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On Two Exponents of Approximation Related to a Real Number and Its Square

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Abstract. For each real number ξ, let $\widehat{\lambda}_2(\xi)$ denote the supremum of all real numbers λ such that, for each sufficiently large X, the inequalities $|x_0| \le X$, $|x_0\xi - x_1| \le X^{-\lambda}$ and $|x_0\xi^2 - x_2| \le X^{-\lambda}$ admit a solution in integers x_0, x_1 and x_2 not all zero, and let $\widehat{\omega}_2(\xi)$ denote the supremum of all real numbers ω such that, for each sufficiently large X, the dual inequalities $|x_0 + x_1\xi + x_2\xi^2| \le X^{-\omega}$, $|x_1| \le X$ and $|x_2| \le X$ admit a solution in integers x_0, x_1 and x_2 not all zero. Answering a question of Y. Bugeaud and M. Laurent, we show that the exponents $\widehat{\lambda}_2(\xi)$ where ξ ranges through all real numbers with $[\mathbb{Q}(\xi):\mathbb{Q}] > 2$ form a dense subset of the interval $[1/2, (\sqrt{5}-1)/2]$ while, for the same values of ξ, the dual exponents $\widehat{\omega}_2(\xi)$ form a dense subset of $[2, (\sqrt{5}+3)/2]$. Part of the proof rests on a result of V. Jarník showing that $\widehat{\lambda}_2(\xi) = 1 - \widehat{\omega}_2(\xi)^{-1}$ for any real number ξ with $[\mathbb{Q}(\xi):\mathbb{Q}] > 2$.

1 Introduction

Let ξ and η be real numbers. Following the notation of Y. Bugeaud and M. Laurent [3], we define $\widehat{\lambda}(\xi,\eta)$ to be the supremum of all real numbers λ such that the inequalities

$$|x_0| \le X, \quad |x_0\xi - x_1| \le X^{-\lambda} \quad \text{and} \quad |x_0\eta - x_2| \le X^{-\lambda}$$

admit a non-zero integer solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$ for each sufficiently large value of X. Similarly, we define $\widehat{\omega}(\xi,\eta)$ to be the supremum of all real numbers ω such that the inequalities

$$|x_0 + x_1\xi + x_2\eta| \le X^{-\omega}, \quad |x_1| \le X \quad \text{and} \quad |x_2| \le X$$

admit a non-zero solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$ for each sufficiently large value of X. An application of Dirichlet box principle shows that we have $1/2 \le \widehat{\lambda}(\xi,\eta)$ and $2 \le \widehat{\omega}(\xi,\eta)$. Moreover, in the (non-degenerate) case where 1, ξ and η are linearly independent over $\mathbb{Q}$, a result of V. Jarník, kindly pointed out to the author by Yann Bugeaud, shows that these exponents are related by the formula

$$\widehat{\lambda}(\xi,\eta) = 1 - \frac{1}{\widehat{\omega}(\xi,\eta)}, \tag{1}$$

with the convention that the right-hand side of this equality is 1 if $\widehat{\omega}(\xi,\eta) = \infty$ (see [7, Theorem 1]).

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In the case where $\eta = \xi^2$, we use the shorter notation $\widehat{\lambda}_2(\xi) := \widehat{\lambda}(\xi, \xi^2)$ and $\widehat{\omega}_2(\xi) := \widehat{\omega}(\xi, \xi^2)$ of [3]. The condition that 1, ξ and ξ^2 are linearly independent over $\mathbb{Q}$ simply means that ξ is not an algebraic number of degree at most 2 over $\mathbb{Q}$, a condition which we also write as $[\mathbb{Q}(\xi):\mathbb{Q}] > 2$. Under this condition, it is known that these exponents satisfy

$$\frac{1}{2} \leq \widehat{\lambda}_2(\xi) \leq \frac{1}{\gamma} = 0.618\ldots \quad \text{and} \quad 2 \leq \widehat{\omega}_2(\xi) \leq \gamma^2 = 2.618\ldots, \tag{2}$$

where $\gamma = (1 + \sqrt{5})/2$ denotes the golden ratio. By virtue of W. M. Schmidt's subspace theorem, the lower bounds in (2) are achieved by any algebraic number ξ of degree at least 3 (see [12, Ch. VI, Corollaries 1C, 1E]). They are also achieved by almost all real numbers ξ, with respect to Lebesgue's measure (see [3, Theorem 2.3]). On the other hand, the upper bounds follow respectively from [5, Theorem 1a] and from [2]. They are achieved in particular by the so-called Fibonacci continued fractions (see [8, §2] or [9, §6]), a special case of the Sturmian continued fractions of [1]. Now, thanks to Jarník's formula (1), we recognize that each set of inequalities in (2) can be deduced from the other one.

Generalizing the approach of [8], Bugeaud and Laurent have computed the exponents $\widehat{\lambda}_2(\xi)$ and $\widehat{\omega}_2(\xi)$ for a general (characteristic) Sturmian continued fraction ξ. They found that, after $1/\gamma$ and γ^2, the next largest values of $\widehat{\lambda}_2(\xi)$ and $\widehat{\omega}_2(\xi)$ for such numbers ξ are, respectively, $2 - \sqrt{2} \simeq 0.586$ and $1 + \sqrt{2} \simeq 2.414$, and they asked if there exists any transcendental real number ξ which satisfies either $2 - \sqrt{2} < \widehat{\lambda}_2(\xi) < 1/\gamma$ or $1 + \sqrt{2} < \widehat{\omega}_2(\xi) < \gamma^2$ (see [3, §8]). Our main result below shows that such numbers exist.

Theorem *The points $(\widehat{\lambda}_2(\xi), \widehat{\omega}_2(\xi))$ where ξ runs through all real numbers with $[\mathbb{Q}(\xi):\mathbb{Q}] > 2$ form a dense subset of the curve $\mathcal{C} = \{(1 - \omega^{-1}, \omega)\,;\, 2 \leq \omega \leq \gamma^2\}$.*

Since $(\widehat{\lambda}_2(\xi), \widehat{\omega}_2(\xi)) = (1/2, 2)$ for any algebraic number ξ of degree at least 3, it follows in particular that $(1/\gamma, \gamma^2)$ is an accumulation point for the set of points $(\widehat{\lambda}_2(\xi), \widehat{\omega}_2(\xi))$ with ξ a transcendental real number. Because of Jarník's formula (1), this theorem is equivalent to either one of the following two assertions.

Corollary *The exponents $\widehat{\lambda}_2(\xi)$ attached to transcendental real numbers ξ form a dense subset of the interval* $[1/2, 1/\gamma]$. *The corresponding dual exponents $\widehat{\omega}_2(\xi)$ form a dense subset of* $[2, \gamma^2]$.

The proof is inspired by the constructions of [9, §6] and [11, §5]. We produce countably many real numbers ξ of "Fibonacci type" (see §7 for a precise definition) for which we show that the exponents $\widehat{\omega}_2(\xi)$ are dense in $[2, \gamma^2]$. By (1), this implies the theorem. One may then reformulate the question of Bugeaud and Laurent by asking if there exist transcendental real numbers ξ not of that type which satisfy $\widehat{\omega}_2(\xi) > 1 + \sqrt{2}$. The work of S. Fischler announced in [6] should shed some light on this question.

2 Notation and Equivalent Definitions of the Exponents

We define the *norm* of a point $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{R}^3$ as its maximum norm

$$\|\mathbf{x}\| = \max_{0 \le i \le 2} |x_i|.$$

Given a second point $\mathbf{y} \in \mathbb{R}^3$, we denote by $\mathbf{x} \wedge \mathbf{y}$ the standard vector product of $\mathbf{x}$ and $\mathbf{y}$, and by $\langle \mathbf{x}, \mathbf{y} \rangle$ their standard scalar product. Given a third point $\mathbf{z} \in \mathbb{R}^3$, we also denote by $\det(\mathbf{x}, \mathbf{y}, \mathbf{z})$ the determinant of the 3×3 matrix whose rows are $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$. Then we have the well-known relation

$$\det(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \langle \mathbf{x}, \mathbf{y} \wedge \mathbf{z} \rangle$$

and we get the following alternative definition of the exponents $\widehat{\lambda}(\xi, \eta)$ and $\widehat{\omega}(\xi, \eta)$.

Lemma 2.1 *Let $\xi, \eta \in \mathbb{R}$, and let $\mathbf{y} = (1, \xi, \eta)$. Then $\widehat{\lambda}(\xi, \eta)$ is the supremum of all real numbers λ such that, for each sufficiently large real number $X \ge 1$, there exists a point $\mathbf{x} \in \mathbb{Z}^3$ with*

$$0 < \|\mathbf{x}\| \le X \quad \text{and} \quad \|\mathbf{x} \wedge \mathbf{y}\| \le X^{-\lambda}.$$

Similarly, $\widehat{\omega}(\xi, \eta)$ is the supremum of all real numbers ω such that, for each sufficiently large real number $X \ge 1$, there exists a point $\mathbf{x} \in \mathbb{Z}^3$ with

$$0 < \|\mathbf{x}\| \le X \quad \text{and} \quad |\langle \mathbf{x}, \mathbf{y} \rangle| \le X^{-\omega}.$$

In the sequel, we will need the following inequalities.

Lemma 2.2 *For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$, we have*

$$\|\langle \mathbf{x}, \mathbf{z} \rangle \mathbf{y} - \langle \mathbf{x}, \mathbf{y} \rangle \mathbf{z}\| \le 2\|\mathbf{x}\|\|\mathbf{y} \wedge \mathbf{z}\|, \tag{3}$$

$$\|\mathbf{y}\|\|\mathbf{x} \wedge \mathbf{z}\| \le \|\mathbf{z}\|\|\mathbf{x} \wedge \mathbf{y}\| + 2\|\mathbf{x}\|\|\mathbf{y} \wedge \mathbf{z}\|. \tag{4}$$

Proof Writing $\mathbf{y} = (y_0, y_1, y_2)$ and $\mathbf{z} = (z_0, z_1, z_2)$, we find

$$\|\langle \mathbf{x}, \mathbf{z} \rangle \mathbf{y} - \langle \mathbf{x}, \mathbf{y} \rangle \mathbf{z}\| = \max_{i=0,1,2} |\langle \mathbf{x}, y_i \mathbf{z} - z_i \mathbf{y} \rangle| \le 2\|\mathbf{x}\|\|\mathbf{y} \wedge \mathbf{z}\|,$$

which proves (3). Similarly, one finds $\|y_i \mathbf{x} \wedge \mathbf{z} - z_i \mathbf{x} \wedge \mathbf{y}\| \le 2\|\mathbf{x}\|\|\mathbf{y} \wedge \mathbf{z}\|$ for $i = 0, 1, 2$, and this implies (4). ∎

For any non-zero point $\mathbf{x}$ of $\mathbb{R}^3$, let $[\mathbf{x}]$ denote the point of $\mathbb{P}^2(\mathbb{R})$ having $\mathbf{x}$ as a set of homogeneous coordinates. Then (4) has a useful interpretation in terms of the projective distance defined for non-zero points $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^3$ by

$$\operatorname{dist}([\mathbf{x}], [\mathbf{y}]) = \operatorname{dist}(\mathbf{x}, \mathbf{y}) = \frac{\|\mathbf{x} \wedge \mathbf{y}\|}{\|\mathbf{x}\|\|\mathbf{y}\|}.$$

Indeed, for any triple of non-zero points $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$, it gives

$$\operatorname{dist}([\mathbf{x}], [\mathbf{z}]) \le \operatorname{dist}([\mathbf{x}], [\mathbf{y}]) + 2\operatorname{dist}([\mathbf{y}], [\mathbf{z}]). \tag{5}$$

3 Fibonacci Sequences in $\mathrm{GL}_2(\mathbb{C})$

A *Fibonacci sequence* in a monoid is a sequence $(\mathbf{w}_i)_{i\geq 0}$ of elements of that monoid such that $\mathbf{w}_{i+2} = \mathbf{w}_{i+1}\mathbf{w}_i$ for each index $i \geq 0$. Clearly, such a sequence is entirely determined by its first two elements $\mathbf{w}_0$ and $\mathbf{w}_1$. We start with the following observation.

Proposition 3.1 *There exists a non-empty Zariski open subset $\mathcal{U}$ of $\mathrm{GL}_2(\mathbb{C})^2$ with the following property. For each Fibonacci sequence $(\mathbf{w}_i)_{i\geq 0}$ with $(\mathbf{w}_0, \mathbf{w}_1) \in \mathcal{U}$, there exists $N \in \mathrm{GL}_2(\mathbb{C})$ such that the matrix*

$$\mathbf{y}_i = \begin{cases} \mathbf{w}_i N & \text{if } i \text{ is even,} \\ \mathbf{w}_i{}^t N & \text{if } i \text{ is odd,} \end{cases} \tag{6}$$

is symmetric for each $i \geq 0$. Any matrix $N \in \mathrm{GL}_2(\mathbb{C})$ such that $\mathbf{w}_0 N$, $\mathbf{w}_1{}^t N$ and $\mathbf{w}_1\mathbf{w}_0 N$ are symmetric satisfies this property. When $\mathbf{w}_0$ and $\mathbf{w}_1$ have integer coefficients, we may take N with integer coefficients.

Proof Let $(\mathbf{w}_i)_{i\geq 0}$ be a Fibonacci sequence in $\mathrm{GL}_2(\mathbb{C})$ and let $N \in \mathrm{GL}_2(\mathbb{C})$. Defining $\mathbf{y}_i$ by (6) for each $i \geq 0$, we find $\mathbf{y}_{i+3} = \mathbf{y}_{i+1}{}^t S \mathbf{y}_i S \mathbf{y}_{i+1}$ with $S = N^{-1}$ if i is even and $S = {}^t N^{-1}$ if i is odd. Thus, $\mathbf{y}_i$ is symmetric for each $i \geq 0$ if and only if it is so for $i = 0, 1, 2$.

Now, for any given point $(\mathbf{w}_0, \mathbf{w}_1) \in \mathrm{GL}_2(\mathbb{C})^2$, the conditions that $\mathbf{w}_0 N$, $\mathbf{w}_1{}^t N$ and $\mathbf{w}_1\mathbf{w}_0 N$ are symmetric represent a system of three linear equations in the four unknown coefficients of N. Let $\mathcal{V}$ be the Zariski open subset of $\mathrm{GL}_2(\mathbb{C})^2$ consisting of all points $(\mathbf{w}_0, \mathbf{w}_1)$ for which this linear system has rank 3. Then, for each $(\mathbf{w}_0, \mathbf{w}_1) \in \mathcal{V}$, the 3×3 minors of this linear system conveniently arranged into a 2×2 matrix provide a non-zero solution N of the system, whose coefficients are polynomials in those of $\mathbf{w}_0$ and $\mathbf{w}_1$ with integer coefficients. Then the condition $\det(N) \neq 0$ in turn determines a Zariski open subset $\mathcal{U}$ of $\mathcal{V}$. To conclude, we note that $\mathcal{U}$ is not empty as a short computation shows that it contains the point formed by $\mathbf{w}_0 = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\mathbf{w}_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. ∎

Definition 3.2 Let $\mathcal{M} = \mathrm{Mat}_{2\times 2}(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{C})$ denote the monoid of 2×2 integer matrices with non-zero determinant. We say that a Fibonacci sequence $(\mathbf{w}_i)_{i\geq 0}$ in $\mathcal{M}$ is *admissible* if there exists a matrix $N \in \mathcal{M}$ such that the sequence $(\mathbf{y}_i)_{i\geq 0}$ given by (6) consists of symmetric matrices.

Since $\mathcal{M}$ is Zariski dense in $\mathrm{GL}_2(\mathbb{C})$, Proposition 3.1 shows that almost all Fibonacci sequences in $\mathcal{M}$ are admissible. The following example is an illustration of this.

Example 3.3 Fix integers a, b, c with $a \geq 2$ and $c \geq b \geq 1$, and define

$$\mathbf{w}_0 = \begin{pmatrix} 1 & b \\ a & a(b+1) \end{pmatrix}, \quad \mathbf{w}_1 = \begin{pmatrix} 1 & c \\ a & a(c+1) \end{pmatrix}$$

and

$$N = \begin{pmatrix} -1 + a(b+1)(c+1) & -a(b+1) \\ -a(c+1) & a \end{pmatrix}.$$

These matrices belong to $\mathcal{M}$ since $\det(\mathbf{w}_0) = \det(\mathbf{w}_1) = a$ and $\det(N) = -a$. Moreover, one finds that

$$\mathbf{w}_0 N = \begin{pmatrix} -1 + a(c+1) & -a \\ -a & 0 \end{pmatrix}, \quad \mathbf{w}_1{}^t N = \begin{pmatrix} -1 + a(b+1) & -a \\ -a & 0 \end{pmatrix}$$

and

$$\mathbf{w}_1 \mathbf{w}_0 N = \begin{pmatrix} -1 + a & -a \\ -a & -a^2 \end{pmatrix}$$

are symmetric matrices. Therefore, the Fibonacci sequence $(\mathbf{w}_i)_{i \geq 0}$ constructed on $\mathbf{w}_0$ and $\mathbf{w}_1$ is admissible with an associated sequence of symmetric matrices $(\mathbf{y}_i)_{i \geq 0}$ given by (6), the first three matrices of this sequence being the above products $\mathbf{y}_0 = \mathbf{w}_0 N$, $\mathbf{y}_1 = \mathbf{w}_1{}^t N$ and $\mathbf{y}_2 = \mathbf{w}_1 \mathbf{w}_0 N$.

4 Fibonacci Sequences of 2×2 Integer Matrices

In the sequel, we identify $\mathbb{R}^3$ (resp., $\mathbb{Z}^3$) with the space of 2×2 symmetric matrices with real (resp., integer) coefficients under the map

$$\mathbf{x} = (x_0, x_1, x_2) \longmapsto \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix}.$$

Accordingly, it makes sense to define the determinant of a point $\mathbf{x} = (x_0, x_1, x_2)$ of $\mathbb{R}^3$ by $\det(\mathbf{x}) = x_0 x_2 - x_1^2$. Similarly, given symmetric matrices $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, we write $\mathbf{x} \wedge \mathbf{y}$, $\langle \mathbf{x}, \mathbf{y} \rangle$ and $\det(\mathbf{x}, \mathbf{y}, \mathbf{z})$ to denote respectively the vector product, scalar product and determinant of the corresponding points.

In this section we look at arithmetic properties of admissible Fibonacci sequences in the monoid $\mathcal{M}$ of Definition 3.2. For this purpose, we define the *content* of an integer matrix $\mathbf{w} \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ or of a point $\mathbf{y} \in \mathbb{Z}^3$ as the greatest common divisor of their coefficients. We say that such a matrix or point is *primitive* if its content is 1.

Proposition 4.1 *Let $(\mathbf{w}_i)_{i \geq 0}$ be an admissible Fibonacci sequence of matrices in $\mathcal{M}$ and let $(\mathbf{y}_i)_{i \geq 0}$ be a corresponding sequence of symmetric matrices in $\mathcal{M}$. For each $i \geq 0$, define $\mathbf{z}_i = \det(\mathbf{w}_i)^{-1} \mathbf{y}_i \wedge \mathbf{y}_{i+1}$. Then, for each $i \geq 0$, we have*
(a) $\mathrm{tr}(\mathbf{w}_{i+3}) = \mathrm{tr}(\mathbf{w}_{i+1})\, \mathrm{tr}(\mathbf{w}_{i+2}) - \det(\mathbf{w}_{i+1})\, \mathrm{tr}(\mathbf{w}_i)$,
(b) $\mathbf{y}_{i+3} = \mathrm{tr}(\mathbf{w}_{i+1}) \mathbf{y}_{i+2} - \det(\mathbf{w}_{i+1}) \mathbf{y}_i$,
(c) $\mathbf{z}_{i+3} = \mathrm{tr}(\mathbf{w}_{i+1}) \mathbf{z}_{i+1} + \det(\mathbf{w}_i) \mathbf{z}_i$,
(d) $\det(\mathbf{y}_i, \mathbf{y}_{i+1}, \mathbf{y}_{i+2}) = (-1)^i \det(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2) \det(\mathbf{w}_2)^{-1} \det(\mathbf{w}_{i+2})$,
(e) $\mathbf{z}_i \wedge \mathbf{z}_{i+1} = (-1)^i \det(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2) \det(\mathbf{w}_2)^{-1} \mathbf{y}_{i+1}$.

Proof For each index $i \geq 0$, let N_i denote the element of $\mathcal{M}$ for which $\mathbf{y}_i = \mathbf{w}_i N_i$. According to (6), we have $N_i = N$ if i is even and $N_i = {}^t N$ if i is odd. We first prove

(b) following the argument of the proof of [10, Lemma 2.5(i)]. Multiplying both sides of the equality $\mathbf{w}_{i+2} = \mathbf{w}_{i+1}\mathbf{w}_i$ on the right by $N_{i+2} = N_i$, we find

$$\mathbf{y}_{i+2} = \mathbf{w}_{i+1}\mathbf{y}_i, \tag{7}$$

which can be rewritten as $\mathbf{y}_{i+2} = \mathbf{y}_{i+1}N_{i+1}^{-1}\mathbf{y}_i$. Taking the transpose of both sides, this gives $\mathbf{y}_{i+2} = \mathbf{y}_i N_i^{-1}\mathbf{y}_{i+1} = \mathbf{w}_i\mathbf{y}_{i+1}$. Replacing i by $i+1$ in the latter identity and combining it with (7), we get

$$\mathbf{y}_{i+3} = \mathbf{w}_{i+1}\mathbf{y}_{i+2} = \mathbf{w}_{i+1}^2\mathbf{y}_i. \tag{8}$$

Then (b) follows from (7) and (8), using the fact that, by the Cayley–Hamilton theorem, we have $\mathbf{w}_{i+1}^2 = \mathrm{tr}(\mathbf{w}_{i+1})\mathbf{w}_{i+1} - \det(\mathbf{w}_{i+1})I$. Multiplying both sides of (b) on the right by N_i^{-1} and taking the trace, we deduce that

$$\mathrm{tr}(\mathbf{y}_{i+3}N_i^{-1}) = \mathrm{tr}(\mathbf{w}_{i+1})\,\mathrm{tr}(\mathbf{w}_{i+2}) - \det(\mathbf{w}_{i+1})\,\mathrm{tr}(\mathbf{w}_i).$$

This gives (a) because $\mathrm{tr}(\mathbf{y}_{i+3}N_i^{-1}) = \mathrm{tr}({}^t\mathbf{y}_{i+3}{}^tN_i^{-1}) = \mathrm{tr}(\mathbf{w}_{i+3})$. Taking the exterior product of both sides of (b) with $\mathbf{y}_{i+1}$, we also find

$$\mathbf{y}_{i+1} \wedge \mathbf{y}_{i+3} = \mathrm{tr}(\mathbf{w}_{i+1})\det(\mathbf{w}_{i+1})\mathbf{z}_{i+1} + \det(\mathbf{w}_{i+1})\det(\mathbf{w}_i)\mathbf{z}_i.$$

Similarly, replacing i by $i+1$ in (b) and taking the exterior product with $\mathbf{y}_{i+3}$ gives

$$\det(\mathbf{w}_{i+3})\mathbf{z}_{i+3} = \det(\mathbf{w}_{i+2})\mathbf{y}_{i+1} \wedge \mathbf{y}_{i+3}.$$

Then (c) follows upon noting that $\det(\mathbf{w}_{i+3}) = \det(\mathbf{w}_{i+2})\det(\mathbf{w}_{i+1})$.

The formula (d) is clearly true for $i = 0$. If we assume that it holds for some integer $i \geq 0$, then using the formula for $\mathbf{y}_{i+3}$ given by (b) and taking into account the multilinearity of the determinant we find

$$\begin{aligned}\det(\mathbf{y}_{i+1}, \mathbf{y}_{i+2}, \mathbf{y}_{i+3}) &= -\det(\mathbf{w}_{i+1})\det(\mathbf{y}_i, \mathbf{y}_{i+1}, \mathbf{y}_{i+2})\\ &= (-1)^{i+1}\det(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2)\frac{\det(\mathbf{w}_{i+3})}{\det(\mathbf{w}_2)}.\end{aligned}$$

This proves (d) by induction on i. Then (e) follows since, for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{Z}^3$, we have $(\mathbf{x} \wedge \mathbf{y}) \wedge (\mathbf{y} \wedge \mathbf{z}) = \det(\mathbf{x}, \mathbf{y}, \mathbf{z})\,\mathbf{y}$ which, in the present case, gives

$$\mathbf{z}_i \wedge \mathbf{z}_{i+1} = \det(\mathbf{w}_{i+2})^{-1}\det(\mathbf{y}_i, \mathbf{y}_{i+1}, \mathbf{y}_{i+2})\,\mathbf{y}_{i+1}. \qquad \blacksquare$$

Corollary 4.2 *The notation being as in the proposition, assume that* $\mathrm{tr}(\mathbf{w}_i)$ *and* $\det(\mathbf{w}_i)$ *are relatively prime for* $i = 0, 1, 2, 3$ *and that* $\det(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2) \neq 0$. *Then for each* $i \geq 0$,

(a) *the points* $\mathbf{y}_i, \mathbf{y}_{i+1}, \mathbf{y}_{i+2}$ *are linearly independent,*
(b) $\mathrm{tr}(\mathbf{w}_i)$ *and* $\det(\mathbf{w}_i)$ *are relatively prime,*
(c) *the matrix* $\mathbf{w}_i$ *is primitive,*

(d) *the content of* $\mathbf{y}_i$ *divides* $\det(\mathbf{y}_2)/\det(\mathbf{w}_2)$,
(e) *the point* $\det(\mathbf{w}_2)\,\mathbf{z}_i$ *belongs to* $\mathbb{Z}^3$ *and its content divides* $\det(\mathbf{y}_2)\det(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2)$.

Proof The assertion (a) follows from Proposition 4.1(d). Since (b) holds by hypothesis for $i = 0, 1, 2, 3$, and since $\det(\mathbf{w}_2)$ and $\det(\mathbf{w}_i)$ have the same prime factors for each $i \geq 2$, the assertion (b) follows, by induction on i, from the fact that Proposition 4.1(a) gives $\mathrm{tr}(\mathbf{w}_{i+1}) \equiv \mathrm{tr}(\mathbf{w}_i)\,\mathrm{tr}(\mathbf{w}_{i-1})$ modulo $\det(\mathbf{w}_2)$ for each $i \geq 3$. Then (c) follows since the content of $\mathbf{w}_i$ divides both $\mathrm{tr}(\mathbf{w}_i)$ and $\det(\mathbf{w}_i)$.

Let $N \in \mathcal{M}$ such that $\mathbf{y}_2 = \mathbf{w}_2 N$. For each i, we have $\mathbf{y}_i = \mathbf{w}_i N_i$ where $N_i = N$ if i is even and $N_i = {}^tN$ if i is odd. This gives $\mathbf{y}_i\,\mathrm{Adj}(N_i) = \det(N)\mathbf{w}_i$ where $\mathrm{Adj}(N_i) \in \mathcal{M}$ denotes the adjoint of N_i. Thus, by (c), the content of $\mathbf{y}_i$ divides $\det(N) = \det(\mathbf{y}_2)/\det(\mathbf{w}_2)$, as claimed in (d).

The fact that $\det(\mathbf{w}_2)\,\mathbf{z}_i$ belongs to $\mathbb{Z}^3$ is clear for $i = 0, 1, 2$ because $\det(\mathbf{w}_0)$ and $\det(\mathbf{w}_1)$ divide $\det(\mathbf{w}_2)$. Then Proposition 4.1(c) shows, by induction on i, that $\det(\mathbf{w}_2)\,\mathbf{z}_i \in \mathbb{Z}^3$ for each $i \geq 0$. Moreover, the content of that point divides that of $\det(\mathbf{w}_2)^2\mathbf{z}_i \wedge \mathbf{z}_{i+1}$ which, by (d) and Proposition 4.1(e), divides $\det(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2)\det(\mathbf{y}_2)$. This proves (e). ■

Example 4.3 Let $(\mathbf{w}_i)_{i\geq 0}$, N and $(\mathbf{y}_i)_{i\geq 0}$ be as in Example 3.3. Since $\mathbf{w}_0$, $\mathbf{w}_1$ and N are congruent to matrices of the form $\left(\begin{smallmatrix} \pm 1 & * \\ 0 & 0 \end{smallmatrix}\right)$ modulo a and have determinant $\pm a$, all matrices $\mathbf{w}_i$ and $\mathbf{y}_i$ are congruent to matrices of the same form modulo a and their determinant is, up to sign, a power of a. Thus these matrices have relatively prime trace and determinant, and so are primitive for each $i \geq 0$. Since $\det(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2) = a^4(c - b)$, Proposition 4.1(e) shows that the points $\mathbf{z}_i = \det(\mathbf{w}_i)^{-1}\mathbf{y}_i \wedge \mathbf{y}_{i+1}$ satisfy $\mathbf{z}_i \wedge \mathbf{z}_{i+1} = (-1)^i a^2(c - b)\mathbf{y}_{i+1}$ for each $i \geq 0$. Moreover, we find that $a^{-1}\mathbf{z}_0 = (0, 0, b - c)$, $a^{-1}\mathbf{z}_1 = (a, -1 + a(b + 1), -b)$ and $a^{-1}\mathbf{z}_2 = (a, -1 + a(c + 1), -c)$ are integer points. Then Proposition 4.1(c) shows, by induction on i, that $a^{-1}\mathbf{z}_i \in \mathbb{Z}^3$ for each $i \geq 0$. In particular, if $c = b + 1$, we deduce from the relation $a^{-1}\mathbf{z}_i \wedge a^{-1}\mathbf{z}_{i+1} = \pm\mathbf{y}_{i+1}$ that $a^{-1}\mathbf{z}_i$ is a primitive integer point for each $i \geq 0$.

5 Growth Estimates

Define the *norm* of a 2×2 matrix $\mathbf{w} = (w_{k,\ell}) \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ as the largest absolute value of its coefficients $\|\mathbf{w}\| = \max_{1\leq k,\ell\leq 2} |w_{k,\ell}|$, and define $\gamma = (1 + \sqrt{5})/2$ as in the introduction. In this section, we provide growth estimates for the norm and determinant of elements of certain Fibonacci sequences in $\mathrm{GL}_2(\mathbb{R})$. We first establish two basic lemmas.

Lemma 5.1 *Let* $\mathbf{w}_0, \mathbf{w}_1 \in \mathrm{GL}_2(\mathbb{R})$. *Suppose that, for* $i = 0, 1$, *the matrix* $\mathbf{w}_i$ *is of the form* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *with* $1 \leq a \leq \min\{b, c\}$ *and* $\max\{b, c\} \leq d$. *Then all matrices of the Fibonacci sequence* $(\mathbf{w}_i)_{i\geq 0}$ *constructed on* $\mathbf{w}_0$ *and* $\mathbf{w}_1$ *have this form and for each* $i \geq 0$, *they satisfy*

$$\|\mathbf{w}_i\|\|\mathbf{w}_{i+1}\| < \|\mathbf{w}_{i+2}\| \leq 2\|\mathbf{w}_i\|\|\mathbf{w}_{i+1}\|. \tag{9}$$

Proof The first assertion follows by recurrence on i and is left to the reader. It implies that $\|\mathbf{w}_i\|$ is equal to the element of index $(2,2)$ of $\mathbf{w}_i$ for each $i \geq 0$. Then (9) follows by observing that, for any 2×2 matrices $\mathbf{w} = (w_{k,\ell})$ and $\mathbf{w}' = (w'_{k,\ell})$ with positive real coefficients, the product $\mathbf{w}'\mathbf{w} = (w''_{k,\ell})$ satisfies $w_{2,2}w'_{2,2} < w''_{2,2} \leq 2\|\mathbf{w}\|\|\mathbf{w}'\|$. ■

Lemma 5.2 *Let $(r_i)_{i\geq 0}$ be a sequence of positive real numbers. Assume that there exist constants $c_1, c_2 > 0$ such that $c_1 r_i r_{i+1} \leq r_{i+2} \leq c_2 r_i r_{i+1}$ for each $i \geq 0$. Then there also exist constants $c_3, c_4 > 0$ such that $c_3 r_i^{\gamma} \leq r_{i+1} \leq c_4 r_i^{\gamma}$ for each $i \geq 0$.*

Proof Define $c_3 = c_1^{\gamma}/(cc_2)$ and $c_4 = cc_2^{\gamma}/c_1$, where $c \geq 1$ is chosen so that the condition $c_3 \leq r_{i+1}/r_i^{\gamma} \leq c_4$ holds for $i = 0$. Assuming that the same condition holds for some index $i \geq 0$, we find

$$\frac{r_{i+2}}{r_{i+1}^{\gamma}} \geq c_1 \frac{r_i}{r_{i+1}^{1/\gamma}} \geq c_1 c_4^{-1/\gamma} = c^{1/\gamma^2} c_3 \geq c_3,$$

and similarly $r_{i+2}/r_{i+1}^{\gamma} \leq c_4$. This proves the lemma by recurrence on i. ■

Proposition 5.3 *Let $(\mathbf{w}_i)_{i\geq 0}$ be a Fibonacci sequence in $\mathrm{GL}_2(\mathbb{R})$. Suppose that there exist real numbers $c_1, c_2 > 0$ such that*

$$c_1\|\mathbf{w}_i\|\|\mathbf{w}_{i+1}\| \leq \|\mathbf{w}_{i+2}\| \leq c_2\|\mathbf{w}_i\|\|\mathbf{w}_{i+1}\| \tag{10}$$

for each $i \geq 0$. Then there exist constants $c_3, c_4 > 0$ for which the inequalities

$$c_3\|\mathbf{w}_i\|^{\gamma} \leq \|\mathbf{w}_{i+1}\| \leq c_4\|\mathbf{w}_i\|^{\gamma}, \quad c_3|\det(\mathbf{w}_i)|^{\gamma} \leq |\det(\mathbf{w}_{i+1})| \leq c_4|\det(\mathbf{w}_i)|^{\gamma} \tag{11}$$

hold for each $i \geq 0$. Moreover, if there exist $\alpha, \beta \geq 0$ such that

$$(c_2\|\mathbf{w}_i\|)^{\alpha} \leq |\det(\mathbf{w}_i)| \leq (c_1\|\mathbf{w}_i\|)^{\beta} \tag{12}$$

holds for $i = 0, 1$, then this relation extends to each $i \geq 0$.

Proof The first assertion of the proposition follows from Lemma 5.2 applied once with $r_i = \|\mathbf{w}_i\|$ and once with $r_i = |\det(\mathbf{w}_i)|$. To prove the second assertion, assume that for some index $j \geq 0$ the condition (12) holds both with $i = j$ and $i = j+1$. We find

$$|\det(\mathbf{w}_{j+2})| = |\det(\mathbf{w}_{j+1})||\det(\mathbf{w}_j)| \geq (c_2\|\mathbf{w}_{j+1}\|)^{\alpha}(c_2\|\mathbf{w}_j\|)^{\alpha} \geq (c_2\|\mathbf{w}_{j+2}\|)^{\alpha}$$

and similarly $|\det(\mathbf{w}_{j+2})| \leq (c_1\|\mathbf{w}_{j+2}\|)^{\beta}$. Therefore, (12) holds with $i = j+2$. By recurrence on i, this shows that (12) holds for each $i \geq 0$ if it holds for $i = 0, 1$. ■

Example 5.4 Let the notation be as in Example 3.3. Since $\mathbf{w}_0$ and $\mathbf{w}_1$ satisfy the hypotheses of Lemma 5.1, the Fibonacci sequence $(\mathbf{w}_i)_{i\geq 0}$ that they generate fulfills for each $i \geq 0$ the condition (10) of Proposition 5.3 with $c_1 = 1$ and $c_2 = 2$. As $\det(\mathbf{w}_0) = \det(\mathbf{w}_1) = a$, we also note that for this choice of c_1 and c_2 the condition (12) holds for $i = 0, 1$ with

$$\alpha = \frac{\log a}{\log(2a(c+1))} \quad \text{and} \quad \beta = \frac{\log a}{\log(a(b+1))}.$$

Then, for an appropriate choice of $c_3, c_4 > 0$, both (11) and (12) hold for each $i \geq 0$. Moreover, the estimates (9) of Lemma 5.1 imply that the sequence $(\mathbf{w}_i)_{i\geq 0}$ is unbounded.

6 Construction of a Real Number

Given sequences of non-negative real numbers with general terms a_i and b_i, we write $a_i \ll b_i$ or $b_i \gg a_i$ if there exists a real number $c > 0$ such that $a_i \leq cb_i$ for all sufficiently large values of i. We write $a_i \sim b_i$ when $a_i \ll b_i$ and $b_i \ll a_i$. With this notation, we now prove the following result (*cf.* [11, §5]).

Proposition 6.1 *Let $(\mathbf{w}_i)_{i\geq 0}$ be an admissible Fibonacci sequence in $\mathcal{M}$ and let $(\mathbf{y}_i)_{i\geq 0}$ be a corresponding sequence of symmetric matrices in $\mathcal{M}$. Assume that $(\mathbf{w}_i)_{i\geq 0}$ is unbounded and satisfies the conditions*

$$(13) \qquad \|\mathbf{w}_{i+1}\| \sim \|\mathbf{w}_i\|^{\gamma}, \quad |\det(\mathbf{w}_{i+1})| \sim |\det(\mathbf{w}_i)|^{\gamma} \quad \textit{and} \quad |\det(\mathbf{w}_i)| \ll \|\mathbf{w}_i\|^{\beta}$$

for a real number β with $0 < \beta < 2$. Viewing each $\mathbf{y}_i$ as a point in $\mathbb{Z}^3$, assume that $\det(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2) \neq 0$ and define $\mathbf{z}_i = (\det(\mathbf{w}_i))^{-1}\mathbf{y}_i \wedge \mathbf{y}_{i+1}$ for each $i \geq 0$. Then we have

$$(14) \qquad \|\mathbf{y}_i\| \sim \|\mathbf{w}_i\|, \quad |\det(\mathbf{y}_i)| \sim |\det(\mathbf{w}_i)|, \quad \|\mathbf{z}_i\| \sim \|\mathbf{w}_{i-1}\|,$$

and there exists a non-zero point $\mathbf{y}$ of $\mathbb{R}^3$ with $\det(\mathbf{y}) = 0$ such that

$$(15) \qquad \|\mathbf{y}_i \wedge \mathbf{y}\| \sim \frac{|\det(\mathbf{w}_i)|}{\|\mathbf{w}_i\|} \quad \textit{and} \quad |\langle \mathbf{z}_i, \mathbf{y}\rangle| \sim \frac{|\det(\mathbf{w}_{i+1})|}{\|\mathbf{w}_{i+2}\|}.$$

If $\beta < 1$, the coordinates of such a point $\mathbf{y}$ are linearly independent over $\mathbb{Q}$ and we may assume that $\mathbf{y} = (1, \xi, \xi^2)$ for some real number ξ with $[\mathbb{Q}(\xi):\mathbb{Q}] > 2$.

Proof For each $i \geq 0$, let N_i denote the element of $\mathcal{M}$ for which $\mathbf{y}_i = \mathbf{w}_i N_i$. Putting $N = N_0$, we have by hypothesis $N_i = N$ when i is even and $N_i = {}^tN$ otherwise. This implies that $\|\mathbf{y}_i\| \sim \|\mathbf{w}_i\|$ and $|\det(\mathbf{y}_i)| \sim |\det(\mathbf{w}_i)|$. In the sequel, we will repeatedly use these relations as well as the hypothesis (13).

We claim that we have

$$(16) \qquad \|\mathbf{y}_i \wedge \mathbf{y}_{i+1}\| \ll |\det(\mathbf{w}_i)|\|\mathbf{w}_{i-1}\|.$$

To prove this, we define $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and note that for each $i \geq 0$ the coefficients of the diagonal of $\mathbf{y}_i J \mathbf{y}_{i+1}$ coincide with the first and third coefficients of $\mathbf{y}_i \wedge \mathbf{y}_{i+1}$ while the sum of the coefficients of $\mathbf{y}_i J \mathbf{y}_{i+1}$ outside of the diagonal is the middle coefficient of $\mathbf{y}_i \wedge \mathbf{y}_{i+1}$ multiplied by -1. This gives

$$\|\mathbf{y}_i \wedge \mathbf{y}_{i+1}\| \leq 2\|\mathbf{y}_i J \mathbf{y}_{i+1}\|. \tag{17}$$

Since $\mathbf{y}_{i+1} = \mathbf{y}_i N_i^{-1} \mathbf{y}_{i-1}$ and since $\mathbf{x} J \mathbf{x} = \det(\mathbf{x}) J$ for any symmetric matrix $\mathbf{x}$, we also find that $\mathbf{y}_i J \mathbf{y}_{i+1} = \det(\mathbf{y}_i) J N_i^{-1} \mathbf{y}_{i-1}$ and therefore $\|\mathbf{y}_i J \mathbf{y}_{i+1}\| \ll |\det(\mathbf{w}_i)| \|\mathbf{w}_{i-1}\|$. Combining this with (17) proves our claim (16), which can also be written in the form

$$\|\mathbf{z}_i\| \ll \|\mathbf{w}_{i-1}\|. \tag{18}$$

As $\|\mathbf{y}_i\| \sim \|\mathbf{w}_i\|$ and $\|\mathbf{y}_{i+1}\| \sim \|\mathbf{w}_i\|^\gamma$, the estimate (16) shows, in the notation of §2, that

$$\operatorname{dist}([\mathbf{y}_i], [\mathbf{y}_{i+1}]) \leq c\delta_i, \quad \text{where} \quad \delta_i = \frac{|\det(\mathbf{w}_i)|}{\|\mathbf{w}_i\|^2} \tag{19}$$

and where c is some positive constant which does not depend on i. Since by hypothesis we have $|\det(\mathbf{w}_i)| \ll \|\mathbf{w}_i\|^\beta$ with $\beta < 2$, we find that $\lim_{i\to\infty} \delta_i = 0$. Since moreover, we have $\delta_{i+1} \sim \delta_i^\gamma$, we deduce that there exists an index $i_0 \geq 1$ such that $\delta_{i+1} \leq \delta_i/4$ for each $i \geq i_0$. Then, using (5), we deduce that

$$\operatorname{dist}([\mathbf{y}_i], [\mathbf{y}_j]) \leq \sum_{k=i}^{j-1} 2^{k-i} \operatorname{dist}([\mathbf{y}_k], [\mathbf{y}_{k+1}]) \leq c \sum_{k=i}^{j-1} 2^{k-i} \delta_k \leq 2c\delta_i \tag{20}$$

for each choice of i and j with $i_0 \leq i < j$. Thus the sequence $([\mathbf{y}_i])_{i\geq 0}$ converges in $\mathbb{P}^2(\mathbb{R})$ to a point $[\mathbf{y}]$ for some non-zero $\mathbf{y} \in \mathbb{R}^3$. Since the ratio $|\det(\mathbf{y}_i)|/\|\mathbf{y}_i\|^2$ depends only on the class $[\mathbf{y}_i]$ of $\mathbf{y}_i$ in $\mathbb{P}^2(\mathbb{R})$ and tends to 0 like δ_i as $i \to \infty$, we deduce by continuity that $|\det(\mathbf{y})|/\|\mathbf{y}\|^2 = 0$ and thus that $\det(\mathbf{y}) = 0$. By continuity, (20) also leads to $\operatorname{dist}([\mathbf{y}_i], [\mathbf{y}]) \leq 2c\delta_i$ for each $i \geq i_0$, and so

$$\|\mathbf{y}_i \wedge \mathbf{y}\| \ll \frac{|\det(\mathbf{w}_i)|}{\|\mathbf{w}_i\|}. \tag{21}$$

Applying (3) together with the above estimates (18) and (21), we find

$$\|\langle \mathbf{z}_i, \mathbf{y}\rangle \mathbf{y}_{i+2} - \langle \mathbf{z}_i, \mathbf{y}_{i+2}\rangle \mathbf{y}\| \leq 2\|\mathbf{z}_i\| \|\mathbf{y}_{i+2} \wedge \mathbf{y}\| \ll \|\mathbf{w}_{i-1}\| \frac{|\det(\mathbf{w}_{i+2})|}{\|\mathbf{w}_{i+2}\|} \ll |\det(\mathbf{w}_{i+1})| \delta_i.$$

Using Proposition 4.1(d), we also get

$$\|\langle \mathbf{z}_i, \mathbf{y}_{i+2}\rangle \mathbf{y}\| = \frac{|\det(\mathbf{y}_i, \mathbf{y}_{i+1}, \mathbf{y}_{i+2})|}{|\det(\mathbf{w}_i)|} \|\mathbf{y}\| \sim |\det(\mathbf{w}_{i+1})|. \tag{22}$$

Combining the above two estimates, we deduce that $\|\langle \mathbf{z}_i, \mathbf{y}\rangle \mathbf{y}_{i+2}\| \sim |\det(\mathbf{w}_{i+1})|$ and therefore that $|\langle \mathbf{z}_i, \mathbf{y}\rangle| \sim |\det(\mathbf{w}_{i+1})|/\|\mathbf{w}_{i+2}\|$. The latter estimate is the second half of (15). It implies

$$\|\langle \mathbf{z}_{i+1}, \mathbf{y}\rangle \mathbf{y}_i\| \sim \frac{|\det(\mathbf{w}_{i+2})|}{\|\mathbf{w}_{i+3}\|}\|\mathbf{w}_i\| \sim |\det(\mathbf{w}_i)|\delta_{i+1}.$$

Since $\langle \mathbf{z}_{i+1}, \mathbf{y}_i\rangle = \det(\mathbf{w}_{i-1})^{-1}\langle \mathbf{z}_i, \mathbf{y}_{i+2}\rangle$, the estimate (22) can also be written in the form $\|\langle \mathbf{z}_{i+1}, \mathbf{y}_i\rangle \mathbf{y}\| \sim |\det(\mathbf{w}_i)|$. Then, applying (3) once again, we find

$$2\|\mathbf{z}_{i+1}\|\|\mathbf{y}_i \wedge \mathbf{y}\| \geq \|\langle \mathbf{z}_{i+1}, \mathbf{y}\rangle \mathbf{y}_i - \langle \mathbf{z}_{i+1}, \mathbf{y}_i\rangle \mathbf{y}\| \gg |\det(\mathbf{w}_i)|.$$

Since, by (18) and (21), we have $\|\mathbf{z}_{i+1}\| \ll \|\mathbf{w}_i\|$ and $\|\mathbf{y}_i \wedge \mathbf{y}\| \ll |\det(\mathbf{w}_i)|/\|\mathbf{w}_i\|$, we conclude from this that $\|\mathbf{z}_{i+1}\| \sim \|\mathbf{w}_i\|$ and $\|\mathbf{y}_i \wedge \mathbf{y}\| \sim |\det(\mathbf{w}_i)|/\|\mathbf{w}_i\|$, which completes the proof of (14) and (15).

Now, assume that $\beta < 1$, and let $\mathbf{u} \in \mathbb{Z}^3$ such that $\langle \mathbf{u}, \mathbf{y}\rangle = 0$. By (3), we have

$$2\|\mathbf{u}\|\|\mathbf{y}_i \wedge \mathbf{y}\| \geq \|\langle \mathbf{u}, \mathbf{y}\rangle \mathbf{y}_i - \langle \mathbf{u}, \mathbf{y}_i\rangle \mathbf{y}\| = |\langle \mathbf{u}, \mathbf{y}_i\rangle|\|\mathbf{y}\| \tag{23}$$

for each $i \geq 0$. Since $\|\mathbf{y}_i \wedge \mathbf{y}\| \sim |\det(\mathbf{w}_i)|/\|\mathbf{w}_i\| \ll \|\mathbf{w}_i\|^{\beta-1}$ tends to 0 as $i \to \infty$, we deduce from (23) that the integer $\langle \mathbf{u}, \mathbf{y}_i\rangle$ must vanish for all sufficiently large values of i. This implies that $\mathbf{u} = 0$ because it follows from the hypothesis $\det(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2) \neq 0$ and the formula in Proposition 4.1(d) that any three consecutive points of the sequence $(\mathbf{y}_i)_{i\geq 0}$ are linearly independent. Thus the coordinates of $\mathbf{y}$ must be linearly independent over $\mathbb{Q}$. In particular, the first coordinate of $\mathbf{y}$ is non-zero and, dividing $\mathbf{y}$ by this coordinate, we may assume that it is equal to 1. Then, upon denoting by ξ the second coordinate of $\mathbf{y}$, the condition $\det(\mathbf{y}) = 0$ implies that $\mathbf{y} = (1, \xi, \xi^2)$ and thus $[\mathbb{Q}(\xi):\mathbb{Q}] > 2$. ■

7 Estimates for the Exponent $\widehat{\omega}_2$

We first prove the following result and then deduce from it our main theorem in §1.

Proposition 7.1 *Let $(\mathbf{w}_i)_{i\geq 0}$ be an admissible Fibonacci sequence in $\mathcal{M}$, and let $(\mathbf{y}_i)_{i\geq 0}$ be a corresponding sequence of symmetric matrices in $\mathcal{M}$. Assume that $(\mathbf{w}_i)_{i\geq 0}$ is unbounded and satisfies*

$$\|\mathbf{w}_{i+1}\| \sim \|\mathbf{w}_i\|^\gamma, \ |\det(\mathbf{w}_{i+1})| \sim |\det(\mathbf{w}_i)|^\gamma, \ \|\mathbf{w}_i\|^\alpha \ll |\det(\mathbf{w}_i)| \ll \|\mathbf{w}_i\|^\beta \tag{24}$$

for real numbers α and β with $0 \leq \alpha \leq \beta < \gamma^{-2}$. Assume moreover that $\mathrm{tr}(\mathbf{w}_i)$ *and* $\det(\mathbf{w}_i)$ *are relatively prime for $i = 0, 1, 2, 3$ and that* $\det(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2) \neq 0$. *Then the real number ξ which comes out from the last assertion of Proposition* 6.1 *satisfies*

$$\gamma^2 - \beta\gamma \leq \widehat{\omega}_2(\xi) \leq \gamma^2 - \alpha\gamma.$$

Proof Put $\mathbf{y} = (1, \xi, \xi^2)$ and define the sequence $(\mathbf{z}_i)_{i\geq 0}$ as in Proposition 4.1. Since $\|\mathbf{y}\| \geq 1$, the inequality (3) combined with the estimates of Proposition 6.1 shows that, for any point $\mathbf{z} \in \mathbb{Z}^3$ and any index $i \geq 1$, we have

$$(25)\quad |\langle \mathbf{z}, \mathbf{y}_i\rangle| \leq \|\mathbf{y}_i\||\langle \mathbf{z}, \mathbf{y}\rangle| + 2\|\mathbf{z}\|\|\mathbf{y}_i \wedge \mathbf{y}\| < c_5 \max\left\{ \|\mathbf{w}_i\||\langle \mathbf{z}, \mathbf{y}\rangle|, \|\mathbf{z}\|\frac{|\det(\mathbf{w}_i)|}{\|\mathbf{w}_i\|}\right\},$$

with a constant $c_5 > 0$ which is independent of $\mathbf{z}$ and i. Suppose that a point $\mathbf{z} \in \mathbb{Z}^3$ satisfies

$$(26)\qquad 0 < \|\mathbf{z}\| \leq Z_i := c_6\|\mathbf{w}_i\| \quad \text{and} \quad |\langle \mathbf{z}, \mathbf{y}\rangle| \leq \frac{|\det(\mathbf{w}_{i+1})|}{\|\mathbf{w}_{i+2}\|},$$

where $c_6 = c_5^{-1}|\det(\mathbf{y}_2)|^{-1}$. Using (25) with i replaced by $i + 1$, we find

$$|\langle \mathbf{z}, \mathbf{y}_{i+1}\rangle| \ll |\det(\mathbf{w}_i)|^{\gamma}\|\mathbf{w}_i\|^{-1/\gamma}.$$

Since $|\det(\mathbf{w}_i)| \ll \|\mathbf{w}_i\|^{\beta}$ with $\beta < \gamma^{-2}$, this gives $|\langle \mathbf{z}, \mathbf{y}_{i+1}\rangle| < 1$ provided that i is sufficiently large. Then the integer $\langle \mathbf{z}, \mathbf{y}_{i+1}\rangle$ must be zero and, by Proposition 4.1(e), we deduce that $\mathbf{z} = a\mathbf{z}_i + b\mathbf{z}_{i+1}$ for some $a, b \in \mathbb{Q}$ where b is given by

$$\mathbf{z}_i \wedge \mathbf{z} = b\mathbf{z}_i \wedge \mathbf{z}_{i+1} = (-1)^i b \det(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2)\det(\mathbf{w}_2)^{-1}\mathbf{y}_{i+1}.$$

Since $\det(\mathbf{w}_2)\mathbf{z}_i \wedge \mathbf{z} \in \mathbb{Z}^3$ and since, by Corollary 4.2(d), the content of $\mathbf{y}_{i+1}$ divides $\det(\mathbf{y}_2)/\det(\mathbf{w}_2)$, this implies that $b\det(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2)\det(\mathbf{y}_2)/\det(\mathbf{w}_2)$ is an integer. So, if b is non-zero, it satisfies the lower bound

$$|b| \geq |\det(\mathbf{w}_2)/(\det(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2)\det(\mathbf{y}_2))|.$$

We note that $\langle \mathbf{z}_i, \mathbf{y}_i\rangle = 0$ and by Proposition 4.1(d) that

$$\langle \mathbf{z}_{i+1}, \mathbf{y}_i\rangle = \frac{\det(\mathbf{y}_i, \mathbf{y}_{i+1}, \mathbf{y}_{i+2})}{\det(\mathbf{w}_{i+1})} = (-1)^i\frac{\det(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2)}{\det(\mathbf{w}_2)}\det(\mathbf{w}_i).$$

Therefore, if $b \neq 0$, the point $\mathbf{z} = a\mathbf{z}_i + b\mathbf{z}_{i+1}$ satisfies

$$|\langle \mathbf{z}, \mathbf{y}_i\rangle| = |b||\langle \mathbf{z}_{i+1}, \mathbf{y}_i\rangle| \geq |\det(\mathbf{y}_2)|^{-1}|\det(\mathbf{w}_i)| = c_5c_6|\det(\mathbf{w}_i)|.$$

However, (25) and (26) give

$$|\langle \mathbf{z}, \mathbf{y}_i\rangle| < c_5 \max\left\{\frac{|\det(\mathbf{w}_{i+1})|\|\mathbf{w}_i\|}{\|\mathbf{w}_{i+2}\|}, c_6|\det(\mathbf{w}_i)|\right\} = c_5c_6|\det(\mathbf{w}_i)|$$

if i is sufficiently large, because the ratio $|\det(\mathbf{w}_{i+1})|\|\mathbf{w}_i\|/\|\mathbf{w}_{i+2}\| \ll \|\mathbf{w}_i\|^{\beta\gamma-\gamma}$ tends to 0 as $i \to \infty$. Comparison with the previous inequality then forces $b = 0$, and so we get $\mathbf{z} = a\mathbf{z}_i$ with $a \neq 0$. Since $\det(\mathbf{w}_2)\mathbf{z}_i$ is, by Corollary 4.2(e), an integer point whose content divides $\det(\mathbf{y}_2)\det(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2)$, we deduce that

$$a\det(\mathbf{y}_2)\det(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2)/\det(\mathbf{w}_2)$$

is a non-zero integer and therefore, using the second part of (15) in Proposition 6.1, we find that

$$|\langle \mathbf{z}, \mathbf{y}\rangle| = |a||\langle \mathbf{z}_i, \mathbf{y}\rangle| \geq \frac{|\det(\mathbf{w}_2)|}{|\det(\mathbf{y}_2)\det(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2)|}|\langle \mathbf{z}_i, \mathbf{y}\rangle| \gg \frac{|\det(\mathbf{w}_{i+1})|}{\|\mathbf{w}_{i+2}\|}.$$

Since this holds for any point $\mathbf{z}$ satisfying (26) with i sufficiently large, we deduce that for any index $i \geq 0$ and any point $\mathbf{z} \in \mathbb{Z}^3$ with $0 < \|\mathbf{z}\| \leq Z_i$ we have

$$|\langle \mathbf{z}, \mathbf{y}\rangle| \gg \frac{|\det(\mathbf{w}_{i+1})|}{\|\mathbf{w}_{i+2}\|} \gg \|\mathbf{w}_i\|^{\gamma\alpha-\gamma^2} \gg Z_i^{\gamma\alpha-\gamma^2}.$$

This shows that $\widehat{\omega}_2(\xi) \leq \gamma^2 - \gamma\alpha$.

Finally, for any real number $Z \geq \|\mathbf{z}_0\|$, there exists an index $i \geq 0$ such that $\|\mathbf{z}_i\| \leq Z < \|\mathbf{z}_{i+1}\|$ and, for such choice of i, we find by Proposition 6.1 that

$$|\langle \mathbf{z}_i, \mathbf{y}\rangle| \ll \frac{|\det(\mathbf{w}_{i+1})|}{\|\mathbf{w}_{i+2}\|} \ll \|\mathbf{w}_i\|^{\beta\gamma-\gamma^2} \sim \|\mathbf{z}_{i+1}\|^{\beta\gamma-\gamma^2} \ll Z^{\beta\gamma-\gamma^2},$$

showing that $\widehat{\omega}_2(\xi) \geq \gamma^2 - \gamma\beta$. ■

Let us say that a real number ξ is of "Fibonacci type" if there exist an unbounded Fibonacci sequence $(\mathbf{w}_i)_{i\geq 0}$ in $\mathcal{M}$ and a real number θ with $\theta > 1/\gamma$ such that $\|(\xi, -1)\mathbf{w}_i\| \leq \|\mathbf{w}_i\|^{-\theta}$ for each sufficiently large index i. There are countably many such numbers, and any real number ξ obtained from Proposition 6.1 with $\beta < \gamma^{-2}$ is of this type. The following corollary shows that the exponents $\widehat{\omega}_2(\xi)$ attached to transcendental numbers of Fibonacci type are dense in the interval $[2, \gamma^2]$. By Jarník's formula (1), this implies our main theorem in §1.

Corollary 7.2 *Let t and ϵ be real numbers with $0 < t < \gamma^{-2}$ and $\epsilon > 0$. Then there exist a transcendental real number ξ and an unbounded Fibonacci sequence $(\mathbf{w}_i)_{i\geq 0}$ in $\mathcal{M}$ which satisfy*

(a) $\|(\xi, -1)\mathbf{w}_i\| \leq \|\mathbf{w}_i\|^{-1+t}$ *for each sufficiently large i,*

(b) $\gamma^2 - t\gamma \leq \widehat{\omega}_2(\xi) \leq \gamma^2 - (t-\epsilon)\gamma$.

Proof Since $t < 1$, there exist integers k and ℓ with $0 < \ell < k$ and $t-\epsilon \leq \ell/(k+2) \leq \ell/k < t$. For such a choice of k and ℓ, consider the Fibonacci sequence $(\mathbf{w}_i)_{i\geq 0}$ of Example 3.3 with parameters $a = 2^\ell$, $b = 2^{k-\ell} - 1$ and $c = 2^{k-\ell}$. According to Example 4.3, $\mathbf{w}_i$ has relatively prime trace and determinant for each $i \geq 0$ and the corresponding sequence of symmetric matrices $(\mathbf{y}_i)_{i\geq 0}$ satisfies $\det(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2) = 2^{4\ell} \neq 0$. Moreover, Example 5.4 shows that $(\mathbf{w}_i)_{i\geq 0}$ is unbounded and satisfies the estimates (24) of Proposition 7.1 with $\alpha = \ell/(k+2)$ and $\beta = \ell/k$ (note that the example provides a slightly larger value for α). So, Proposition 7.1 applies and shows that the corresponding real number ξ constructed by Proposition 6.1 satisfies the above condition (b). In particular, ξ is transcendental since $\widehat{\omega}_2(\xi) > 2$. Moreover, since $\|(\xi, -1)\mathbf{w}_i\| \sim \|(\xi, -1)\mathbf{y}_i\| \sim \|\mathbf{y}_i \wedge \mathbf{y}\|$, the first estimate in (15) leads to (a). ■

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