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Matrices Whose Coefficients Are Linear Forms in Logarithms

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In this paper, we obtain a result which allows us to give a lower bound for the rank of the matrices whose coefficients are linear forms in logarithms. We give several applications of this result, one of them a generalization of the six exponentials theorem. © 1992 Academic Press, Inc.

INTRODUCTION

Let $\bar{\mathbf{Q}}$ be an algebraic closure of $\mathbf{Q}$, and let K be the field $\mathbf{C}$ or $\mathbf{C}_p$ obtained by taking the completion of $\bar{\mathbf{Q}}$ with respect to the absolute value of $\bar{\mathbf{Q}}$ which extends an archimedean or a p-adic absolute value of $\mathbf{Q}$. Also, let L be the $\mathbf{Q}$-vector subspace of K consisting of the logarithms of the non-zero elements of $\bar{\mathbf{Q}}$. According to Schanuel's conjecture, elements of L which are linearly independent over $\mathbf{Q}$ should be algebraically independent over $\mathbf{Q}$. This statement is still unproved; it is not even known whether or not there exist two elements of L which are algebraically independent over $\mathbf{Q}$. Nevertheless, there are several results supporting this conjecture. For instance, a theorem of A. Baker (Theorem 2.1 of [B1]), extended to the p-adic case by A. Brumer (Theorem 1 of [B3]), tells us that the sum $\mathbf{Q}+L$ is direct and that elements of $\mathbf{Q}+L$ which are linearly independent over $\mathbf{Q}$ are also linearly independent over $\bar{\mathbf{Q}}$ in K. Also, M. Waldschmidt has obtained a result (Theorems 1.1 and 1.1.p of [W1]) which allowed him to give a lower bound for the rank of the matrices with coefficients in L, by taking into account only the eventual relations of linear dependence over $\mathbf{Q}$ between their coefficients (Corollary 7.2 and Theorem 2.1.p of [W1]).

We generalize here the above-mentioned result of M. Waldschmidt. This allows us to give a lower bound for the rank of the matrices with coefficients in the $\bar{\mathbf{Q}}$-vector subspace $\mathscr{L}$ of K generated by $\mathbf{Q}+L$, by taking into account only the eventual relations of linear dependence over $\bar{\mathbf{Q}}$

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between their coefficients. We call linear forms in logarithms the elements of $\mathscr{L}$. A consequence of this lower bound is that a 2×3 matrix with coefficients in $\mathscr{L}$ has rank 2 if its rows as well as its columns are linearly independent over $\bar{\mathbf{Q}}$. This result, which contains the six exponentials theorem, was previously obtained by M. Waldschmidt in the special case of a 2×3 matrix with coefficients in $\bar{\mathbf{Q}}+L$ (Corollary 2.1 of [W3]). An equivalent statement is that, if x_1, x_2 (resp. y_1, y_2, y_3) are elements of K which are linearly independent over $\bar{\mathbf{Q}}$, then at least one of the six products $x_i y_j$ $(i=1, 2; j=1, 2, 3)$ does not belong to $\mathscr{L}$.

We also give two other applications of our result. The first one consists in establishing an upper bound for $\dim_{\mathbf{Q}}(V\cap \mathscr{L}^d)$, where V is a subspace of K^d. This type of study was initiated by M. Emsalem. Using the above-mentioned result of M. Waldschmidt, he showed that, for a subspace V of K^d, the dimension of $V\cap L^d$ over $\mathbf{Q}$ is finite if and only if $V\cap \mathbf{Q}^d=0$, in which case it is bounded above by nd, where $n=\dim_K(V)$ (Theorems 1 and 2 of [E2]). M. Waldschmidt showed afterwards that the bound nd could be replaced by $n(n+1)$ (Theorem 1.1 of [W2]). The remarks he makes in Section 6 of [W2] suggest refining this bound again, assuming that V is not contained in any hyperplane of K^d which is rational over $\mathbf{Q}$. This is what we do here, in our more general context where $\mathbf{Q}$ is replaced by $\bar{\mathbf{Q}}$ and L by $\mathscr{L}$. We show that the bound obtained is essentially the best up to a factor 2.

The second application consists in proving a theorem of M. Laurent (Theorem 1′ of [L1]) without a certain restrictive assumption. Using this theorem, M. Laurent has confirmed Leopoldt's conjecture in many new cases (Sect. 6 of [L1]). His result also supports a generalization of Leopoldt's conjecture due to J.-F. Jaulent. The fact that we can avoid that restriction allows us to recover a result of M. Emsalem according to which a "sufficiently big" multiplicative group satisfies Jaulent's conjecture (Corollary 2 of [E1]).

As for the proof of our main result, it rests on a recent trancendence theorem of M. Waldschmidt (Theorem 4.1 of [W3]) applied to the linear algebraic groups. This theorem is also the one M. Laurent used in proving his above-mentioned result. As a last remark, observe that Schanuel's conjecture for the logarithms permits us to compute a priori the rank of each matrix with coefficients in $\mathscr{L}$. If one could show that the number we get in this way is always equal to the rank of the matrix, this conjecture would be proved (see Proposition 4 of [R1]).

This paper is organized as follows. In Section 1, we state the transcendence result of M. Waldschmidt on which our argument is based, and we state a consequence of it which, in fact, makes Waldschmidt's result more precise. This consequence is proved in Section 3 using the language of categories which enlightens the structure of the proof; Section 2 is devoted to the construction of an appropriate category. In Section 4, we use this consequence of Waldschmidt's theorem to establish our main result. We deduce from it a lower bound for the rank of the matrices with coefficients in $\mathscr{L}$. Finally, Sections 5 and 6 are devoted to the two other applications mentioned above.

Notations

We denote by $\bar{\mathbf{Q}}$ an algebraic closure of $\mathbf{Q}$, and by K the field $\mathbf{C}$ (resp. $\mathbf{C}_p$) obtained by taking the completion of $\bar{\mathbf{Q}}$ with respect to its absolute value extending the usual archimedean absolute value of $\mathbf{Q}$ (resp. the p-adic absolute value of $\mathbf{Q}$ for which $|p| = p^{-1}$). We write $|\ |$ to denote the absolute value of K which extends by continuity the one chosen on $\bar{\mathbf{Q}}$. Then, the usual series of the logarithm defines a continuous mapping $\log: \mathscr{U} \to K$ from the open set $\mathscr{U}$ of elements x of K satisfying $|x-1| < 1$, to the field K. We denote by L the $\mathbf{Q}$-vector subspace of K generated by $\log(\bar{\mathbf{Q}} \cap \mathscr{U})$, and by ω the element of L equal to $2\pi i$ if $K = \mathbf{C}$, equal to 0 otherwise. If $K = \mathbf{C}$, L is the set of the logarithms of the non-zero elements of $\bar{\mathbf{Q}}$.

Let $F' \subset F$ be two fields, and let V be a vector space over F. Then V is also a vector space over F'. If S is a subset of V, we denote by $F' \cdot S$ the F'-vector subspace of V generated by S. By an *F'-structure on V*, we mean, as in Bourbaki (Sect. 8, No. 1, of [B2]), an F'-vector subspace V' of V such that any basis of V' over F' is a basis of V over F. Suppose V endowed with an F'-structure V'. Then, we say that an F-vector subspace T of V is *rational over F'* if it is generated (over F) by elements of V' (Sect. 8, No. 2, of [B2]). Given F-vector spaces V_1, V_2 endowed respectively with F'-structures V'_1, V'_2, we say that an F-linear mapping $f: V_1 \to V_2$ is *rational over F'* if $f(V'_1) \subset V'_2$ (Sect. 8, No. 3, of [B2]). For each integer $d \geqslant 0$, we put on the K-vector space K^d the $\mathbf{Q}$-structure $\mathbf{Q}^d$ and the $\bar{\mathbf{Q}}$-structure $\bar{\mathbf{Q}}^d$. This gives immediately the notions of a K-vector subspace of K^d which is rational over $\mathbf{Q}$ (resp. over $\bar{\mathbf{Q}}$) and of a K-linear mapping $f: K^{d_1} \to K^{d_2}$ which is rational over $\mathbf{Q}$ (resp. over $\bar{\mathbf{Q}}$).

Finally, we let $\mathscr{L} = \bar{\mathbf{Q}} + \bar{\mathbf{Q}} \cdot L$. The theorems of Baker and Brumer mentioned in the Introduction (Theorem 2.1 of [B1] and Theorem 1 of [B3]) can be stated by saying that the sum $\mathbf{Q} + L$ is direct, and that it gives a $\mathbf{Q}$-structure on the $\bar{\mathbf{Q}}$-vector space $\mathscr{L}$. We shall make use of this result under the name of Baker's theorem.

1. A Transcendence Theorem of M. Waldschmidt

We begin by expressing the result of M. Waldschmidt on which our work is based. It is Theorem 4.1 of [W3] applied to a linear algebraic group $\mathbf{G}_{\mathrm{a}}^{d_0} \times \mathbf{G}_{\mathrm{m}}^{d_1}$. In our formulation, we identify the tangent space at the neutral element of this group with $K^{d_0} \times K^{d_1}$. Then, we state a second result which specifies M. Waldschmidt's result. Its proof constitutes the object of Sections 2 and 3.

THEOREM 1 (M. Waldschmidt). *Let d_0, d_1 be integers $\geqslant 0$,*

Y be a finite dimensional $\mathbf{Q}$-vector subspace of $K^{d_0} \times K^{d_1}$ contained in $\bar{\mathbf{Q}}^{d_0} \times L^{d_1}$,

W be a K-vector subspace of $K^{d_0} \times K^{d_1}$ which is rational over $\bar{\mathbf{Q}}$,

V be a K-vector subspace of $K^{d_0} \times K^{d_1}$ containing Y and W.

If $V \neq K^{d_0} \times K^{d_1}$, there exists a surjective K-linear mapping $s\colon K^{d_0} \times K^{d_1} \to K^{d_0'} \times K^{d_1'}$ satisfying

$$s(\bar{\mathbf{Q}}^{d_0} \times 0) \subset \bar{\mathbf{Q}}^{d_0'} \times 0 \qquad \textit{and} \qquad s(0 \times \mathbf{Q}^{d_1}) \subset 0 \times \mathbf{Q}^{d_1'},$$

such that, letting $Y' = s(Y)$, $W' = s(W)$, $V' = s(V)$, $\Omega = 0 \times \omega \mathbf{Q}^{d_1}$, and $\Omega' = 0 \times \omega \mathbf{Q}^{d_1'}$, we have $W' \neq K^{d_0'} \times K^{d_1'}$ and

$$\frac{d_1' - \dim_{\mathbf{Q}}(Y' \cap \Omega') + \dim_{\mathbf{Q}}(Y')}{d_0' + d_1' - \dim_K(W')} \leqslant \frac{d_1 - \dim_{\mathbf{Q}}(Y \cap \Omega)}{d_0 + d_1 - \dim_K(V)}.$$

Theorem 1 asserts the existence of a linear mapping s with certain properties. The following result, which we prove in Section 3, points out a possible choice of s.

THEOREM 2. *Let d_0, d_1, Y, W, V be as in Theorem* 1, *with $V \neq K^{d_0} \times K^{d_1}$. Consider the set of all surjective K-linear mappings $s\colon K^{d_0} \times K^{d_1} \to K^{d_0'} \times K^{d_1'}$ satisfying*

$$s(\bar{\mathbf{Q}}^{d_0} \times 0) \subset \bar{\mathbf{Q}}^{d_0'} \times 0, \qquad s(0 \times \mathbf{Q}^{d_1}) \subset 0 \times \mathbf{Q}^{d_1'},$$
$$s(V) \neq K^{d_0'} \times K^{d_1'}.$$

In this set, there exists at least one mapping s for which the ratio

$$\frac{d_1'}{d_0' + d_1' - \dim_K(s(V))}$$

is minimal, and for which $s(V) \cap (\bar{\mathbf{Q}}^{d_0'} \times 0) = 0$. For such an s, we have

$$\frac{d_1' + \dim_{\mathbf{Q}}(s(Y))}{d_0' + d_1' - \dim_K(s(W))} \leqslant \frac{d_1'}{d_0' + d_1' - \dim_K(s(V))}$$
$$\leqslant \frac{d_1 - \dim_{\mathbf{Q}}(Y \cap \Omega)}{d_0 + d_1 - \dim_K(V)},$$

letting $\Omega = 0 \times \omega \mathbf{Q}^{d_1}$.

2. Construction of a Category

In this section, we build up a category $\mathscr{C}$ adapted to the context of Section 1, and we provide it with some functions defined on the set of all objects of $\mathscr{C}$, denoted by $\mathrm{Ob}(\mathscr{C})$, taking values in the set $\mathbf{N}$ of integers $\geqslant 0$. This allows us to express Theorems 1 and 2 in terms of objects and morphisms of $\mathscr{C}$. We establish also certain properties of the morphisms of $\mathscr{C}$ and of the functions from $\mathrm{Ob}(\mathscr{C})$ to $\mathbf{N}$ attached to $\mathscr{C}$. These are used in Section 3 to deduce Theorem 2 from Theorem 1.

The category $\mathscr{C}$ is defined as follows. Its objects are the families $(K^{d_0} \times K^{d_1},\ Y, W, V)$ where d_0, d_1, Y, W, V are as in Theorem 1. Its morphisms from an object $X_1 = (K^{d_{01}} \times K^{d_{11}},\ Y_1, W_1, V_1)$ to an object $X_2 = (K^{d_{02}} \times K^{d_{12}},\ Y_2, W_2, V_2)$ are the triples (X_1, X_2, f), where f is a K-linear mapping from $K^{d_{01}} \times K^{d_{11}}$ to $K^{d_{02}} \times K^{d_{12}}$ such that

$$f(\bar{\mathbf{Q}}^{d_{01}} \times 0) \subset \bar{\mathbf{Q}}^{d_{02}} \times 0, \qquad f(0 \times \mathbf{Q}^{d_{11}}) \subset 0 \times \mathbf{Q}^{d_{12}},$$
$$f(Y_1) \subset Y_2, \qquad f(W_1) \subset W_2, \qquad f(V_1) \subset V_2.$$

A morphism $g: X_1 \to X_2$ of $\mathscr{C}$ can thus be written $g = (X_1, X_2, f)$ for some linear mapping f, which we will call *its underlying linear mapping*. The composition of morphisms in $\mathscr{C}$ is given by the composition of the underlying mappings:

$$(X_2, X_3, g) \circ (X_1, X_2, f) = (X_1, X_3, g \circ f).$$

The reader can verify that this really defines a category (cf. I, 1 of [M1]).

Let objects of $\mathscr{C}$,

$$\begin{aligned} X^* &= (K^{d_0^*} \times K^{d_1^*},\ Y^*, W^*, V^*), \\ X &= (K^{d_0} \times K^{d_1},\ Y, W, V), \\ X' &= (K^{d_0'} \times K^{d_1'},\ Y', W', V'), \end{aligned} \tag{1}$$

be given. We say that a morphism (X^*, X, i) from X^* to X is a *kernel* of $\mathscr{C}$ if the linear mapping i is injective and satisfies

$$Y^* = i^{-1}(Y), \qquad W^* = i^{-1}(W), \qquad V^* = i^{-1}(V).$$

We say that a morphism (X, X', s) from X to X' is a *cokernel* of $\mathscr{C}$ if the linear mapping s is surjective and satisfies

$$Y' = s(Y), \qquad W' = s(W), \qquad V' = s(V).$$

Last, given a kernel (X^*, X, i) and a cokernel (X, X', s) in $\mathscr{C}$, we say that (X^*, X, i) is a *kernel* of (X, X', s), or that (X, X', s) is a *cokernel* of (X^*, X, i), if $\mathrm{Im}(i) = \ker(s)$. The reader can verify that this is in accordance with the categorical notions of kernel and cokernel in $\mathscr{C}$ (cf. VIII, 1 of [M1]). Moreover, we have:

PROPOSITION 1. *Any kernel of $\mathscr{C}$ admits a cokernel in $\mathscr{C}$ and, vice versa, any cokernel of $\mathscr{C}$ admits a kernel in $\mathscr{C}$. The set of all kernels of $\mathscr{C}$ and the set of its cokernels are closed under composition.*

We also define mappings from $\mathrm{Ob}(\mathscr{C})$ to $\mathbf{N}$ by putting, for each object $X=(K^{d_0}\times K^{d_1}, Y, W, V)$ of $\mathscr{C}$,

$$\begin{aligned}
&a(X)=d_1-\dim_{\mathbf{Q}}(Y\cap\Omega), \qquad \text{where } \Omega=0\times\omega\mathbf{Q}^{d_1},\\
&b(X)=d_0+d_1-\dim_K(V),\\
&c(X)=\dim_{\mathbf{Q}}(Y),\\
&d(X)=\dim_K(V/W),\\
&r(X)=d_0+d_1, \qquad d_0(X)=d_0, \qquad d_1(X)=d_1.
\end{aligned}$$

In this new formulation, Theorems 1 and 2 can be expressed as follows:

THEOREM 1^{bis}. *For each object X of $\mathscr{C}$ with $b(X)\neq 0$, there exists a cokernel $s\colon X\to X'$ of $\mathscr{C}$ with domain X such that*

$$b(X')+d(X')\neq 0 \qquad \textit{and} \qquad \frac{a(X')+c(X')}{b(X')+d(X')}\leqslant\frac{a(X)}{b(X)}.$$

THEOREM 2^{bis}. *Let X be an object of $\mathscr{C}$ with $b(X)\neq 0$. Consider, among the set of cokernels $s\colon X\to X'$ of $\mathscr{C}$ with domain X, with $b(X')\neq 0$, those for which the ratio $d_1(X')/b(X')$ is minimal. This subset is not empty and contains at least one cokernel $s\colon X\to X'$ for which there does not exist in $\mathscr{C}$ any kernel $i\colon X^*\to X'$ with codomain X' such that $d_1(X^*)=b(X^*)=0$ and $r(X^*)\neq 0$. For such a cokernel, we have*

$$\frac{d_1(X')+c(X')}{b(X')+d(X')}\leqslant\frac{d_1(X')}{b(X')}\leqslant\frac{a(X)}{b(X)}.$$

We prove Theorem 2^{bis} in Section 3 as a consequence of Theorem 1^{bis}. For this purpose, we shall need certain properties of the functions from $\mathrm{Ob}(\mathscr{C})$ to $\mathbf{N}$ introduced above. To formulate them, we first set a definition.

DEFINITION. *We say that a function $f\colon \mathrm{Ob}(\mathscr{C})\to\mathbf{N}$ is additive (resp. lower additive, resp. upper additive) if it satisfies*

$$f(X)=f(X^*)+f(X') \qquad (\textit{resp.}\, f(X)\leqslant f(X^*)+f(X'),\ \textit{resp.}\, f(X)\geqslant f(X^*)+f(X'))$$

for each triple (X^*, X, X') of objects of $\mathscr{C}$ for which there exists a kernel from X^* to X which admits as cokernel a morphism from X to X'.

PROPOSITION 2. *The function a is upper additive while b, c, d, r, d_0, and d_1 are additive. These functions vanish on each object on which r vanishes.*

Proof. Let X^*, X, X' be objects of $\mathscr{C}$ given as in (1). Suppose that there exist a kernel (X^*, X, i) from X^* to X and a cokernel (X, X', s) from X to X' such that (X, X', s) is a cokernel of (X^*, X, i). Then, the sequence of K-linear mappings

$$0 \to K^{d_0^*} \times K^{d_1^*} \xrightarrow{i} K^{d_0} \times K^{d_1} \xrightarrow{s} K^{d_0'} \times K^{d_1'} \to 0$$

is exact and induces, by restriction, exact sequences of K-linear mappings

$$0 \to K^{d_0^*} \times 0 \to K^{d_0} \times 0 \to K^{d_0'} \times 0 \to 0,$$

$$0 \to 0 \times K^{d_1^*} \to 0 \times K^{d_1} \to 0 \times K^{d_1'} \to 0,$$

$$0 \to W^* \to W \to W' \to 0,$$

$$0 \to V^* \to V \to V' \to 0,$$

and exact sequences of $\mathbf{Q}$-linear mappings

$$0 \to \Omega^* \to \Omega \to \Omega' \to 0,$$

$$0 \to Y^* \to Y \to Y' \to 0,$$

where $\Omega^* = 0 \times \omega \mathbf{Q}^{d_1^*}$, $\Omega = 0 \times \omega \mathbf{Q}^{d_1}$, and $\Omega' = 0 \times \omega \mathbf{Q}^{d_1'}$. From this we deduce that the functions b, c, d, r, d_0, d_1 are additive. We also get the relations

$$i^{-1}(Y \cap \Omega) = Y^* \cap \Omega^* \qquad \text{and} \qquad s(Y \cap \Omega) \subset Y' \cap \Omega'.$$

These imply that the function $d_1 - a$ is lower additive. Therefore, the function a, which can be written $d_1 - (d_1 - a)$, is upper additive. This proves the first assertion of the proposition. The last one is straightforward.

PROPOSITION 3. *The function a is bounded above by d_1. For each object X of $\mathscr{C}$, there exists a cokernel $s\colon X \to X'$ with domain X such that $d_1(X') \leqslant a(X)$ and $b(X') = b(X)$.*

Proof. The first assertion is clear. Let $X = (K^{d_0} \times K^{d_1}, Y, W, V)$ be an object of $\mathscr{C}$, and let $s\colon K^{d_0} \times K^{d_1} \to K^{d_0} \times K^{d_1'}$ be a surjective K-linear mapping, of kernel $K\cdot(V\cap(0\times\mathbf{Q}^{d_1}))$, satisfying $s(\bar{\mathbf{Q}}^{d_0}\times 0)=\bar{\mathbf{Q}}^{d_0}\times 0$ and $s(0\times\mathbf{Q}^{d_1})=0\times\mathbf{Q}^{d'_1}$. We put

$$X'=(K^{d_0}\times K^{d'_1}, s(Y), s(W), s(V)).$$

Then X' is an object of $\mathscr{C}$, and the triple (X, X', s) constitutes a cokernel of $\mathscr{C}$. This cokernel has the required properties. In fact, we find

$$\begin{aligned} d_1(X') = d'_1 &= d_1 - \dim_{\mathbf{Q}}(V\cap(0\times\mathbf{Q}^{d_1})) \\ &\leqslant d_1 - \dim_{\mathbf{Q}}(Y\cap(0\times\omega\mathbf{Q}^{d_1})) = a(X). \end{aligned}$$

Since $\ker(s)\subset V$, we also get $b(X')=b(X)$.

3. Four Equivalent Statements

The notions of kernel and cokernel are defined in any category which contains a zero object. Let $\mathscr{C}$ be such a category. A property of these notions is that, given a kernel i and a cokernel s of $\mathscr{C}$, it is equivalent to say that i is a kernel of s or that s is a cokernel of i (VIII, 1 of [M1]). We say that $\mathscr{C}$ is *admissible* if it satisfies the statement of Proposition 1. We denote by Ob($\mathscr{C}$) the set of all objects of $\mathscr{C}$ and, when $\mathscr{C}$ is admissible, we define the notions of additive, lower additive, and upper additive functions from Ob($\mathscr{C}$) to $\mathbf{N}$ as in the preceding section. In this general context, we show equivalences between four statements. Then, specializing the category and the functions as in Section 2, we prove Theorem 2^{bis}.

THEOREM 3. *Let $\mathscr{C}$ be an admissible category, and let a, b, c, d, r be functions defined on* Ob($\mathscr{C}$) *taking values in* $\mathbf{N}$. *Assume that a and d are upper additive, that b, c, and r are additive, and that a, b, c, d vanish on each object on which r vanishes. Then the following statements are equivalent*:

STATEMENT 1. *For each object X of $\mathscr{C}$ such that $b(X)\neq 0$, there exists a cokernel $s\colon X\to X'$ with domain X satisfying*

$$b(X')+d(X')\neq 0 \qquad \textit{and} \qquad \frac{a(X')+c(X')}{b(X')+d(X')}\leqslant\frac{a(X)}{b(X)}.$$

STATEMENT 2. *Let X be an object of $\mathscr{C}$. Assume that $b(X)\neq 0$, $c(X)\neq 0$, and that, for each kernel $i\colon X^*\to X$ with codomain X, with $c(X^*)\neq 0$, we have $d(X)/c(X)\leqslant d(X^*)/c(X^*)$. Assume also that there does not exist a cokernel $s\colon X\to X'$ with domain X such that $c(X')=d(X')=0$ and $r(X')\neq 0$. Then we have*

$$a(X)\neq 0 \qquad \textit{and} \qquad \frac{d(X)}{c(X)}\geqslant\frac{b(X)}{a(X)}.$$

STATEMENT 1′. *For each object X of $\mathscr{C}$ such that $c(X)\neq 0$, there exists a kernel $i: X^* \to X$ with codomain X satisfying*

$$a(X^*)+c(X^*)\neq 0 \qquad \textit{and} \qquad \frac{b(X^*)+d(X^*)}{a(X^*)+c(X^*)} \leqslant \frac{d(X)}{c(X)}.$$

STATEMENT 2′. *Let X be an object of $\mathscr{C}$. Assume that $b(X)\neq 0$, $c(X)\neq 0$, and that, for each cokernel $s: X \to X'$ with domain X, with $b(X')\neq 0$, we have $a(X)/b(X) \leqslant a(X')/b(X')$. Assume also that there does not exist a kernel $i: X^* \to X$ with codomain X such that $a(X^*)=b(X^*)=0$ and $r(X^*)\neq 0$. Then we have*

$$d(X)\neq 0 \qquad \textit{and} \qquad \frac{a(X)}{b(X)} \geqslant \frac{c(X)}{d(X)}.$$

Proof of the Implication: *Statement* 1 $\Rightarrow$ *Statement* 2. Let X be an object of $\mathscr{C}$ which fulfils the conditions of Statement 2, and let E be the set of cokernels $s: X \to X'$ with domain X such that $b(X')\neq 0$. The set E is not empty since it contains the identity morphism of X. Moreover, for each cokernel $s: X \to X'$ with domain X, we have $b(X') \leqslant b(X)$ since b is additive and s admits a kernel. This implies that there exists in E a cokernel $s': X \to X'$ for which the ratio $a(X')/b(X')$ is minimal. As $b(X')\neq 0$, Statement 1 applies to X'. It asserts the existence of a cokernel $s'': X' \to X''$ with domain X' satisfying

$$b(X'')+d(X'')\neq 0 \qquad \text{and} \qquad \frac{a(X'')+c(X'')}{b(X'')+d(X'')} \leqslant \frac{a(X')}{b(X')}. \tag{1}$$

Let $s = s'' \circ s'$ be the composite morphism from X to X''. Since it is a cokernel, the choice of s' leads to

$$\frac{a(X'')}{b(X'')} \geqslant \frac{a(X')}{b(X')} \qquad \text{if} \quad b(X'')\neq 0. \tag{2}$$

If $d(X'')$ were zero, we would have $b(X'')\neq 0$ because of (1), and the comparison between (1) and (2) would imply $c(X'')=0$. We would also have $r(X'')\neq 0$ since $b(X'')\neq 0$. The relations $c(X'')=d(X'')=0$ and $r(X'')\neq 0$ would then contradict the last assumption of Statement 2, whence $d(X'')\neq 0$. If $b(X'')\neq 0$, the relations (1) and (2) then imply

$$\frac{c(X'')}{d(X'')} \leqslant \frac{a(X')}{b(X')}. \tag{3}$$

If $b(X'')=0$, this inequality is still valid and follows directly from (1). Since the identity morphism of X belongs to E, the choice of s' induces also

$$\frac{a(X')}{b(X')} \leqslant \frac{a(X)}{b(X)}. \tag{4}$$

Let $i: X^* \to X$ be a kernel of s. Since c is additive and d is upper additive, we get $c(X)=c(X^*)+c(X'')$ and $d(X) \geqslant d(X^*)+d(X'')$. As $d(X'') \neq 0$, this implies $d(X) \neq 0$ and

$$\frac{c(X)}{d(X)} \leqslant \frac{c(X^*)+c(X'')}{d(X^*)+d(X'')}. \tag{5}$$

Moreover, the first assumption of Statement 2 gives

$$\frac{d(X)}{c(X)} \leqslant \frac{d(X^*)}{c(X^*)} \quad \text{if} \quad c(X^*) \neq 0.$$

Then, from (5), we deduce

$$\frac{c(X)}{d(X)} \leqslant \frac{c(X'')}{d(X'')}, \tag{6}$$

whether $c(X^*)$ is zero or not. Combining the inequalities (3), (4), and (6), we get

$$\frac{c(X)}{d(X)} \leqslant \frac{a(X)}{b(X)}.$$

Since $c(X) \neq 0$, this shows that X satisfies the conclusion of Statement 2.

Proof of the Implication: *Statement* 2 $\Rightarrow$ *Statement* 1′. Let X be an object of $\mathscr{C}$ such that $c(X) \neq 0$. The set E of all kernels $i: X^* \to X$ with codomain X such that $c(X^*) \neq 0$ is not empty since it contains the identity morphism of X. For such a kernel, we have $c(X^*) \leqslant c(X)$ because c is additive and each kernel admits a cokernel. The set of ratios $d(X^*)/c(X^*)$ attached to these kernels therefore possesses a minimum, and the set E_0 of elements in E for which this minimum is reached is not empty.

Let $i: X^* \to X$ be an element of E_0 for which $r(X^*)$ is minimal. If $b(X^*) \neq 0$, then the object X^* fulfils the conditions of Statement 2. This is clear concerning the first condition. To verify that it satisfies the last one, suppose the existence of a cokernel $s: X^* \to X'$ with domain X^* such that $c(X')=d(X')=0$. We have to show that $r(X')=0$. Let $i^*: X^{**} \to X^*$ be a kernel of s. Since c is additive and d is upper additive, we get $c(X^*)=c(X^{**})$ and $d(X^*) \geqslant d(X^{**})$. The composite morphism $i \circ i^*: X^{**} \to X$ thus belongs to E_0. Because of the choice of i, this implies $r(X^{**}) \geqslant r(X^*)$. Since r is additive, we then deduce $r(X') = 0$, as announced. So, if $b(X^*) \neq 0$, Statement 2 can be applied to X^*, and this gives

$$a(X^*) \neq 0 \quad \text{and} \quad \frac{b(X^*)}{a(X^*)} \leqslant \frac{d(X^*)}{c(X^*)}.$$

Since the identity morphism of X belongs to E, the choice of i implies on the other hand

$$\frac{d(X^*)}{c(X^*)} \leqslant \frac{d(X)}{c(X)},$$

whatever $b(X^*)$ is. If $b(X^*) \neq 0$, these two inequalites lead to

$$\frac{b(X^*) + d(X^*)}{a(X^*) + c(X^*)} \leqslant \frac{d(X^*)}{c(X^*)} \leqslant \frac{d(X)}{c(X)}.$$

If $b(X^*) = 0$, this comes simply from the last one. Thus X satisfies Statement 1′ for the choice of the kernel i.

Proof of the Implications: *Statement* 1′ ⇒ *Statement* 2′ ⇒ *Statement* 1. Consider the opposite category $\mathscr{C}^{\mathrm{op}}$ (II, 2 of [M1]). In this category, a zero object of $\mathscr{C}$ remains a zero object, the kernels of $\mathscr{C}$ become the cokernels of $\mathscr{C}^{\mathrm{op}}$, and its cokernels become the kernels of $\mathscr{C}^{\mathrm{op}}$. This category is thus admissible, and the additive, lower additive, and upper additive functions from $\mathrm{Ob}(\mathscr{C})$ to $\mathbf{N}$ remain so considered as functions from $\mathrm{Ob}(\mathscr{C}^{\mathrm{op}})$ to $\mathbf{N}$. The chain of implications

$$\text{Statement 1} \Rightarrow \text{Statement 2} \Rightarrow \text{Statement 1}'$$

proved above thus applies also to the family $(\mathscr{C}^{\mathrm{op}}, d, c, b, a, r)$ instead of $(\mathscr{C}, a, b, c, d, r)$. In terms of the category $\mathscr{C}$, this gives the required chain of implications.

Remark. Theorem 3 remains valid if, instead of assuming that the functions b and c are additive, we assume only that they are lower additive and bounded above by upper additive functions. The proof is similar.

Proof of Theorem 2^{bis}. Consider the category $\mathscr{C}$ and the functions a, b, c, d, r defined in Section 2. By Propositions 1 and 2, they fulfil the conditions of Theorem 3. Moreover, for this choice of category and functions, Statement 1 is true since it is Theorem 1^{bis}. Therefore all statements contained in Theorem 3 are true for the same category and functions. In particular, Statement 2′ is true.

Let X be an object of $\mathscr{C}$ such that $b(X) \neq 0$. The set E of all cokernels $s: X \to X'$ with domain X with $b(X') \neq 0$ is not empty since it contains the identity morphism of X. For such a cokernel, we have $b(X') \leqslant b(X)$ because b is additive and each cokernel admits a kernel. The set of ratios $a(X')/b(X')$ attached to these cokernels thus possesses a minimum μ_0, and the set E_0 of all elements in E for which this minimum is achieved is not empty. Likewise, for the function d_1 defined in Section 2, the set of all ratios $d_1(X')/b(X')$ attached to the cokernels $s: X \to X'$ of E possesses a minimum μ_1, and the set E_1 of all elements in E for which this minimum is achieved is not empty.

By Proposition 3, there exists, for each element $s: X \to X'$ of E, a cokernel $s': X' \to X''$ of $\mathscr{C}$ with domain X' such that

$$d_1(X'') \leqslant a(X') \qquad \text{and} \qquad b(X'') = b(X').$$

Then, the composite morphism $s' \circ s: X \to X''$ also belongs to E. Applying this argument to an element s of E_0, we get $\mu_1 \leqslant \mu_0$. Applying it to an element s of E_1, and taking into account the inequality $a(X') \leqslant d_1(X')$, we get $a(X') = d_1(X')$. This last result together with the inequality $\mu_1 \leqslant \mu_0$ gives $\mu_1 = \mu_0$ and $E_1 \subset E_0$.

The first assertion of Theorem 2^{bis} is that there exists a cokernel $s: X \to X'$ in E_1, for which there does not exist any kernel $i: X^* \to X'$ of $\mathscr{C}$ with codomain X' such that $d_1(X^*) = b(X^*) = 0$ and $r(X^*) \neq 0$. To establish this assertion, we choose in E_1 a cokernel $s: X \to X'$ for which $r(X')$ is minimal, and we show that s possesses the required property. In fact, let $i: X^* \to X'$ be a kernel with codomain X' such that $d_1(X^*) = b(X^*) = 0$, and let $s': X' \to X''$ be a cokernel of i. Since d_1 and b are additive, we get $d_1(X'') = d_1(X')$ and $b(X'') = b(X')$. Thus the composite morphism $s' \circ s: X \to X''$ belongs to E_1. Given the choice of s, this implies $r(X'') \geqslant r(X')$. Then, r being additive, we get $r(X^*) = 0$ as announced.

Let $s: X \to X'$ be an element of E_1 for which there does not exist any kernel $i: X^* \to X'$ of $\mathscr{C}$ with codomain X' such that $d_1(X^*) = b(X^*) = 0$ and $r(X^*) \neq 0$. We first show that X' fulfils the conditions of Statement 2′ if $c(X') \neq 0$. Since $E_1 \subset E_0$, we have $s \in E_0$. This ensures the first condition. Let $i: X^* \to X'$ be a kernel of $\mathscr{C}$ with codomain X' such that $b(X^*) = 0$. To verify the remaining condition, it suffices to show $r(X^*) = 0$. For this purpose, let $s': X' \to X''$ be a cokernel of i. Since b is additive, we have $b(X'') = b(X')$. Thus $s' \circ s$ belongs to E. Since $s \in E_1$, this implies $d_1(X'') \geqslant d_1(X')$, whence $d_1(X^*) = 0$ because d_1 is additive. The choice of s therefore implies $r(X^*) = 0$ as requested. If $c(X') \neq 0$, Statement 2′ thus applies to X', and gives

$$d(X') \neq 0 \qquad \text{and} \qquad \frac{c(X')}{d(X')} \leqslant \frac{a(X')}{b(X')}.$$

From this we deduce

$$\frac{a(X')+c(X')}{b(X')+d(X')}\leqslant\frac{a(X')}{b(X')},$$

whatever $c(X')$ is. Since the identity morphism of X belongs to E, we also have

$$\frac{a(X')}{b(X')}\leqslant\frac{a(X)}{b(X)}.$$

Finally, since $s\in E_1$, we have $a(X')=d_1(X')$. This equality together with the last two inequalities proves the second assertion of Theorem 2^{bis} for the object X and the cokernel s.

4. The Main Result

Using Theorem 2, we establish here our main result. Afterwards, we apply it to give a lower bound for the rank of matrices with coefficients in $\mathscr{L}$.

Theorem 4. *Let d be a positive integer, Z be a finite dimensional $\bar{\mathbf{Q}}$-vector subspace of $\mathscr{L}^d$, and U be a K-vector subspace of K^d containing Z. Among the set of all surjective K-linear mappings $t\colon K^d\to K^{d'}$ which are rational over $\bar{\mathbf{Q}}$ and non-zero, we choose one for which the ratio $(\dim_K(t(U)))/d'$ is minimal. Then, letting $Z'=t(Z)$ and $U'=t(U)$, we have*

$$\frac{\dim_{\bar{\mathbf{Q}}}(Z')}{d'+\dim_{\bar{\mathbf{Q}}}(Z')}\leqslant\frac{\dim_K(U')}{d'}\leqslant\frac{\dim_K(U)}{d}. \tag{1}$$

Proof. By construction, d' is a positive integer, Z' is a finite dimensional $\bar{\mathbf{Q}}$-vector subspace of $\mathscr{L}^{d'}$, and U' is a K-vector subspace of $K^{d'}$ which contains Z'. If $d'<d$, this allows us, by induction on d, to assume the theorem true for the triple (d', Z', U'). The choice of the application t attached to (d, Z, U) justifies the right inequality in (1). Also, it allows us to choose the identity mapping of $K^{d'}$ in applying the theorem to (d', Z', U'), and this gives the left inequality in (1) if $d'<d$.

This brings us to proving the theorem in the case $d'=d$. In this case, t is an isomorphism. We have $\dim_{\bar{\mathbf{Q}}}(Z')=\dim_{\bar{\mathbf{Q}}}(Z)$ and $\dim_K(U')=\dim_K(U)$. Therefore, U satisfies

$$\frac{\dim_K(t_1(U))}{d_1}\geqslant\frac{\dim_K(U)}{d}, \tag{2}$$

for each surjective K-linear mapping $t_1: K^d \to K^{d_1}$ which is rational over $\bar{\mathbf{Q}}$ and non-zero. From this, we shall deduce that

$$\frac{\dim_{\bar{\mathbf{Q}}}(Z)}{d+\dim_{\bar{\mathbf{Q}}}(Z)} \leqslant \frac{\dim_K(U)}{d}. \tag{3}$$

It is clear if $U = K^d$. We thus suppose $U \neq K^d$.

Since Z is of finite dimension over $\bar{\mathbf{Q}}$, there exist a subfield k of $\bar{\mathbf{Q}}$ of finite degree over $\mathbf{Q}$ and a k-vector subspace Z_1 of $(k + k \cdot L)^d$ such that $Z = \bar{\mathbf{Q}} \cdot Z_1$. Let $\omega_1, \ldots, \omega_m$ be a basis of k over $\mathbf{Q}$. Consider the surjective K-linear mapping

$$\phi: K^d \times (K^d)^m \to K^d$$

$$(x, (y_1, \ldots, y_m)) \mapsto x + \omega_1 y_1 + \cdots + \omega_m y_m.$$

It gives by restriction a surjective $\mathbf{Q}$-linear mapping from $k^d \times (L^d)^m$ to $(k + k \cdot L)^d$. We put

$$Y = \phi^{-1}(Z_1) \cap (k^d \times (L^d)^m), \quad W = \phi^{-1}(0), \quad V = \phi^{-1}(U).$$

Since $U \neq K^d$, we have $V \neq K^d \times (K^d)^m$. Therefore Theorem 2 applies to the family $(K^d \times (K^d)^m, Y, W, V)$.

Let us show that we can choose the identity mapping of $K^d \times (K^d)^m$ to apply this theorem. This amounts on the one hand to showing

$$V \cap (\bar{\mathbf{Q}}^d \times 0) = 0, \tag{4}$$

and on the other hand to showing

$$\frac{d_1'}{d_0' + d_1' - \dim_K(s(V))} \geqslant \frac{md}{d + md - \dim_K(V)}, \tag{5}$$

for each surjective K-linear mapping $s: K^d \times (K^d)^m \to K^{d_0'} \times K^{d_1'}$ satisfying

$$s(\bar{\mathbf{Q}}^d \times 0) \subset \bar{\mathbf{Q}}^{d_0'} \times 0, \qquad s(0 \times (\mathbf{Q}^d)^m) \subset 0 \times \mathbf{Q}^{d_1'}, \qquad s(V) \neq K^{d_0'} \times K^{d_1'}.$$

We begin by establishing (5) for a fixed s. The above conditions on s show that its kernel is a product $S_0 \times S_1$, where S_0 is a subspace of K^d which is rational over $\bar{\mathbf{Q}}$, and where S_1 is a subspace of $(K^d)^m$ which is rational over $\mathbf{Q}$. In terms of S_0 and S_1, the last condition on s reads $V + (S_0 \times S_1) \neq K^d \times (K^d)^m$, and the surjectivity of s implies

$$d_1' = md - \dim_K(S_1),$$

$$d_0' + d_1' - \dim_K(s(V)) = \dim_K((K^d \times (K^d)^m)/(V + (S_0 \times S_1))).$$

We put $T = \phi(S_0 \times S_1)$ and we choose a surjective K-linear mapping $t_1: K^d \to K^{d_1}$ which is rational over $\bar{\mathbf{Q}}$, with kernel T. This is possible since

T is a subspace of K^d which is rational over $\bar{\mathbf{Q}}$. We also put $S = S_1 \cap (\mathbf{Q}^d)^m$. Since S_1 is rational over $\mathbf{Q}$, and since ϕ induces by restriction a $\mathbf{Q}$-linear isomorphism from $0 \times (\mathbf{Q}^d)^m$ to k^d, we find

$$\begin{aligned} \dim_K(S_1) &= \dim_{\mathbf{Q}}(S) = \dim_{\mathbf{Q}}(\phi(0 \times S)) \\ &\leqslant m \dim_k(T \cap k^d) \leqslant m \dim_K(T), \end{aligned}$$

whence

$$d'_1 \geqslant m(d - \dim_K(T)) = md_1.$$

Moreover, since V contains the kernel of ϕ, we have

$$\begin{aligned} \dim_K((K^d \times (K^d)^m)/V) &= \dim_K(\phi(K^d \times (K^d)^m)/\phi(V)) \\ &= \dim_K(K^d/U), \end{aligned}$$

$$\begin{aligned} &\dim_K((K^d \times (K^d)^m)/(V + (S_0 \times S_1)) \\ &\quad = \dim_K(\phi(K^d \times (K^d)^m)/\phi(V + (S_0 \times S_1))) \\ &\quad = \dim_K(K^d/(U + T)), \end{aligned}$$

whence

$$\begin{aligned} d + md - \dim_K(V) &= d - \dim_K(U), \\ d'_0 + d'_1 - \dim_K(s(V)) &= d_1 - \dim_K(t_1(U)). \end{aligned} \tag{6}$$

The inequality (5) thus follows from the inequality

$$\frac{md_1}{d_1 - \dim_K(t_1(U))} \geqslant \frac{md}{d - \dim_K(U)},$$

which we get from (2) applied to our choice of t_1. We now prove the relation (4). Since ϕ is injective on $\bar{\mathbf{Q}}^d \times 0$ with image $\bar{\mathbf{Q}}^d$, this amounts to showing

$$U \cap \bar{\mathbf{Q}}^d = 0.$$

Let $t_1 : K^d \to K^{d_1}$ be a surjective K-linear mapping which is rational over $\bar{\mathbf{Q}}$, with kernel $T = K \cdot (U \cap \bar{\mathbf{Q}}^d)$. Since $T \subset U$ and $U \neq K^d$, we have

$$d_1 = d - \dim_K(T) \neq 0 \qquad \text{and} \qquad \dim_K(t_1(U)) = \dim_K(U) - \dim_K(T).$$

Then the inequality (2) applied to this choice of t_1 gives $\dim_K(T) = 0$, thus $U \cap \bar{\mathbf{Q}}^d = 0$.

If for s the identity mapping of $K^d \times (K^d)^m$ is chosen, Theorem 2 gives

$$\frac{md + \dim_{\mathbf{Q}}(Y)}{d + md - \dim_K(W)} \leqslant \frac{md}{d + md - \dim_K(V)}.$$

Since $\phi(Y) = Z_1$ and $\bar{\mathbf{Q}} \cdot Z_1 = Z$, we have

$$\dim_{\mathbf{Q}}(Y) \geqslant \dim_{\mathbf{Q}}(Z_1) = m \dim_k(Z_1) \geqslant m \dim_{\bar{\mathbf{Q}}}(Z).$$

Since W is the kernel of ϕ, we also have

$$d + md - \dim_K(W) = d.$$

Making use of (6), we then get

$$\frac{d + \dim_{\bar{\mathbf{Q}}}(Z)}{d} \leqslant \frac{d}{d - \dim_K(U)},$$

from which the inequality (3) follows.

Remarks. (i) Let F be a subfield of $\bar{\mathbf{Q}}$. Theorem 4 remains valid if we substitute everywhere in its statement F to $\bar{\mathbf{Q}}$ and $F + F \cdot L$ to $\mathscr{L}$. The proof is the same provided that we make the same substitutions.

(ii) Under the assumptions of Theorem 4, one can moreover choose the mapping t in such a way that its kernel T satisfies $\sigma(T) = T$ for all K-linear automorphisms σ of K^d which are rational over $\bar{\mathbf{Q}}$ and such that $\sigma(U) = U$. This can be proved as Lemma 2 of Section 6.

(iii) One can deduce Baker's theorem from Theorem 1 (Corollary 3.3 of [W3]), but I do not know if one can recover Theorem 1 from Theorem 4 and Baker's theorem.

To each matrix M with coefficients in K, of size $d \times l$ with $d, l > 0$, we assign a number $\vartheta(M)$ analogous to the number $\theta(M)$ defined by M. Waldschmidt (Sect. 7 of [W1]). We define it as the minimum of all ratios l'/d', when (d', l') runs among the couples of integers satisfying $0 < d' \leqslant d$ and $0 \leqslant l' \leqslant l$, for which there exists matrices $P \in \mathrm{GL}_d(\bar{\mathbf{Q}})$ and $Q \in \mathrm{GL}_l(\bar{\mathbf{Q}})$ such that the product PMQ can be written

$$\begin{pmatrix} M' & 0 \\ N & M'' \end{pmatrix}$$

with M' of size $d' \times l'$. This number thus depends only on the eventual relations of linear dependence over $\bar{\mathbf{Q}}$ between the coefficients of M. The definition of $\theta(M)$ is the same provided that we read everywhere $\mathbf{Q}$ instead of $\bar{\mathbf{Q}}$. One can show $\theta(M) = \vartheta(M)$ for any matrix M with coefficients in L, but we will not do it here. Making use of this definition, we can give a lower bound for the rank of the matrices with coefficients in $\mathscr{L}$ as M. Waldschmidt has done for matrices with coefficients in L (Corollary 7.2 of [W1]):

COROLLARY 1. *Let M be a matrix with coefficients in $\mathscr{L}$, of size $d \times l$ with $d, l > 0$, and let n be its rank. We have*

$$n \geqslant \frac{\vartheta(M)}{1+\vartheta(M)} \cdot d.$$

Proof. Let $\phi \colon K^l \to K^d$ be the K-linear mapping given by $\phi(x) = Mx$ for all $x \in K^l$. We put $Z = \phi(\bar{\mathbf{Q}}^l)$ and $U = K \cdot Z = \phi(K^l)$. Then d, Z, and U fulfil the conditions of Theorem 4. Since $\dim_K(U) = n$, this theorem asserts the existence of a surjective K-linear mapping $t \colon K^d \to K^{d'}$ which is rational over $\bar{\mathbf{Q}}$, and which, letting $l' = \dim_{\bar{\mathbf{Q}}}(t(Z))$, satisfies

$$n \geqslant \frac{l'}{d'+l'} \cdot d. \tag{7}$$

Since t is surjective and rational over $\bar{\mathbf{Q}}$, there exists a basis $(u_1, ..., u_d)$ of K^d over K, made of elements of $\bar{\mathbf{Q}}^d$, whose $d - d'$ last elements form a basis of $\ker(t)$ over K. Since $Z' = (t \circ \phi)(\bar{\mathbf{Q}}^l)$ is of dimension l' over $\bar{\mathbf{Q}}$, there also exists a basis $(v_1, ..., v_l)$ of K^l over K, made of elements of $\bar{\mathbf{Q}}^l$, whose $l - l'$ last elements belong to $\ker(t \circ \phi)$. Relative to these bases of K^d and K^l, the matrix of ϕ can be written as a lower triangular block matrix $\left(\begin{smallmatrix} M' & 0 \\ N & M'' \end{smallmatrix}\right)$, with M' of size $d' \times l'$. Since M is the matrix of ϕ with respect to the canonical bases of K^d and K^l, and since the base-change matrices have their coefficients in $\bar{\mathbf{Q}}$, this implies $\vartheta(M) \leqslant l'/d'$. The announced inequality follows from this upper bound combined with (7).

This corollary leads to the generalization of the six exponentials theorem announced in the Introduction:

COROLLARY 2. *Let M be a 2×3 matrix with coefficients in $\mathscr{L}$. Assume that its rows are linearly independent over $\bar{\mathbf{Q}}$ and that its columns are linearly independent over $\bar{\mathbf{Q}}$. Then the rank of M is 2.*

Proof. We first observe that, for each matrix M of size $d \times l$ with $d, l > 0$, of rank 1, whose rows and columns are linearly independent over $\bar{\mathbf{Q}}$, we have $\vartheta(M) = l/d$. In our situation, if the rank of M were 1, we would thus have $\vartheta(M) = \frac{3}{2}$, and this would contradict Corollary 1.

5. POINTS WHOSE COORDINATES ARE LINEAR FORMS IN LOGARITHMS

To each couple of integers (n, d) with $0 < n < d$, we attach a number $\phi(n, d)$ defined as the maximum of all sums

$$\sum_{i=1}^{k} \frac{n_i d_i}{d_i - n_i},$$

corresponding to all possible decompositions of (n, d) as a sum of couples of integers $(n_1, d_1), ..., (n_k, d_k)$ satisfying $0 < n_i < d_i$ for $i = 1, ..., k$. Using this definition, we prove the result below; then we show that the provided upper bound is essentially the best up to a factor 2.

THEOREM 5. *Let d be a positive integer, and let U be a K-vector subspace of K^d such that $U \cap \bar{\mathbf{Q}}^d = 0$. Suppose that K^d is the smallest subspace of K^d which is rational over $\bar{\mathbf{Q}}$ and which contains U. Then, the dimension of U is an integer n satisfying $0 < n < d$, and we have*

$$\dim_{\bar{\mathbf{Q}}}(U \cap \mathscr{L}^d) \leqslant \phi(n, d).$$

Proof. Let Z be a $\bar{\mathbf{Q}}$-vector subspace of $U \cap \mathscr{L}^d$ of finite dimension l, and let $t: K^d \to K^{d'}$ be a surjective K-linear mapping which is rational over $\bar{\mathbf{Q}}$ and non-zero, for which the ratio $(\dim_K(t(U)))/d'$ is minimal. We put

$$U' = t(U), \qquad n' = \dim_K(U'), \qquad Z' = t(Z), \qquad l' = \dim_{\bar{\mathbf{Q}}}(Z').$$

Then Theorem 4 gives

$$\frac{l'}{d' + l'} \leqslant \frac{n'}{d'} \leqslant \frac{n}{d}. \tag{1}$$

Since d is positive, the assumption $U \cap \bar{\mathbf{Q}}^d = 0$ implies $n < d$. We also have $n' > 0$; otherwise U would be contained in the kernel of t, which is rational over $\bar{\mathbf{Q}}$ and distinct from K^d. Making use of the inequalities (1), we deduce $0 < n' < d'$ and

$$l' \leqslant \frac{n'd'}{d' - n'} \leqslant \phi(n', d'). \tag{2}$$

If $d' = d$, we also have $l' = l$ and $n' = n$, because t is then an isomorphism. In this case, we get $0 < n < d$ and $l \leqslant \phi(n, d)$. Now, assume $d' < d$. We put $d^* = d - d'$, and we choose an injective K-linear mapping $i: K^{d^*} \to K^d$ which is rational over $\bar{\mathbf{Q}}$, whose image is the kernel of t. Again we put

$$U^* = i^{-1}(U), \qquad n^* = \dim_K(U^*), \qquad Z^* = i^{-1}(Z), \qquad l^* = \dim_{\bar{\mathbf{Q}}}(Z^*).$$

Let T_1 be the smallest subspace of K^{d^*} which is rational over $\bar{\mathbf{Q}}$ and which contains U^*. To show $T_1 = K^{d^*}$, we consider a surjective K-linear mapping $t_1: K^d \to K^{d_1}$ which is rational over $\bar{\mathbf{Q}}$, with kernel $i(T_1)$. It satisfies

$$U \cap \ker(t) \subset \ker(t_1) \subset \ker(t).$$

It is thus non-zero and satisfies $\dim_K(t_1(U)) = \dim_K(t(U))$. Because of the choice of t, this implies $d_1 \leqslant d'$, thus $T_1 = K^{d^*}$. We also have $U^* \cap \bar{\mathbf{Q}}^{d^*} = 0$, since i is injective and maps $U^* \cap \bar{\mathbf{Q}}^{d^*}$ in $U \cap \bar{\mathbf{Q}}^d$. Since $d^* < d$, we may suppose, by induction on d, that the theorem is true for the subspace U^* of K^{d^*}. Since Z^* is a $\bar{\mathbf{Q}}$-vector subspace of $U^* \cap \mathscr{L}^{d^*}$, we get in this way $0 < n^* < d^*$ and

$$l^* \leqslant \phi(n^*, d^*).$$

Moreover, the choice of i implies

$$l = l^* + l', \qquad n = n^* + n', \qquad d = d^* + d'.$$

Together with (2), these relations imply $0 < n < d$ and

$$l \leqslant \phi(n^*, d^*) + \phi(n', d') \leqslant \phi(n, d).$$

The inequality $l \leqslant \phi(n, d)$ being true in all cases, and the choice of Z being arbitrary, this proves the theorem.

To show that the bound $\phi(n, d)$ is essentially the best up to a factor 2, we shall need the following lemma, which allows us to compute $\phi(n, d)$.

LEMMA. *For integers* $0 < n < d$, *we have*

$$\phi(n, d) = \begin{cases} \phi(1, d-n) + \phi(n-1, n) & \textit{if} \quad n \geqslant 2 \quad \textit{and} \quad d-n \geqslant 2, \\ nd/(d-n) & \textit{if} \quad n = 1 \quad \textit{or} \quad d-n = 1. \end{cases}$$

Proof. Let (n, d) be a couple of integers satisfying $0 < n < d$. We put $\delta = d - n$. The equality

$$\frac{nd}{d-n} = n + \frac{n^2}{\delta},$$

valid for all these couples, shows that $\phi(n, d) - n$ is the maximum of all sums

$$\frac{n_1^2}{\delta_1} + \cdots + \frac{n_k^2}{\delta_k}$$

corresponding to all possible decompositions of (n, δ) as a sum of couples of positive integers $(n_1, \delta_1), ..., (n_k, \delta_k)$. If $n = 1$ or $\delta = 1$, there is only one such decomposition. Then, we get

$$\phi(n, d) = n + \frac{n^2}{\delta} = \frac{nd}{d-n}.$$

Otherwise, we have $n \geqslant 2$ and $\delta \geqslant 2$. Then, the relation

$$\frac{n^2}{\delta} \leqslant \frac{(n-1)^2}{1} + \frac{1^2}{\delta - 1},$$

which requires only $n \geqslant 2$ and $\delta \geqslant 2$, shows that there exists a decomposition $(n, \delta) = (n_1, \delta_1) + \cdots + (n_k, \delta_k)$ for which the maximum is achieved and such that, for each i, one of the numbers n_i or δ_i is equal to 1. Such a decomposition cannot contain two couples $(n_i, 1)$, $(n_j, 1)$ with $n_i, n_j \geqslant 2$; otherwise we would have

$$\frac{n_i^2}{1} + \frac{n_j^2}{1} < \frac{(n_i + n_j - 1)^2}{1} + \frac{1^2}{1},$$

and the decomposition would not give the maximum. Neither can such a decomposition contain two couples $(1, \delta_i)$, $(1, \delta_j)$ with $\delta_i, \delta_j \geqslant 2$; otherwise we would have

$$\frac{1^2}{\delta_i} + \frac{1^2}{\delta_j} < \frac{1^2}{1} + \frac{1^2}{\delta_i + \delta_j - 1}.$$

It also satisfies $k \geqslant 2$, so that $k = 2$ and we have

$$\phi(n, d) - n = \frac{1^2}{d - n - 1} + \frac{(n-1)^2}{1},$$

which gives $\phi(n, d) = \phi(1, d - n) + \phi(n - 1, n)$.

THEOREM 6. *For each couple of integers (n, d) such that $0 < n < d$, there exists a subspace U of K^d, of dimension n, satisfying the following conditions*:

(i) $U \cap \bar{\mathbf{Q}}^d = 0$,

(ii) K^d *is the smallest subspace of* K^d *which is rational over* $\bar{\mathbf{Q}}$ *and which contains* U,

(iii) $\dim_{\mathbf{Q}}(U \cap \mathscr{L}^d) \geqslant \frac{1}{2}\phi(n, d)$.

Proof. Let $0 < n < d$ be integers, and $\lambda_1, \ldots, \lambda_d$ be elements of $\mathscr{L}$ which are linearly independent over $\bar{\mathbf{Q}}$. If $n = 1$, the subspace U of K^d spanned by $(\lambda_1, \ldots, \lambda_d)$ is of dimension n; it fulfils the conditions (i) and (ii), and satisfies

$$\dim_{\mathbf{Q}}(U \cap \mathscr{L}^d) \geqslant 1 \geqslant \tfrac{1}{2}\phi(n, d).$$

If $d - n = 1$, the subspace U of K^d formed by all points $(x_1, \ldots, x_d) \in K^d$ satisfying $\lambda_1 x_1 + \cdots + \lambda_d x_d = 0$ is also of dimension n and fulfils the conditions (i) and (ii). Moreover, letting $(e_1, \ldots, e_d)$ be the canonical basis of K^d, it contains the points $\lambda_i e_j - \lambda_j e_i$ $(1 \leqslant i < j \leqslant d)$. As they belong to $\mathscr{L}^d$ and are linearly independent over $\bar{\mathbf{Q}}$, we get

$$\dim_{\mathbf{Q}}(U \cap \mathscr{L}^d) \geqslant \tfrac{1}{2}d(d-1) = \tfrac{1}{2}\phi(n, d).$$

Finally, if $n \geqslant 2$ and $d-n \geqslant 2$, we identify K^d with $K^{d-n} \times K^n$, and we consider the product $U = U_1 \times U_2$, where U_1 is the subspace of K^{d-n} spanned by $(\lambda_1, ..., \lambda_{d-n})$, and where U_2 is the subspace of K^n formed by all points $(x_1, ..., x_n) \in K^n$ satisfying $\lambda_1 x_1 + \cdots + \lambda_n x_n = 0$. This is a subspace of K^d of dimension n, which fulfils the conditions (i) and (ii). The preceding considerations show

$$\dim_{\mathbf{Q}}(U_1 \cap \mathscr{L}^{d-n}) \geqslant \tfrac{1}{2}\phi(1, d-n) \quad \text{and} \quad \dim_{\mathbf{Q}}(U_2 \cap \mathscr{L}^n) \geqslant \tfrac{1}{2}\phi(n-1, n).$$

From the lemma, we then deduce

$$\dim_{\mathbf{Q}}(U \cap \mathscr{L}^d) \geqslant \tfrac{1}{2}\phi(1, d-n) + \tfrac{1}{2}\phi(n-1, n) = \tfrac{1}{2}\phi(n, d).$$

The theorem is proved.

6. Lower Bound for the p-adic Rank

Let p be a prime number and let $k \subset \mathbf{C}_p$ be a finite Galois extension of $\mathbf{Q}$ with group G. We consider the multiplicative group $k^\times$ of k as a $\mathbf{Z}[G]$-module. We denote by $\mathscr{U}$ the open ball of $\mathbf{C}_p$ consisting of all elements x of $\mathbf{C}_p$ satisfying $|x-1| < 1$, and by k_1 the $\mathbf{Z}[G]$-submodule of $k^\times$ formed by the elements of k whose conjugates all belong to $\mathscr{U}$. Then the mapping $\theta: k_1 \to \mathbf{C}_p[G]$ defined by

$$\theta(\alpha) = \sum_{\sigma \in G} \log(\sigma\alpha) \cdot \sigma^{-1} \qquad (\alpha \in k_1)$$

is a homomorphism of $\mathbf{Z}[G]$-modules.

Let M be a $\mathbf{Z}[G]$-submodule of k_1 of finite type. Its image $\theta(M)$ is a $\mathbf{Z}[G]$-submodule of $\mathbf{C}_p[G]$. The subspace $\mathbf{C}_p \cdot \theta(M)$ it generates is thus a left ideal of $\mathbf{C}_p[G]$. For each absolutely irreducible character ϕ of G, we denote by d_ϕ, r_ϕ, and ρ_ϕ the respective multiplicities of ϕ in the characters of the $\mathbf{C}_p[G]$-modules $\mathbf{C}_p[G]$, $M \otimes_{\mathbf{Z}} \mathbf{C}_p$, and $\mathbf{C}_p \cdot \theta(M)$. Then, the p-adic rank of M, equal to the dimension of $\mathbf{C}_p \cdot \theta(M)$ over $\mathbf{C}_p$, is given by $\sum_\phi \rho_\phi d_\phi$. On the basis of Schanuel's conjecture, J.-F. Jaulent has shown (Theorem 2 of [J1]) that we should have

$$\rho_\phi = \min\{r_\phi, d_\phi\}. \tag{1}$$

Here, we propose to prove the following lower bound:

Theorem 7. *The notations being as above, we have, for each absolutely irreducible character ϕ of G,*

$$\rho_\phi \geqslant \frac{r_\phi d_\phi}{r_\phi + d_\phi}.$$

This result was proved by M. Laurent under the assumption $r_\phi \leqslant d_\phi$ for all ϕ (Theorem 1′ of [L1]), but in fact this condition is not necessary. Together with the upper bound $\rho_\phi \leqslant \min\{r_\phi, d_\phi\}$, the lower bound of Theorem 7 implies equality (1) for some values of r_ϕ and d_ϕ. This allowed M. Laurent to prove Leopoldt's conjecture in some new cases (Sect. 6 of [L1]). Also, if we require $r_\phi \geqslant d_\phi^2$ for all ϕ, we get $\rho_\phi = d_\phi$ for all ϕ, and then the p-adic rank of M is equal to the degree of k over $\mathbf{Q}$. This strengthens a result of M. Emsalem (Corollary 2 of [E1]). More generally, Theorem 7 allows us to give a lower bound for the p-adic rank of any $\mathbf{Z}[G]$-submodule of k_1 of finite type.

To prove Theorem 7, we use three lemmas, among which the first two are of a general nature. We first need the following consequence of Theorem 4.

LEMMA 1. *Let V be a non-zero K-vector space of finite dimension, endowed with a $\bar{\mathbf{Q}}$-structure V'. Let us denote by $L \cdot V'$ the $\bar{\mathbf{Q}}$-vector subspace of V generated by the products λx with $\lambda \in L$ and $x \in V'$. Let Z be a finite dimensional $\bar{\mathbf{Q}}$-vector subspace of $V' + L \cdot V'$, and let U be a K-vector subspace of V containing Z. If $U \neq V$, then there exists a K-vector subspace T of V which is rational over $\bar{\mathbf{Q}}$ and different from V, such that*

$$\frac{\dim_{\bar{\mathbf{Q}}}((Z+T)/T)}{\dim_K(V/T)} \leqslant \frac{\dim_K(U)}{\dim_K(V/U)}. \tag{2}$$

Proof. Let $\psi\colon V \to K^d$ be a K-linear isomorphism which is rational over $\bar{\mathbf{Q}}$. It maps Z into a $\bar{\mathbf{Q}}$-vector subspace of $\mathscr{L}^d$, and U into a K-vector subspace of K^d containing $\psi(Z)$. Moreover, since $V \neq 0$, we have $d \neq 0$. Then Theorem 4 guarantees the existence of a surjective K-linear mapping $t\colon K^d \to K^{d'}$ which is non-zero and rational over $\bar{\mathbf{Q}}$, such that

$$\frac{\dim_{\bar{\mathbf{Q}}}(t(\psi(Z)))}{d' + \dim_{\bar{\mathbf{Q}}}(t(\psi(Z)))} \leqslant \frac{\dim_K(\psi(U))}{d}.$$

Let T be the kernel of the composite mapping $t \circ \psi\colon V \to K^{d'}$. Since $t \circ \psi$ is a surjective K-linear mapping which is rational over $\bar{\mathbf{Q}}$, T is a subspace of V which is rational over $\bar{\mathbf{Q}}$, and we have

$$d' = \dim_K(V/T) \qquad \text{and} \qquad \dim_{\bar{\mathbf{Q}}}(t(\psi(Z))) = \dim_{\bar{\mathbf{Q}}}((Z+T)/T).$$

Moreover, since t is non-zero, we have $d' \neq 0$, thus $T \neq V$. Last, since ψ is an isomorphism, we have also

$$d = \dim_K(V) \qquad \text{and} \qquad \dim_K(\psi(U)) = \dim_K(U).$$

The inequality (2) follows easily from this if $U \neq V$.

LEMMA 2. *Let V be a non-zero vector space of finite dimension over K endowed with a $\bar{\mathbf{Q}}$-structure, and let Z be a $\bar{\mathbf{Q}}$-vector subspace of V of finite dimension. Among the K-vector subspaces T of V which are rational over $\bar{\mathbf{Q}}$ and different from V, for which the ratio*

$$\frac{\dim_{\bar{\mathbf{Q}}}((Z+T)/T)}{\dim_K(V/T)}$$

is minimal, we choose one of minimal dimension which we denote again by T. Then, T is the smallest subspace of V which is rational over $\bar{\mathbf{Q}}$ and which contains $Z \cap T$. Moreover, T satisfies $\sigma(T) = T$ for all K-linear automorphisms σ of V which are rational over $\bar{\mathbf{Q}}$ and which satisfy $\sigma(Z) = Z$.

Proof. Let T_1 be the smallest subspace of V which is rational over $\bar{\mathbf{Q}}$ and which contains $Z \cap T$. Since T is rational over $\bar{\mathbf{Q}}$, we have $T_1 \subset T$, whence $Z \cap T_1 = Z \cap T$, and thus

$$\dim_{\bar{\mathbf{Q}}}((Z+T_1)/T_1) = \dim_{\bar{\mathbf{Q}}}((Z+T)/T).$$

Then, the choice of T implies $\dim_K(V/T_1) \leqslant \dim_K(V/T)$, which, together with the inclusion $T_1 \subset T$, gives $T_1 = T$. This proves the first assertion of the lemma.

Let σ be a K-linear automorphism of V which is rational over $\bar{\mathbf{Q}}$, such that $\sigma(Z) = Z$. It remains to show $\sigma(T) = T$. To this end, we put

$$R = T \cap \sigma(T) \qquad \text{and} \qquad S = T + \sigma(T).$$

These are subspaces of V which are rational over $\bar{\mathbf{Q}}$. They satisfy

$$\dim_K(R) + \dim_K(S) = \dim_K(T) + \dim_K(\sigma(T)) = 2\dim_K(T),$$

so that

$$\dim_K(V/R) + \dim_K(V/S) = 2\dim_k(V/T). \tag{3}$$

We have also the relations

$$Z \cap R = (Z \cap T) \cap (Z \cap \sigma(T)) \qquad \text{and} \qquad Z \cap S \supset (Z \cap T) + (Z \cap \sigma(T)).$$

Since $\sigma(Z \cap T) = Z \cap \sigma(T)$, they imply

$$\dim_{\bar{\mathbf{Q}}}(Z \cap R) + \dim_{\bar{\mathbf{Q}}}(Z \cap S) \geqslant 2\dim_{\bar{\mathbf{Q}}}(Z \cap T),$$

whence

$$\dim_{\bar{\mathbf{Q}}}((Z+R)/R) + \dim_{\bar{\mathbf{Q}}}((Z+S)/S) \leqslant 2\dim_{\bar{\mathbf{Q}}}((Z+T)/T). \tag{4}$$

Therefore, in defining $\mu \in \mathbf{Q}$ by $\dim_{\mathbf{Q}}((Z+T)/T) = \mu \dim_K(V/T)$, the relations (3) and (4) give

$$\dim_{\mathbf{Q}}((Z+R)/R) + \dim_{\mathbf{Q}}((Z+S)/S)$$
$$\leqslant \mu(\dim_K(V/R) + \dim_K(V/S));$$

but the choice of T and the definition of μ imply $\dim_{\mathbf{Q}}((Z+S)/S) \geqslant \geqslant \mu \dim_K(V/S)$, whence $\dim_{\mathbf{Q}}((Z+R)/R) \leqslant \mu \dim_K(V/R)$. Since R is contained in T, this cannot hold unless $R = T$, i.e., unless $T = \sigma(T)$.

Finally, we shall need the following lemma, inspired by Lemma 5 of [L1].

LEMMA 3. *Let $\mathbf{C}_p[G]$ be given the $\bar{\mathbf{Q}}$-structure $\bar{\mathbf{Q}}[G]$ and let S be a subset of $\bar{\mathbf{Q}} \cdot \theta(k_1)$. Then, the smallest subspace of $\mathbf{C}_p[G]$ which is rational over $\bar{\mathbf{Q}}$ and which contains S is a right ideal of $\mathbf{C}_p[G]$.*

Proof. We may suppose that S consists of one element z. It suffices to show that if z belongs to a hyperplane H of $\mathbf{C}_p[G]$ which is rational over $\bar{\mathbf{Q}}$, then z is contained in $H\tau$ for all $\tau \in G$. So let us assume

$$z \in H = \left\{ \sum_\sigma x_\sigma \sigma^{-1} \in \mathbf{C}_p[G]; \sum_\sigma a_\sigma x_\sigma = 0 \right\}$$

for some $a_\sigma \in \bar{\mathbf{Q}}$ ($\sigma \in G$) not all zero. If we write

$$z = \sum_{i=1}^{t} b_i \theta(\alpha_i) = \sum_{i,\sigma} b_i \log(\sigma\alpha_i) \cdot \sigma^{-1}$$

with $\alpha_1, ..., \alpha_t \in k_1$ and $b_1, ..., b_t \in \bar{\mathbf{Q}}$, the relation $z \in H$ reads

$$\sum_{i,\sigma} a_\sigma b_i \log(\sigma\alpha_i) = 0. \tag{5}$$

Let E be the subspace of L generated over $\mathbf{Q}$ by the numbers $\log(\alpha)$ with $\alpha \in k_1$, and let τ be an element of G. Since τ determines by restriction an automorphism of the multiplicative group k_1, and that log determines by restriction a homomorphism of finite kernel from k_1 to the additive group E, there exists a unique $\mathbf{Q}$-linear automorphism of E which maps $\log(\alpha)$ to $\log(\tau^{-1}\alpha)$ for all $\alpha \in k_1$. This automorphism extends to a $\bar{\mathbf{Q}}$-linear automorphism of $\bar{\mathbf{Q}} \cdot E$, since, by Baker's theorem, any basis of E over $\mathbf{Q}$ is also a basis $\bar{\mathbf{Q}} \cdot E$ over $\bar{\mathbf{Q}}$. Applying this automorphism to both sides of (5), we get

$$\sum_{i,\sigma} a_\sigma b_i \log(\tau^{-1}\sigma\alpha_i) = 0.$$

This means $z\tau^{-1} \in H$, thus $z \in H\tau$.

Proof of Theorem 7. We fix the choice of ϕ, and, for the sake of simplicity, we put $d = d_\phi$, $r = r_\phi$, and $\rho = \rho_\phi$. We denote by V' the simple subalgebra of $\bar{\mathbf{Q}}[G]$ with character $d\phi$, and by e its unit element, so that $V' = \bar{\mathbf{Q}}[G]e$. We put

$$Z = (\bar{\mathbf{Q}} \cdot \theta(M))e, \qquad U = (\mathbf{C}_p \cdot \theta(M))e, \qquad V = \mathbf{C}_p[G]e = \mathbf{C}_p \cdot V'.$$

Then Z is the sum of all $\bar{\mathbf{Q}}[G]$-submodules of $\bar{\mathbf{Q}} \cdot \theta(M)$ isomorphic to a submodule of V', U is the $\mathbf{C}_p[G]$-submodule of $\mathbf{C}_p \cdot \theta(M)$ with character $\rho\phi$, and V is the simple subalgebra of $\mathbf{C}_p[G]$ with character $d\phi$. It remains to determine the character of Z. Since the kernel of θ consists of the roots of unity of $\mathbf{C}_p$ contained in k_1, there exists an isomorphism of $\mathbf{Q}[G]$-modules from $M \otimes_{\mathbf{Z}} \mathbf{Q}$ to $\mathbf{Q} \cdot \theta(M)$ which sends $x \otimes 1$ to $\theta(x)$ for all $x \in M$. Since $\mathbf{Q} \cdot \theta(M)$ is contained in $\sum_{\sigma \in G} L\sigma$, it extends, by virtue of Baker's theorem, to an isomorphism of $\bar{\mathbf{Q}}[G]$-modules, from $M \otimes_{\mathbf{Z}} \bar{\mathbf{Q}}$ to $\bar{\mathbf{Q}} \cdot \theta(M)$. This implies that the character of the $\bar{\mathbf{Q}}[G]$-module Z is $r\phi$. Since the degree of ϕ is d, we get

$$\dim_{\bar{\mathbf{Q}}}(Z) = rd, \qquad \dim_{\mathbf{C}_p}(U) = \rho d, \qquad \dim_{\mathbf{C}_p}(V) = d^2.$$

Let us give $\mathbf{C}_p[G]$ the $\bar{\mathbf{Q}}$-structure $\bar{\mathbf{Q}}[G]$. Since $V = \mathbf{C}_p \cdot V'$ with $V' \subset \bar{\mathbf{Q}}[G]$, the subspace V of $\mathbf{C}_p[G]$ is rational over $\bar{\mathbf{Q}}$, and V' defines on V an induced $\bar{\mathbf{Q}}$-structure. With respect to this $\bar{\mathbf{Q}}$-structure of V, the subspaces of V which are rational over $\bar{\mathbf{Q}}$ are the subspaces of $\mathbf{C}_p[G]$ which are rational over $\bar{\mathbf{Q}}$ and contained in V. Let us define $L \cdot V'$ as in Lemma 1. Since Z is contained in $L \cdot V'$, and since U contains Z, we are in a position to apply this lemma.

If $U = V$, we get $\rho = d$, and the inequality of the theorem is verified. If $U \neq V$, Lemma 1 asserts the existence of a subspace T of V which is rational over $\bar{\mathbf{Q}}$ and different from V, such that

$$\frac{\dim_{\bar{\mathbf{Q}}}((Z+T)/T)}{\dim_{\mathbf{C}_p}(V/T)} \leqslant \frac{\dim_{\mathbf{C}_p}(U)}{\dim_{\mathbf{C}_p}(V/U)}.$$

By Lemma 2, we can choose T in such a way that T is the smallest subspace of V which is rational over $\bar{\mathbf{Q}}$ and which contains $Z \cap T$, and that T is fixed under any $\mathbf{C}_p$-linear automorphism of V which is rational over $\bar{\mathbf{Q}}$ and which fixes Z. Then, T is also the smallest subspace of $\mathbf{C}_p[G]$ which is rational over $\bar{\mathbf{Q}}$ and which contains $Z \cap T$. Since $Z \subset \bar{\mathbf{Q}} \cdot \theta(M)$, this implies, by Lemma 3, that T is a right ideal of V. On the other hand, since, for all $\sigma \in G$, the left multiplication by σ in V is a $\mathbf{C}_p$-linear automorphism of V which is rational over $\bar{\mathbf{Q}}$ and which fixes Z, T is stable under these automorphisms and is thus also a left ideal of V. Thus T is a two-sided ideal of V. Since it is different from V, and since V is a simple algebra, this implies $T=0$. So, the above inequality becomes

$$\frac{\dim_{\mathbf{Q}}(Z)}{\dim_{\mathbf{C}_p}(V)} \leqslant \frac{\dim_{\mathbf{C}_p}(U)}{\dim_{\mathbf{C}_p}(V/U)}.$$

The inequality stated in the theorem follows from this one by substituting to the dimensions which appear in it the values calculated above.

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