SIMULTANEOUS APPROXIMATION TO VALUES OF THE EXPONENTIAL FUNCTION OVER THE ADELES

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Abstract. We show that Hermite's approximations to values of the exponential function at given algebraic numbers are nearly optimal when considered from an adelic perspective. We achieve this by taking into account the ratio of these values whenever they make sense in the various completions (Archimedean or p-adic) of a number field containing these algebraic numbers.

1. Introduction

We know by Euler that the number e admits a continued fraction expansion consisting of intertwined arithmetic progressions

$$e = [2, (1, 2n, 1)_{n=1}^{\infty}] = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, \dots].$$

Euler, Sundman and Hurwitz also obtained similar expansions for the numbers $e^{2/m}$ where m is a non-zero integer [13, §§31-32]. Consequently, one may derive very good measures of rational approximations to these numbers (see for example the fully explicit results of Bundschuch [6, Satz 2], in the case where m is even). This is the aspect that interests us here. We propose the following heuristic explanation: the ratios $2/m$ with $m \in \mathbb{Z} \setminus \{0\}$ are the only non-zero rational numbers z for which the usual power series

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \tag{1.1}$$

converges only as a real number. Indeed, let p be a prime number and let $\mathbb{C}_p$ denote the completion of the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ for the p-adic absolute value of $\mathbb{Q}$ extended to $\overline{\mathbb{Q}}$, with $|p|_p = p^{-1}$. We know that, for $z \in \mathbb{C}_p$, the series (1.1) converges in $\mathbb{C}_p$ if and only if $|z|_p < p^{-1/(p-1)}$. In particular, for a rational number z, viewed as an element of $\mathbb{C}_p$, this series converges if and only if the numerator of z is divisible by p when $p \neq 2$, and by 4 when $p = 2$.

This phenomenon also extends to algebraic numbers. Indeed, let K be a number field, namely an algebraic extension of $\mathbb{Q}$ of finite degree. Then any absolute value on K induces the same topology on K as an absolute value coming from an embedding from K into $\mathbb{C}$ or

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into $\mathbb{C}_p$ for a prime number p. We say that such embeddings define the same place v of K if they induce the same absolute value on K denoted $|\ |_v$. We then denote by K_v the completion of K for this absolute value. When the place v comes from an embedding of K into $\mathbb{C}$, the place v is called Archimedean and we write $v \mid \infty$. Otherwise it is called ultrametric, and we write $v \mid p$ if it comes from an embedding of K into $\mathbb{C}_p$. When $\alpha \in K$ is non-zero, the series for e^α converges in each Archimedean completion of K but only in a finite number of ultrametric completions. In particular, when K admits a single Archimedean place, which happens when $K = \mathbb{Q}$ or when K is quadratic imaginary, then it may occur that e^α has a meaning only for this place. Then, we obtain the following estimate where $\mathcal{O}_K$ denotes the ring of integers of K.

Proposition 1.1. *Let $K \subset \mathbb{C}$ be the field $\mathbb{Q}$ or a quadratic imaginary extension of $\mathbb{Q}$, and let α be a non-zero element of K such that $|\alpha|_v \ge p^{-1/(p-1)}$ for each prime number p and each place v of K with $v \mid p$. Then, for any $x, y \in \mathcal{O}_K$ with $|x| > 1$, we have*

$$|x|\,|xe^\alpha - y| \ge c(\log|x|)^{-2g-1}$$

where g stands for the number of places v of K with $v \mid \infty$ or $|\alpha|_v \ne 1$, and where $c > 0$ is a constant depending only on α and K.

For example if $K = \mathbb{Q}(\sqrt{-2})$, we may take $\alpha = 2(1 \pm \sqrt{-2})/m$ where $m \in \mathcal{O}_K \setminus \{0\}$. If $K = \mathbb{Q}(\sqrt{-23})$, we may take $\alpha = (1 \pm \sqrt{-23})/(2m)$ where $m \in \mathcal{O}_K \setminus \{0\}$. We do not know what is the best possible exponent for $\log|x|$ in the above measure of approximation to e^α. Note that, in some cases, e^α admits a generalized continued fraction expansion similar to the one of e (with partial quotients in $\mathcal{O}_K$) but we do not consider this question here.

More generally, let $\alpha_1, \dots, \alpha_s$ be distinct elements of a number field $K \subset \mathbb{C}$. Lindemann-Weierstrass theorem [18] tells us that their exponentials $e^{\alpha_1}, \dots, e^{\alpha_s} \in \mathbb{C}$ are linearly independent over K and the classical proof, in all variants (see [11, Appendix]), is based on Hermite's approximations which we recall in the next section. Our goal is to show that these approximations are nearly optimal in the context of geometry of numbers over the adeles of K, when taking into account all places v of K and all pairs of indices i, j with $1 \le i < j \le s$ for which the series for $e^{\alpha_i - \alpha_j}$ converges in K_v. It is possible that this observation reflects a much wider property of the values of the exponential function.

For example the series for e^3 converges in $\mathbb{R}$ and in $\mathbb{Q}_3$ but not in any $\mathbb{Q}_p$ for a prime number $p \ne 3$. Then our approach leads to the following result.

Proposition 1.2. *For any integer $n \ge 1$, we define a convex body $\mathcal{C}_n$ of $\mathbb{R}^2$ and a lattice Λ_n of $\mathbb{R}^2$ by*

$$\begin{aligned}
\mathcal{C}_n &= \left\{ (x,y) \in \mathbb{R}^2\,; \quad |x| \le \frac{(2n)!}{n!3^{n/2}}\,, \quad |xe^3 - y| \le \left(\frac{3}{2}\right)^{2n} \frac{1}{n!3^{n/2}} \right\},\\
\Lambda_n &= \left\{ (x,y) \in \mathbb{Z}^2\,; \quad |xe^3 - y|_3 \le 3^{-n} \right\}.
\end{aligned}$$

For $i = 1, 2$, let $\lambda_i(\mathcal{C}_n, \Lambda_n)$ denote the i-th minimum of $\mathcal{C}_n$ with respect to Λ_n, that is the smallest $\lambda > 0$ such that $\lambda\mathcal{C}_n$ contains at least i elements of Λ_n which are linearly independent over $\mathbb{Q}$. Then we have

$$(cn^2)^{-1} \le \lambda_1(\mathcal{C}_n, \Lambda_n) \le \lambda_2(\mathcal{C}_n, \Lambda_n) \le cn^2,$$

for a constant $c > 1$ that does not depend on n.

Since $3^n\mathbb{Z}^2 \subset \Lambda_n$ and $\lambda_1(\mathcal{C}_n, \Lambda_n) \ge (cn^2)^{-1}$, one deduces that $\lambda_1(\mathcal{C}_n, \mathbb{Z}^2) \ge (cn^2 3^n)^{-1}$ for any integer $n \ge 1$. Consequently, for each $\epsilon > 0$, there exists a constant $c_\epsilon > 0$ such that

$$|x|\,|xe^3 - y| \ge c_\epsilon |x|^{-\epsilon}$$

for all $(x, y) \in \mathbb{Z}^2$ with $x \ne 0$. One may even derive slightly sharper estimates (see [6, Satz 1]). However, numerical computations described in Section 12 yield

$$|x|\,|xe^3 - y| \ge (3\,\log|x|\,\log\log|x|)^{-1} \quad \text{if} \quad 4 \le |x| \le 10^{500\,000}. \tag{1.2}$$

If true whenever $|x| \ge 4$, this would be slightly better than what we expect for almost all real numbers with respect to Lebesgue measure. More involved computations which we do not describe here even suggest the existence of a real number $g > 0$ such that

$$|x_1|\,|x_1e^3 - x_2|\,|x_1e^3 - x_2|_3 \ge (\log|x_1|)^{-g}$$

for any $(x_1, x_2, x_3) \in \mathbb{Z}^3$ with $|x_1|$ large enough. Finally, an important result of Baker [2] shows that if $\alpha_2, \dots, \alpha_s \in \mathbb{Q}$ are distinct non-zero rational numbers then, for each $\epsilon > 0$, there exists a constant $c_\epsilon > 0$ such that

$$|x_1|\,|x_1e^{\alpha_2} - x_2| \cdots |x_1e^{\alpha_s} - x_s| \ge c_\epsilon |x_1|^{-\epsilon}$$

for each $(x_1, \dots, x_s) \in \mathbb{Z}^s$ with $|x_1| \ne 0$. The properties of Hermite's approximations suggest that the right hand side $c_\epsilon|x_1|^{-\epsilon}$ in this inequality could be remplaced by $(\log|x_1|)^{-g}$ for a constant $g > 0$ depending only on $(\alpha_2, \dots, \alpha_s)$, when $|x_1|$ is large enough.

In this paper, $\mathbb{N}$ stands for the set of non-negative integers and $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ for the set of positive integers. A French translation is available on the arXiv server under the identifier arXiv:1905.00343 [math.NT].

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2. Statement of the main result

Let K be a number field, let $\mathcal{O}_K$ be its ring of integers, let $d = [K : \mathbb{Q}]$ be its degree over $\mathbb{Q}$, and let $s \in \mathbb{N}_+$. For any ultrametric place v of K, we denote by $\mathcal{O}_v = \{x \in K_v\,;\, |x|_v \le 1\}$ the ring of integers of K_v and by $d_v = [K_v : \mathbb{Q}_p]$ the local degree of K_v, where p stands for the prime number below v (notation $v \mid p$), namely the prime number p for which $|\ |_v$ extends the

p-adic absolute value on $\mathbb{Q}$. Following McFeat [12, §2.2], we denote by μ_v the Haar measure on K_v normalized so that $\mu_v(\mathcal{O}_v) = 1$. For an Archimedean place (notation $v \mid \infty$), we again denote by $d_v = [K_v : \mathbb{R}]$ the local degree of K_v, and define μ_v as the Lebesgue measure on K_v (this field is $\mathbb{R}$ or $\mathbb{C}$). We denote by r_1 (resp. r_2) the number of places $v \mid \infty$ with $d_v = 1$ (resp. $d_v = 2$), so that $d = r_1 + 2r_2$.

The ring of adeles of K is the product $K_{\mathbb{A}} = \prod_v K_v$ running over all places v of K, with the restricted topology. This is a locally compact ring that we equip with the Haar measure μ, product of the μ_v. We identify K as a subfield of $K_{\mathbb{A}}$ via the diagonal embedding. Then K becomes a discrete subgroup of $K_{\mathbb{A}}$ and, with the above normalization, we have

$$\mu(K_{\mathbb{A}}/K) = 2^{-r_2}|D(K)|^{1/2},$$

where $D(K)$ stands for the discriminant of K. By abuse of notation, we also write μ for the product measure of s copies of μ on $K_{\mathbb{A}}^s$. Similarly, for each place v of K, we also write μ_v for the product measure of s copies of μ_v on K_v^s. With our normalization of the absolute value on K_v, if $T\colon K_v^s \to K_v^s$ is a K_v-linear map and if $\mathcal{E}$ is a measurable subset of K_v^s, the set $T(\mathcal{E})$ is measurable with measure $\mu_v(T(\mathcal{E})) = |\det T|_v^{d_v}\mu_v(\mathcal{E})$.

2.1. **Minima of adelic convex bodies.** An *adelic convex body* of K^s is a product

$$\mathcal{C} = \prod_v \mathcal{C}_v \subset K_{\mathbb{A}}^s,$$

indexed by all places v of K, which satisfies the following properties:

(i) if $v \mid \infty$, then $\mathcal{C}_v$ is a *convex body* of K_v^s, namely a compact convex neighborhood of 0 in K_v^s such that $\alpha\,\mathcal{C}_v = \mathcal{C}_v$ for any $\alpha \in K_v$ with $|\alpha|_v = 1$;
(ii) if $v \nmid \infty$, then $\mathcal{C}_v$ is a finite type (thus free) sub-$\mathcal{O}_v$-module of K_v^s of rank s;
(iii) $\mathcal{C}_v = \mathcal{O}_v^s$ for all but finitely many places v of K with $v \nmid \infty$.

Suppose that $\mathcal{C}$ is such a product. For each $i = 1, \dots, s$, we define its i-th minimum $\lambda_i(\mathcal{C})$ as the smallest $\lambda > 0$ for which the adelic convex body

$$\lambda\mathcal{C} = \prod_{v\mid\infty} \lambda\mathcal{C}_v \prod_{v\nmid\infty} \mathcal{C}_v$$

contains at least i linearly independent elements of K^s over K. With this notation and our normalization of measures, the adelic version of Minkowski's theorem reads as follows.

Theorem 2.1 (McFeat, Bombieri and Vaaler)**.** *For any adelic convex body $\mathcal{C}$ of K^s, we have*

$$2^{sr_1}(s!)^{-d} \le \left(\lambda_1(\mathcal{C}) \cdots \lambda_s(\mathcal{C})\right)^d \mu(\mathcal{C}) \le 2^{s(r_1+r_2)}|D(K)|^{s/2}.$$

We refer the reader to [12, Theorem 5] and [4, Theorem 3] for the upper bound on the product of the minima (see also the upper bound of Thunder in [16, Theorem 1 and Corollary]). The lower bound given here is taken from [12, Theorem 6]; it is slightly weaker than the one of [4, Theorem 6].

2.2. **Hermite's approximations.** Suppose from now on that $s \geq 2$ and let $\alpha_1, \dots, \alpha_s$ be distinct elements of K. For each s-tuple $\mathbf{n} := (n_1, \dots, n_s) \in \mathbb{N}^s$, we define polynomials of $K[z]$ by

$$f_{\mathbf{n}}(z) = (z-\alpha_1)^{n_1} \cdots (z-\alpha_s)^{n_s} \quad \text{and} \quad P_{\mathbf{n}}(z) = \sum_{k=0}^{N} f_{\mathbf{n}}^{(k)}(z)$$

where

$$N = n_1 + \cdots + n_s$$

represents the degree of $f_{\mathbf{n}}$, and where $f_{\mathbf{n}}^{(k)}$ denotes the k-th derivative of $f_{\mathbf{n}}$ for each integer $k \geq 0$. We then form the point

$$a_{\mathbf{n}} := \big(P_{\mathbf{n}}(\alpha_1), \dots, P_{\mathbf{n}}(\alpha_s)\big) \in K^s.$$

We call it the *Hermite approximation of order* $\mathbf{n}$ *for the* s*-tuple* $(\alpha_1, \dots, \alpha_s)$. Our goal is to give a precise meaning to the term "approximation", by working in the adeles of K.

We first recall some properties of these points. For simplicity, we start by assuming that $K \subseteq \mathbb{C}$. We find

$$\frac{d}{dz}\big(P_{\mathbf{n}}(z)e^{-z}\big) = \big(P'_{\mathbf{n}}(z) - P_{\mathbf{n}}(z)\big)e^{-z} = -f_{\mathbf{n}}(z)e^{-z}. \tag{2.1}$$

So, for any pair $i, j \in \{1, \dots, s\}$, we obtain

$$P_{\mathbf{n}}(\alpha_i)e^{-\alpha_i} - P_{\mathbf{n}}(\alpha_j)e^{-\alpha_j} = \int_{\alpha_i}^{\alpha_j} f_{\mathbf{n}}(z)e^{-z}\,dz\,,$$

independently of the path of integration from α_i to α_j in $\mathbb{C}$. Upon integrating along the line segment $[\alpha_i, \alpha_j]$ joining those two points and observing that

$$\max_{z\in[\alpha_i,\alpha_j]} |f_{\mathbf{n}}(z)| \leq R^N \quad \text{with} \quad R = \max_{1\leq k,\ell\leq s} |\alpha_k - \alpha_\ell|,$$

we deduce that

$$\big|P_{\mathbf{n}}(\alpha_i)e^{-\alpha_i} - P_{\mathbf{n}}(\alpha_j)e^{-\alpha_j}\big| \leq c_1 R^N$$

for a constant $c_1 > 0$ that is independent of the choice of i, j and $\mathbf{n}$. Similarly, for $i = 1, \dots, s$, the formula (2.1) yields

$$P_{\mathbf{n}}(\alpha_i) = \int_0^\infty f_{\mathbf{n}}(z+\alpha_i)e^{-z}\,dz\,,$$

by integrating along $[0, \infty) \subset \mathbb{R}$. Since $|f_{\mathbf{n}}(t+\alpha_i)| \leq (t+R)^N$ for all $t \geq 0$, we deduce that

$$|P_{\mathbf{n}}(\alpha_i)| \leq \int_0^\infty (t+R)^N e^{-t}\,dt = e^R \int_R^\infty t^N e^{-t}\,dt \leq e^R \int_0^\infty t^N e^{-t}\,dt = e^R N!\,.$$

More generally, let v be any Archimedean place of K. Put

$$R_v = \max_{1\leq k,\ell\leq s} |\alpha_k - \alpha_\ell|_v \tag{2.2}$$

and choose an embedding $\sigma\colon K \to \mathbb{C}$ such that $|\alpha|_v = |\sigma(\alpha)|$ for all $\alpha \in K$. Then, for any pair of indices $i, j \in \{1, \dots, s\}$, the above computations yield

$$\big|P_{\mathbf{n}}(\alpha_i)e^{-\alpha_i} - P_{\mathbf{n}}(\alpha_j)e^{-\alpha_j}\big|_v = \left|\int_{\sigma(\alpha_i)}^{\sigma(\alpha_j)} f_{\mathbf{n}}^{\sigma}(z)e^{-z}dz\right| \le c_v R_v^N\,, \tag{2.3}$$

$$|P_{\mathbf{n}}(\alpha_i)|_v \le e^{R_v} N!\,, \tag{2.4}$$

where $f_{\mathbf{n}}^{\sigma}$ denotes the image of $f_{\mathbf{n}}$ under the ring homomorphism from $K[z]$ to $\mathbb{C}[z]$ which fixes z and coincides with σ on K, and where $c_v > 0$ depends only on v and $\alpha_1, \dots, \alpha_s$. Thus, $a_{\mathbf{n}}$ is a projective approximation to $(e^{\alpha_1}, \dots, e^{\alpha_s})$ at each Archimedean place of K.

In this paper, we establish an upper bound for the integral in (2.3) which is sharper than $c_v R_v^N$ for each Archimedean place v of K. We also provide analogs of (2.3) and of (2.4) for the ultrametric places v of K whenever their left hand side makes sense in K_v. More precisely, as $e^{\alpha_j - \alpha_i}$ could make sense in K_v without e^{α_i} and e^{α_j} making sense, we consider instead the quantities $|P_{\mathbf{n}}(\alpha_i)e^{\alpha_j - \alpha_i} - P_{\mathbf{n}}(\alpha_j)|_v$. Here again, we will need sharp estimates while usually the ultrametric places are treated in an expeditious manner. In general, one chooses a common denominator b of $\alpha_1, \dots, \alpha_s$, that is an integer $b \ge 1$ such that $b\alpha_1, \dots, b\alpha_s \in \mathcal{O}_K$. Then the polynomial $g(z) := b^N f(z/b)$ has coefficients in $\mathcal{O}_K$ and, for each $i = 1, \dots, s$, we find

$$\frac{b^N}{(n_i)!} P_{\mathbf{n}}(\alpha_i) = \sum_{k=n_i}^{N} \frac{b^N}{(n_i)!} f^{(k)}(\alpha_i) = \sum_{k=n_i}^{N} \frac{b^k k!}{(n_i)!} \cdot \frac{g^{(k)}(b\alpha_i)}{k!} \in \mathcal{O}_K\,.$$

For example, if $n_1 = \cdots = n_s = n$, this implies that $(b^N/n!)a_{\mathbf{n}} \in \mathcal{O}_K^s$.

The above estimates are key-ingredients in the classical proof of the Lindemann-Weiertrass theorem asserting that $e^{\alpha_1}, \dots, e^{\alpha_s}$ are linearly independent over K. However, two more ingredients are missing. The first one is a reduction step of Weierstrass which is explained in [11, Appendix, §3] (see also [3, Chapter 1, §3]). The second one is the existence of families of s linearly independent approximations over K. Hermite himself noticed this problem and solved it in order to prove the transcendence of e. We will use here the following remarkable result of Mahler.

Theorem 2.2 (Mahler)**.** *Suppose that* $\mathbf{n} = (n_1, \dots, n_s) \in \mathbb{N}_+^s$ *has positive coordinates. Let* $\mathbf{e}_1 = (1, 0, \dots, 0), \dots, \mathbf{e}_s = (0, \dots, 0, 1)$ *denote the canonical basis elements of* $\mathbb{Z}^s$*. Then, we have*

$$\Delta_{\mathbf{n}} := \det(a_{\mathbf{n}-\mathbf{e}_1}, \dots, a_{\mathbf{n}-\mathbf{e}_s}) = \prod_{i=1}^{s} \left((n_i - 1)! \prod_{k \ne i} (\alpha_i - \alpha_k)^{n_k} \right) \ne 0. \tag{2.5}$$

The proof of Mahler is clever. It is presented in [10, §8] and again in [11, Appendix, §16]. In the case where $n_1 = \cdots = n_s$, the result is due to Hermite [9]. Hermite's proof is different. It is based on the recurrence relations satisfied by the points $\mathbf{a}_{\mathbf{n}}$ which we generalize in Appendix A.

2.3. **Statement of the main result.** With the above notation, let $\mathcal{S}$ be the finite set consisting of all Archimedean places of K together with the ultrametric places v of K such that $|\alpha_i - \alpha_j|_v \neq 1$ for at least one pair of indices $i, j \in \{1, \dots, s\}$ with $i \neq j$. For each s-tuple $\mathbf{n} = (n_1, \dots, n_s) \in \mathbb{N}_+^s$, we let N denote its sum and we define an adelic convex body $\mathcal{C}_\mathbf{n} = \prod_v \mathcal{C}_{\mathbf{n},v}$ of K^s as follows.

(i) If $v \mid \infty$ is the place attached to an embedding $\sigma\colon K \hookrightarrow \mathbb{C}$, we define R_v by (2.2). Then $\mathcal{C}_{\mathbf{n},v}$ is the set of points $(x_1, \dots, x_s) \in K_v^s$ which satisfy

$$|x_i|_v \le e^{R_v}(N-1)! \quad \text{and} \quad |x_i e^{\alpha_j - \alpha_i} - x_j|_v \le \max_{1\le k\le s} \left| \int_{\sigma(\alpha_i)}^{\sigma(\alpha_j)} f^\sigma_{\mathbf{n}-\mathbf{e}_k}(z) e^{\sigma(\alpha_j)-z}\, dz \right| \tag{2.6}$$

for each pair of indices $i, j \in \{1, \dots, s\}$ with $i \neq j$.

(ii) If $v \in \mathcal{S}$ and if $v \mid p$ for a prime number p, then $\mathcal{C}_{\mathbf{n},v}$ is the set of points $(x_1, \dots, x_s)$ in K_v^s which satisfy

$$|x_i|_v \le p^3 N \prod_{1\le k\le s} \max\left\{|\alpha_i - \alpha_k|_v,\, p^{-1/(p-1)}\right\}^{n_k} \tag{2.7}$$

for $i = 1, \dots, s$, as well as

$$|x_i e^{\alpha_j - \alpha_i} - x_j|_v \le p^3 N \prod_{1\le k\le s} \max\left\{|\alpha_i - \alpha_k|_v, |\alpha_j - \alpha_k|_v\right\}^{n_k} \tag{2.8}$$

for each pair of integers $i, j \in \{1, \dots, s\}$ such that $0 < |\alpha_j - \alpha_i|_v < p^{-1/(p-1)}$.

(iii) Finally, if $v \notin \mathcal{S}$, then $\mathcal{C}_{\mathbf{n},v}$ is the set of points $(x_1, \dots, x_s) \in K_v^s$ satisfying

$$|x_i|_v \le |(n_i - 1)!|_v$$

for $i = 1, \dots, s$.

The crucial feature of these adelic convex bodies $\mathcal{C}_\mathbf{n}$ is that the linear forms which define them involve only the complex or p-adic values of the exponential function at the points $\alpha_j - \alpha_i$.

In view of the estimates in §2.2, the points $a_{\mathbf{n}-\mathbf{e}_1}, \dots, a_{\mathbf{n}-\mathbf{e}_s}$ satisfy the conditions in (i) and so they belong to $\mathcal{C}_{\mathbf{n},v}$ for each Archimedean place v of K. Likewise, we will see in the next section that the conditions in (ii) or (iii) are also satisfied by these points (in fact, they are designed for that purpose). So these points also belong to $\mathcal{C}_{\mathbf{n},v}$ for each non-Archimedean place v of K. This yields the first assertion in the following result.

Theorem 2.3. *Let $\mathbf{n} = (n_1, \dots, n_s) \in \mathbb{N}_+^s$. Then the adelic convex body $\mathcal{C}_\mathbf{n}$ contains the points $a_{\mathbf{n}-\mathbf{e}_1}, \dots, a_{\mathbf{n}-\mathbf{e}_s}$. Moreover, upon setting $N = n_1 + \cdots + n_s$, we have the following volume estimates.*

(i) *If $v \mid \infty$, then*

$$(s!)^{-1} |\Delta_\mathbf{n}|_v \le \mu_v(\mathcal{C}_{\mathbf{n},v})^{1/d_v} \le c_v N^{2s-2} |\Delta_\mathbf{n}|_v$$

for a constant $c_v > 0$ depending only on $\alpha_1, \dots, \alpha_s$ and v.

(ii) *If* $v \in \mathcal{S}$ *and if* $v \mid p$ *for a prime number* p, *then*

$$|\Delta_{\mathbf{n}}|_v \le \mu_v(\mathcal{C}_{\mathbf{n},v})^{1/d_v} \le (p^3N)^s|\Delta_{\mathbf{n}}|_v.$$

(iii) *If* $v \notin \mathcal{S}$, *then* $\mu_v(\mathcal{C}_{\mathbf{n},v})^{1/d_v} = |\Delta_{\mathbf{n}}|_v$.

Note that, for each place v of K, these estimates enclose the volume of $\mathcal{C}_{\mathbf{n},v}$ between limits whose ratio is a polynomial in N while these limits themselves grow like $|\Delta_{\mathbf{n}}|_v$, that is roughly like an exponential in N if $v \nmid \infty$ or like $N!$ if $v \mid \infty$. When $v \mid \infty$, we give an explicit value for the constant c_v in Theorem 8.1.

The lower bounds for $\mu_v(\mathcal{C}_{\mathbf{n},v})$ follow easily from the definition of $\Delta_{\mathbf{n}}$ as a determinant in (2.5), if we take for granted the fact that $\mathcal{C}_{\mathbf{n},v}$ contains the points $\mathbf{a}_{\mathbf{n}-\mathbf{e}_i}$ for $i = 1, \dots, s$. Indeed, let $T\colon K_v^s \to K_v^s$ be the K_v-linear map defined by

$$T(x_1, \dots, x_s) = x_1\mathbf{a}_{\mathbf{n}-\mathbf{e}_1} + \cdots + x_s\mathbf{a}_{\mathbf{n}-\mathbf{e}_s}$$

for each $(x_1, \dots, x_s) \in K_v^s$. Then $\mathcal{C}_{\mathbf{n},v}$ contains $T(\mathcal{E}_v)$ where $\mathcal{E}_v$ is given by

$$\begin{aligned} \mathcal{E}_v &= \{(x_1, \dots, x_s) \in K_v^s\,;\, |x_1|_v + \cdots + |x_s|_v \le 1\} && \text{if } v \mid \infty,\\ \mathcal{E}_v &= \mathcal{O}_v^s && \text{if } v \nmid \infty. \end{aligned}$$

As $|\det T|_v = |\Delta_{\mathbf{n}}|_v$, we have $\mu_v(T(\mathcal{E}_v)) = |\Delta_{\mathbf{n}}|_v^{d_v}\mu_v(\mathcal{E}_v)$. If $v \mid \infty$, we also have $\mu_v(\mathcal{E}_v) \ge (s!)^{-d_v}$, thus $\mu_v(\mathcal{C}_{\mathbf{n},v})^{1/d_v} \ge (s!)^{-1}|\Delta_{\mathbf{n}}|_v$. If $v \nmid \infty$, we simply have $\mu_v(\mathcal{E}_v) = 1$, thus $\mu_v(\mathcal{C}_{\mathbf{n},v})^{1/d_v} \ge |\Delta_{\mathbf{n}}|_v$.

Our main contribution therefore lies in the upper bounds for the volume of the components $\mathcal{C}_{\mathbf{n},v}$, and we explain our strategy below. These upper bounds in turn yield an upper bound for the volume of $\mathcal{C}_{\mathbf{n}}$ from which we derive the following conclusion thanks to the adelic Minkowski theorem.

Corollary 2.4. *In the notation of Theorem 2.3, we have*

$$cN^{-g} \le \lambda_1(\mathcal{C}_{\mathbf{n}}) \le \cdots \le \lambda_s(\mathcal{C}_{\mathbf{n}}) \le 1 \quad \textit{where} \quad g = s - 2 + s\sum_{v\in\mathcal{S}}\frac{d_v}{d},$$

and where $c > 0$ *is a constant depending only on* $\alpha_1, \dots, \alpha_s$.

Proof. Since $\prod_v |\Delta_{\mathbf{n}}|_v^{d_v} = 1$ and since $\mathcal{S}$ contains all Archimedean places of K, we find

$$\mu(\mathcal{C}_n) = \prod_v \mu_v(\mathcal{C}_{\mathbf{n},v}) \le \prod_{v\mid\infty}\Big(c_v^{d_v}N^{(2s-2)d_v}\Big)\prod_{v\in\mathcal{S},\, v\mid p}(p^3N)^{sd_v} = c_1^dN^{gd}$$

where $c_1 > 0$ is independent of $\mathbf{n}$. Since $\mathcal{C}_{\mathbf{n}}$ contains the points $a_{\mathbf{n}-\mathbf{e}_1}, \dots, a_{\mathbf{n}-\mathbf{e}_s}$ of K^s and since, by Theorem 2.2, these points are linearly independent over K, we also have

$$\lambda_1(\mathcal{C}_{\mathbf{n}}) \le \cdots \le \lambda_s(\mathcal{C}_{\mathbf{n}}) \le 1.$$

Thus, by Theorem 2.1, we obtain

$$(s!)^{-1} \le \lambda_1(\mathcal{C}_{\mathbf{n}})\cdots\lambda_s(\mathcal{C}_{\mathbf{n}})\mu(\mathcal{C}_{\mathbf{n}})^{1/d} \le \lambda_1(\mathcal{C}_{\mathbf{n}})c_1N^g,$$

so $\lambda_1(\mathcal{C}_{\mathbf{n}}) \ge cN^{-g}$ with $c = 1/(c_1s!)$. ∎

The proof of Theorem 2.3 uses general results on univariate polynomials $f(z) \in \mathbb{C}[z]$ which we could not find in the literature. Suppose that f has degree $N \geq 1$. Let A be its set of roots in $\mathbb{C}$ and let B be the set of roots of its derivative f' which do not belong to A. In Section 5, we consider the paths of steepest descent for $|f|$ starting from an arbitrary point β of $\mathbb{C}$. These paths necessarily end in an element of A. We show that they are contained in the convex hull of $A \cup \{\beta\}$, with length at most πRN where R is the radius of any disk containing $A \cup \{\beta\}$. In Section 6, for each $\beta \in B$, we denote by $m(\beta)$ the multiplicity of β as a root of f' and, starting from β, we choose $m(\beta) + 1$ paths of steepest descent for $|f|$ which are locally distinct in a neighborhood of β. These paths draw a graph on $A \cup B$ and we show that this graph is in fact a tree. We extract from it a sub-graph G on A which is also a tree with edges indexed by B. Then, for each edge of G with end points $\alpha, \alpha' \in A$, indexed by $\beta \in B$, we obtain a path joining α to α' passing through β, with length at most $2\pi RN$, along which $|f|$ is maximal at the point β.

For the proof of Theorem 2.3 (i), we may assume that the given place $v \mid \infty$ comes from an inclusion $K \subset \mathbb{C}$. We then apply the above construction, choosing f to be the gcd of the polynomials $f_{\mathbf{n}-\mathbf{e}_1}, \dots, f_{\mathbf{n}-\mathbf{e}_s}$. If the coordinates of $\mathbf{n} \in \mathbb{N}_+^s$ are all ≥ 2, we thus obtain a tree G on $A = \{\alpha_1, \dots, \alpha_s\}$. Then, for each edge of G with end points α_i, α_j, we bound from above the integrals in (2.6) as a function of $|f(\beta)|$ where $\beta \notin A$ is the corresponding root of f'. From this, we deduce in Section 8 an upper bound for the volume of the convex body $\mathcal{C}_{\mathbf{n},v}$ in terms of the product of the values $|f(\beta)|^{m(\beta)}$ with $\beta \in B$, this being the Chudnovsky semi-resultant of f and f'. The upper bound for $\mu_v(\mathcal{C}_{\mathbf{n},v})$ then follows thanks to the computation of this semi-resultant in Section 7. The general case where at least one coordinate of $\mathbf{n}$ is equal to 1 requires a slight adjustment.

The treatment of the ultrametric places $v \nmid \infty$ is simpler. In Section 3, we show that $\mathcal{C}_{\mathbf{n},v}$ contains the points $\mathbf{a}_{\mathbf{n}-\mathbf{e}_1}, \dots, \mathbf{a}_{\mathbf{n}-\mathbf{e}_s}$. Afterwards, in Section 9, we construct a rooted forest on $\{\alpha_1, \dots, \alpha_s\}$ associated with the place v. This allows us to select s inequalities among (2.7) and (2.8) and to deduce from them the required upper bound on the volume of $\mathcal{C}_{\mathbf{n},v}$ in Section 10. The relevant notions from graph theory are recalled in Section 4.

In Section 11, we restrict to "diagonal" approximations to two exponentials, namely to the case $s = 2$ and $n_1 = n_2$. In this situation, we provide a refined form of our main result whose proof relies only on the estimates from Sections 2.2 and 3. We then use it to prove Propositions 1.1 and 1.2 from the introduction.

We conclude in Section 12 by explaining how Hermite's recurrence formulas recalled in Appendix A can be used to compute efficiently the partial quotients in the continued fraction expansion of e^3. This in turn permits to validate the inequalities (1.2) in less than two hours of computation on a small desk computer.

3. Ultrametric estimates

Let v be a place of K above a prime number p. In this section, we complete the proof of the first assertion in Theorem 2.3 by showing that the component $\mathcal{C}_{\mathbf{n},v}$ of $\mathcal{C}_{\mathbf{n}}$ contains the points $\mathbf{a}_{\mathbf{n}-\mathbf{e}_1}, \dots, \mathbf{a}_{\mathbf{n}-\mathbf{e}_s}$ for each $\mathbf{n} \in \mathbb{N}_+^s$. To this end, we use the following notation and results.

For each $a \in \mathbb{C}_p$ and each $r > 0$, we denote by

$$B(a,r) = \{z \in \mathbb{C}_p\,;\ |z-a|_p \le r\}$$

the closed disk of $\mathbb{C}_p$ with center a and radius r (both closed and open in $\mathbb{C}_p$). For such a disk $B = B(a,r)$ and for any analytic function $g\colon B \to \mathbb{C}_p$, we define

$$|g|_B = \sup\{|g(z)|_p\,;\ z \in B\}.$$

This quantity can also be computed from the Taylor series expansion of g around the point a via the formula

$$|g|_B = \sup_{k\in\mathbb{N}} \left|\frac{g^{(k)}(a)}{k!}\right|_p r^k,$$

which yields the p-adic form of Cauchy's inequalities

$$|g^{(k)}(a)|_p \le |k!|_p r^{-k} |g|_B \quad (k \in \mathbb{N})$$

(see [14, §1.5]). For the computations, we also use the estimates

$$\delta^k \le |k!|_p \le k\delta^{k-p} \le p^2 k \delta^k \quad (k \in \mathbb{N}), \quad \text{where} \quad \delta = p^{-1/(p-1)}, \tag{3.1}$$

which follow from the formula $|k!|_p = p^{-m}$ where $m = \sum_{\ell=1}^{\infty} \lfloor k/p^\ell \rfloor$.

Lemma 3.1. *Let $\mathbf{n} = (n_1, \dots, n_s) \in \mathbb{N}^s$, let $N = n_1 + \cdots + n_s$, and let $i, j \in \{1, \dots, s\}$. Then, we have*

$$|P_{\mathbf{n}}(\alpha_i)|_v \le p^2 N \prod_{k=1}^{s} \max\{|\alpha_i - \alpha_k|_v,\ \delta\}^{n_k}. \tag{3.2}$$

If $|\alpha_i - \alpha_k|_v \le 1$ for $k = 1, \dots, s$, we also have

$$|P_{\mathbf{n}}(\alpha_i)|_v \le |n_i!|_v. \tag{3.3}$$

Finally, if $\rho = |\alpha_i - \alpha_j|_v$ satisfies $0 < \rho < \delta$, we have

$$|P_{\mathbf{n}}(\alpha_i) e^{\alpha_j - \alpha_i} - P_{\mathbf{n}}(\alpha_j)|_v \le \frac{\rho}{\delta} p^2 N \prod_{k=1}^{s} \max\{|\alpha_i - \alpha_k|_v,\ |\alpha_j - \alpha_k|_v\}^{n_k}. \tag{3.4}$$

Proof. To simplify, we may assume that $K \subset \mathbb{C}_p$ and that $|\alpha|_v = |\alpha|_p$ for each $\alpha \in K$. Then, the polynomial $f_{\mathbf{n}}(z) \in K[z]$ can be viewed as an analytic function $f_{\mathbf{n}}\colon \mathbb{C}_p \to \mathbb{C}_p$. To estimate $|P_{\mathbf{n}}(\alpha_i)|_v = |P_{\mathbf{n}}(\alpha_i)|_p$, we set

$$B = B(\alpha_i, \delta) \quad \text{and} \quad M = |f_{\mathbf{n}}|_B.$$

For $k = 0, 1, \dots, N$, Cauchy's inequalities together with (3.1) yield

$$|f_{\mathbf{n}}^{(k)}(\alpha_i)|_p \le |k!|_p \delta^{-k} M \le p^2 k M \le p^2 N M,$$

thus

$$|P_{\mathbf{n}}(\alpha_i)|_v = \left| \sum_{k=0}^{N} f_{\mathbf{n}}^{(k)}(\alpha_i) \right|_p \le p^2 N M.$$

This proves (3.2) since

$$M \le \prod_{k=1}^{s} \sup\{|z - \alpha_k|_p \,;\, z \in B\}^{n_k} = \prod_{k=1}^{s} \max\{|\alpha_i - \alpha_k|_v,\, \delta\}^{n_k}.$$

If $|\alpha_i - \alpha_k|_v \le 1$ for each k, a similar computation yields $|f_{\mathbf{n}}|_B \le 1$ with $B = B(\alpha_i, 1)$. Then Cauchy's inequalities give $|f_{\mathbf{n}}^{(k)}(\alpha_i)|_p \le |k!|_p$ for each $k \in \mathbb{N}$. Since we have $f_{\mathbf{n}}^{(k)}(\alpha_i) = 0$ for $k = 0, \dots, n_i - 1$, we deduce that $|f_{\mathbf{n}}^{(k)}(\alpha_i)|_v \le |n_i!|_v$ for each $k \in \mathbb{N}$ and the upper bound (3.3) follows.

Suppose now that $0 < \rho = |\alpha_i - \alpha_j|_p < \delta$. To prove (3.4), we use instead

$$B = B(\alpha_j, \rho) \quad \text{and} \quad M = |f_{\mathbf{n}}|_B.$$

Since $\rho < \delta$, the function $g\colon B \to \mathbb{C}_p$ given by

$$g(z) = P_{\mathbf{n}}(z) e^{\alpha_j - z} - P_{\mathbf{n}}(\alpha_j) \quad (z \in B)$$

is analytic with $g(\alpha_j) = 0$ and

$$g'(z) = -f_{\mathbf{n}}(z) e^{\alpha_j - z} \quad (z \in B). \tag{3.5}$$

For each integer $\ell = 0, 1, \dots, N$, we have

$$|f_{\mathbf{n}}^{(\ell)}(\alpha_j)|_p \le |\ell!|_p \rho^{-\ell} M \le p^2 \ell (\delta/\rho)^{\ell} M \le p^2 N (\delta/\rho)^{\ell} M.$$

Since $f_{\mathbf{n}}^{(\ell)} = 0$ for $\ell > N$, this remains valid for each $\ell \in \mathbb{N}$. Then, by (3.5), Leibniz formula for the derivative of a product yields, for each integer $k \ge 1$,

$$|g^{(k)}(\alpha_j)|_p \le \max_{0 \le \ell < k} |f_{\mathbf{n}}^{(\ell)}(\alpha_j)|_p \le p^2 N (\delta/\rho)^{k-1} M.$$

Since $\alpha_i \in B$ and $g(\alpha_j) = 0$, we deduce that

$$|P_{\mathbf{n}}(\alpha_i) e^{\alpha_j - \alpha_i} - P_{\mathbf{n}}(\alpha_j)|_v = |g(\alpha_i)|_p \le |g|_B = \sup_{k \ge 1} \left| \frac{g^{(k)}(\alpha_j)}{k!} \right|_p \rho^k \le p^2 N (\rho/\delta) M.$$

The upper bound (3.4) follows since

$$M \le \prod_{k=1}^{s} \sup\{|z - \alpha_k|_p \,;\, z \in B\}^{n_k} = \prod_{k=1}^{s} \max\{|\alpha_i - \alpha_k|_v,\, |\alpha_j - \alpha_k|_v\}^{n_k}. \qquad \square$$

Theorem 3.2. *Let $\mathbf{n} = (n_1, \dots, n_s) \in \mathbb{N}_+^s$. Then the subset $\mathcal{C}_{\mathbf{n},v}$ of K_v^s defined in Section 2.3 contains the points $\mathbf{a}_{\mathbf{n}-\mathbf{e}_1}, \dots, \mathbf{a}_{\mathbf{n}-\mathbf{e}_s}$.*

Proof. Fix an integer $\ell \in \{1, \dots, s\}$ and put $P = P_{\mathbf{n}-\mathbf{e}_\ell}$. To show that $\mathcal{C}_{\mathbf{n},v}$ contains the point $\mathbf{a}_{\mathbf{n}-\mathbf{e}_\ell} = (P(\alpha_1), \dots, P(\alpha_s))$, we fix arbitrary $i, j \in \{1, \dots, s\}$. Since $\max\{|\alpha_i - \alpha_\ell|_v, \delta\} \geq \delta \geq 1/p$, the inequality (3.2) of Lemma 3.1 applied to $\mathbf{n} - \mathbf{e}_\ell$ instead of $\mathbf{n}$ provides

$$\bigl|P(\alpha_i)\bigr|_v \leq p^2(N-1)\frac{1}{\delta}\prod_{k=1}^{s}\max\{|\alpha_i - \alpha_k|_v,\, \delta\}^{n_k} \leq p^3 N \prod_{k=1}^{s}\max\{|\alpha_i - \alpha_k|_v,\, \delta\}^{n_k}.$$

If $|\alpha_i - \alpha_k|_v = 1$ for each $k = 1, \dots, s$ with $k \neq i$, the inequality (3.3) of the same lemma also provides

$$\bigl|P(\alpha_i)\bigr|_v \leq |(n_i - 1)!|_v.$$

Finally, if $\rho = |\alpha_j - \alpha_i|_v$ satisfies $0 < \rho < \delta$, then, since $\max\{|\alpha_i - \alpha_\ell|_v,\, |\alpha_j - \alpha_\ell|_v\} \geq \rho$, the inequality (3.4) with $\mathbf{n}$ replaced by $\mathbf{n} - \mathbf{e}_\ell$ yields

$$\begin{aligned} |P(\alpha_i)e^{\alpha_j - \alpha_i} - P(\alpha_j)|_v &\leq \frac{\rho}{\delta}p^2(N-1)\cdot\frac{1}{\rho}\prod_{k=1}^{s}\max\{|\alpha_i - \alpha_k|_v,\, |\alpha_j - \alpha_k|_v\}^{n_k} \\ &\leq p^3 N \prod_{k=1}^{s}\max\{|\alpha_i - \alpha_k|_v,\, |\alpha_j - \alpha_k|_v\}^{n_k}. \end{aligned}$$ □

4. Preliminaries of graph theory

A *graph* G is a pair (V, E) where V is a finite non-empty set and E is a possibly empty set consisting of subsets of V with two elements. The elements of V are called the *vertices* of G and those of E the *edges* of G in agreement with the usual graphic representation.

Let $G = (V, E)$ be a graph. An *elementary chain* in G is a sequence $(\alpha_1, \dots, \alpha_m)$ of $m \geq 2$ distinct elements of V such that $\{\alpha_i, \alpha_{i+1}\} \in E$ for $i = 1, \dots, m-1$. We say that G is *connected* if, for each pair of distinct elements α, β of V, there exists at least one elementary chain $(\alpha_1, \dots, \alpha_m)$ in G with $\alpha_1 = \alpha$ and $\alpha_m = \beta$. We say that G is a *tree* if there exists exactly one such chain for each choice of $\alpha, \beta \in V$ with $\alpha \neq \beta$. When G is connected, we have $|V| \leq |E| + 1$ with equality if and only if G is a tree.

In general, for a graph $G = (V, E)$, there exists one and only one choice of integer $r \geq 1$ and partitions $V = V_1 \cup \cdots \cup V_r$ and $E = E_1 \cup \cdots \cup E_r$ of V and E into r disjoint subsets such that $G_i = (V_i, E_i)$ is a connected graph for $i = 1, \dots, r$. We say that $G_1, \dots, G_r$ are the *connected components* of G. If these are trees, we say that G is a *forest*. When G admits r connected components, we have $|V| \leq |E| + r$ with equality if and only if G is a forest.

A *rooted forest* is a triple $G = (R, V, E)$ where (V, E) is a forest and where R is a subset of V containing exactly one vertex from each connected component of (V, E). We say that R is the set of *roots* of G. Then, for each $\beta \in V \setminus R$, there is a unique elementary chain $(\alpha_1, \dots, \alpha_m)$ with $\alpha_1 \in R$ and $\alpha_m = \beta$. So we obtain a partial ordering on V by defining $\alpha < \beta$ if $\beta \notin R \cup \{\alpha\}$ and if the elementary chain which links β to an element of R contains α. In particular, any edge $\{\alpha, \beta\} \in E$ can be ordered so that $\alpha < \beta$. The resulting pairs (α, β) are called the *directed edges* of G. For fixed $\alpha \in V$, we say that $D_G(\alpha) = \{\beta \in V\,;\, \alpha < \beta\}$ is the set of *descendants* of α. The set $S_G(\alpha)$ of minimal elements of $D_G(\alpha)$ is called the set of *successors* of α. Note that the pairs $(\alpha, \beta) \in V \times V$ with $\beta \in S_G(\alpha)$ are exactly the directed edges of G. Moreover, any $\beta \in V \setminus R$ is the successor of a unique $\alpha \in V$. This allows us to formulate the following result.

Proposition 4.1. *Let $G = (R, V, E)$ be a rooted forest, let K be a field, let $(x_\alpha)_{\alpha \in V}$ be a family of indeterminates over K indexed by V, and let $\varphi\colon E \to K$ be a function. For each $\beta \in V$, we define*

$$L_\beta = \begin{cases} x_\beta & \text{if } \beta \in R, \\ x_\beta - \varphi(\{\alpha, \beta\}) x_\alpha & \text{if } \beta \in S_G(\alpha) \text{ with } \alpha \in V. \end{cases}$$

Then, upon extending the partial ordering on V to a total ordering, the matrix of the linear forms $(L_\beta)_{\beta \in V}$ with respect to the basis $(x_\alpha)_{\alpha \in V}$ is lower triangular with 1 everywhere on the diagonal.

5. Paths of steepest ascent

In this section, we fix a non-constant monic polynomial $f(z) \in \mathbb{C}[z]$, a compact convex subset $\mathcal{K}$ of $\mathbb{C}$ containing all the roots of f, and a closed disk D of $\mathbb{C}$ containing $\mathcal{K}$. We denote by N the degree of f, and by R the radius of D. The main goal of this section is to prove the following result.

Theorem 5.1. *Let $\beta \in \mathcal{K}$. There exists a root α of f and a path $\gamma\colon [0,1] \to \mathbb{C}$ linking $\gamma(0) = \alpha$ to $\gamma(1) = \beta$, such that $f(\gamma(t)) = tf(\beta)$ for each $t \in [0,1]$. The image of such a path is contained in $\mathcal{K}$, with length at most πRN.*

By a *path* we mean here a continuous piecewise differentiable map $\gamma\colon I \to \mathbb{C}$ on a closed subinterval I of $\mathbb{R}$. For a path γ as in the statement of the theorem, $\gamma(0)$ is necessarily a root of f and we have $\max\{|f(\gamma(t))|\,;\, 0 \le t \le 1\} = |f(\beta)|$. We will see that, in fact, γ is a path of steepest ascent for $|f|$.

For the proof, we consider the polynomial f as a covering of Riemann surfaces $f\colon \mathbb{C} \to \mathbb{C}$ of degree N, ramified in a finite number of points. Then any path $\gamma\colon [0,1] \to \mathbb{C}$ lifts into N paths $\gamma_1, \dots, \gamma_N\colon [0,1] \to \mathbb{C}$ such that $f^{-1}(\gamma(t)) = \{\gamma_1(t), \dots, \gamma_N(t)\}$ for all $t \in [0,1]$. The latter are not unique in general, because of ramification, and are constructed by pasting as in the proof of [8, Theorem 4.14]. For a path γ of the form $\gamma(t) = tf(\beta)$ with $f(\beta) \neq 0$, this leads to the following statement.

Lemma 5.2. *Let $\beta \in \mathbb{C}$ with $f(\beta) \neq 0$, and let $m = m(\beta) \ge 0$ denote the order of the derivative of f at β. Then, there exist $\delta \in (0,1)$ and $m+1$ paths $\gamma_0, \dots, \gamma_m$ from $[0,1]$ to $\mathbb{C}$ such that*

(i) $\gamma_0(1) = \cdots = \gamma_m(1) = \beta$,
(ii) $f(\gamma_0(t)) = \cdots = f(\gamma_m(t)) = tf(\beta)$ *for each* $t \in [0,1]$,

(iii) *$\gamma_0(t),\dots,\gamma_m(t)$ are $m+1$ distinct numbers for each $t\in(1-\delta,1)$.*

Moreover, for each $j=0,1,\dots,m$ and each $t\in(0,1)$ such that $f'(\gamma_j(t))\neq 0$, the function γ_j is analytic at t and its derivative $\gamma_j'(t)$ heads in the direction where the norm $|f|$ of f grows fastest.

The last assertion of the lemma means that $\gamma_0,\dots,\gamma_m$ are paths of steepest ascent for the norm of f. This is true in fact for any path γ such that $f(\gamma(t))=ct$ $(0\le t\le 1)$ with a fixed $c\in\mathbb{C}\setminus\{0\}$ because the image of the map $t\mapsto ct$ with $t\ge 0$ is a half line that is orthogonal to the circles centered at the origin. As the map $f\colon\mathbb{C}\to\mathbb{C}$ is conformal outside of the ramification points, the preimage γ of this curve is orthogonal to the level curves of $|f|$ outside of these points. We will revisit the construction of the paths γ_j in Lemma 6.3.

Proof of Theorem 5.1. If $f(\beta)=0$, the constant path $\gamma(t)=\beta$ for each $t\in[0,1]$ is the only possible choice and it has the required properties. Suppose from now on that $f(\beta)\neq 0$. Then the preceding lemma provides a path γ of the required type linking β to a root of f. Fix such a path. For the computations, we denote by $\alpha_1,\dots,\alpha_s$ the distinct roots of f in $\mathbb{C}$ and by $n_1,\dots,n_s$ their respective multiplicities so that

$$f(z)=(z-\alpha_1)^{n_1}\cdots(z-\alpha_s)^{n_s}.$$

We also denote by B the set of zeros of the derivative f' of f.

By Gauss-Lucas theorem the set B is contained in the convex hull of the roots of f, thus $B\subset\mathcal{K}$. The fact that the image of γ is contained in $\mathcal{K}$ admits a similar proof. Indeed, suppose by contradiction that the image escapes from $\mathcal{K}$. Then, since $\mathcal{K}$ is convex, there exists a half-plane containing $\mathcal{K}$ but not the image of γ. More precisely, there exist $a,b\in\mathbb{C}$ with $|a|=1$ such that $\mathrm{Re}(az+b)\le 0$ for each $z\in\mathcal{K}$ and $\mathrm{Re}(a\gamma(t)+b)>0$ for at least one $t\in[0,1]$. Choose $t_0\in[0,1]$ for which $\mathrm{Re}(a\gamma(t_0)+b)$ is maximal, and set $z_0=\gamma(t_0)$. Since $\mathrm{Re}(az_0+b)>0$, we have $z_0\notin\mathcal{K}$, thus $t_0\in(0,1)$ and $z_0\notin B$. Therefore γ is differentiable at t_0 with $\mathrm{Re}(a\gamma'(t_0))=0$. However, by differentiating both sides of the equality $f(\gamma(t))=tf(\beta)$ at $t=t_0$, we obtain

$$a\gamma'(t_0)=\frac{af(\beta)}{f'(\gamma(t_0))}=\frac{af(z_0)}{t_0f'(z_0)}=\left(\sum_{\ell=1}^{s}\frac{t_0n_\ell}{a(z_0-\alpha_\ell)}\right)^{-1}.$$

As $\mathrm{Re}(a(z_0-\alpha_\ell))=\mathrm{Re}(az_0+b)-\mathrm{Re}(a\alpha_\ell+b)\ge\mathrm{Re}(az_0+b)>0$ for $\ell=1,\dots,s$, we deduce that $\mathrm{Re}(a\gamma'(t_0))>0$, a contradiction.

To estimate the length $L(\gamma)$ of γ, we use the Cauchy-Crofton formula

$$L(\gamma)=\frac{1}{4}\int_0^{2\pi}A(\theta)\,d\theta\quad\text{where}\quad A(\theta)=\int_{-\infty}^{\infty}N(r,\theta)\,dr\,,$$
$$\text{and}\quad N(r,\theta)=\mathrm{Card}\{t\in[0,1]\,;\,\mathrm{Re}(\gamma(t)e^{-i\theta})=r\}$$

(see for example the beautiful proof of [1]). Fix $r, \theta \in \mathbb{R}$ and consider the polynomial

$$g_{r,\theta}(u) = \mathrm{Im}\left(\frac{f((r+iu)e^{i\theta})}{f(\beta)}\right) \in \mathbb{R}[u].$$

If $t_0 \in [0,1]$ satisfies $\mathrm{Re}(\gamma(t_0)e^{-i\theta}) = r$, we may write $\gamma(t_0) = (r+iu_0)e^{i\theta}$ for some $u_0 \in \mathbb{R}$. Then we have $f((r+iu_0)e^{i\theta}) = t_0 f(\beta)$ and consequently $g_{r,\theta}(u_0) = 0$. As γ is injective on $[0,1]$ (because $f \circ \gamma$ is), this means that $N(r,\theta)$ is at most equal to the number of real roots of $g_{r,\theta}$. But, as f has degree N, the polynomial $g_{r,\theta}(u)$ has degree at most N and its coefficient of u^N is $\mathrm{Im}((ie^{i\theta})^N/f(\beta))$. Thus, except possibly for the $2N$ values of $\theta \in [0,2\pi)$ for which this coefficient vanishes, we have $g_{r,\theta} \neq 0$ and thus $N(r,\theta) \leq N$.

For fixed θ, the set $\{\mathrm{Re}(ze^{-i\theta})\,;\, z \in D\}$ is an interval I_θ of $\mathbb{R}$ of length $2R$. As the image of γ is contained in $\mathcal{K} \subset D$, we have $N(r,\theta) = 0$ if $r \notin I_\theta$. We conclude that $A(\theta) \leq 2RN$ except for at most $2N$ values of $\theta \in [0,2\pi)$, and thus $L(\gamma) \leq \pi RN$.

6. A tree of paths between complex roots

As in the preceding section, we fix a non-constant monic polynomial $f(z) \in \mathbb{C}[z]$. We denote by N its degree, by $A = \{\alpha_1, \dots, \alpha_s\}$ the set of its complex roots, by $\mathcal{K}$ the convex hull of A, and by R the radius of a closed disk D containing A. We also denote by $B = \{\beta_1, \dots, \beta_p\}$ the set of roots of $f'(z)$ which are not roots of $f(z)$, that is the set of zeros of the logarithmic derivative $f'(z)/f(z)$. Then we may write

$$f(z) = (z-\alpha_1)^{n_1} \cdots (z-\alpha_s)^{n_s}, \tag{6.1}$$

$$f'(z) = N(z-\alpha_1)^{n_1-1} \cdots (z-\alpha_s)^{n_s-1}(z-\beta_1)^{m_1} \cdots (z-\beta_p)^{m_p}, \tag{6.2}$$

for integers $n_1, \dots, n_s \geq 1$ with sum N, and integers $m_1, \dots, m_p \geq 1$ with sum $s-1$.

For each $\beta \in \mathbb{C}$, we denote by $m(\beta)$ the order of $f'(z)$ at β. With this notation, we have $m_j = m(\beta_j)$ for $j = 1, \dots, p$. The goal of this section is to prove the following result.

Theorem 6.1. *There exists a tree G with the following properties:*

(i) *Its set of vertices is A.*
(ii) *It has $s-1$ edges, each one indexed by an element of B.*
(iii) *For each $\beta \in B$, there are exactly $m(\beta)$ edges indexed by β.*
(iv) *If $\{\alpha, \alpha'\}$ is an edge of G indexed by β, there exists a path $\gamma\colon [0,1] \to \mathbb{C}$ of length at most $2\pi RN$, contained in $\mathcal{K}$, linking $\gamma(0) = \alpha$ to $\gamma(1) = \alpha'$, such that*

$$\gamma(1/2) = \beta \quad \text{and} \quad \max_{0 \leq t \leq 1} |f(\gamma(t))| = |f(\beta)|.$$

When all the roots of $f(z)$ are real, we have $f(z) \in \mathbb{R}[z]$ and we can give a very simple proof of the theorem. To this end, we may assume that the roots are labelled in increasing order $\alpha_1 < \cdots < \alpha_s$. Then, in each interval $[\alpha_j, \alpha_{j+1}]$ with $1 \leq j \leq s-1$, the function $|f(z)|$ achieves its maximum in a zero β_j of $f'(z)$ with $\alpha_j < \beta_j < \alpha_{j+1}$. Since B has cardinality

$p \leq s-1$, this exhausts all the elements of B: we have $p = s-1$ and $m_1 = \cdots = m_{s-1} = 1$. We take for G the graph with set of vertices A, whose edges are the pairs $\{\alpha_j, \alpha_{j+1}\}$ indexed by β_j for $j = 1, \ldots, s-1$. Then G is a tree and, for each $j = 1, \ldots, s-1$, the piecewise affine linear path γ_j with $\gamma_j(0) = \alpha_j$, $\gamma_j(1/2) = \beta_j$ and $\gamma_j(1) = \alpha_{j+1}$ fulfills the conditions in (iv). Moreover its length is $\alpha_{j+1} - \alpha_j \leq 2R$.

Step 1. The proof of the general case requires several lemmas. For each $\beta \in B$, we choose once for all $m(\beta)+1$ paths $\gamma_{\beta,0}, \ldots, \gamma_{\beta,m(\beta)}$ with end point β as in Lemma 5.2. Then we have $\gamma_{\beta,j}(0) \in A$ for $j = 0, \ldots, m(\beta)$. Our goal is to show that these $m(\beta)+1$ points of A are distinct and that the graph G with vertices $\alpha_1, \ldots, \alpha_s$ and edges $\{\gamma_{\beta,0}(0), \gamma_{\beta,j}(0)\}$ with $\beta \in B$ and $1 \leq j \leq m(\beta)$ satisfies the properties (i) to (iv) from the theorem. Note that this graph is independent of the choices if and only if $m(\beta) = 1$ for each $\beta \in B$ and no pair of elements of B can be connected by a path of steepest ascent. We start with property (iv).

Lemma 6.2. *Let $\beta \in B$ and $j \in \{1, \ldots, m(\beta)\}$. Then the path $\widetilde{\gamma}$ from $\gamma_{\beta,0}(0)$ to $\gamma_{\beta,j}(0)$ given by*

$$\widetilde{\gamma}(t) = \begin{cases} \gamma_{\beta,0}(2t) & \text{if } 0 \leq t \leq 1/2, \\ \gamma_{\beta,j}(2-2t) & \text{if } 1/2 \leq t \leq 1, \end{cases}$$

is contained in $\mathcal{K}$, with length at most $2\pi RN$. Moreover, it satisfies

$$\widetilde{\gamma}(1/2) = \beta \quad \text{and} \quad \max_{0 \leq t \leq 1} |f(\widetilde{\gamma}(t)| = |f(\beta)|.$$

Proof. We have $B \subset \mathcal{K}$ by Gauss-Lucas theorem. Then, for each $\beta \in B$, Theorem 5.1 shows that the paths $\gamma_{\beta,0}$ and $\gamma_{\beta,j}$ are contained in $\mathcal{K}$ with length at most πRN. The conclusion follows since these are paths of steepest ascent for $|f|$. $\square$

Step 2. We first prove the following result where $S = \mathbb{C} \cup \{\infty\}$ stands for the Riemann sphere with its usual topology. Afterwards, we use it to construct a tree H on $A \cup B$.

Lemma 6.3. *Let $\beta \in B$ and let $m = m(\beta)$. There exist $\delta > 0$ and $m+1$ continuous functions $\gamma_0^+, \ldots, \gamma_m^+$ from $[1,\infty]$ to $S = \mathbb{C} \cup \{\infty\}$ such that*

(i) $\gamma_0^+(1) = \cdots = \gamma_m^+(1) = \beta$,
(ii) $f(\gamma_0^+(t)) = \cdots = f(\gamma_m^+(t)) = tf(\beta)$ *for each* $t \in [1,\infty]$,
(iii) $\gamma_0^+(t), \ldots, \gamma_m^+(t)$ *are* $m+1$ *distinct numbers for each* $t \in (1, 1+\delta)$.

Then, the curves $\Gamma_0^+ = \gamma_0^+([1,\infty]), \ldots, \Gamma_m^+ = \gamma_m^+([1,\infty])$ meet only at the points β and ∞ on S. Moreover, their complement $S \setminus (\Gamma_0^+ \cup \cdots \cup \Gamma_m^+)$ is the union of $m+1$ disjoint connected open subsets $\mathcal{R}_0, \ldots, \mathcal{R}_m$ of $\mathbb{C}$ such that $\gamma_{\beta,j}([0,1)) \subseteq \mathcal{R}_j$ for $j = 0, \ldots, m$.

The proof is based on Jordan curve theorem and is illustrated in Figure 1.

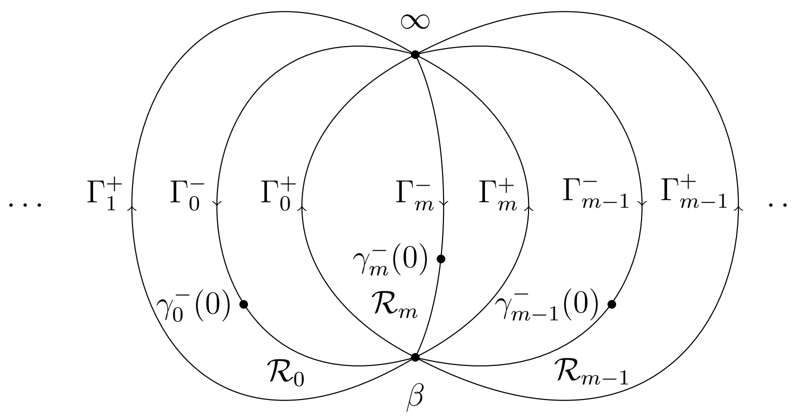

FIGURE 1. Illustration for the proof of Lemma 6.3.

Proof. Upon putting $\ell = m+1$, we may write $f(z) = f(\beta)(1 + (z-\beta)^\ell g(z))$ where $g(z)$ is a polynomial with $g(\beta) \neq 0$. Then, for sufficiently small $\epsilon > 0$, there exist an open neighborhood V of β and a biholomorphic function h from V to $B(0,\epsilon) = \{z \in \mathbb{C}\,;\, |z| < \epsilon\}$ satisfying $h(\beta) = 0$ and

$$f(z) = f(\beta)(1 + h(z)^\ell)$$

for each $z \in V$. Fix such a choice of ϵ, V and h, and set $\delta = \epsilon^\ell$ and $\rho = e^{\pi i/\ell}$. For $j = 0, \dots, m$, we define a continous function $\gamma_j^+ \colon [1, 1+\delta) \to V$ by

$$\gamma_j^+(t) = h^{-1}\left(\rho^{2j}(t-1)^{1/\ell}\right) \quad (1 \le t < 1+\delta). \tag{6.3}$$

Then, for fixed $t \in (1, 1+\delta)$, the numbers $z = \gamma_0^+(t), \dots, \gamma_m^+(t)$ are the ℓ distinct solutions of $f(z) = tf(\beta)$ with $z \in V$. In particular, $\gamma_0^+, \dots, \gamma_m^+$ satisfy Conditions (i) and (iii) of the lemma, as well as (ii) for each $t \in [1, 1+\delta)$. For $j = 0, \dots, m$, we extend γ_j^+ to a continuous function $\gamma_j^+ \colon [1,\infty] \to S$ satisfying $f(\gamma_j^+(t)) = tf(\beta)$ for each $t \in [1,\infty]$.

Similarly, for $j = 0, \dots, m$, we define a continuous function $\gamma_j^- \colon (1-\delta, 1] \to V$ by

$$\gamma_j^-(t) = h^{-1}\left(\rho^{2j+1}(1-t)^{1/\ell}\right) \quad (1-\delta < t \le 1).$$

For fixed $t \in (1-\delta, 1)$, the numbers $z = \gamma_0^-(t), \dots, \gamma_m^-(t)$ are the ℓ distinct solutions of $f(z) = tf(\beta)$ with $z \in V$, thus they form a permutation of $\gamma_{\beta,0}(t), \dots, \gamma_{\beta,m}(t)$. This permutation being independent of t, there is no loss of generality in assuming that γ_j^- is the restriction of $\gamma_{\beta,j}$ to $(1-\delta, 1]$ for $j = 0, \dots, m$. Then we extend each $\gamma_{\beta,j} \colon [0,1] \to \mathbb{C}$ to a continuous function $\gamma_j^- \colon [-\infty, 1] \to S$ such that $f(\gamma_j^-(t)) = tf(\beta)$ for each $t \in [-\infty, 1]$.

Put $\Gamma_j^- = \gamma_j^-([-\infty, 1])$ and $\Gamma_j^+ = \gamma_j^+([1,\infty])$ for $j = 0, \dots, m$, and fix $j, k \in \{0, 1, \dots, m\}$. The curves Γ_j^- and Γ_k^+ meet only at the points β and ∞ because if $\gamma_j^-(t) = \gamma_k^+(u)$ for some $t \in [-\infty, 1]$ and $u \in [1,\infty]$, then $tf(\beta) = uf(\beta)$, thus $t = u = 1$ or $-t = u = \infty$. Suppose now that $j < k$. As the curves Γ_j^+ and Γ_k^+ meet at infinity, there exists a smallest $r \in [1+\delta, \infty]$ such that $\gamma_j^+(r) = \gamma_k^+(r)$. For this choice of r, the union $\gamma_j^+([1,r]) \cup \gamma_k^+([1,r])$ is a simple closed curve Γ. By Jordan curve theorem, its complement in S is thus the union of two connected open sets $\mathcal{R}$ and $\mathcal{R}'$ with boundary Γ. On the other hand, the map $h\colon V \to B(0,\epsilon)$ is a homeomorphism and, in view of (6.3), we find that

$$V \cap \Gamma = \gamma_j^+([1, 1+\delta)) \cup \gamma_k^+([1, 1+\delta)) = h^{-1}(P) \quad \text{where} \quad P = [0,\epsilon)\rho^{2j} \cup [0,\epsilon)\rho^{2k}.$$

As P is the union of two rays in $B(0,\epsilon)$ making angles $2\pi j/\ell$ and $2\pi k/\ell$ with respect to the real axis, its complement $B(0,\epsilon) \setminus P$ is the union of two disjoint connected open sets $\mathcal{U}$ and $\mathcal{U}'$ which are open sectors of the disk $B(0,\epsilon)$. One of them, say $\mathcal{U}$, contains the rays $(0,\epsilon)\rho^{2i+1}$ with $j \le i < k$ while the other $\mathcal{U}'$ contains those with $0 \le i < j$ or $k \le i \le m$. As h is a homeomorphism, $h^{-1}(\mathcal{U})$ and $h^{-1}(\mathcal{U}')$ are disjoint connected open subsets of S whose union is $V \setminus \Gamma$. We may assume that $h^{-1}(\mathcal{U}) \subset \mathcal{R}$ and $h^{-1}(\mathcal{U}') \subset \mathcal{R}'$. Then, we obtain

$$\gamma_i^-((1-\delta, 1)) = h^{-1}((0,\epsilon)\rho^{2i+1}) \subseteq \begin{cases} \mathcal{R} & \text{if } j \le i < k, \\ \mathcal{R}' & \text{else.} \end{cases}$$

However, $\mathcal{R}$ and $\mathcal{R}'$ share the same boundary, contained in $\Gamma_j^+ \cup \Gamma_k^+$. Thus none of the sets $\Gamma_i^- \setminus \{\beta, \infty\} = \gamma_i^-((-\infty, 1))$ meet this boundary. As these are connected curves, we conclude that $\Gamma_i^- \setminus \{\beta, \infty\}$ is contained in $\mathcal{R}$ if $j \le i < k$ and in $\mathcal{R}'$ otherwise. In particular, none of the open subsets $\mathcal{R}$ and $\mathcal{R}'$ of $\mathbb{C}$ is bounded and consequently we must have $r = \infty$. This means that Γ_j^+ and Γ_k^+ meet only at β and ∞.

With the above notation, we define $\mathcal{R}_j = \mathcal{R}$ for the choice of $j \in \{0, \dots, m-1\}$ and $k = j+1$. We also define $\mathcal{R}_m = \mathcal{R}'$ for the choice of $j = 0$ and $k = m$. These are connected open subsets of $\mathbb{C}$ with $\gamma_{\beta,j}([0,1)) \subset \Gamma_j^- \setminus \{\beta, \infty\} \subset \mathcal{R}_j$ for $j = 0, \dots, m$. It remains to show that $\mathcal{R}_0, \dots, \mathcal{R}_m$ pairwise disjoint. To this end, we first note that if $j \ne k$, then $\mathcal{R}_j \not\subseteq \mathcal{R}_k$ since $\Gamma_j^- \setminus \{\beta, \infty\}$ is contained in $\mathcal{R}_j$ but not in $\mathcal{R}_k$. So if $\mathcal{R}_j$ and $\mathcal{R}_k$ intersect, then $\mathcal{R}_j$ meets the boundary of $\mathcal{R}_k$. Then $\mathcal{R}_j$ contains at least one point of $\Gamma_i^+ \setminus \{\beta, \infty\}$ for some $i \in \{0, 1, \dots, m\}$. However, by the choice of $\mathcal{R}_j$, we have $\gamma_i^+(t) \notin \mathcal{R}_j$ for each $t \in (1, 1+\delta)$. Thus the curve $\Gamma_i^+ \setminus \{\beta, \infty\}$ is not fully contained in $\mathcal{R}_j$ and, as it is a connected set, it meets the boundary of $\mathcal{R}_j$ without being fully contained in it. This is impossible because that boundary is the union of two curves among $\Gamma_0^+, \dots, \Gamma_m^+$. □

Lemma 6.4. *For each $\beta \in B$, the $m(\beta)+1$ points $\gamma_{\beta,j}(0) \in A$ with $0 \le j \le m(\beta)$ are distinct. Moreover, let H be the graph whose set of vertices is $A \cup B$ and whose edges are the pairs $\{\beta, \gamma_{\beta,j}(0)\}$ with $\beta \in B$ and $0 \le j \le m(\beta)$. Then H is a tree.*

Proof. The first assertion is a direct consequence of the preceding lemma because, for $\beta \in B$ and $m = m(\beta)$, this lemma provides disjoint connected open sets $\mathcal{R}_0, \dots, \mathcal{R}_m$ such that $\gamma_{\beta,j}(0) \in \mathcal{R}_j$ for $j = 0, \dots, m$.

To begin with, suppose that H is not a forest. Then H contains a simple cycle: an elementary chain $(a_1, \dots, a_k)$ with $k \ge 3$ such that $\{a_k, a_1\}$ is an edge of H. Then, k is an even integer and the a_i's belong alternatively to A or B according to the parity of i. By permuting cyclicly the elements of this chain if necessary, we may assume that $a_1 \in B$ and that $|f(a_1)| \ge |f(a_i)|$ for $i = 1, \dots, k$. Let $m = m(a_1)$ and let $\mathcal{R}_0, \dots, \mathcal{R}_m$ be the connected open sets associated to the point $a_1 \in B$ by Lemma 6.3. For each point $z \neq a_1$ outside of these open sets, we have $f(z) = tf(a_1)$ for a real number $t > 1$, thus $|f(z)| > |f(a_1)|$. We set $a_{k+1} = a_1$ and, for $i = 1, \dots, k$, we denote by γ_i the path of the form $\gamma_{\beta,j}$ which links a_i and a_{i+1}. For each $t \in [0, 1]$, we have $f(\gamma_i(t)) = tf(a_i)$ if i is odd and $f(\gamma_i(t)) = tf(a_{i+1})$ if i is even. In both cases, this yields $|f(\gamma_i(t))| \leq |f(a_1)|$, with the strict inequality if $t \neq 1$. As $a_1, \dots, a_k$ are distinct and as $\gamma_i(1) \in \{a_3, \dots, a_{k-1}\}$ when $2 \leq i \leq k-1$, we deduce that the curve

$$\Gamma = \gamma_1([0,1)) \cup \gamma_2([0,1]) \cup \cdots \cup \gamma_{k-1}([0,1]) \cup \gamma_k([0,1))$$

is contained in $\mathcal{R}_0 \cup \cdots \cup \mathcal{R}_m$. As this is a connected subset of $\mathbb{C}$, it is therefore fully contained in $\mathcal{R}_j$ for some j. Since $\gamma_1(1) = \gamma_k(1) = a_1$, this implies that $\gamma_1 = \gamma_k$, thus $a_2 = \gamma_1(0) = \gamma_k(0) = a_k$, which is impossible.

So H is a forest. Therefore, its number of connected components is equal to its number of vertices minus its number of edges, that is

$$|A \cup B| - \sum_{\beta \in B} (m(\beta) + 1) = s - \sum_{\beta \in B} m(\beta) = 1.$$

Thus H is connected and so it is a tree. $\square$

Step 3. Proof of Theorem 6.1. Let G be the graph whose set of vertices is A and whose edges are the pairs

$$\{\gamma_{\beta,0}(0), \gamma_{\beta,j}(0)\} \quad \big(\beta \in B,\ 1 \leq j \leq m(\beta)\big). \tag{6.4}$$

Since H is connected, so is the graph G. Since G possesses $s = |A|$ vertices and since $\sum_{\beta \in B} m(\beta) = s - 1$, we deduce that the $s-1$ edges (6.4) are distinct and that G is a tree. In particular, for each $\beta \in B$, there are exactly $m(\beta)$ edges of G indexed by β and Lemma 6.2 shows that, for each of them, there exists a path satisfying Condition (iv) of the theorem.

7. Computation of a semi-resultant

We first prove the following formula.

Proposition 7.1. *With the notation of the preceding section, we have*

$$N^N \prod_{j=1}^{p} f(\beta_j)^{m_j} = \prod_{i=1}^{s} \Big(n_i^{n_i} \prod_{k \neq i} (\alpha_i - \alpha_k)^{n_k} \Big).$$

The left hand side of this equality is the semi-resultant of $f(z)$ and $f'(z)$ in the sense of Chudnovsky [5, 7].

Proof. The formula for the derivative of a product applied to the factorization (6.1) of $f(z)$ yields

$$f'(z) = (z - \alpha_1)^{n_1 - 1} \cdots (z - \alpha_s)^{n_s - 1} g(z)$$

where

$$g(z) = \sum_{k=1}^{s} n_k \prod_{i\neq k} (z-\alpha_i).$$

By comparison with the factorization (6.2) of $f'(z)$, we also find that

$$g(z) = N(z-\beta_1)^{m_1}\cdots(z-\beta_p)^{m_p}.$$

Upon evaluating both expressions for $g(z)$ at $z=\alpha_k$, we obtain

$$N\prod_{j=1}^{p}(\alpha_k-\beta_j)^{m_j} = n_k\prod_{i\neq k}(\alpha_k-\alpha_i) \quad (1\le k\le s).$$

Since $m_1+\cdots+m_p = s-1$, these equalities may be rewritten as

$$N\prod_{j=1}^{p}(\beta_j-\alpha_k)^{m_j} = n_k\prod_{i\neq k}(\alpha_i-\alpha_k) \quad (1\le k\le s).$$

As stated, this yields

$$\begin{aligned} N^N\prod_{j=1}^{p} f(\beta_j)^{m_j} &= N^N\prod_{j=1}^{p}\Big(\prod_{k=1}^{s}(\beta_j-\alpha_k)^{n_k}\Big)^{m_j} \\ &= \prod_{k=1}^{s}\Big(N\prod_{j=1}^{p}(\beta_j-\alpha_k)^{m_j}\Big)^{n_k} \\ &= \prod_{k=1}^{s}\Big(n_k\prod_{i\neq k}(\alpha_i-\alpha_k)\Big)^{n_k} = \prod_{i=1}^{s}\Big(n_i^{n_i}\prod_{k\neq i}(\alpha_i-\alpha_k)^{n_k}\Big). \end{aligned}$$

□

Corollary 7.2. *With the same notation, we have*

$$N!\prod_{j=1}^{p}\big|f(\beta_j)\big|^{m_j} \le \prod_{i=1}^{s}\Big(n_i!\prod_{k\neq i}|\alpha_i-\alpha_k|^{n_k}\Big).$$

Proof. Since $N = n_1+\cdots+n_s$, we find

$$\frac{N!}{n_1!\cdots n_s!}\Big(\frac{n_1}{N}\Big)^{n_1}\cdots\Big(\frac{n_s}{N}\Big)^{n_s} \le \Big(\frac{n_1}{N}+\cdots+\frac{n_s}{N}\Big)^{N} = 1.$$

This yields $N!\prod_{i=1}^{s} n_i^{n_i} \le N^N\prod_{i=1}^{s} n_i!$, and the conclusion follows. □

8. Volume of the Archimedean components

We are now ready to prove the upper bound estimate in Theorem 2.3 (i). The notation is as in Section 2.

Theorem 8.1. *Let v be an Archimedean place of K and let $\mathcal{C}_{\mathbf{n},v}$ be the convex body of K_v^s defined in Section 2.3 for the choice of an s-tuple $\mathbf{n} = (n_1,\dots,n_s)\in\mathbb{N}_+^s$. Then, we have*

$$\mu_v(\mathcal{C}_{\mathbf{n},v})^{1/d_v} \le c_v N^{2s-2}|\Delta_{\mathbf{n}}|_v \quad \text{with} \quad c_v = 2^s e^{sR_v}(2\pi R_v^s)^{s-1}|\Delta_{\mathbf{1}}|_v^{-1},$$

where $N = n_1+\cdots+n_s$, $R_v = \max_{1\le i<j\le s}|\alpha_i-\alpha_j|_v$, and $\mathbf{1} = (1,\dots,1)$.

Proof. To simplify, we may assume that $K \subset \mathbb{C}$ and that $|\alpha|_v = |\alpha|$ for each $\alpha \in K$. By permuting $\alpha_1, \dots, \alpha_s$ if necessary, we may also assume that $n_1 \geq \cdots \geq n_s$ form a non-increasing sequence. We denote by D the closed disk of radius R_v and center $(\alpha_1 + \cdots + \alpha_s)/s$ in $\mathbb{C}$. As this disk contains $\alpha_1, \dots, \alpha_s$, it also contains the convex hull $\mathcal{K}$ of these points.

Suppose first that $n_1 \geq 2$ and let r be the largest index such that $n_r \geq 2$. We form the polynomial

$$f(z) = \frac{f_{\mathbf{n}}(z)}{(z-\alpha_1)\cdots(z-\alpha_s)} = \prod_{i=1}^{r}(z-\alpha_i)^{n_i-1}.$$

The set of its roots is $A = \{\alpha_1, \dots, \alpha_r\}$ and its degree is $N - s$. Its derivative factors as

$$f'(z) = (N-s)(z-\alpha_1)^{n_1-2}\cdots(z-\alpha_r)^{n_r-2}(z-\beta_1)^{m_1}\cdots(z-\beta_p)^{m_p}$$

where $B = \{\beta_1, \dots, \beta_p\}$ is the set of roots of $f'(z)$ outside of A, and where m_j is the multiplicity of β_j for $j = 1, \dots, p$. We choose a tree G as in Theorem 6.1 for this polynomial $f(z)$. By construction, the set of vertices of G is A. We now extend G to a graph $\widetilde{G}$ on $\{\alpha_1, \dots, \alpha_s\}$ in the following way. For each $j = r+1, \dots, s$, we choose a path $\gamma_j \colon [0,1] \to \mathbb{C}$ such that $\gamma_j(1) = \alpha_j$ and $f(\gamma_j(t)) = tf(\alpha_j)$ as in Theorem 5.1. Then $\gamma_j(0)$ is a root of f, thus an element of A, and we add the edge $\{\gamma_j(0), \alpha_j\}$ to the graph G. Finally, we choose $\alpha_1 \in A$ as a root of the resulting tree $\widetilde{G}$. As explained in Section 4, this turns $\widetilde{G}$ into an oriented graph. Let $\widetilde{E}$ denote the set of oriented edges of $\widetilde{G}$. By construction, $\mathcal{C}_{\mathbf{n},v}$ is contained in the set $\widetilde{\mathcal{C}}_{\mathbf{n},v}$ of all points $(x_1, \dots, x_s) \in K_v^s$ satisfying

$$|x_1|_v \leq e^{R_v}(N-1)!$$

as well as

$$|x_i e^{\alpha_j - \alpha_i} - x_j|_v \leq b_{i,j} := \max_{1 \leq k \leq s}\left|\int_{\alpha_i}^{\alpha_j} f_{\mathbf{n}-\mathbf{e}_k}(z)e^{\alpha_j - z}dz\right|$$

for each directed edge (α_i, α_j) in $\widetilde{E}$. By Proposition 4.1, the s linear forms defining $\widetilde{\mathcal{C}}_{\mathbf{n},v}$ are linearly independent with determinant 1, in some ordering. Thus $\widetilde{\mathcal{C}}_{\mathbf{n},v}$ is a convex body of K_v^s with

$$\mu_v(\mathcal{C}_{\mathbf{n},v})^{1/d_v} \leq \mu_v(\widetilde{\mathcal{C}}_{\mathbf{n},v})^{1/d_v} \leq 2^s e^{R_v}(N-1)! \prod_{(\alpha_i,\alpha_j)\in\widetilde{E}} b_{i,j}. \tag{8.1}$$

For now, fix $(\alpha_i, \alpha_j) \in \widetilde{E}$ and $k \in \{1, \dots, s\}$. By construction, we have $i \leq r$, that is $\alpha_i \in A$. If $j \leq r$, we also have $\alpha_j \in A$, and $\{\alpha_i, \alpha_j\}$ is an edge of G. Then, Theorem 6.1 associates to this edge a point $\beta \in B$ and a path $\gamma \colon [0,1] \to \mathbb{C}$ of length at most $2\pi R_v N$, contained in $\mathcal{K}$, joining α_i and α_j, such that

$$\max_{0 \leq t \leq 1}|f(\gamma(t))| = |f(\beta)|.$$

Using a hat to indicate that a factor is omitted in a product, this yields

$$\begin{aligned}\left|\int_{\alpha_i}^{\alpha_j} f_{\mathbf{n}-\mathbf{e}_k}(z)e^{\alpha_j-z}dz\right| &= \left|\int_{\alpha_i}^{\alpha_j} f(z)(z-\alpha_1)\cdots\widehat{(z-\alpha_k)}\cdots(z-\alpha_s)e^{\alpha_j-z}dz\right|\\ &\le 2\pi R_v N|f(\beta)| \max_{z\in\mathcal{K}}\left|(z-\alpha_1)\cdots\widehat{(z-\alpha_k)}\cdots(z-\alpha_s)e^{\alpha_j-z}\right|\\ &\le 2\pi R_v^s e^{R_v} N|f(\beta)|\,,\end{aligned}$$

since $|z-\alpha_\ell| \le R_v$ for any $z \in \mathcal{K}$ and $\ell = 1,\dots,s$. Finally, if $j > r$, we have $\alpha_i = \gamma_j(0)$ for the path γ_j chosen earlier. By Theorem 5.1, the image of γ_j is contained in $\mathcal{K}$, of length at most $\pi R_v N \le 2\pi R_v N$. Thus the same computation as above yields

$$\left|\int_{\alpha_i}^{\alpha_j} f_{\mathbf{n}-\mathbf{e}_k}(z)e^{\alpha_j-z}dz\right| \le 2\pi R_v^s e^{R_v} N|f(\alpha_j)|\,.$$

Since each β_j is associated to m_j edges of G and since $\widetilde{E}$ has cardinality $s-1$, we deduce from (8.1) that

$$\mu_v(\mathcal{C}_{\mathbf{n},v})^{1/d_v} \le 2^s e^{R_v}(N-1)!\big(2\pi R_v^s e^{R_v} N\big)^{s-1} \prod_{j=1}^{p} |f(\beta_j)|^{m_j} \prod_{j=r+1}^{s} |f(\alpha_j)|\,. \tag{8.2}$$

As $n_k = 1$ for $k > r$, Corollary 7.2 gives

$$(N-s)!\prod_{j=1}^{p}\big|f(\beta_j)\big|^{m_j} \le \prod_{i=1}^{r}\Big((n_i-1)!\prod_{k\neq i}|\alpha_i-\alpha_k|^{n_k-1}\Big).$$

For $i = r+1,\dots,s$, we also have $n_i = 1$ and so

$$|f(\alpha_i)| = \prod_{k=1}^{r}|\alpha_i-\alpha_k|^{n_k-1} = (n_i-1)!\prod_{k\neq i}|\alpha_i-\alpha_k|^{n_k-1}.$$

Using (2.5), this implies that

$$(N-s)!\prod_{j=1}^{p}|f(\beta_j)|^{m_j}\prod_{j=r+1}^{s}|f(\alpha_j)| \le \prod_{i=1}^{s}\Big((n_i-1)!\prod_{k\neq i}|\alpha_i-\alpha_k|^{n_k-1}\Big) = \frac{|\Delta_{\mathbf{n}}|_v}{|\Delta_{\mathbf{1}}|_v}.$$

Substituting this upper bound in (8.2), we conclude that $\mu_v(\mathcal{C}_{\mathbf{n},v})^{1/d_v} \le c_v N^{2s-2}|\Delta_{\mathbf{n}}|_v$, as in the statement of the theorem. ☐

9. A forest at ultrametric places

Let v be an ultrametric place of K. In this section we use the terminology for graphs explained in Section 4 to build a rooted forest on an arbitrary non-empty finite subset of K_v. We start with a preliminary construction.

Proposition 9.1. *Let A be a non-empty finite subset of K_v and let $\alpha_0 \in A$. There exists a tree G rooted in α_0 having A as its set of vertices, such that, for each $\alpha, \beta, \gamma \in A$ with $\beta \in S_G(\alpha)$, we have*

$$\gamma \in D_G(\beta) \iff |\alpha-\beta|_v > |\beta-\gamma|_v > 0. \tag{9.1}$$

Proof. We proceed by induction on the cardinality $|A|$ of A. If $|A| = 1$, there is nothing to prove. Suppose that $|A| \geq 2$. Let ρ be the largest distance between two elements of A, and let $\{\alpha_0, \dots, \alpha_k\}$ be a maximal subset of A containing α_0, whose elements are at mutual distance $|\alpha_i - \alpha_j|_v = \rho$ for $0 \leq i < j \leq k$. Since v is ultrametric, we have $k \geq 1$ and the sets

$$A_i := \{\beta \in A\,;\, |\alpha_i - \beta|_v < \rho\} \quad (0 \leq i \leq k)$$

form a partition of A. For $i = 0, \dots, k$, we have $\alpha_i \in A_i$ and $|A_i| < |A|$, thus we may assume the existence of a rooted tree $G_i = (\alpha_i, A_i, E_i)$ which fulfils Condition (9.1) for each choice of $\alpha, \beta, \gamma \in A_i$ with $\beta \in S_{G_i}(\alpha)$. We set

$$E = E_0 \cup \cdots \cup E_k \cup \{\{\alpha_0, \alpha_1\}, \dots, \{\alpha_0, \alpha_k\}\}.$$

Then $G = (\alpha_0, A, E)$ is a rooted tree. Let $\alpha, \beta, \gamma \in A$ with $\beta \in S_G(\alpha)$, and let i be the index for which $\alpha \in A_i$. If $\beta \in A_i$, then $\beta \in S_{G_i}(\alpha)$ and $D_G(\beta) = D_{G_i}(\beta)$, thus

$$\gamma \in D_G(\beta) \iff \gamma \in D_{G_i}(\beta) \iff |\alpha - \beta|_v > |\beta - \gamma|_v > 0.$$

If instead $\beta \in A_j$ for some $j \neq i$, then we must have $i = 0$, $\alpha = \alpha_0$ and $\beta = \alpha_j$. Then $|\alpha - \beta|_v = \rho$ and $D_G(\beta) = A_j \setminus \{\alpha_j\}$. So we find

$$\gamma \in D_G(\beta) \iff \rho > |\alpha_j - \gamma|_v > 0 \iff |\alpha - \beta|_v > |\beta - \gamma|_v > 0.$$

Thus G has the required property. □

As the proof shows, the graph G constructed in this way is not unique in general (since the choice $\alpha_1, \dots, \alpha_k \in A$ is not unique). This leads to the following construction which in general is not unique either.

Theorem 9.2. *Let A be a non-empty finite subset of K_v, let $\delta > 0$, and let R be a maximal subset of A whose elements are at mutual distance at least δ. Then, there exists a rooted forest G having A as its set of vertices and R as its set of roots, which satisfies the following properties:*

(i) *for any $\beta \in R$ and $\gamma \in A$, we have*

$$\gamma \in D_G(\beta) \iff \delta > |\beta - \gamma|_v > 0;$$

(ii) *for any $\alpha, \beta, \gamma \in A$ with $\beta \in S_G(\alpha)$, we have*

$$\gamma \in D_G(\beta) \iff |\alpha - \beta|_v > |\beta - \gamma|_v > 0.$$

Proof. For each $\rho \in R$, we define

$$A^{(\rho)} = \{\alpha \in A\,;\, |\alpha - \rho|_v < \delta\},$$

and we choose a rooted tree $G^{(\rho)} = (\rho, A^{(\rho)}, E^{(\rho)})$ as in Proposition 9.1. Since the sets $A^{(\rho)}$ with $\rho \in R$ form a partition of A, the union of these graphs constitutes a rooted forest $G = (R, A, E)$ where $E = \cup_{\rho \in R} E^{(\rho)}$. By construction, it satisfies Condition (i). To show that Condition (ii) is also fulfilled, fix $\alpha, \beta, \gamma \in A$ with $\beta \in S_G(\alpha)$, and let $\rho \in R$ such that $\alpha \in A^{(\rho)}$. Since $\beta \in S_G(\alpha)$, we have $\beta \in A^{(\rho)}$ and $D_G(\beta) = D_{G^{(\rho)}}(\beta)$. Moreover, if γ satisfies

$|\alpha-\beta|_v > |\beta-\gamma|_v$ then $|\beta-\gamma|_v < \delta$ and so $\gamma \in A^{(\rho)}$. Thus Condition (ii) for α, β, γ is satisfied in G since it is satisfied in $G^{(\rho)}$. □

In terms of elementary chains, Conditions (i) and (ii) of the theorem can be reformulated as follows: given $\gamma \in A$, a sequence $(\gamma_1, \dots, \gamma_k)$ in G, with $k \geq 1$ and $\gamma_k \neq \gamma$, starting on a root $\gamma_1 \in R$, can be extended to an elementary chain $(\gamma_1, \dots, \gamma_\ell)$ ending on $\gamma_\ell = \gamma$ if and only if either we have $k = 1$ and $\delta > |\gamma_1 - \gamma|_v > 0$ or the sequence $(\gamma_1, \dots, \gamma_k)$ is an elementary chain with $k \geq 2$ and $|\gamma_{k-1} - \gamma_k|_v > |\gamma_k - \gamma|_v > 0$.

10. Volume of the ultrametric components

We now complete the proof of Theorem 2.3 by proving the remaining estimates in parts (ii) and (iii). The notation is as in Section 2.

Theorem 10.1. *Let v be a place of K above a prime number p, let $\mathbf{n} = (n_1, \dots, n_s) \in \mathbb{N}_+^s$ and let $N = n_1 + \cdots + n_s$. Then the sub-$\mathcal{O}_v$-module $\mathcal{C}_{\mathbf{n},v}$ of K_v^s defined in Section 2.3 satisfies*

$$\mu_v(\mathcal{C}_{\mathbf{n},v})^{1/d_v} \leq (p^3 N)^s |\Delta_{\mathbf{n}}|_v \,.$$

Moreover, if $|\alpha_i - \alpha_j|_v = 1$ for each $i, j \in \{1, \dots, s\}$ with $i \neq j$, then we also have

$$\mu_v(\mathcal{C}_{\mathbf{n},v})^{1/d_v} = |\Delta_{\mathbf{n}}|_v.$$

Proof. We apply Theorem 9.2 to the set $A = \{\alpha_1, \dots, \alpha_s\}$ with $\delta = p^{-1/(p-1)}$. It provides a rooted forest G with set of roots R, set of vertices A, and set of edges E (possibly empty). For each $\alpha \in A$, we define $x_\alpha = x_i$ and $n_\alpha = n_i$ where i is the index for which $\alpha = \alpha_i$. Then, $\mathcal{C}_{\mathbf{n},v}$ is contained in the set $\widetilde{\mathcal{C}}_{\mathbf{n},v}$ of points $(x_1, \dots, x_s) \in K_v^s$ satisfying

$$|x_\beta|_v \leq p^3 N \prod_{\gamma \in A} \max\{|\beta - \gamma|_v,\, \delta\}^{n_\gamma}$$

for each root $\beta \in R$, as well as

$$|x_\alpha e^{\beta-\alpha} - x_\beta|_v \leq p^3 N \prod_{\gamma \in A} \max\{|\alpha - \gamma|_v,\, |\beta - \gamma|_v\}^{n_\gamma}$$

for each directed edge $(\alpha, \beta) \in E$ or equivalently for each pair $\{\alpha, \beta\}$ with $\beta \in S_G(\alpha)$ (since we then have $|\beta - \alpha|_v < \delta$). By Proposition 4.1, the above s linear forms are linearly independent with determinant 1, in some ordering. So $\widetilde{\mathcal{C}}_{\mathbf{n},v}$ is a free sub-$\mathcal{O}_v$-module of K_v^s of rank s with

$$\mu_v(\mathcal{C}_{\mathbf{n},v})^{1/d_v} \leq \mu_v(\widetilde{\mathcal{C}}_{\mathbf{n},v})^{1/d_v} \leq (p^3 N)^s \Delta' \Delta''$$

where

$$\Delta' = \prod_{\substack{\beta \in R \\ \gamma \in A}} \max\{|\beta - \gamma|_v,\, \delta\}^{n_\gamma} \quad \text{and} \quad \Delta'' = \prod_{\substack{\alpha,\beta,\gamma \in A \\ \beta \in S_G(\alpha)}} \max\{|\alpha - \gamma|_v,\, |\beta - \gamma|_v\}^{n_\gamma}.$$

Let $\beta, \gamma \in A$. If $\beta \in R$, Theorem 9.2 (i) yields

$$\max\{|\beta-\gamma|_v,\, \delta\} = \begin{cases} \delta & \text{if } \gamma \in D_G(\beta) \cup \{\beta\}, \\ |\beta-\gamma|_v & \text{else.} \end{cases} \tag{10.1}$$

Otherwise, there exists a unique $\alpha \in A$ such that $\beta \in S_G(\alpha)$ and, since

$$|\alpha-\gamma|_v > |\beta-\gamma|_v \quad \Longleftrightarrow \quad |\alpha-\beta|_v > |\beta-\gamma|_v,$$

Theorem 9.2 (ii) yields

$$\max\{|\alpha-\gamma|_v,\, |\beta-\gamma|_v\} = \begin{cases} |\alpha-\gamma|_v & \text{if } \gamma \in D_G(\beta) \cup \{\beta\}, \\ |\beta-\gamma|_v & \text{else.} \end{cases} \tag{10.2}$$

Since $D_G(\beta) \cup \{\beta\}$ runs through all connected components of G as β runs through R and since we have $\sum_{\gamma \in A} n_\gamma = N$, the equality (10.1) implies that

$$\Delta' = \delta^N \prod_{\substack{\beta \in R \\ \gamma \notin D_G(\beta) \cup \{\beta\}}} |\beta-\gamma|_v^{n_\gamma}.$$

Furthermore, the equality (10.2) implies that

$$\Delta'' = \Bigg(\prod_{\substack{\alpha \in A \\ \gamma \in D_G(\alpha)}} |\alpha-\gamma|_v^{n_\gamma} \Bigg) \Bigg(\prod_{\substack{\beta \notin R \\ \gamma \notin D_G(\beta) \cup \{\beta\}}} |\beta-\gamma|_v^{n_\gamma} \Bigg)$$

As a result we obtain

$$\Delta'\Delta'' = \delta^N \prod_{\beta \in A} \prod_{\gamma \in A \setminus \{\beta\}} |\beta-\gamma|_v^{n_\gamma}.$$

Since $\delta^N = \prod_{\beta \in A} \delta^{n_\beta} \le \prod_{\beta \in A} |n_\beta!|_v \le \prod_{\beta \in A} |(n_\beta - 1)!|_v$, we conclude that

$$\mu_v(\mathcal{C}_{\mathbf{n},v})^{1/d_v} \le (p^3 N)^s \prod_{\beta \in A} \bigg| (n_\beta - 1)! \prod_{\gamma \ne \beta} (\beta-\gamma)^{n_\gamma} \bigg|_v = (p^3 N)^s |\Delta_{\mathbf{n}}|_v.$$

Finally, if $|\alpha_i - \alpha_j|_v = 1$ for each $i, j \in \{1, \dots, s\}$ with $i \ne j$, then $\mathcal{C}_{\mathbf{n},v}$ consists of all points $(x_1, \dots, x_s) \in K_v^s$ satisfying

$$|x_i|_v \le |(n_i - 1)!|_v$$

for $i = 1, \dots, s$, thus

$$\mu(\mathcal{C}_{\mathbf{n},v})^{1/d_v} = \prod_{i=1}^{s} |(n_i - 1)!|_v = |\Delta_{\mathbf{n}}|_v. \qquad \square$$

11. A special case

The adelic convex bodies $\mathcal{C}_{\mathbf{n}}$ associated to a point $(\alpha_1, \dots, \alpha_s) \in K^s$ depend only on the differences $\alpha_j - \alpha_i$ with $1 \le i < j \le s$. So, we may always assume that $\alpha_1 = 0$. Then for $s = 2$, we simply have a point $(0, \alpha) \in K^2$. The proposition below is an explicit form of Corollary 2.4 for such a point and for diagonal pairs $\mathbf{n} = (n, n) \in \mathbb{N}_+^2$. In this statement, the adelic convex body is rescaled so that its v-adic component is contained in $\mathcal{O}_v^2$ for each ultrametric place v of K. We use it afterwards to prove Propositions 1.1 and 1.2 from the introduction. The notation is the same as in Section 2.

Proposition 11.1. *Let $\alpha \in K \setminus \{0\}$, and let $\mathcal{S}$ be the finite set of places v of K with $v \mid \infty$ or $|\alpha|_v \neq 1$. For each place v of K with $v \nmid \infty$, we set $B_v = \min\{1, p^{1/(p-1)}|\alpha|_v\}$ where p is the prime number below v. We also set*

$$g = \sum_{v \in \mathcal{S}} \frac{d_v}{d} \quad \textit{and} \quad B = \prod_{v \nmid \infty} B_v^{-d_v/d}.$$

Finally, for each $n \in \mathbb{N}_+$, we denote by $\widetilde{\mathcal{C}}_n$ the adelic convex body of K^2 whose components $\widetilde{\mathcal{C}}_{n,v}$ are defined as follows.

(i) *If $v \mid \infty$, then $\widetilde{\mathcal{C}}_{n,v}$ is the set of points $(x, y) \in K_v^2$ such that*

$$|x|_v \leq n^{g-1} \frac{B^n (2n)!}{|\alpha|_v^n \, n!} \quad \textit{and} \quad |xe^\alpha - y|_v \leq n^g \frac{B^n |\alpha|_v^n}{4^n \, n!}.$$

(ii) *If $v \mid p$ for a prime number p and if $|\alpha|_v < p^{-1/(p-1)}$, then $\widetilde{\mathcal{C}}_{n,v}$ consists of the points $(x, y) \in K_v^2$ such that*

$$|x|_v \leq 1 \quad \textit{and} \quad |xe^\alpha - y|_v \leq B_v^{2n}.$$

(iii) *If $v \mid p$ for a prime number p and if $|\alpha|_v \geq p^{-1/(p-1)}$, then $\widetilde{\mathcal{C}}_{n,v} = \mathcal{O}_v^2$.*

Then we have

$$c_4 n^{-2g+1} \leq \lambda_1(\widetilde{\mathcal{C}}_n) \leq \lambda_2(\widetilde{\mathcal{C}}_n) \leq c_3 \tag{11.1}$$

for constants $c_3, c_4 > 0$ that depend only on α and K.

Proof. Let $n \in \mathbb{N}_+$. We consider the adelic convex body $\mathcal{C}_{\mathbf{n}}$ constructed in Section 2.3 for the choice of $\alpha_1 = 0$, $\alpha_2 = \alpha$ and $\mathbf{n} = (n, n)$. For an Archimedean place v of K associated to an embedding $\sigma \colon K \hookrightarrow \mathbb{C}$ and for $k = 1, 2$, we find

$$\begin{aligned} \left| \int_0^{\sigma(\alpha)} f_{\mathbf{n}-\mathbf{e}_k}^\sigma(z) e^{\sigma(\alpha)-z} dz \right| &\leq |\sigma(\alpha)| e^{|\sigma(\alpha)|} \max_{t \in [0,1]} \left| f_{\mathbf{n}-\mathbf{e}_k}^\sigma(\sigma(\alpha)t) \right| \\ &\leq e^{|\sigma(\alpha)|} |\sigma(\alpha)|^{2n} \max_{t \in [0,1]} t^{n-1}(1-t)^{n-1} \\ &= 4e^{|\alpha|_v} (|\alpha|_v/2)^{2n}. \end{aligned}$$

Thus the points (x, y) of $\mathcal{C}_{\mathbf{n},v}$ satisfy

$$|x|_v \leq e^{|\alpha|_v}(2n-1)! \quad \text{and} \quad |xe^\alpha - y|_v \leq 4e^{|\alpha|_v}(|\alpha|_v/2)^{2n}.$$

This implies that $a_v \mathcal{C}_{\mathbf{n},v} \subseteq \widetilde{\mathcal{C}}_{n,v}$ for

$$a_v = \frac{n^{g-1} B^n}{4e^{|\alpha|_v} \alpha^n (n-1)!} \in K_v^\times.$$

For each prime number p and each place v of K with $v \mid p$, we also find that $a_v \mathcal{C}_{\mathbf{n},v} \subseteq \widetilde{\mathcal{C}}_{n,v}$ for

$$a_v = \frac{p^{t_v}}{\alpha^n (n-1)!} \in K_v^\times$$

where t_v is the integer for which

$$2np^3 B_v^{-n} \le p^{t_v} < 2np^4 B_v^{-n}$$

if $v \in \mathcal{S}$, and $t_v = 0$ otherwise. This computation is based simply on the fact that $|(n-1)!|_v \ge |n!|_v \ge p^{-n/(p-1)}$. Thus we obtain $a\,\mathcal{C}_{\mathbf{n}} \subseteq \widetilde{\mathcal{C}}_n$ for the idele $a = (a_v)_v \in K_{\mathbb{A}}^\times$.

The product $\mathcal{D} = \prod_v \{x \in K_v\,;\, |x|_v \le |a_v|_v\} \subset K_{\mathbb{A}}$ is an adelic convex body of K. By the product formula applied to the principal idele $\alpha^n(n-1)! \in K^\times$, we find that the volume of $\mathcal{D}$ is

$$\mu(\mathcal{D}) = 2^{r_1}\pi^{r_2} \prod_v |a_v|_v^{d_v} = 2^{r_1}\pi^{r_2} \prod_{v\mid\infty} \Big(\frac{n^{g-1}B^n}{4e^{|\alpha|_v}}\Big)^{d_v} \prod_{p,\, v\mid p} p^{-t_v d_v}.$$

Since $\prod_{v\mid\infty} B^{d_v} = B^d = \prod_{v\nmid\infty} B_v^{-d_v}$, this can be rewritten as

$$\mu(\mathcal{D}) = c_1 n^{d(g-1)} \prod_{p,\, v\mid p} (p^{t_v} B_v^n)^{-d_v},$$

with $c_1 = 2^{r_1}\pi^{r_2} \prod_{v\mid\infty} (4e^{|\alpha|_v})^{-d_v}$. Since $p^{t_v}B_v^n = 1$ if $v \notin \mathcal{S}$ and $p^{t_v}B_v^n < 2np^4$ if $v \in \mathcal{S}$ and $v \mid p$, this yields

$$\mu(\mathcal{D}) \ge c_2 n^{d(g-1)} \prod_{v\in\mathcal{S}'} n^{-d_v} = c_2 n^{dg} \prod_{v\in\mathcal{S}} n^{-d_v} = c_2,$$

where $\mathcal{S}' = \{v \in \mathcal{S}\,;\, v \nmid \infty\}$ and $c_2 = c_1 \prod_{v\in\mathcal{S}'} (2p^4)^{-d_v}$. By Theorem 2.1 (with $s = 1$), we thus have $\lambda_1(\mathcal{D}) \le c_3$ where $c_3 = (2^{r_1+r_2}|D(K)|^{1/2}c_2^{-1})^{1/d}$. This means that there exists $\beta \in K^\times$ satisfying $|\beta|_v \le c_3|a_v|_v$ for all Archimedean places v of K and $|\beta|_v \le |a_v|_v$ for all other places. So, we obtain

$$\beta\mathcal{C}_{\mathbf{n}} \subseteq c_3\widetilde{\mathcal{C}}_n,$$

which yields

$$\lambda_1(\widetilde{\mathcal{C}}_n) \le \lambda_2(\widetilde{\mathcal{C}}_n) \le c_3$$

since $\beta\mathcal{C}_{\mathbf{n}}$ contains the K-linearly independent points $\beta\mathbf{a}_{\mathbf{n}-\mathbf{e}_1}, \beta\mathbf{a}_{\mathbf{n}-\mathbf{e}_2}$ of K^2. By Theorem 2.1 (with $s = 2$), this implies that

$$\lambda_1(\widetilde{\mathcal{C}}_n) \ge (2c_3)^{-1}\mu(\widetilde{\mathcal{C}}_n)^{-1/d}.$$

Finally, for each place v of K, we find that

$$\mu_v(\widetilde{\mathcal{C}}_{n,v})^{1/d_v} \le \begin{cases} 4n^{2g-1}B^{2n} & \text{if } v \mid \infty, \\ B_v^{2n} & \text{else.} \end{cases}$$

Since $B^d \prod_{v\nmid\infty} B_v^{d_v} = 1$, this implies that $\mu(\widetilde{\mathcal{C}}_n)^{1/d} \le 4n^{2g-1}$, and so (11.1) follows with $c_4 = (8c_3)^{-1}$. $\square$

Proof of Proposition 1.1. Under the hypotheses of this proposition, the field K admits a single Archimedean place ∞, induced by the inclusion $K \subset \mathbb{C}$. Moreover, in the notation of Proposition 11.1, the choice of α leads to $B_v = 1$ for any other place v of K. Thus, for each $n \in \mathbb{N}_+$, we obtain

$$\widetilde{\mathcal{C}}_n = \widetilde{\mathcal{C}}_{n,\infty} \times \prod_{v \neq \infty} \mathcal{O}_v^2,$$

where $\widetilde{\mathcal{C}}_{n,\infty}$ consists of all points (x, y) of $K_\infty^2 \subseteq \mathbb{C}^2$ satisfying

$$|x| \le n^{g-1} \frac{(2n)!}{|\alpha|^n\, n!} \quad \text{and} \quad |xe^\alpha - y| \le n^g \frac{|\alpha|^n}{4^n\, n!}.$$

Moreover, by (11.1), we have $\lambda_1(\widetilde{\mathcal{C}}_n) \ge c_4 n^{-2g+1}$ for a constant $c_4 > 0$ depending only on α and K.

Let $(x, y) \in \mathcal{O}_K^2$ with $x \neq 0$. The above implies that, for each $n \in \mathbb{N}_+$,

$$\text{if} \quad |x| < h(n) := c_4 n^{-g} \frac{(2n)!}{|\alpha|^n\, n!} \quad \text{then} \quad |xe^\alpha - y| \ge c_4 n^{-g+1} \frac{|\alpha|^n}{4^n\, n!}\,.$$

If $|x|$ is large enough, we can find an integer $n \ge 2$ such that $e^n \le h(n-1) \le |x| < h(n)$. Then we have $n \le \log |x|$ and we obtain

$$\begin{aligned} |x|\,|xe^\alpha - y| &\ge h(n-1) c_4 n^{-g+1} \frac{|\alpha|^n}{4^n\, n!} \\ &\ge c_4^2 |\alpha| n^{-2g} \binom{2n-2}{n-1} 4^{-n} \ge c_5 n^{-2g-1} \ge c_5 (\log |x|)^{-2g-1}, \end{aligned}$$

with $c_5 = c_4^2 |\alpha|/8$. Since $\mathcal{O}_K$ is a discrete subset of $\mathbb{C}$, this leaves out a finite number of values of x. To include them in the final lower bound, it suffices to replace c_5 by a sufficiently small constant $c > 0$. □

Proof of Proposition 1.2. We apply Proposition 11.1 with $K = \mathbb{Q}$ and $\alpha = 3$. In this context, we have $g = 2$ and $B = B_3^{-1} = 3^{1/2}$. For a given $n \in \mathbb{N}_+$, a simple computation shows that the Archimedean component $\widetilde{\mathcal{C}}_{n,\infty}$ of the adelic convex body $\widetilde{\mathcal{C}}_n$ satisfies

$$n\mathcal{C}_n \subseteq \widetilde{\mathcal{C}}_{n,\infty} \subseteq n^2 \mathcal{C}_n, \tag{11.2}$$

where $\mathcal{C}_n$ is the convex body of $\mathbb{R}^2$ defined in Proposition 1.2. For its ultrametric components, we find that

$$\widetilde{\mathcal{C}}_{n,3} = \{(x, y) \in \mathbb{Z}_3^2\,;\, |xe^3 - y|_3 \le 3^{-n}\}$$

and $\widetilde{\mathcal{C}}_{n,p} = \mathbb{Z}_p^2$ for each prime number $p \neq 3$. Thus the points of $\mathbb{Q}^2$ which belong to the latter components are exactly those of the lattice Λ_n in Proposition 1.2. Therefore, the minima of $\widetilde{\mathcal{C}}_n$ with respect to $\mathbb{Q}^2$ in the adelic sense are also the minima of $\widetilde{\mathcal{C}}_{n,\infty}$ with respect to Λ_n in the classical sense. In view of the inclusions (11.2), this implies that $c_4 n^{-2} \le \lambda_1(\mathcal{C}_n, \Lambda) \le \lambda_2(\mathcal{C}_n, \Lambda) \le c_3 n^2$ for the constants c_3 and c_4 given by Proposition 11.1. □

12. Numerical computations

The formulas in Appendix A allow us to compute recursively the diagonal Hermite approximations to $(1, e^3)$. In this last section, we explain how they can be used to compute efficiently the partial quotients in the continued fraction expansion of $e^3 \in \mathbb{R}$, and then to verify the inequalities (1.2) from the introduction. Our reference for continued fractions is [15, Ch. I].

Let $e^3 = [a_0, a_1, a_2, \dots]$ denote the continued fraction expansion of e^3. Its first terms are

$$e^3 = [20, 11, 1, 2, 4, 3, 1, 5, 1, 2, 16, \dots],$$

without any noticeable regularity. For each integer $n \geq 0$, we form the n-th convergent of e^3

$$\frac{p_n}{q_n} = [a_0, a_1, \dots, a_n]$$

with $p_n \in \mathbb{Z}$, $q_n \in \mathbb{N}_+$ and $\gcd(p_n, q_n) = 1$. The table below lists all integers $n \geq 1$ with $q_{n-1} \leq 10^{500\,000}$ for which

$$a_n = \max\{a_1, a_2, \dots, a_n\}.$$

For each of those integers, it provides the corresponding value of a_n as well as the value of $\log(q_{n-1})$ truncated at the first decimal place.

n	1	10	31	87	133	211	244	388	2708	8055
a_n	11	16	68	189	492	739	2566	5885	6384	10409
$\log(q_{n-1})$	0.0	9.4	34.5	97.9	151.1	256.6	297.6	475.0	3307.2	9614.8

n	9437	29508	30939	43482	91737	196440	476544
a_n	19362	21981	46602	51140	315466	546341	569869
$\log(q_{n-1})$	11258.4	34996.8	36750.6	51515.4	109063.1	233261.9	566111.1

To show how this implies the estimations (1.2), define $\psi(x) = 3\log(x)\log(\log(x))$ for each $x \geq e$. For each pair $(p, q) \in \mathbb{Z}^2$ with $q \geq 1$, there exists an integer $n \geq 1$ such that $q_{n-1} \leq q < q_n$. Then, by the property of best approximation of the convergents (Theorem of Lagrange [15, Chapter I, Theorem 5E]), we have

$$|qe^3 - p| \geq |q_{n-1}e^3 - p_{n-1}| \geq \frac{1}{q_n + q_{n-1}} \geq \frac{1}{(a_n + 2)q_{n-1}}.$$

Assuming $q \geq 3$, this implies that

$$\psi(q)q\,|qe^3 - p| \geq \frac{\psi(q_{n-1})}{a_n + 2}. \tag{12.1}$$

It is easy to check that the right hand side of (12.1) is ≥ 1 for all entries n of the table with $n \geq 10$. Thus it is also ≥ 1 for each integer $n \geq 10$ with $q_{n-1} \leq 10^{500\,000}$. A quick computation shows that this is also true for $n = 2, \dots, 9$. Thus the left hand side of (12.1) is ≥ 1 if $11 \leq q \leq 10^{500\,000}$. Finally, one checks that this is still true when $4 \leq q \leq 10$.

To compute the partial quotients a_n, put

$$C_n = \begin{pmatrix} 2n-4 & 2n-1 \\ 2n-1 & 2n+2 \end{pmatrix} \quad \text{and} \quad A_n = C_n \cdots C_1$$

for each $n \geq \mathbb{N}_+$. By Corollary A.3 in the Appendix, the rows of $(n-1)!A_n$ are Hermite's approximations $\mathbf{a}_{n-1,n}$ and $\mathbf{a}_{n,n-1}$ to $(1, e^3)$. Thus we have

$$\lim_{n\to\infty} A_n \begin{pmatrix} e^3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{12.2}$$

We also note that, for each $n \geq 2$, the matrices C_n and A_n belong to the set

$$\mathcal{M} = \left\{ \begin{pmatrix} t & u \\ t' & u' \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})\,;\, 0 \leq t < u,\ 0 \leq t' < u' \quad \text{and} \quad tu' \neq t'u \right\}.$$

This is clear for the matrices C_n. For the matrices A_n, this follows from the fact that $\mathcal{M}$ is closed under matrix multiplication.

In general, if $A = \begin{pmatrix} t & u \\ t' & u' \end{pmatrix} \in \mathcal{M}$, the ratios t/u and t'/u' admit unique continued fraction expansions

$$\frac{t}{u} = [a_0, a_1, \dots, a_\ell] \quad \text{and} \quad \frac{t'}{u'} = [a'_0, a'_1, \dots, a'_{\ell'}]$$

with $a_0 = a'_0 = 0$, $a_\ell \geq 2$ if $\ell \geq 1$, and $a'_{\ell'} \geq 2$ if $\ell' \geq 1$. Let $(a_0, \dots, a_k)$ be the common initial part of the sequences $(a_0, \dots, a_\ell)$ and $(a'_0, \dots, a'_{\ell'})$. When $k = 0$, that is when $t = 0$ or $t' = 0$ or $\lfloor u/t \rfloor \neq \lfloor u'/t' \rfloor$, we say that A is *reduced*. Then, we find that

$$A = R \begin{pmatrix} 0 & 1 \\ 1 & a_k \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix}$$

where $R \in \mathcal{M}$ is reduced, with the convention that the right hand side is R when $k = 0$. In particular, for each $n \geq 2$, we obtain

$$A_n = R_n \begin{pmatrix} 0 & 1 \\ 1 & a_{k(n)} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix}$$

for a reduced matrix $R_n \in \mathcal{M}$, integers $0 \leq k(1) \leq k(2) \leq \cdots$ and positive integers $a_1, a_2, \dots$ such that

$$C_{n+1} R_n = R_{n+1} \begin{pmatrix} 0 & 1 \\ 1 & a_{k(n+1)} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_{k(n)+1} \end{pmatrix}, \tag{12.3}$$

with the convention that the product on the right is R_{n+1} when $k(n+1) = k(n)$. By (12.2), the integers $k(n)$ go to infinity with n and so we conclude that

$$e^{-3} = [0, a_1, a_2, \dots] \quad \text{and} \quad e^3 = [a_1, a_2, \dots]$$

are the respective continued fraction expansions of e^{-3} and e^3. Therefore, to compute their partial quotients a_k, it suffices to compute recursively the matrices R_n whose coefficients are in practice much smaller then those of A_n (we may also at each step factor out the power of 3 dividing R_n). To further save computation time we do not compute exactly the integers

q_n but keep only a floating point approximation of them (in practice we use 10 significative decimal digits). In this way, it takes slightly above an hour of CPU time to produce the table using MAPLE software with a 64 bits intel i5 processor.

Appendix A. Recurrence relations

The notation being as in Section 2.2 we extend the definition of $f_{\mathbf{n}}(z)$, $P_{\mathbf{n}}(z)$ and $a_{\mathbf{n}}$ to any s-tuple $\mathbf{n} \in \mathbb{Z}^s$ by setting

$$f_{\mathbf{n}}(z) = P_{\mathbf{n}}(z) = 0 \quad \text{and} \quad a_{\mathbf{n}} = (0, \dots, 0) \quad \text{if} \quad \mathbf{n} \notin \mathbb{N}^s.$$

For each $\mathbf{n} \in \mathbb{N}_+^s$, we denote by $A_{\mathbf{n}}$ the matrix whose ℓ-th row is $\mathbf{a}_{\mathbf{n}-\mathbf{e}_\ell}$ for $\ell = 1, \dots, s$. In [9, §§IX-X], Hermite provides a recurrence formula linking $A_{\mathbf{n}+\mathbf{1}}$ to $A_{\mathbf{n}}$ where $\mathbf{1} = (1, \dots, 1)$. Here we give more general recurrence relations based on the same principle. The formula (A.1) below is due to Hermite [9, §IX, p. 230] when $\mathbf{n} \in \mathbb{N}_+^s$.

Proposition A.1. *Let* $\mathbf{n} = (n_1, \dots, n_s) \in \mathbb{N}^s$. *We have*

$$a_{\mathbf{n}} = (f_{\mathbf{n}}(\alpha_1), \dots, f_{\mathbf{n}}(\alpha_s)) + \sum_{j=1}^{s} n_j a_{\mathbf{n}-\mathbf{e}_j}\,. \tag{A.1}$$

Moreover, if $k, \ell \in \{1, \dots, s\}$ *with* $n_k \geq 1$, *we also have*

$$a_{\mathbf{n}+\mathbf{e}_\ell-\mathbf{e}_k} = a_{\mathbf{n}} + (\alpha_k - \alpha_\ell) a_{\mathbf{n}-\mathbf{e}_k}. \tag{A.2}$$

Proof. Leibniz formula for the derivative of a product gives

$$f'_{\mathbf{n}}(z) = \sum_{j=1}^{s} n_j f_{\mathbf{n}-\mathbf{e}_j}(z).$$

Taking the sum of all derivatives on both sides of this equality, we obtain

$$P_{\mathbf{n}}(z) = f_{\mathbf{n}}(z) + \sum_{j=1}^{s} n_j P_{\mathbf{n}-\mathbf{e}_j}(z)$$

and (A.1) follows. The formula (A.2) is trivial if $k = \ell$. Suppose that $k \neq \ell$ and $n_k \geq 1$ so that $\mathbf{n} - \mathbf{e}_k \in \mathbb{N}^s$. Then we find

$$f_{\mathbf{n}+\mathbf{e}_\ell-\mathbf{e}_k}(z) - f_{\mathbf{n}}(z) = (z - \alpha_\ell) f_{\mathbf{n}-\mathbf{e}_k}(z) - (z - \alpha_k) f_{\mathbf{n}-\mathbf{e}_k}(z) = (\alpha_k - \alpha_\ell) f_{\mathbf{n}-\mathbf{e}_k}(z).$$

Taking again the sum of the derivatives, this yields

$$P_{\mathbf{n}+\mathbf{e}_\ell-\mathbf{e}_k}(z) = P_{\mathbf{n}}(z) + (\alpha_k - \alpha_\ell) P_{\mathbf{n}-\mathbf{e}_k}(z)$$

and (A.2) follows. □

Corollary A.2. *Let* $\mathbf{n} = (n_1, \dots, n_s) \in \mathbb{N}_+^s$ *and* $\ell \in \{1, \dots, s\}$. *Then we have*

$$A_{\mathbf{n}+\mathbf{e}_\ell} = M_{\mathbf{n},\ell} A_{\mathbf{n}}$$

where

$$M_{\mathbf{n},\ell} = \begin{pmatrix} n_1 + (\alpha_1 - \alpha_\ell) & n_2 & \cdots & n_s \\ n_1 & n_2 + (\alpha_2 - \alpha_\ell) & \cdots & n_s \\ \vdots & \vdots & \ddots & \vdots \\ n_1 & n_2 & \cdots & n_s + (\alpha_s - \alpha_\ell) \end{pmatrix}.$$

Proof. As the entries of $\mathbf{n}$ are positive, the polynomial $f_{\mathbf{n}}$ vanishes at all points $\alpha_1, \dots, \alpha_s$ and the formulas of Proposition A.1 yield

$$a_{\mathbf{n}+\mathbf{e}_\ell-\mathbf{e}_k} = (\alpha_k - \alpha_\ell) a_{\mathbf{n}-\mathbf{e}_k} + \sum_{j=1}^{s} n_j a_{\mathbf{n}-\mathbf{e}_j} \quad (1 \le k \le s). \qquad \square$$

When $s = 2$, this provides a quick way of computing the matrices $A_{n,n}$.

Corollary A.3. *Suppose that $s = 2$, $\alpha_1 = 0$ and $\alpha_2 = \alpha \in K \setminus \{0\}$. Then, for each $n \in \mathbb{N}_+$, we have*

$$A_{n,n} = \begin{pmatrix} P_{n-1,n}(0) & P_{n-1,n}(\alpha) \\ P_{n,n-1}(0) & P_{n,n-1}(\alpha) \end{pmatrix} = (n-1)! C_n C_{n-1} \cdots C_1 \tag{A.3}$$

where

$$C_i = \begin{pmatrix} 2i - 1 - \alpha & 2i - 1 \\ 2i - 1 & 2i - 1 + \alpha \end{pmatrix} \quad (i \in \mathbb{N}_+).$$

Proof. We find that $P_{0,1}(z) = z + 1 - \alpha$ and $P_{1,0}(z) = z + 1$, thus $A_{1,1} = C_1$. In general, for an integer $n \ge 1$, the formulas of the preceding corollary give

$$A_{n+1,n+1} = \begin{pmatrix} n & n+1 \\ n & n+1+\alpha \end{pmatrix} \begin{pmatrix} n - \alpha & n \\ n & n \end{pmatrix} A_{n,n} = n C_{n+1} A_{n,n}$$

and the conclusion follows by induction on n. $\square$

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