two positive constants $c(n)$ and $\lambda(k) = \left(9(3k)^{(k+1)}\right)^k$ such that*

$$\hat{\mu}^{\mathsf{ess}}(V) \geq \frac{c(n)}{\omega_{\overline{\mathbb{Q}}}(V)} \left(\log 3\omega_{\overline{\mathbb{Q}}}(V)\right)^{-\lambda(k)}.$$

4.2 Overview of the methods

The proofs of the main results of [Amo-Dav 1999] and [Amo-Dav 2003] (theorems 4.2 and 4.4) require several technical tools. By contradiction, we assume in both proofs that the essential minimum is sufficiently small. We then start following the usual steps of a transcendence proof: interpolation (construction of an auxiliary function), extrapolation and zero estimates.

Concerning the zero lemma, in both cases these proofs become very technical. In diophantine analysis a classical zero lemma (as [Phi 1986]) is normally enough to conclude the proof. On the contrary, in [Amo-Dav 1999] we need a more complicated zero lemma. As a consequence, this force to extrapolate over different set of primes. In Dobrowolski's proof one construct, using Siegel's Lemma, an auxiliary function F which vanishes on α. Then we extrapolate by proving that F must also vanish on α^p at least for small primes p. In the proof of theorem 4.2 (in the 0 dimensional case which is the hardest one) we construct an auxiliary function vanishing on $\boldsymbol{\alpha}$ and then we extrapolate by proving that F must also vanish on $\boldsymbol{\alpha}^{p_1 \cdots p_n}$ for p_j small primes. The zero lemma we alluded before shows that for some $l = p_1 \cdots p_n$

the obstruction index $\omega_{\mathbb{Q}}(\boldsymbol{\alpha}^l)$ is pathologically small than $\omega_{\mathbb{Q}}(\boldsymbol{\alpha})$. Unfortunately, it seems hard to find lower bound for $\omega_{\mathbb{Q}}(\boldsymbol{\alpha}^l)$ in terms of $\omega_{\mathbb{Q}}(\boldsymbol{\alpha})$. Thus, we cannot conclude easily the proof. To avoid this problem, we start again the whole construction replacing $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}^l$. To ensure that the process end at some moment, we need a cumbersome induction ("descent step").

The situation is quite similar in the original proof of the geometric result (theorem 4.4). We construct again an auxiliary function vanishing on V and then we extrapolate by proving that F must also vanish on $\ker[p_1 \cdots p_n]V$ for p_j small primes. We need again a variant of a zero lemma which use the fact that our set of translation (the union of $\ker[p_1 \cdots p_n]$) is actually an union of big subgroups. Using this new zero lemma we succeed to show that again for some $l = p_1 \cdots p_n$ the obstruction index $\omega_{\mathbb{Q}}([l]V)$ is pathologically small than $\omega_{\mathbb{Q}}(V)$. As in the arithmetic situation, we cannot conclude easily and we need again a cumbersome descent step.

Very recently in [Amo-Via 2008] we succeed to drastically simplify the proof of the geometric result. The new proof code the classical diophantine analysis in an inequality involving some parameters, the essential minimum of a subvariety of $\mathbb{G}_{\mathrm{m}}^n$ and two Hilbert's functions. The new key idea to decode the diophantine information is to use sharp estimates for the Hilbert function. The upper bound is a variant of the main result of [Cha 1988]. It is proved in [Amo-Dav 2003], lemma 2.5. The lower bound is a deep result of M. Chardin and P. Philippon [Cha-Phi 1999], corollary 3. Using these tools in [Amo-Via 2008] we prove the following result.

Theorem 4.5 *Let V be an irreducible subvariety of $\mathbb{G}_{\mathrm{m}}^n$ of codimension k which is not a translate of a torus. Let*

$$\theta_0 = \delta_0(V)\big(27n^2 \log(n^2\delta_0(V))\big)^{kn}.$$

Then $V(\theta_0^{-1})$ is contained in a hypersurface Z of degree at most θ_0 which does not contain V. In particular, $\hat{\mu}^{\mathsf{ess}}(V) \geq \theta_0^{-1}$.

In this theorem $\delta_0(V)$ is the minimal degree δ_0 such that V is a component of an intersection of hypersurfaces of degree $\leq \delta_0$. In the sequel we also need a third invariant, closely relate to ω and δ_0. Let V be a (not necessarily irreducible) variety of $\mathbb{G}_m^n$. We define $\delta(V)$ as the minimal degree δ such that V is, as a set, intersection of hypersurfaces of degree $\leq \delta$. If V is irreducible, then

$$\omega(V) \leq \delta_0(V) \leq \delta(V) \leq \deg(V) \leq \delta_0(V)^{\mathrm{codim}(V)}. \qquad (4.2.1)$$

The first three inequalities are immediate. The last one follows from [Phi 1986] proposition 3.3 with $p = 1$, $N_1 = n$ and $D_1 = \delta_0(V)$.

A priori, it is difficult to compare theorem 4.5 with theorem 4.4. On the one hand, in theorem 4.5 we do not assume that V is transverse, but only that V is not a translate of a torus. On the other hand, the bound in theorem 4.5 depends on $\delta_0(V)$ which could potentially be equal to the degree of V, while

$$\omega(V) \leq n \deg(V)^{1/\operatorname{codim}(V)}.$$

A new reduction process applied to *each* variety involved, allows us to deduce from theorem 4.5 a new simple proof of theorem 4.4.

In the next sections we shall describe in detail this new method and we sketch a new proof of theorem 4.4. Hopefully, this method also apply in the arithmetic case: this is still a work in progress.

4.3 Hilbert function

Let $I \subset \overline{\mathbb{Q}}[\mathbf{x}]$ be a homogeneous reduced ideal. For $\nu \in \mathbb{N}$ we denote by $H(I;\nu)$ the Hilbert function $\dim[\overline{\mathbb{Q}}[\mathbf{x}]/I]_\nu$. Let T be a positive integer and $I \subset \overline{\mathbb{Q}}[\mathbf{x}]$ be a homogeneous reduced ideal. We denote by $I^{(T)}$ the T-symbolic power of I, *i.e.* the ideal of polynomials vanishing on the variety defined by I with multiplicity at least T. Let V be a variety of $\mathbb{G}_{\mathrm{m}}^n$, defined in $\mathbb{P}_n$ by a reduced ideal I. By abuse of notations, we set $H(V;\nu) = H(I;\nu)$ and $H(V,T;\nu) = H(I^{(T)};\nu)$.

We recall that *for large* ν the Hilbert function $H(V;\nu)$ is actually a polynomial of degree $\dim(V)$ and leading coefficient $\deg(V)/\dim(V)!$. To prove theorem 4.5 we need a sharp lower bound for the Hilbert Function. This is a deep result proved by M. Chardin and P. Philippon ([Cha-Phi 1999], corollary 3). We formulate here a simplified statement.

Theorem 4.6 *Let $V \subseteq \mathbb{P}_n$ be an equidimensional variety of dimension d and codimension $k = n - d$. Define $m = k\big(\delta_0(V) - 1\big)$. Then, for any $\nu > m$, we have*

$$H(V;\nu) \geq \binom{\nu + d - m}{d} \deg(V).$$

We also need an upper bound for $H(V,T;\nu)$. The proposition below follows (see lemma 2.5 of [Amo-Dav 2003] for details) from a result of M. Chardin [Cha 1988].

Proposition 4.7 *Let $V \subseteq \mathbb{P}_n$ be a reduced equidimensional variety of dimension d and codimension $k = n - d$. Let ν, T be positive integers. Then*

$$H(V, T; \nu) \leq \binom{T-1+k}{k}\binom{\nu+d}{d}\deg(V).$$

4.4 Diophantine analysis: encoding the information

The original proof of theorem 4.4 relies on the fact that V is p-adically close to $\boldsymbol{\zeta} V$ for all p-torsion points $\boldsymbol{\zeta}$ and for all "small" primes p. This follows by the following simple fact. For any p-root of unity ζ and for any place $v \mid p$ we have $|1-\zeta|_v \leq p^{-1/p}$. But also *all* the translates of V by p-torsion points are p-adically close to each other. This gives a first simplification: we replace the vanishing principle used in [Amo-Dav 2003] by a *symmetric* vanishing principle. For technical reasons, it is more convenient to use an interpolation determinant than an auxiliary function.

Lemma 4.8 *Let ν, T be positive integers. Let $W = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_L\} \subseteq \mathbb{G}_{\mathrm{m}}^n(\mathbb{C})$ be a finite set and $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_L$ be multi-indexes of weight ν. Define*

$$T_0 := \big(L - H(W, T; \nu)\big)T.$$

Then the multi-homogeneous polynomial

$$F(\mathbf{x}_1, \ldots, \mathbf{x}_L) = \det(\mathbf{x}_i^{\boldsymbol{\lambda}_j})_{1 \leq i,j \leq L}.$$

vanishes on $(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_L) \in W^L$ with multiplicity at least T_0.

Proof. We assume $\boldsymbol{\lambda}_i \neq \boldsymbol{\lambda}_j$ for $i \neq j$. Otherwise F is identically zero and the proof is clear. If $H(W, T; \nu) \geq L$ the assertion is obvious. Assume $H(W, T; \nu) < L$ and let $L_0 = L - H(W, T; \nu)$. Then, for $k = 1, \ldots, L_0$ there exist linearly independent polynomials

$$G_k = \sum_{j=1}^{L} g_{kj} \mathbf{x}^{\boldsymbol{\lambda}_j}$$

vanishing on W with multiplicity $\geq T$. By elementary operations we replace the last L_0 columns of the matrix $(\mathbf{x}_i^{\boldsymbol{\lambda}_j})$ by

$${}^{\tau}\big(G_k(\mathbf{x}_1), \ldots, G_k(\mathbf{x}_L)\big), \qquad k = 1, \ldots, L_0.$$

Let $F'(\mathbf{x}_1, \dots, \mathbf{x}_L)$ be the determinant of the new matrix. Then

$$F'(\mathbf{x}_1, \dots, \mathbf{x}_L) = cF(\mathbf{x}_1, \dots, \mathbf{x}_L)$$

for some $c \in \mathbb{C}^*$. The polynomials G_k vanish on W with multiplicity $\geq T$. Developing $F'(\mathbf{x}_1, \dots, \mathbf{x}_L)$ with respect to the last L_0 columns we see that $F'(\mathbf{x}_1, \dots, \mathbf{x}_L)$ vanishes on $(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_L) \in \mathbb{P}_n(\mathbb{C})^L$ with multiplicity $\geq T_0$.

$\square$

Proposition 4.9 *Let ν, T be positive integers and let p be a prime number. Let V be a subvariety of $\mathbb{G}_{\mathrm{m}}^n$. Then*

$$\hat{\mu}^{\mathsf{ess}}(V) \geq \left(1 - \frac{H(V,T;\nu)}{H(\ker[p]V;\nu)}\right)\frac{T\log p}{p\nu} - \frac{n}{2\nu}\log(\nu+1).$$

Proof. Choose any real ε such that $\varepsilon > \hat{\mu}^{\mathsf{ess}}(V)$. For semplicity we define $S = V(\varepsilon)$. Then S is Zariski dense in V. We consider the (potentially infinite) matrix

$$(\boldsymbol{\beta}^{\boldsymbol{\lambda}})_{\substack{\boldsymbol{\beta}\in\ker[p]S \\ |\boldsymbol{\lambda}|\leq\nu}}$$

of rank $L = H(\ker[p]V;\nu)$. We select $\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_L \in \ker[p]S$ and $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_L$ with $|\boldsymbol{\lambda}_j| \leq \nu$ such that

$$\det(\boldsymbol{\beta}_i^{\boldsymbol{\lambda}_j})_{i,j=1,\dots,L} \neq 0$$

Consider $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_L \in S$ such that $\boldsymbol{\beta}_j \in \ker[p]\boldsymbol{\alpha}_j$. We set

$$F(\mathbf{x}_1, \dots, \mathbf{x}_L) = \det(\mathbf{x}_i^{\boldsymbol{\lambda}_j})_{i,j=1,\dots,L} \in \mathbb{Z}[\mathbf{x}_1, \dots, \mathbf{x}_L].$$

It follows $F(\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_L) \neq 0$. By lemma 4.8, F vanishes on $(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_L)$ with multiplicity at least

$$T_0 := \big(L - H(\{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_L\}, T;\nu)\big)T \geq \big(L - H(V,T;\nu)\big)T.$$

Let v be a place dividing p. Using the inequality $|1-\zeta_p|_v \leq p^{-1/(p-1)}$ we get

$$|\boldsymbol{\alpha}_{j,k} - \boldsymbol{\beta}_{j,k}|_v \leq p^{-1/(p-1)}|\boldsymbol{\alpha}_{j,k}|_v$$

for $j = 1, \dots, L$ and $k = 1, \dots, n$. Thus, by Taylor expansion of F around $(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_L)$

$$|F(\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_L)|_v \leq p^{-T_0/(p-1)} \prod_{j=1}^{L} |\boldsymbol{\alpha}_k|_v^{\nu}.$$

where $|\boldsymbol{\alpha}_k|_v = \max\{1, |\alpha_{j,1}|_v, \dots, |\alpha_{j,n}|_v\}$.

By the ultrametric inequality for $v \nmid \infty$ and by the Hadamard inequality for $v \mid \infty$ we obtain that, for an arbitrary place v,

$$|F(\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_L)|_v \leq \begin{cases} |\boldsymbol{\beta}_k|_v, & \text{if } v \nmid \infty \\ L^{L/2}|\boldsymbol{\beta}_k|_v, & \text{if } v \mid \infty. \end{cases}$$

Since $\boldsymbol{\alpha}_k$ is a translate of $\boldsymbol{\beta}_k$ by a torsion point, $|\boldsymbol{\beta}_k|_v = |\boldsymbol{\alpha}_k|_v$. We apply the Product formula:

$$0 \leq -\frac{T_0 \log p}{p-1} + \frac{L}{2} \log L + \nu \sum_{j=1}^{L} h(\boldsymbol{\alpha}_j) \leq -\frac{T_0 \log p}{p} + \frac{L}{2} \log L + \nu L \varepsilon.$$

Moreover $L \leq (\nu + 1)^n$. Thus

$$\varepsilon \geq \frac{T_0 \log p}{L p \nu} - \frac{n}{2\nu} \log(\nu + 1).$$

Taking the limit for ε which tends to $\hat{\mu}^{\mathsf{ess}}(V)$ we obtain the wished bound.

$\square$

4.5 Diophantine analysis: decoding the information

Let V be an irreducible variety of $\mathbb{G}_{\mathrm{m}}^n \subseteq \mathbb{P}_n$ and let p be a prime number. In order to prove theorem 4.5, we shall apply theorem 4.6 to $V' = \ker[p]V$. Therefore, we need an upper bound for $\delta_0(V')$ and a lower bound for $\deg(V')$. These bounds are the object of the following technical lemma whose proof is omitted:

Lemma 4.10 *Let V be an irreducible variety of $\mathbb{G}_{\mathrm{m}}^n$. Let $G \subseteq \mathbb{G}_{\mathrm{m}}^n$ be a finite group and t be the number of irreducible components of $V' = GV$. Then*

$$\deg(V') = t \deg(V) \quad \textit{and} \quad \delta_0(V') \leq t \delta_0(V).$$

A technical part of the proof of theorem 4.5 is devoted to the computation of the constant involved. In order to simplify things and to improve the comprehension, we restrict ourself to sketch a proof the following non explicit version of this theorem:

Theorem 4.11 *Let V be an irreducible subvariety of $\mathbb{G}_{\mathrm{m}}^n$ of codimension k which is not a translate of a torus. Then, there exists a constant $c(n)$ with the following property. Let*

$$\theta_0 = \delta_0(V)\big(c(n)\log(n^2\delta_0(V))\big)^{kn}.$$

Then $V(\theta_0)$ is contained in a hypersurface Z of degree at most θ_0 which does not contain V. In particular, $\hat{\mu}^{\mathsf{ess}}(V) \geq \theta_0^{-1}$.

Proof. Let $d = n - k = \dim(V)$ and $\delta_0 = \delta_0(V)$. By (3.1.1) and (4.2.1),

$$\deg(\mathrm{Stab}(V)) \leq \deg(V)^{\dim(V)+1} \leq \delta_0^{nk}.$$

Thus

$$\log[\mathrm{Stab}(V) : \mathrm{Stab}(V)^0] \leq nk\log(\delta_0). \tag{4.5.1}$$

Let[3]

$$N \approx (\log(n^2\delta_0))^k.$$

If for any prime p with $N/2 \leq p \leq N$ we have $p \mid [\mathrm{Stab}(V) : \mathrm{Stab}(V)^0]$ then

$$\log[\mathrm{Stab}(V) : \mathrm{Stab}(V)^0] \geq \sum_{N/2\leq p\leq N} \log p \gg N. \tag{4.5.2}$$

Equations (4.5.1) and (4.5.2) are not consistent. We conclude that there exists a prime $p \nmid [\mathrm{Stab}(V) : \mathrm{Stab}(V)^0]$ satisfying $N/2 \leq p \leq N$. By the remark at the en of section 3.1, the variety $V' = \ker[p]V$ is a union of $p^{\mathrm{codim}(\mathrm{Stab}\,V)}$ distinct components which are translated of V by a p-torsion point. Since V is not a translate of a torus,

$$k + 1 \leq \mathrm{codim}(\mathrm{Stab}\,V) \leq n.$$

By lemma 4.10 ii),

$$\deg(V') \geq p^{k+1}\deg(V) \qquad \text{and} \qquad \delta_0(V') \leq p^n\delta_0. \tag{4.5.3}$$

[3]The symbols $\approx$, $\ll$ and $\gg$ have the same meaning as in section 2.5

We shall apply proposition 4.7 to V and theorem 4.6 to V'. As in the statement of theorem 4.6, let $m = k\delta_0(V')$. The upper bound for $\delta_0(V')$ in (4.5.3) gives

$$m \leq kp^n\delta_0.$$

Choose

$$\theta_0 = md + m \quad \text{and} \quad T \approx p^{1+1/k}.$$

We have

$$\theta_0 \ll \log(n^2\delta_0)^{kn}\delta_0.$$

Let W be the Zariski closure of the set $V(\theta_0^{-1})$ and let $W' = \ker[p]W$. Then,

$$\hat{\mu}^{\mathsf{ess}}(W) \leq \theta_0^{-1}. \tag{4.5.4}$$

Furthermore, as $W \subseteq V$ and $W' \subseteq V'$,

$$H(W, T; \theta_0) \leq H(V, T; \theta_0) \quad \text{and} \quad H(W'; \theta_0) \leq H(V'; \theta_0).$$

We shall show that $H(W'; \theta_0) < H(V'; \theta_0)$. Assume by contradiction that

$$H(W'; \theta_0) = H(V'; \theta_0).$$

Apply theorem 4.6 to the variety V' and proposition 4.7 to the variety V. Then

$$\frac{H(W, T; \theta_0)}{H(W'; \theta_0)} \leq \frac{H(V, T; \theta_0)}{H(V'; \theta_0)} \leq \frac{\binom{T-1+k}{k}\binom{\theta_0+d}{d}\deg(V)}{\binom{\theta_0+d-m}{d}\deg(V')}.$$

By the lower bound for $\deg(V')$ given in (4.5.3) and by the choice of θ_0,

$$\frac{H(W, T; \theta_0)}{H(W'; \theta_0)} \leq \frac{\binom{T-1+k}{k}\binom{\theta_0+d}{d}}{\binom{\theta_0+d-m}{d}p^{k+1}} \ll \frac{T^k}{p^{k+1}}\left(1 + \frac{m}{\theta_0 - m}\right)^d < \frac{1}{2},$$

say. By proposition 4.9 (with V replaced by W).

$$\begin{aligned}\hat{\mu}^{\mathsf{ess}}(W) &\geq \left(1 - \frac{H(W, T; \theta_0)}{H(W'; \theta_0)}\right)\frac{T\log p}{p\theta_0} - \frac{n}{2\theta_0}\log(\theta_0 + 1) \\ &\geq \left(\frac{T\log p}{2p} - \frac{n}{2}\log(\theta_0 + 1)\right)\theta_0^{-1}.\end{aligned}$$

We have

$$\frac{T\log p}{2p} \gg p^{1/k}\log p \gg \log(n^2\delta_0)$$

and

$$\log(\theta_0 + 1) \ll \log(n^2\delta_0).$$

Thus, choosing in an appropriate way the implicit constants in the parameters,

$$\hat{\mu}^{\mathsf{ess}}(W) > \theta_0^{-1}.$$

This contradicts (4.5.4) and shows that

$$H(W'; \theta_0) < H(V'; \theta_0).$$

Equivalently, there exists a homogeneous polynomial F of degree $\leq \theta_0$ vanishing on W' but not on V'. Replacing $F(\mathbf{x})$ by $F(\boldsymbol{\zeta}\mathbf{x})$ for a suitable $\boldsymbol{\zeta} \in \ker[p]$, we can assume $F \neq 0$ on V (recall that W' is invariant by translation by p torsion points).

□

4.6 From $\delta_0(V)$ to $\omega(V)$.

In this section we deduce from theorem 4.5 a new simple proof of an improved and explicit version of the main result of [Amo-Dav 2003] (theorem 4.4). We follows closely [Amo-Via 2008].

Theorem 4.12 *Let $V \subset \mathbb{G}_{\mathrm{m}}^n$ be a transverse variety of codimension k. Then*

$$\hat{\mu}^{\mathsf{ess}}(V) \geq \omega(V)^{-1}\big(300n^5\log(n^2\omega(V))\big)^{-nk^2}.$$

Proof. We simply denote $\omega = \omega(V)$ and

$$\theta = \omega\big(200n^5\log(n^2\omega)\big)^{nk^2}.$$

We assume by contradiction that the lower bound for the essential minimum does not hold or, equivalently, that $V(\theta^{-1})$ is Zariski dense in V.

For $r \in \{1, \ldots, k\}$ we define

$$D_r = \omega\big(200n^5\log(n^2\omega)\big)^{rkn}.$$

Since $r \leq k$, we have $D_r \leq \theta$.

Using an inductive process on r, we are going to construct a chain of varieties

$$X_1 \supseteq \cdots \supseteq X_r \supseteq \cdots \supseteq X_k$$

satisfying:

Claim 4.13

i) $V \subseteq X_r$.

ii) Each irreducible component of X_r *containing* V *has codimension* $\geq r$.

iii) $\delta(X_r) \leq D_r$.

Theorem 4.12 is proved if we show that claim 4.13 for $r = k$. Indeed, for $r = k$, claim 4.13 imply that V is an irreducible component of X, thus $\delta_0(V) \leq \delta(X) \leq D_k = \theta$. Then, in view of theorem 4.5, $V(\theta^{-1}) \subset V(\delta_0(V)^{-1})$ can not be dense. This gives a contradiction.

We now define X_r and prove claim 4.13 by induction on r.

• For $r = 1$, we choose X_1 to be a geometrically irreducible hypersurface containing V of degree ω. This is possible by definition of ω. Furthermore, as X_1 is a hypersurface, $\delta_0(X_1) = \delta(X_1) = \deg X_1 = \omega$. Assertions i), ii) and iii) clearly hold.

• We assume that claim 4.13 holds for some $r \in \{1, \dots, k-1\}$ and we prove that it still holds for $r+1$. Let X_r be a Zariski closed set satisfying conditions i), ii) and iii) for r. Since $V \subseteq X_r$ there exists at least one irreducible component of X_r which contains V. We distinguish the irreducible components of X_r which contain V (and by ii) they all have codimension $\geq r$) and the irreducible components of X_r which do not contain V. Let $1 \leq s \leq t$ be integers and let $W_1, \dots, W_s, W_{s+1}, \dots, W_t$ be irreducible components such that

$$X_r = W_1 \cup \cdots \cup W_s \cup W_{s+1} \cup \cdots \cup W_t$$

and

I) for $j = 1, \dots, s$, $V \subseteq W_j$ and $r \leq \operatorname{codim}(W_j) \leq k$.

II) for $j = s+1, \dots, t$, $V \not\subseteq W_j$.

Let $j \in \{1, \dots, s\}$. Since $\delta(X_r) \leq D_r$, the variety W_j is an irreducible component of an intersection of hypersurfaces of degree $\leq D_r$. Thus $\delta_0(W_j) \leq D_r$. Moreover V is transverse and $V \subseteq W_j$. Thus W_j is not a translate of a subtorus. Let

$$\theta_0 = D_r \Big(27n^2 \log\big(n^2 D_r\big)\Big)^{kn}.$$

By theorem 4.5 the set $W_j(\theta_0^{-1})$ is contained in a hypersurface Z_j which does not contain W_j and $\deg Z_j \le \theta_0$. Furthermore

$$\theta_0 \ll \delta\big(\log(n^2\delta)\big)^{rk_0n+k_0n} \ll D_{r+1}$$

and indeed a computation shows that $\theta_0 \le D_{r+1}$. Since $V \subseteq W_j$ and $\theta_0 \le D_{r+1} \le \theta$, we have $V(\theta^{-1}) \subseteq W_j(\theta_0^{-1})Z_j$. By assumption $V(\theta^{-1})$ is Zariski dense in V. Thus, $V \subseteq Z_j$ for $j = 1, \dots, s$ and

$$V \subseteq \bigcap_{j=1}^{s} Z_j.$$

Let

$$X_{r+1} = X_r \cap Z_1 \cap \dots \cap Z_s.$$

Then $V \subseteq X_{r+1}$. Recall that $\deg Z_j \le \theta_0 \le D_{r+1}$. Then $\delta(X_{r+1}) \le \max\{\delta(X_r), D_{r+1}\} \le \max\{D_r, D_{r+1}\} = D_{r+1}$.

We decompose

$$X_{r+1} = W_1' \cup \dots \cup W_s' \cup W_{s+1}' \cup \dots \cup W_t',$$

where $W_j' = W_j \cap Z_1 \cap \dots \cap Z_s$.

Let $j \in \{s+1, \dots, t\}$. Since $V \not\subseteq W_j$, the variety V is not contained in any irreducible component of W_j'.

For $j \in \{1, \dots, s\}$, $W_j \not\subseteq Z_j$. Thus every irreducible component of W_j' has codimension $> \operatorname{codim}(W_j) + 1 \ge r + 1$.

We conclude that X_{r+1} satisfies claim 4.13 for $r + 1$.

□

4.7 Some more conjectures and results on hypersurfaces

Let V be a hypersurface in $\mathbb{G}_{\mathrm{m}}^n$. Then the stabilizer of V has dimension s if and only if there exist a monoidal morphism $\varphi\colon \mathbb{G}_{\mathrm{m}}^n \to \mathbb{G}_{\mathrm{m}}^{n-s}$ and a hypersurface W in $\mathbb{G}_{\mathrm{m}}^{n-s}$ such that $V = \varphi^{-1}(W)$. In this case $\hat{h}(V) = \hat{h}(W)$, as we have remarked in section 3.1.

Conjecture 4.3 and Zhang's inequality (4.1.1) predict

$$\hat{h}(V) \ge c(n) > 0 \tag{4.7.1}$$

for hypersurfaces with stabilizer of dimension $< n - 1$. By the remarks above this is equivalent to ask that (4.7.1) holds for any $n \ge 2$ and for any hypersurface $V \subset \mathbb{G}_{\mathrm{m}}^n$ with discrete stabilizer.

We formulate a more optimistic conjecture.

Conjecture 4.14 *There exists a positive constant c such that for any integer $n \geq 2$ and for any irreducible hypersurface $V \subset \mathbb{G}_{\mathrm{m}}^n$ with discrete stabilizer we have $\hat{h}(V) > c$.*

In [Amo 2008] we shows that any eventual counterexample to this conjecture must have a very high degree with respect to n.

Theorem 4.15 *Let $n \in \mathbb{N}$ and let $V \subset \mathbb{G}_{\mathrm{m}}^n$ be a hypersurface with discrete stabilizer and such that*

$$\deg(V) \leq 3^{2^n} .$$

Then

$$\hat{h}(V) \geq \frac{1}{23} .$$

The proof of this theorem follows from a multi-homogeneous version of proposition 4.9.

One can formulate even more optimistic conjectures. Following Smyth ([Smy 1981]) we say that an irreducible polynomial

$$f = \sum_{\boldsymbol{\lambda}} f_{\boldsymbol{\lambda}} \mathbf{x}^{\boldsymbol{\lambda}} \in \mathbb{Z}[\mathbf{x}]$$

has dimension n if the convex-hull of the set $\{\boldsymbol{\lambda} \text{ s.t. } f_{\boldsymbol{\lambda}} \neq 0\}$ has dimension n. It is easy to see that this equivalent to require that the irreducible components of $\{f = 0\}$ have discrete stabilizer. In [Boy 1981], Boyd asked whether the function

$$m(n) = \inf\{M(f) \text{ s.t. } f \in \mathbb{Z}[\mathbf{x}] \text{ is irreducible and } \dim f = n\}$$

tends to infinity with n. This could suggest:

Conjecture 4.16 *There exists a positive function $c(n)$ which tends to infinity and such that for any $n \in \mathbb{N}$ and for any irreducible hypersurface $V \subset \mathbb{G}_{\mathrm{m}}^n$ with discrete stabilizer we have $\hat{h}(V) > c(n)$.*

Concerning this problem, the best known sequence of polynomials is the simplest one: $f_n(x) = x_1 + \cdots + x_n$. For these polynomials we have

$$\log M(f_n) \sim \frac{1}{2} \log n$$

(see [Smy 1981]). It would be of great interest to find non-trivial examples.

4.8 Localization of small points

Let V be a non-torsion variety of $\mathbb{G}_m^n$ and define

$$V^* = V \backslash \bigcup_{\substack{B \subseteq V \\ B \text{ torsion}}} B.$$

By the former Manin-Mumford conjecture, V^* is a Zariski open set, since $V \backslash V^*$ is a finite union of torsion varieties.

As mentioned in the introduction, an equivalent version of theorem 1.2 says that the height on $V^*(\overline{\mathbb{Q}})$ is bounded from below by a positive quantity:

$$\hat{\mu}^*(V) = \inf_{\boldsymbol{\alpha} \in V^*} \hat{h}(\boldsymbol{\alpha}) > 0.$$

Remark that obviously $\hat{\mu}^*(V) \leq \hat{\mu}^{\mathsf{ess}}(V)$. Hence one could hope, in analogy to conjecture 4.1, that

$$\hat{\mu}^*(V) \geq \frac{c(n)}{\omega_{\mathbb{Q}}(V)}$$

for some constant $c(n) > 0$. This lower bound is false, as the following example shows. Let α_k be a sequence of algebraic numbers whose height is positive and tends to zero as $k \to +\infty$. Let us consider

$$V_k = \{(\alpha_k, x_2, x_3) \in \mathbb{G}_m^3 \text{ s.t. } \alpha_k^2 + \alpha_k^3 - x_2 - x_3 = 0\}.$$

One checks that V_k is not torsion, the height of $\boldsymbol{\alpha}_k = (\alpha_k, \alpha_k^2, \alpha_k^3) \in V_k \backslash V_k^*$ tends to zero and $\omega_{\mathbb{Q}}(V) \leq 3$, since $V_k \subseteq \{x_1^2 + x_1^3 - x_2 - x_3 = 0\}$.

Let K be any subfield of $\overline{\mathbb{Q}}$; we let $\delta_K(V)$ be the minimum integer δ such that V is the intersection of hypersurfaces $Z_1, \ldots, Z_r$ defined over K and of degree $\leq \delta$. Thus $\delta_{\overline{\mathbb{Q}}} = \delta$.

In [Amo-Dav 2004] we formulate the following conjecture:

Conjecture 4.17 *Let V be a non-torsion variety of $\mathbb{G}_m^n$; then there exists a constant $c(n) > 0$ such that*

$$\hat{\mu}^*(V) \geq \frac{c(n)}{\delta_{\mathbb{Q}}(V)}\ .$$

In [Amo-Dav 2004] we prove conjecture 4.17 up to a logarithmic factor.

Theorem 4.18 *Let V be a non-torsion variety of $\mathbb{G}_m^n$; then there exist two positive constants $c(n)$ and $\kappa(n)$ such that*

$$\hat{\mu}^*(V) \geq \frac{c(n)}{\delta_{\mathbb{Q}}(V)} (\log 3\delta_{\mathbb{Q}}(V))^{-\kappa(n)}.$$

Theorem 4.18 follows by an inductive argument from the following generalization of theorem 4.2.

Theorem 4.19 *Let $\boldsymbol{\alpha} \in \mathbb{G}_m^n$ et let K be a cyclotomic extension. Let also $\boldsymbol{\alpha} \in \mathbb{G}_m^n$. Then there exist three positive constants $c(n)$, $\kappa(n)$ and $\lambda(n)$ such that if*

$$\hat{h}(\boldsymbol{\alpha}) < \frac{c(n)}{\omega_K(\boldsymbol{\alpha})}\Big(\log(3[K:\mathbb{Q}]\omega_K(\boldsymbol{\alpha}))\Big)^{-\kappa(n)},$$

then $\boldsymbol{\alpha}$ belongs to a variety B defined over $\mathbb{Q}$ which is an union of torsion variety and such that

$$(\deg B)^{1/\mathrm{codim}(B)} \leq c(n)^{-1}\,\omega_K(\boldsymbol{\alpha})\,\Big(\log(3[K:\mathbb{Q}]\omega_K(\boldsymbol{\alpha}))\Big)^{\mu(n)}.$$

In this theorem $\omega_K(\boldsymbol{\alpha})$ is the minimum degree of a hypersurface defined over K containing $\boldsymbol{\alpha}$. Theorem 4.19 can be viewed as a partial generalization of the "relative" Dobrowolski's theorem of section 2.6.

Very recently Delsinne ([Del 2008]) prove a full generalization of the relative Dobrowolski theorem, removing the extra factor $[K:\mathbb{Q}]$ in the previous formula.

We make a similar analysis in the geometric case. Let V be a tranverse subvariety of $\mathbb{G}_m^n$ and define, as in [Bom-Zan 1995],

$$V^0 = V \setminus \bigcup_{B \subseteq V} B.$$

where the union is now on the set of translates B of tori of dimension 1. Again $V \setminus V^0$ is an open set (see [Bom-Zan 1995] and [Sch 1996]); Bombieri and Zannier prove that, outside a finite set, the height on V^0 is bounded from below by a positive quantity depending only on the ideal of definition of V and not on its field of definition. More precisely, Schmidt [Sch 1996] proves that the set of points $\boldsymbol{\alpha} \in V^0$ such that $\hat{h}(\boldsymbol{\alpha}) < q^{-1}$ is finite, of cardinality $\leq q$, where

$$q = \exp\left(n^{\delta_{\overline{\mathbb{Q}}}(V)^n}\right).$$

David and Philippon (see [Dav-Phi 1999]) improve this result. They show that the above assertion still hold choosing:

$$q = (\deg(V)\log\deg(V))^{4^{\dim(V)}}.$$

More precisely, they prove that the set $V(q^{-3/4})$ is contained in a finite union of translates B_j of tori such that $B_j \subseteq V$ and $\sum \deg(B_j) \leq q$. In [Amo-Dav 2006], the following lower bound is conjectured.

Conjecture 4.20 *Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be an irreducible variety. There exists $c(n) > 0$ such that, for all but finitely many $\boldsymbol{\alpha} \in V^0(\overline{\mathbb{Q}})$,*

$$h(\boldsymbol{\alpha}) \geq c(n)\delta(V)^{-1}. \tag{4.8.1}$$

More precisely, there exist $c_1(n)$, $c_2(n) > 0$ and $l \in \mathbb{N}$ such that

$$V\left(c_1(n)\delta(V)^{-1}\right) \subseteq \bigcup_{j=1}^{l} B_j$$

where the $B_j \subseteq V$ are translates of tori and

$$\sum_{j=1}^{l} \deg(B_j) \leq c_2(n)\delta(V)^n. \tag{4.8.2}$$

Using a variant of the main result of [Amo-Dav 2003] (theorem 4.4) and an additional induction, in [Amo-Dav 2006] we deduce a bound of the type (4.8.1) up to a logarithmic factor. More precisely, in theorem 1.5 of this article, we prove[4] that, for all but finitely many $\boldsymbol{\alpha} \in V^0(\overline{\mathbb{Q}})$,

$$\hat{h}(\boldsymbol{\alpha}) \geq c(n)\delta(V)^{-1}\left(\log(3\delta(V))\right)^{-\lambda(n-1)},$$

where $c(n) > 0$ and $\lambda(k) = \left(9(3k)^{(k+1)}\right)^k$.

Without any additional effort, the method of the proof of theorem 4.12 can be easily modified to get the following result (see [Amo-Via 2008]):

Theorem 4.21 *Let $V_0 \subseteq V_1$ be subvarieties of $\mathbb{G}_{\mathrm{m}}^n$ of codimensions k_0 and k_1 respectively. Assume that V_0 is irreducible. Let*

$$\theta = \delta(V_1)\left(200n^5 \log(n^2\delta(V_1))\right)^{(k_0-k_1+1)k_0 n}.$$

Then,

[4]In that theorem $\delta(V)$ is defined as the minimal degree δ such that V is, as a set, a component of the intersection of hypersurfaces of degree $\leq \delta$. Unfortunately there is a mistake in the proof. This can be corrected, defining $\delta(V)$ as we have done here.

- *either there exists a translate B of a torus such that $V_0 \subseteq B \subseteq V_1$ and $\delta_0(B) \leq \theta$,*

- *or there exists a hypersurface Z of degree at most θ such that $V_0 \not\subseteq Z$ and $V_0(\theta^{-1}) \subseteq Z$.*

Note that theorem 4.12 becomes a corollary of this theorem (choose $V_0 = V$ and V_1 an irreducible hypersurface of degree $\omega(V)$ containing V). Moreover, theorem 4.12 immediately implies an improved and explicit version of theorem 1.5 of [Amo-Dav 2006]. Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be an irreducible variety of codimension k. Define

$$\theta = \delta(V)\big(200n^5 \log(n^2\delta(V))\big)^{k^2 n}.$$

Let V_0 be one of the finitely many irreducible components of

$$W = \overline{V(\theta^{-1})}.$$

Then $\overline{V_0(\theta^{-1})} = V_0$. Apply theorem 4.21 to the component V_0 and to $V_1 = V$. Then, V_0 is contained in a translate B of a torus such that $B \subseteq V$ and $\delta_0(B) \leq \theta$. Varing V_0 over all components of W, we conclude that $W \subseteq \cup B_j$ where $B_j \subseteq V$ are translates of tori with $\delta_0(B_j) \leq \theta$.

We remark that the method of [Amo-Dav 2006] can not produce a bound of the shape (4.8.2) for the sum of the degrees of the translates. A close inspection of their proof shows indeed that one can only bound the degree of *each* translate B by a constant (depending on n) times $\delta(V)^{2^{\mathrm{codim}(B)}}$.

A refined induction based on theorem 4.21 leads us to a complete quantitative description of the small points of a variety V. The following is a simplified version of the main result of [Amo-Via 2008].

Theorem 4.22 *Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be an irreducible variety of codimension k. Let*

$$\theta(V) = \delta(V)\big(200n^5 \log(n^2\delta(V))\big)^{(n-k)n(n-1)}.$$

Then,

$$\overline{V(\theta^{-1})} = G_k \cup \cdots \cup G_n$$

where G_j is either the empty set or a finite union of translates $B_{j,i}$ of tori of codimension j such that $\delta_0(B_{j,i}) \leq \theta$. Moreover

$$\theta^{n-k} \deg G_k + \cdots + \theta \deg G_{n-1} + \deg G_n \leq \theta^{n-k} \deg V \leq \theta^n.$$

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