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Interpolation formulas and auxiliary functions

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Abstract. We prove an interpolation formula for "semi-cartesian products" and use it to study several constructions of auxiliary functions. We get in this way a criterion for the values of the exponential map of an elliptic curve E defined over $\mathbf{Q}$. It reduces the analogue of Schanuel's conjecture for the elliptic logarithms of E to a statement of the form of a criterion of algebraic independence. We also consider a construction of auxiliary function related to the four exponentials conjecture and show that it is essentially optimal. For analytic functions vanishing on a semi-cartesian product, we get a version of the Schwarz lemma in which the exponent involves a condition of distribution reminiscent of the so-called technical hypotheses in algebraic independence. We show by two examples that such a condition is unavoidable.

Introduction

A proof of transcendence or algebraic independence of values of analytic functions requires an analytic argument to exploit the analytic nature of the given numbers, and an algebraic or arithmetic argument to reach the desired conclusion. Usually, the first step is the construction of a so-called auxiliary function: a polynomial in several basic analytic functions taking small values (or even vanishing) at many points in the common domain of these functions. In some instances, this construction is the main analytic part of the whole argument. A remarkable example of this is the proof of the theorem of the algebraic subgroup of M. Waldschmidt (Theorem 1.1 of [14]). There the polynomial is chosen so that the corresponding auxiliary function f has small supremum norm in a large ball B of $\mathbf{C}^n$ centered at the origin. The rest of the argument is based on this and is essentially arithmetic. It uses the fact that this ball B contains a set S where f takes values in a finitely generated extension K of $\mathbf{Q}$. This suggests the following two questions which motivate this paper. Does the fact that f takes small values on S fully reflect the hypotheses? Is it possible to construct f so that it admits a smaller supremum norm on a larger disk?

In a preceding paper [9], we studied Waldschmidt's construction of auxiliary function for analytic subgroups of the group $\mathbf{G_a}\times\mathbf{G_m}$. We showed that it can be used to characterize those points of $(\mathbf{G_a}\times\mathbf{G_m})(\mathbf{C}) = \mathbf{C}\times\mathbf{C}^\times$ which have finite order modulo the subgroup of points of the form (z, e^z) with $z \in \mathbf{C}$, and we used this fact to present Schanuel's conjecture

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in the form of a criterion of algebraic independence. Here, we extend this idea to the group $\mathbf{G_a} \times E$ where E is an elliptic curve defined over $\mathbf{Q}$. We get the same kind of results with the usual exponential function e^z replaced by the exponential map of the elliptic curve $\exp_E\colon \mathbf{C} \to E(\mathbf{C})$. This provides another example where the first question gets a positive answer. It suggests an approach to proving the analogue of Schanuel's conjecture for the elliptic logarithms of E.

For the second question, we consider Waldschmidt's construction for one-dimensional analytic subgroups of $\mathbf{G_m^2}$, and show that it is essentially optimal, up to a constant. To motivate this, let us state his result and recall briefly how it implies the so-called six exponentials theorem of S. Lang and K. Ramachandra. Let λ be an irrational complex number. On denoting by $B_1(0,r)$ the closed disk of $\mathbf{C}$ of center 0 and radius r, and by $|f|_r$ the supremum norm of a continuous function $f\colon B_1(0,r) \to \mathbf{C}$, Theorem 3.1 of [13] shows the existence of a constant $c_0 \geq 1$ with the following property:

> *For any real number c and any integer N with $N \geq c \geq c_0$, there is a non-zero polynomial $P \in \mathbf{Z}[X_1, X_2]$ of degree $\leq N$ whose coefficients are in absolute value $\leq \exp(c^{-3/2}N^2)$ such that the entire function $f(z) = P(e^z, e^{\lambda z})$ satisfies $|f|_{c^{-2}N} \leq \exp(-c^{-1}N^2)$.*

For fixed c and N as above, the corresponding function f takes algebraic values at each point $z \in \mathbf{C}$ such that both z and λz are logarithms of algebraic numbers. Let $\mathcal{L}_\lambda$ be the $\mathbf{Q}$-vector subspace of $\mathbf{C}$ consisting of all those points. If $x_1, \dots, x_\ell$ are $\mathbf{Q}$-linearly independent elements of $\mathcal{L}_\lambda$, then, provided that c is large enough (as a function of $x_1, \dots, x_\ell$ solely), Liouville's inequality shows that f vanishes at each point of the form $m_1x_1 + \cdots + m_\ell x_\ell$ with integers $m_1, \dots, m_\ell$ in absolute value at most $c^{-3}N$. According to a zero-estimate of D. Masser (Theorem 2 of [6]), this condition, translated in terms of the polynomial P, implies that the degree of P is at least $(c^{-3}N)^{\ell/2}$ if $\ell \geq 2$. Since N can be taken arbitrarily large, this shows $\ell \leq 2$ and thus $\dim_{\mathbf{Q}}(\mathcal{L}_\lambda) \leq 2$, which is one of the multiple forms of the six exponentials theorem. The four exponentials conjecture claims that we should in fact have $\dim_{\mathbf{Q}}(\mathcal{L}_\lambda) \leq 1$, and the above seems close to producing this. However, assuming $\mathcal{L}_\lambda \neq \{0\}$, the results of section 8 below show that the hopes in this direction are limited since, for an appropriate choice of $c_0 \geq 1$, we have:

(∗) *For any integer $N \geq c_0$, and any non-zero polynomial $P \in \mathbf{Z}[X_1, X_2]$ of degree $\leq N$, the entire function $f(z) = P(e^z, e^{\lambda z})$ satisfies $|f|_{c_0 N} \geq \exp(-c_0^3 N^2)$.*

We prove this as a consequence of a stronger result. We also look from the same point of view at a similar construction for the group $\mathbf{G_a} \times \mathbf{G_m}$.

The common tool at the basis of these results is an interpolation formula which generalizes Theorem 2 of [9]. The purpose of an interpolation formula is to control the growth of an analytic function on an open set of $\mathbf{C}^n$ in terms of its values and some of its derivatives on a finite subset S of this open set. A criterion of M. Waldschmidt and J.-C. Moreau reduces this problem to the case of polynomials. Thus, we have satisfactory interpolation formulas when S is a Cartesian product or satisfies a condition of good distribution (for the second case, see Theorem 7.4.13 of [12] deduced from Theorem A of [5], and Proposition 2.6 of [7] deduced from Theorem 2.1 of [7]). In the case that we consider here, the set S is a sort of semi-cartesian product where the points are spread along parallel lines. For functions vanishing on S, this provides new examples where Conjecture 7.1.10 of [12] is verified except for a weak condition of distribution on the points of S, and we show by two examples that such a condition is necessary.

This paper is organized as follows. Our interpolation formula is stated in §1 along with two corollaries. It is proved in §6 using the results previously established in §§2–5. In §6, we also conjecture a more general interpolation formula in the spirit of Conjecture 7.1.10 of [12] for the Schwarz lemma. We show in §7 that it requires a Diophantine condition even for functions vanishing on the given set. The last two sections §§8–9 provide applications for our interpolation formula. In §8, we use it to study constructions of auxiliary functions on $\mathbf{G}_{\mathbf{m}}^2$ and we also look at the case of $\mathbf{G}_{\mathbf{a}} \times \mathbf{G}_{\mathbf{m}}$ using a Theorem of R. Tijdeman. Finally, §9 gives a criterion for the values of the exponential map of an elliptic curve.

1. An interpolation formula

Let m be a positive integer. We define the *norm* of a point $\mathbf{z} = (z_1, \ldots, z_m) \in \mathbf{C}^m$ to be its maximum norm

$$\|\mathbf{z}\| = \max_{1 \le j \le m} |z_j|.$$

For a point $\mathbf{a} \in \mathbf{C}^m$ and a real number $r \ge 0$, we denote by

$$B_m(\mathbf{a}, r) = \{\, \mathbf{z} \in \mathbf{C}^m \ ;\ \|\mathbf{z} - \mathbf{a}\| \le r \,\}$$

the closed polydisk of $\mathbf{C}^m$ of center $\mathbf{a}$ and radius r. We also denote by $\mathcal{H}_r$ the Banach space of continuous functions $f \colon B_m(0, r) \to \mathbf{C}$ which are holomorphic in the interior of $B_m(0, r)$, with the supremum norm denoted $|\ |_r$. For a point $\mathbf{w} = (w_1, \ldots, w_m) \in \mathbf{C}^m$, we write

$$D_{\mathbf{w}} = w_1 \frac{\partial}{\partial z_1} + \cdots + w_m \frac{\partial}{\partial z_m}$$

for the directional derivative along $\mathbf{w}$. We also use the symbol $[x]$ to denote the integral part of a real number x. In order to state our main result, we need three more definitions.

Definition 1.1. Let $\Sigma = \{\gamma_1, \ldots, \gamma_\ell\}$ be a finite non-empty subset of $\mathbf{C}^m$ of cardinality ℓ. We define the *rank* of Σ, denoted $\operatorname{rank}(\Sigma)$, to be the rank of the subgroup of $\mathbf{C}^m$ generated by Σ. For any integer $N \ge 0$, we also define

$$\Sigma(N) = \{\, n_1\gamma_1 + \cdots + n_\ell\gamma_\ell \ ;\ n_1, \ldots, n_\ell \in \mathbf{Z} \text{ and } \max_{1 \le j \le \ell} |n_j| \le N \,\}.$$

Note that alternative definitions could have been employed for $\Sigma(N)$, but this particular choice happens to produce simpler statements in our context.

Definition 1.2. Let Σ be a finite non-empty subset of $\mathbf{C}^m$, let $\mathcal{B} = \{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ be a linearly independent subset of $\mathbf{C}^m$, let $N \geq 0$ be an integer and let $S \geq 0$ be a real number. For any $\mathbf{C}$-valued function f defined and holomorphic in an open subset of $\mathbf{C}^m$ containing $\Sigma(N)$, we denote by $\Phi\big(f\,;\,\Sigma, N\,;\,\mathcal{B}, S\big)$ the maximum of the numbers

$$\frac{1}{\tau_1! \cdots \tau_k!} \big|\big(D_{\mathbf{w}_1}^{\tau_1} \cdots D_{\mathbf{w}_k}^{\tau_k} f\big)(\gamma)\big|\, N^{\tau_1 + \cdots + \tau_k}$$

with $\gamma \in \Sigma(N)$ and $\underline{\tau} = (\tau_1, \dots, \tau_k) \in \mathbf{N}^k$ of norm $\|\,\underline{\tau}\,\| = \max\{\tau_1, \dots, \tau_k\} \leq S$. When $\mathcal{B} = \emptyset$, we agree that this is just $\max_{\gamma \in \Sigma(N)} |f(\gamma)|$.

Definition 1.3. Let Γ be a finitely generated subgroup of $\mathbf{C}^m$, let W be a subspace of $\mathbf{C}^m$, let $s \geq 0$ be a real number, and let $(L_N)_{N \geq 1}$ be a sequence of non-negative integers. We say that the triple (Γ, W, s) admits a sequence of interpolation formulas with exponents $(L_N)_{N \geq 1}$ if the following condition holds:

(I.1) *There exist a finite subset $\Sigma \neq \emptyset$ of Γ, a basis $\mathcal{B}$ of W, and positive constants c_0, c_1, c_2 such that, for any integer $N \geq 1$, any real numbers r and R with $R \geq r \geq c_0 N$, and any function $f \in \mathcal{H}_R$, the set $\Sigma(N)$ is contained in the interior of $B_m(0, R)$ and we have*

$$|f|_r \leq \left(\frac{c_1 r}{N}\right)^{L_N} \Phi\big(f\,;\,\Sigma, N\,;\,\mathcal{B}, N^s\big) + \left(\frac{c_2 r}{R}\right)^{L_N} |f|_R.$$

When $W = \{0\}$, the above condition is independent of s. So, if it holds with $W = \{0\}$, we simply say that Γ admits a sequence of interpolation formulas with exponents $(L_N)_{N \geq 1}$.

Note that, in the condition (I.1), we need $\Sigma(N)$ to be contained in the interior of $B_m(0, R)$ in order for $\Phi\big(f\,;\,\Sigma, N\,;\,\mathcal{B}, N^s\big)$ to make sense. Since any element of $\Sigma(N)$ is a sum of at most N elements of $\Sigma(1)$, this amounts to choosing c_0 such that $\Sigma(1)$ is contained in the interior of $B_m(0, c_0)$, in view of the constraint $R \geq c_0 N$. Our main result is the following:

Theorem 1.4. *Let m be a positive integer, let Γ be a finitely generated subgroup of $\mathbf{C}^m$, let W be a subspace of $\mathbf{C}^m$, let $s \geq 0$ be a non-negative real number, and let*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_m = \mathbf{C}^m$$

be a strictly increasing sequence of subspaces of $\mathbf{C}^m$ with $\dim(V_j) = j$ for $j = 0, \dots, m$. For $j = 1, \dots, m$, we choose a linear map $\pi_j\colon V_j \to \mathbf{C}$ with kernel V_{j-1}, and put

$$\Gamma_j = \pi_j(\Gamma \cap V_j) \quad \text{and} \quad W_j = \pi_j(W \cap V_j).$$

We also choose a finite subset $\Sigma_j \neq \emptyset$ of Γ_j. For each integer $N \geq 1$, we define

$$\mu_N = \min_{1\leq j\leq m} \Big\{ \operatorname{rank}(\Sigma_j) - \operatorname{rank}\big(\Sigma_j(N) \cap B_1(0, e^{-N})\big) + s\dim(W_j) \Big\}$$

and put $L_N = [N^{\mu_N}]$. Then the triple (Γ, W, s) admits a sequence of interpolation formulas with exponents $(L_N)_{N\geq 1}$. More precisely, the condition (I.1) is satisfied for any choice of basis $\mathcal{B}$ of W and any finite subset Σ of Γ with $\Sigma_j \subseteq \pi_j(\Sigma \cap V_j)$ for $j = 1, \dots, m$.

For the applications in sections 8 and 9 of this paper, we will use the following corollaries of Theorem 1.4 in dimension $m = 2$.

Corollary 1.5. *Let $0 \to \mathbf{C} \xrightarrow{\iota} \mathbf{C}^2 \xrightarrow{\pi} \mathbf{C} \to 0$ be an exact sequence of $\mathbf{C}$-linear maps, let Σ_1, Σ_2 be non-empty finite subsets of $\mathbf{C}$ and let Σ be a finite subset of $\mathbf{C}^2$ with $\iota(\Sigma_1) \subseteq \Sigma$ and $\Sigma_2 \subseteq \pi(\Sigma)$. Then, there is a constant $c \geq 1$ with the following property. For any integer $N \geq 1$, any real numbers r and R with $R \geq r \geq cN$, and any entire function $f\colon \mathbf{C}^2 \to \mathbf{C}$, we have*

$$|f|_r \leq \left(\frac{cr}{N}\right)^{L_N} \max_{\gamma\in\Sigma(N)} |f(\gamma)| + \left(\frac{cr}{R}\right)^{L_N} |f|_R$$

where $L_N = \big[N^{\mu_N}\big]$ and $\mu_N = \min_{j=1,2}\big\{ \operatorname{rank}(\Sigma_j) - \operatorname{rank}\big(\Sigma_j(N) \cap B_1(0, e^{-N})\big)\big\}$.

Proof. Let Γ be the subgroup of $\mathbf{C}^2$ generated by Σ. We apply Theorem 1.4 with $W = \{0\}$ and $V_1 = \iota(\mathbf{C})$, taking $\pi_2 = \pi$ and defining π_1 as the inverse of the restriction $\iota\colon \mathbf{C} \to V_1$. ∎

The second corollary generalizes Theorem 2 of [9].

Corollary 1.6. *Let Σ be a non-empty finite subset of $\mathbf{C}^2$, let $\mathbf{w}$ be a non-zero element of $\mathbf{C}^2$, let $s \geq 0$ be a real number, and let $\pi\colon \mathbf{C}^2 \to \mathbf{C}$ be a $\mathbf{C}$-linear map whose kernel is the subspace W of $\mathbf{C}^2$ generated by $\mathbf{w}$. We put $\Sigma_2 = \pi(\Sigma)$. Then there exists a constant $c \geq 1$ with the following property. For any given integer $N \geq 1$, any real numbers r and R with $R \geq r \geq cN$, and any entire function $f\colon \mathbf{C}^2 \to \mathbf{C}$, we have*

$$|f|_r \leq \left(\frac{cr}{N}\right)^{L_N} \max_{\substack{\gamma\in\Sigma(N)\\ 0\leq\tau\leq N^s}} \frac{1}{\tau!}\big|(D_{\mathbf{w}}^{\tau} f)(\gamma)\big| N^{\tau} + \left(\frac{cr}{R}\right)^{L_N} |f|_R.$$

where $L_N = \big[N^{\mu_N}\big]$ and $\mu_N = \min\big\{ s\,, \operatorname{rank}(\Sigma_2) - \operatorname{rank}\big(\Sigma_2(N) \cap B_1(0, e^{-N})\big)\big\}$.

Proof. Let Γ be the subgroup of $\mathbf{C}^2$ generated by Σ. We apply Theorem 1.4 with $V_1 = W$ and $\pi_2 = \pi$, so that $W_2 = \{0\}$. For π_1, we take any $\mathbf{C}$-linear isomorphism $\pi_1\colon W \to \mathbf{C}$, and we choose $\Sigma_1 = \{0\}$. ∎

2. Preliminary Lemmas

Throughout this section, we fix a positive integer m, a non-empty finite subset Σ of $\mathbf{C}^m$ and a linearly independent subset $\mathcal{B} = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ of $\mathbf{C}^m$. We will need the following four lemmas.

Lemma 2.1. *Let $\psi\colon \mathbf{C}^m \to \mathbf{C}^m$ be an invertible $\mathbf{C}$-linear map. For any integer $N \geq 0$, any real number $S \geq 0$ and any function f holomorphic in an open neighborhood of $\Sigma(N)$, we have*

$$\Phi\big(f \circ \psi^{-1}\,;\, \psi(\Sigma), N\,;\, \psi(\mathcal{B}), S\big) = \Phi\big(f\,;\, \Sigma, N\,;\, \mathcal{B}, S\big)$$

Proof. This follows simply from the formula

$$D^{\tau_1}_{\psi(\mathbf{w}_1)} \cdots D^{\tau_k}_{\psi(\mathbf{w}_k)} (f \circ \psi^{-1}) = \big(D^{\tau_1}_{\mathbf{w}_1} \cdots D^{\tau_k}_{\mathbf{w}_k} f\big) \circ \psi^{-1},$$

valid for any $(\tau_1, \ldots, \tau_k) \in \mathbf{N}^k$, and the fact that $\big(\psi(\Sigma)\big)(N) = \psi\big(\Sigma(N)\big)$. ∎

Lemma 2.2. *There exists a positive constant c_0, depending only on Σ and $\mathcal{B}$, with the following property. For any integer $N \geq 1$, any real numbers $S \geq 0$ and $R \geq c_0 N$, and any function $f \in \mathcal{H}_R$, the set $\Sigma(N)$ is contained in the interior of $B_m(0, R)$ and we have*

$$\Phi\big(f\,;\, \Sigma, N\,;\, \mathcal{B}, S\big) \leq |f|_R.$$

Proof. Let ψ be a $\mathbf{C}$-linear automorphism of $\mathbf{C}^m$ mapping $\mathcal{B}$ into a subset of the canonical basis $\mathcal{C}$ of $\mathbf{C}^m$, and let c_0 be a positive constant such that

$$\Sigma(1) + \psi^{-1}\big(B_m(0,1)\big) \subseteq B_m(0, c_0).$$

Suppose that N, S, R and f are as in the statement of the lemma, according to this choice of c_0. We put $g = f \circ \psi^{-1}$, $\Delta = \psi(\Sigma)$ and $K = \psi\big(B_m(0, R)\big)$. Then, $\Sigma(N)$ is contained in the interior of $B_m(0, R)$ and Lemma 2.1 gives

$$\Phi\big(f\,;\, \Sigma, N\,;\, \mathcal{B}, S\big) \leq \Phi\big(g\,;\, \Delta, N\,;\, \mathcal{C}, S\big).$$

On the other hand, for $\delta \in \Delta(N)$, the choice of c_0 and R implies

$$B_m(\delta, N) \subseteq \psi\big(B_m(0, c_0 N)\big) \subseteq K.$$

Since g is continuous on K and holomorphic inside, Cauchy's inequalities together with the maximum modulus principle give

$$\frac{1}{\tau_1! \cdots \tau_m!} \left| \Big(\frac{\partial^{\tau_1}}{\partial z_1^{\tau_1}} \cdots \frac{\partial^{\tau_m}}{\partial z_m^{\tau_m}} g\Big)(\delta) \right| N^{\tau_1 + \cdots + \tau_m} \leq \sup_{\mathbf{z} \in K} |g(\mathbf{z})| = |f|_R$$

for any $\underline{\tau} = (\tau_1, \ldots, \tau_m) \in \mathbf{N}^m$. Thus $\Phi\big(g\,;\, \Delta, N\,;\, \mathcal{C}, S\big) \leq |f|_R$, and the conclusion follows. ∎

Lemma 2.3. *Let $d, \ell, N \geq 0$ be integers with $\ell \neq 0$, let $S \geq 0$ be a real number, and let $Q\colon \mathbf{C}^m \to \mathbf{C}$ be a polynomial map of degree $\leq d$. Then, we have*

$$\Phi\big(Q^\ell\,;\,\Sigma, N\,;\,\mathcal{B}, S\big) \leq (k+1)^{d\ell}\Phi\big(Q\,;\,\Sigma, N\,;\,\mathcal{B}, S\big)^\ell.$$

Proof. For any $\underline{\tau} = (\tau_1, \dots, \tau_k) \in \mathbf{N}^k$, we define

$$D^{[\underline{\tau}]} = \frac{1}{\tau_1! \cdots \tau_k!} D_{\mathbf{w}_1}^{\tau_1} \cdots D_{\mathbf{w}_k}^{\tau_k}.$$

With this notation, Leibniz formula for the derivative of a product gives

$$D^{[\underline{\tau}]}Q^\ell = \sum_{\underline{\tau}_1+\cdots+\underline{\tau}_\ell=\underline{\tau}} \big(D^{[\underline{\tau}_1]}Q\big)\cdots\big(D^{[\underline{\tau}_\ell]}Q\big)$$

where the sum ranges over all decompositions of $\underline{\tau}$ into a sum of ℓ elements $\underline{\tau}_1, \dots, \underline{\tau}_\ell$ of $\mathbf{N}^k$. Since Q has degree $\leq d$, we may restrict ourselves to the decompositions where the sum of the k components of $\underline{\tau}_j$ is $\leq d$ for $j = 1, \dots, \ell$. Therefore the number of non-zero terms in this sum is at most $(k+1)^{d\ell}$. The conclusion follows. ∎

The last result is a consequence of the maximum modulus principle which we will use repeatedly (see for example Lemma 1.5 of [7] or Lemma 2 of [5]):

Lemma 2.4. *Let $L \geq 0$ be an integer and let $P\colon \mathbf{C}^m \to \mathbf{C}$ be a polynomial map of degree $\leq L$. For any pair of real numbers r and R with $r \leq R$, we have*

$$|P|_R \leq \Big(\frac{R}{r}\Big)^L |P|_r.$$

Proof. Choose $\mathbf{z}_0 \in B_m(0, r)$ such that $|P(\mathbf{z}_0)| = |P|_r$. Applying the maximum modulus principle to the polynomial in one variable $f(z) = z^L P(z^{-1}\mathbf{z}_0)$, one gets $|f|_1 \leq |f|_{R/r}$ and the conclusion follows. ∎

3. Criterion of Waldschmidt–Moreau

The next proposition is a slight generalization of Proposition 1.6 of J.-C. Moreau in [7], itself formalizing an argument of M. Waldschmidt in Theorems 7.3.5 and 7.4.13 of [12]. The differences in the proof are explained in [10] for the case $m = 2$. The general case is similar. We will use this result to reformulate the condition (I.1) in Definition 1.3.

Proposition 3.1. *Let κ and r_0 be positive real numbers, let L be a positive integer and let $\mathcal{S}$ be a set of linear functionals on $\mathcal{H}_{r_0}$ of norm ≤ 1. Then, the following conditions are equivalent:*

(i) for any polynomial map $P\colon \mathbf{C}^m \to \mathbf{C}$ of degree $< L$ we have $|P|_{r_0} \leq \kappa \sup_{\varphi\in\mathcal{S}} |\varphi(P)|$;

(ii) *for any pair of real numbers r and R with $R \geq r \geq r_0$, and any function $f \in \mathcal{H}_R$, we have*

$$|f|_r \leq \kappa \left(\frac{r}{r_0}\right)^{L-1} \sup_{\varphi \in \mathcal{S}} |\varphi(f)| + \left(1 + \kappa \frac{r_0}{r}\right) \left(\frac{e^{m/2} r}{R}\right)^{L} |f|_R.$$

Corollary 3.2. *Let the notation be as in Definition 1.3. The condition (I.1) in this definition is equivalent to:*

(I.2) *There exist a finite subset $\Sigma \neq \emptyset$ of Γ, a basis $\mathcal{B}$ of W, and a positive constant c such that, for any integer $N \geq 1$ with $L_N \geq 1$ and for any polynomial map $P\colon \mathbf{C}^m \to \mathbf{C}$ of degree $< L_N$, we have*

$$|P|_N \leq c^{L_N} \Phi\big(P\,;\, \Sigma, N\,;\, \mathcal{B}, N^s\big).$$

Moreover, if one of the conditions (I.1) or (I.2) holds for some subset Σ of Γ and some basis $\mathcal{B}$ of W, then both of them hold for the same choice of Σ and $\mathcal{B}$.

Proof. If (I.1) holds, we may assume $c_0 \geq 1$. Then, for any integer $N \geq 1$ with $L_N \geq 1$ and any polynomial map P of degree $< L_N$, we get

$$|P|_N \leq |P|_{c_0 N} \leq (c_0 c_1)^{L_N} \Phi\big(P\,;\, \Sigma, N\,;\, \mathcal{B}, N^s\big)$$

by taking $r = c_0 N$ and letting R tend to ∞. Therefore, (I.2) holds for the same choice of Σ and $\mathcal{B}$, with $c = c_0 c_1$.

Conversely, assume that (I.2) holds, and fix an integer $N \geq 1$. Lemma 2.2 shows that there is a constant $c_0 > 0$ depending only on Σ and $\mathcal{B}$ such that, for $r_0 = c_0 N$, the set $\Sigma(N)$ is contained in the interior of $B_m(0, r_0)$ and the linear functionals on $\mathcal{H}_{r_0}$ which enter into the definition of $\Phi\big(f\,;\, \Sigma, N\,;\, \mathcal{B}, N^s\big)$ all have norm ≤ 1. Assume $L_N \geq 1$. Then by (I.2), the condition (i) of Proposition 3.1 is satisfied with $L = L_N$ and $\kappa = c^{L_N}$, for this value of r_0 and this set $\mathcal{S}$ of linear functionals on $\mathcal{H}_{r_0}$. Therefore, the condition (ii) also holds. Thus (I.1) is verified for the same choice of Σ and $\mathcal{B}$, with $c_1 = c/c_0$ and $c_2 = (1+c)e^{m/2}$, when $L_N \geq 1$. The maximum modulus principle shows that it is also verified when $L_N = 0$. ∎

4. General results

Throughout this section, we fix a positive integer m, a finitely generated subgroup Γ of $\mathbf{C}^m$, a subspace W of $\mathbf{C}^m$, a real number $s \geq 0$, and a sequence $(L_N)_{N \geq 1}$ of non-negative integers.

Proposition 4.1. *Assume that (Γ, W, s) admits a sequence of interpolation formulas with exponents $(L_N)_{N\geq 1}$, for some finite subset $\Sigma \neq \emptyset$ of Γ and some basis $\mathcal{B}$ of W. Let $(L'_N)_{N\geq 1}$ be a sequence of integers with $0 \leq L'_N \leq L_N$ for each $N \geq 1$. Then, (Γ, W, s) also admits* *a sequence of interpolation formulas with exponents $(L'_N)_{N\geq 1}$, for the same choice of Σ and $\mathcal{B}$.*

Proof. Choose a constant $c > 0$ satisfying the condition (I.2) of Corollary 3.2 for the given choice of Σ and $\mathcal{B}$. Fix an integer $N \geq 1$ with $L'_N \geq 1$ and put $\ell = [L_N/L'_N]$. Then, for any polynomial map $Q\colon \mathbf{C}^m \to \mathbf{C}$ of degree $< L'_N$, the polynomial $P = Q^\ell$ has degree $< L_N$ and the condition (I.2) implies

$$|Q^\ell|_N \leq c^{L_N} \Phi\big(Q^\ell\,;\, \Sigma, N\,;\, \mathcal{B}, N^s\big).$$

Put $k = \dim(W)$. By Lemma 2.3, we also have

$$\Phi\big(Q^\ell\,;\, \Sigma, N\,;\, \mathcal{B}, N^s\big) \leq (k+1)^{\ell L'_N} \Phi\big(Q\,;\, \Sigma, N\,;\, \mathcal{B}, N^s\big)^\ell.$$

Since $|Q^\ell|_N = |Q|_N^\ell$ and since $L_N \leq (\ell+1)L'_N \leq 2\ell L'_N$, we conclude that

$$|Q|_N \leq \big(c^2(k+1)\big)^{L'_N} \Phi\big(Q\,;\, \Sigma, N\,;\, \mathcal{B}, N^s\big).$$

This shows that the condition (I.2) also holds with the sequence $(L_N)_{N\geq 1}$ replaced by $(L'_N)_{N\geq 1}$ and the constant c replaced by $c^2(k+1)$. ∎

Proposition 4.2. *Let the notation be as in Proposition 4.1 and let $\psi\colon \mathbf{C}^m \to \mathbf{C}^m$ be a $\mathbf{C}$-linear invertible map. Then, the triple $\big(\psi(\Gamma), \psi(W), s\big)$ admits a sequence of interpolation formulas with exponents $(L_N)_{N\geq 1}$, for the subset $\psi(\Sigma)$ of $\psi(\Gamma)$ and the basis $\psi(\mathcal{B})$ of $\psi(W)$.*

Proof. Again, choose $c > 0$ satisfying the condition (I.2), and choose a constant $c_0 \geq 1$ such that $B_m(0,1) \subseteq \psi\big(B_m(0,c_0)\big)$. Then, for any index $N \geq 1$ with $L_N \geq 1$, and any polynomial map $P\colon \mathbf{C}^m \to \mathbf{C}$ of degree $< L_N$, we have

$$\begin{aligned}
|P|_N \leq |P\circ\psi|_{c_0 N} \leq c_0^{L_N} |P\circ\psi|_N \qquad & \text{by Lemma 2.4,}\\
\leq (c_0 c)^{L_N} \Phi\big(P\circ\psi\,;\, \Sigma, N\,;\, \mathcal{B}, N^s\big) \qquad & \text{by hypothesis,}\\
= (c_0 c)^{L_N} \Phi\big(P\,;\, \psi(\Sigma), N\,;\, \psi(\mathcal{B}), N^s\big) \qquad & \text{by Lemma 2.1.}
\end{aligned}$$

So the condition (I.2) is also verified for the triple $\big(\psi(\Gamma), \psi(W), s\big)$ and the same sequence of exponents $(L_N)_{N\geq 1}$, with the subset $\psi(\Sigma)$ of $\psi(\Gamma)$, the basis $\psi(\mathcal{B})$ of $\psi(W)$, and the constant $c_0 c$. ∎

Proposition 4.3. *Suppose $m \geq 2$. Let m', m'' be positive integers with sum m, let*

$$0 \longrightarrow \mathbf{C}^{m'} \xrightarrow{\ \iota\ } \mathbf{C}^m \xrightarrow{\ \pi\ } \mathbf{C}^{m''} \longrightarrow 0$$

be an exact sequence of **C***-linear maps, and let*

$$\Gamma' = \iota^{-1}(\Gamma), \quad W' = \iota^{-1}(W), \quad \Gamma'' = \pi(\Gamma) \quad \text{and} \quad W'' = \pi(W).$$

Assume that both triples (Γ', W', s) and (Γ'', W'', s) admit sequences of interpolation formulas with exponents $(L_N)_{N\geq 1}$, for some finite non-empty subsets Σ' of Γ' and Σ'' of Γ'' and some bases $\mathcal{B}'$ of W' and $\mathcal{B}''$ of W''. Then, (Γ, W, s) also admits a sequence of interpolation formulas with exponents $(L_N)_{N\geq 1}$, for any finite subset Σ of Γ with $\iota(\Sigma') \subseteq \Sigma$ and $\Sigma'' \subseteq \pi(\Sigma)$ and the basis $\mathcal{B}$ of W for which $\iota(\mathcal{B}') \subseteq \mathcal{B}$ and $\mathcal{B}'' \subseteq \pi(\mathcal{B})$.

Proof. Let Σ and $\mathcal{B}$ be as in the conclusion of the Proposition. The conditions on Σ allow us to write $\Sigma = \Sigma_1 \cup \Sigma_2$ where Σ_1 and Σ_2 are disjoint sets with $\iota(\Sigma') \subseteq \Sigma_1$ and $\Sigma'' \subseteq \pi(\Sigma_2) \cup \{0\}$. Similarly, if we write $\mathcal{B}' = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ and $\mathcal{B}'' = \{\mathbf{v}_1, \ldots, \mathbf{v}_\ell\}$, the conditions on $\mathcal{B}$ imply $\mathcal{B} = \{\mathbf{w}_1, \ldots, \mathbf{w}_{k+\ell}\}$ where $\mathbf{w}_j = \iota(\mathbf{u}_j)$ for $j = 1, \ldots, k$ and $\pi(\mathbf{w}_{k+j}) = \mathbf{v}_j$ for $j = 1, \ldots, \ell$. Choose an injective linear map $\psi\colon \mathbf{C}^{m''} \to \mathbf{C}^m$ with $\psi(\mathbf{v}_j) = \mathbf{w}_{k+j}$ for $j = 1, \ldots, \ell$ such that $\pi \circ \psi$ is the identity map of $\mathbf{C}^{m''}$. By hypothesis, the condition (I.2) is satisfied for the triples (Γ', W', s) and (Γ'', W'', s) with the subsets Σ' of Γ' and Σ'' of Γ'', the bases $\mathcal{B}'$ of W' and $\mathcal{B}''$ of W'', and some constants $c', c'' > 0$. We now show that it holds for the triple (Γ, W, s) for the given choice of Σ and $\mathcal{B}$.

Let $N \geq 1$ be an index with $L_N \geq 1$ and let $P\colon \mathbf{C}^m \to \mathbf{C}$ be a polynomial map of degree $< L_N$. We choose a point $\mathbf{z}_0 \in \mathbf{C}^m$ with $\|\mathbf{z}_0\| \leq N$ such that $|P|_N = |P(\mathbf{z}_0)|$. Since $\mathbf{C}^m = \iota(\mathbf{C}^{m'}) \oplus \psi(\mathbf{C}^{m''})$, we can write $\mathbf{z}_0 = \iota(\mathbf{x}_0) + \psi(\mathbf{y}_0)$ for some points $\mathbf{x}_0 \in \mathbf{C}^{m'}$ and $\mathbf{y}_0 = \pi(\mathbf{z}_0) \in \mathbf{C}^{m''}$ of norms $\leq c_1 N$, with a constant $c_1 \geq 1$ depending only on the choice of ι and π. Consider the polynomial map $G\colon \mathbf{C}^{m''} \to \mathbf{C}$ given by $G(\mathbf{y}) = P(\iota(\mathbf{x}_0) + \psi(\mathbf{y}))$ for each $\mathbf{y} \in \mathbf{C}^{m''}$. Since it has degree $< L_N$, we find, with the help of Lemma 2.4,

$$|P|_N = |G(\mathbf{y}_0)| \leq |G|_{c_1 N} \leq c_1^{L_N} |G|_N \leq (c_1 c'')^{L_N} \Phi\big(G\,;\, \Sigma'', N\,;\, \mathcal{B}'', N^s\big). \qquad (4.1)$$

Choose $\eta \in \Sigma''(N)$ and $\underline{\beta} = (\beta_1, \ldots, \beta_\ell) \in \mathbf{N}^\ell$ with $\|\underline{\beta}\| \leq N^s$ such that

$$\Phi\big(G\,;\, \Sigma'', N\,;\, \mathcal{B}'', N^s\big) = \frac{1}{\underline{\beta}!} \big|\big(D_{\mathbf{v}_1}^{\beta_1} \cdots D_{\mathbf{v}_\ell}^{\beta_\ell} G\big)(\eta)\big|\, N^{|\underline{\beta}|}$$

where, for shortness, we put $\underline{\beta}! = \beta_1! \cdots \beta_\ell!$ and $|\underline{\beta}| = \beta_1 + \cdots + \beta_\ell$. Going back to the definition of G, we find

$$\big(D_{\mathbf{v}_1}^{\beta_1} \cdots D_{\mathbf{v}_\ell}^{\beta_\ell} G\big)(\eta) = \big(D_{\mathbf{w}_{k+1}}^{\beta_1} \cdots D_{\mathbf{w}_{k+\ell}}^{\beta_\ell} P\big)(\iota(\mathbf{x}_0) + \psi(\eta)).$$

Since $\eta \in \Sigma''(N)$, there exists $\gamma_2 \in \Sigma_2(N)$ such that $\pi(\gamma_2) = \eta$. So, we can write $\psi(\eta) = \iota(\theta) + \gamma_2$ for a point $\theta \in \mathbf{C}^{m'}$ of norm $\|\theta\| \leq c_2 N$, where $c_2 > 0$ depends only on the choice of ι, ψ and Σ. Consider now the polynomial map $F\colon \mathbf{C}^{m'} \to \mathbf{C}$ given by

$$F(\mathbf{x}) = \frac{1}{\underline{\beta}!} \big(D_{\mathbf{w}_{k+1}}^{\beta_1} \cdots D_{\mathbf{w}_{k+\ell}}^{\beta_\ell} P\big)(\iota(\mathbf{x}) + \gamma_2) N^{|\underline{\beta}|}$$

for each $\mathbf{x} \in \mathbf{C}^{m'}$. By construction, we have

$$\Phi\big(G\,;\,\Sigma'',N\,;\,\mathcal{B}'',N^s\big) = |F(\mathbf{x}_0+\theta)| \le |F|_{(c_1+c_2)N}. \tag{4.2}$$

Since F has degree $< L_N$, we also find, with the help of Lemma 2.4,

$$|F|_{(c_1+c_2)N} \le (c_1+c_2)^{L_N}|F|_N \le \big((c_1+c_2)c'\big)^{L_N}\Phi\big(F\,;\,\Sigma',N\,;\,\mathcal{B}',N^s\big). \tag{4.3}$$

Choose $\xi \in \Sigma'(N)$ and $\underline{\alpha} = (\alpha_1,\dots,\alpha_k) \in \mathbf{N}^k$ with $\|\underline{\alpha}\| \le N^s$ such that

$$\Phi\big(F\,;\,\Sigma',N\,;\,\mathcal{B}',N^s\big) = \frac{1}{\underline{\alpha}!}\left|\big(D^{\alpha_1}_{\mathbf{u}_1}\cdots D^{\alpha_k}_{\mathbf{u}_k}F\big)(\xi)\right| N^{|\underline{\alpha}|},$$

where $\underline{\alpha}! = \alpha_1!\cdots\alpha_k!$ and $|\underline{\alpha}| = \alpha_1+\cdots+\alpha_k$. Put $\gamma_1 = \iota(\xi)$. By definition of F, the right hand side of this equality is also equal to

$$\frac{1}{\underline{\alpha}!\underline{\beta}!}\left|\big(D^{\alpha_1}_{\mathbf{w}_1}\cdots D^{\alpha_k}_{\mathbf{w}_k}D^{\beta_1}_{\mathbf{w}_{k+1}}\cdots D^{\beta_\ell}_{\mathbf{w}_{k+\ell}}P\big)(\gamma_1+\gamma_2)\right| N^{|\underline{\alpha}|+|\underline{\beta}|}.$$

Since $\gamma_1+\gamma_2 \in \Sigma_1(N)+\Sigma_2(N) = \Sigma(N)$, and since $(\underline{\alpha},\underline{\beta}) \in \mathbf{N}^{k+\ell}$ has norm $\|(\underline{\alpha},\underline{\beta})\| \le N^s$, we deduce

$$\Phi\big(F\,;\,\Sigma',N\,;\,\mathcal{B}',N^s\big) \le \Phi\big(P\,;\,\Sigma,N\,;\,\mathcal{B},N^s\big). \tag{4.4}$$

Combining the inequalities (4.1) to (4.4), we finally get

$$|P|_N \le c^{L_N}\Phi\big(P\,;\,\Sigma,N\,;\,\mathcal{B},N^s\big)$$

with $c = c_1(c_1+c_2)c'c''$. □

5. The case of dimension one

In dimension one, there are satisfactory interpolation formulas. The following one formalizes an argument of K. Mahler in [4]:

Lemma 5.1. *Let B be a non-empty finite subset of $\mathbf{C}$, and let $(\ell_\beta)_{\beta\in B}$ a sequence of positive integers indexed by B. Put*

$$L = \sum_{\beta\in B}\ell_\beta, \qquad b = \max_{\beta\in B}\max\big\{1,|\beta|\big\},$$

$$b_1 = \min_{\beta\in B}\prod_{\substack{\beta'\in B\\ \beta'\neq\beta}}|\beta-\beta'|^{\ell_{\beta'}/L}, \qquad b_2 = \min_{\substack{\beta,\beta'\in B\\ \beta\neq\beta'}}\min\big\{1,|\beta-\beta'|\big\}^{\ell_{\beta'}/L},$$

with the convention that $b_1 = b_2 = 1$ when B consists of only one point. Then, for any pair of real numbers r and R with $R \geq 2r$ and $r \geq 2b$, and for any function $f \in \mathcal{H}_R$, we have

$$|f|_r \leq \Big(\frac{6r}{b_1 b_2}\Big)^L \max_{\substack{\beta\in B\\ 0\leq\tau<\ell_\beta}} \frac{1}{\tau!}\Big|\frac{d^\tau f}{dz^\tau}(\beta)\Big| \;+\; \Big(\frac{6r}{R}\Big)^L |f|_R.$$

Proof. This is essentially the inequality (15) on page 91 of [4] except for the following few changes. First of all, the exponential polynomial E considered by Mahler is replaced here by the holomorphic function f, and the point $z \in \mathbf{C}$ is chosen so that $|z| = r$ and $|f(z)| = |f|_r$. Moreover, the integer which Mahler denotes by r is here denoted by L. Using Hermite's interpolation formula, we get $f(z) = J - \sum_{\beta\in B} J_\beta$, where J and $(J_\beta)_{\beta\in B}$ are the complex path integrals defined on pages 89 and 90 of [4]. Taking absolute values and using the fact that the cardinality of B is at most $L \leq 2^{L-1}$, this gives

$$|f|_r = |f(z)| \leq |J| + 2^{L-1} \max_{\beta\in B} |J_\beta|.$$

Instead of the inequality (11) on page 89 of [4], we get

$$|J| \leq 2\Big(\frac{3r}{R}\Big)^L |f|_R \leq \Big(\frac{6r}{R}\Big)^L |f|_R$$

using the fact that, for any $\beta \in B$ and any $\zeta \in \mathbf{C}$ with $|\zeta| = R$, we have $|z - \beta| \leq r + b \leq (3/2)r$, $|\zeta - \beta| \geq R - b \geq (1/2)R$ and $|\zeta - z| \geq R - r \geq (1/2)R$. Similarly, instead of the inequality (14) on page 91 of [4], we get that, for any $\beta \in B$, we have

$$|J_\beta| \leq 2\Big(\frac{3r}{b_1 b_2}\Big)^L \max_{0\leq\tau<\ell_\beta} \frac{1}{\tau!}\Big|\frac{d^\tau f}{dz^\tau}(\beta)\Big|.$$

The conclusion follows by combining the above three inequalities. ∎

In the next lemma and the rest of the section, we denote by $|E|$ the cardinality of a finite set E.

Lemma 5.2. *Let Σ be a finite subset of $\mathbf{C}$ containing at least one non-zero element x. Suppose that N is a positive integer with $\Sigma(N) \cap B_1(0, e^{-N}) = \{0\}$. Then, for any $\beta \in \Sigma(N)$, we have*

$$\prod_{\substack{\beta'\in\Sigma(N)\\ \beta'\neq\beta}} |\beta - \beta'| \geq (cN)^{|\Sigma(N)|},$$

where $c = (2e^3)^{-1/2} \min\{1, |x|\}$.

Proof. Fix a point $\beta \in \Sigma(N)$. Denote by Γ the subgroup of $\mathbf{C}$ generated by Σ and by Δ the subgroup generated by x. The cosets of Δ in Γ form a partition of Γ. Let $\mathcal{P}$ be the trace of this partition on $\Sigma(N)$. Choose $C \in \mathcal{P}$. Then C is a subset of $\Sigma(N)$ of cardinality at least $2N+1$. Moreover, the numbers $\beta - \beta'$ with $\beta' \in C$ differ from one another by a multiple of x. Thus, for any positive integer k, there are at most k elements β' of C such that $|\beta - \beta'| < k|x|/2$. Moreover, if $\beta \notin C$, the hypothesis tells us that $|\beta - \beta'| \geq e^{-N}$ for any $\beta' \in C$. This implies

$$\prod_{\substack{\beta' \in C \\ \beta' \neq \beta}} |\beta - \beta'| \geq e^{-N} \prod_{k=1}^{M-1} \frac{k|x|}{2},$$

where $M = |C|$. Since $(M-1)! \geq ((M-1)/e)^{M-1} \geq (2N/e)^{M-1} \geq (\sqrt{2}N/e)^M$, we deduce that

$$\prod_{\substack{\beta' \in C \\ \beta' \neq \beta}} |\beta - \beta'| \geq (cN)^{|C|}.$$

where c is as in the statement of the Lemma. The conclusion follows by multiplying term by term these inequalities for all sets C in the partition $\mathcal{P}$ of $\Sigma(N)$. ∎

Lemma 5.3. *Let Σ be a non-empty finite subset of $\mathbf{C}$, and let $N \geq 2$ be an integer. Define*

$$\lambda = \operatorname{rank}(\Sigma) - \operatorname{rank}\big(\Sigma(N) \cap B(0, e^{-N})\big),$$

and assume $\lambda \geq 1$. Then, there exists a subset B of $\Sigma(N)$ with $N^\lambda \leq |B| \leq (N+1)^\lambda$ such that

$$\min_{\substack{\beta, \beta' \in B \\ \beta \neq \beta'}} |\beta - \beta'| \geq e^{-|B|} \quad \text{and} \quad \min_{\beta \in B} \prod_{\substack{\beta' \in B \\ \beta' \neq \beta}} |\beta - \beta'| \geq (cN)^{|B|},$$

where c is a positive constant depending only on Σ.

Note that, for a given set Σ, the condition $\lambda \geq 1$ is automatically satisfied for any sufficiently large integer N provided simply that Σ contains at least one non-zero element.

Proof. Denote by Δ the subgroup of $\mathbf{C}$ generated by $\Sigma(N) \cap B(0, e^{-N})$. Then, there exists a subset Σ' of Σ consisting of λ elements $x_1, \ldots, x_\lambda$ which are linearly independent modulo Δ. Define $M = [N/2]$ and $B = \Sigma'(M)$. Then, B is a subset of $\Sigma(N)$ of cardinality $(2M+1)^\lambda$, and so $N^\lambda \leq |B| \leq (N+1)^\lambda$. Now, fix a choice of $\beta \in B$. Then, for any $\beta' \in B$ with $\beta' \neq \beta$, we have

$$|\beta - \beta'| \geq e^{-N} \geq e^{-|B|}$$

since $\beta - \beta' \in \Sigma'(N)$ is non-zero and $\Sigma'(N) \cap B(0, e^{-N}) \subseteq \Sigma'(N) \cap \Delta = \{0\}$. Moreover, Lemma 5.2 gives

$$\prod_{\substack{\beta' \in B \\ \beta' \neq \beta}} |\beta - \beta'| \geq \big((2e^3)^{-1/2} c_1 M\big)^{|B|}.$$

where c_1 denotes the minimum of 1 and all the absolute values of the non-zero elements of Σ. Since $M \geq N/3$, the set B has all the required properties for the choice of $c = c_1/20$. $\square$

Proposition 5.4. *Let Σ be a non-empty finite subset of $\mathbf{C}$, and let ℓ and N be positive integers. Define λ as in Lemma 5.3. Then, for any polynomial map $P\colon \mathbf{C} \to \mathbf{C}$ of degree $< \ell N^\lambda$, we have*

$$|P|_N \leq c^{\ell N^\lambda} \max_{\substack{\gamma\in\Sigma(N)\\ 0\leq\tau<\ell}} \frac{1}{\tau!}\left|\frac{d^\tau P}{dz^\tau}(\gamma)\right| N^\tau \tag{5.1}$$

where c is a positive constant depending only on Σ.

Proof. Let $P\colon \mathbf{C} \to \mathbf{C}$ be a polynomial map of degree $< \ell N^\lambda$. If $\lambda \geq 1$ and $N \geq 2$, we apply Lemma 5.1 to the function P with the subset B of $\Sigma(N)$ provided by Lemma 5.3 and with $\ell_\beta = \ell$ for any $\beta \in B$. For the corresponding quantities, we have

$$\ell N^\lambda \leq L = \ell|B| \leq \ell(2N)^\lambda, \quad b \leq c_0 N \quad b_1 \geq c_1 N \quad \text{and} \quad b_2 \geq e^{-1}$$

for some constants c_0, c_1 depending only on Σ, with $c_0 \geq 1 \geq c_1 > 0$. Putting $r = 2c_0N$ and letting R tend to ∞, we get:

$$|P|_{2c_0N} \leq \left(\frac{12c_0}{e^{-1}c_1}\right)^L \max_{\substack{\beta\in B\\ 0\leq\tau<\ell}} \frac{1}{\tau!}\left|\frac{d^\tau P}{dz^\tau}(\beta)\right|.$$

Since $|P|_N \leq |P|_{2c_0N}$ and since $B \subseteq \Sigma(N)$, this gives (5.1) with $c = (12ec_0/c_1)^{|\Sigma(1)|}$ on noting that $2^\lambda \leq |\Sigma(1)|$. If $\lambda = 0$ or if $N = 1$, we have $N^\lambda = 1$ and (5.1) still holds with the same choice of c because $0 \in \Sigma(N)$ and

$$|P|_N \leq \sum_{0\leq\tau<\ell} \frac{1}{\tau!}\left|\frac{d^\tau P}{dz^\tau}(0)\right| N^\tau \leq 2^\ell \max_{0\leq\tau<\ell} \frac{1}{\tau!}\left|\frac{d^\tau P}{dz^\tau}(0)\right| N^\tau. \qquad \square$$

Corollary 5.5. *Let Γ be a subgroup of $\mathbf{C}$, let Σ be a finite non-empty subset of Γ, let W be a subspace of $\mathbf{C}$, and let s be a positive real number. For any positive integer N, define*

$$\mu_N = \operatorname{rank}(\Sigma) - \operatorname{rank}\left(\Sigma(N) \cap B(0, e^{-N})\right) + s\dim(W)$$

and $L_N = [N^{\mu_N}]$. Then, (Γ, W, s) admits a sequence of interpolation formulas with exponents $\left(L_N\right)_{N\geq1}$, for the given subset Σ of Γ and any basis $\mathcal{B}$ of W.

Proof. Put $t = \dim W$. For each integer $N \geq 1$, define ℓ_N as the smallest integer $\geq N^{st}$, and put $\lambda_N = \mu_N - st$. Then Proposition 5.4 shows that there is a sequence of interpolation formulas for (Γ, W, s) with exponents $\left(\ell_N N^{\lambda_N}\right)_{N\geq1}$. This uses the basis $\{1\}$ for W if $W = \mathbf{C}$, but any other basis would do as well provided that we adjust the constant c. Since $L_N \leq \ell_N N^{\lambda_N}$ for any $N \geq 1$, the conclusion follows from Proposition 4.1. $\square$

6. Proof of the interpolation formula

Proof of Theorem 1.4. Choose a finite subset Σ of Γ with $\Sigma_j \subseteq \pi_j(\Sigma \cap V_j)$ for $j = 1, \dots, m$. We first observe that, by Corollary 5.5 and Proposition 4.1, the triple (Γ_m, W_m, s) admits a sequence of interpolation formulas with exponents $(L_N)_{N\geq 1}$, for the subset Σ_m of Γ_m and for any basis of W_m. We now prove the theorem by induction on m.

If $m = 1$, then $\pi_1 \colon \mathbf{C} \to \mathbf{C}$ is an isomorphism and $(\Gamma, W, s) = (\pi_1^{-1}(\Gamma_1), \pi_1^{-1}(W_1), s)$. The conclusion then follows from Proposition 4.2.

Assume now $m \geq 2$ and choose a $\mathbf{C}$-linear injective map $\iota \colon \mathbf{C}^{m-1} \to \mathbf{C}^m$ whose image is V_{m-1}. Put

$$\Gamma' = \iota^{-1}(\Gamma),\ \Sigma' = \iota^{-1}(\Sigma),\ W' = \iota^{-1}(W) \text{ and } V'_j = \iota^{-1}(V_j) \text{ for } j = 0, \dots, m-1.$$

Then, $0 = V'_0 \subset \cdots \subset V'_{m-1} = \mathbf{C}^{m-1}$ is a strictly increasing sequence of subspaces of $\mathbf{C}^{m-1}$ with $\dim V'_j = j$ for $j = 0, \dots, m-1$. For $j = 1, \dots, m-1$, denote by $\pi'_j \colon V'_j \to \mathbf{C}$ the linear map with kernel V'_{j-1} given by $\pi'_j(v) = \pi_j(\iota(v))$ for any $v \in V'_j$. We find $\pi'_j(\Gamma' \cap V'_j) = \Gamma_j$, $\Sigma_j \subseteq \pi'_j(\Sigma' \cap V'_j)$ and $\pi'_j(W' \cap V'_j) = W_j$ for all these values of j. Thus, by induction on m, we may assume, using Proposition 4.1, that (Γ', W', s) admits a sequence of interpolation formulas with exponents $(L_N)_{N\geq 1}$, for the subset Σ' of Γ' and any basis $\mathcal{B}'$ of W'. Applying Proposition 4.3 with $\pi = \pi_m$, we deduce that (Γ, W, s) admits a sequence of interpolation formulas with the same exponents, for the given subset Σ of Γ and some basis $\mathcal{B}$ of W. When $W \neq \{0\}$, we have $\mu_N \geq s$ for any $N \geq 1$, and thus we may replace $\mathcal{B}$ by any other basis of W. $\square$

It is tempting to speculate on a possible form of the sequence of exponents $(L_N)_{N\geq 1}$ for a general triple (Γ, W, s). In the spirit of M. Waldschmidt's conjecture 7.1.10 in [12], we propose:

Conjecture 6.1. *Let Γ be a subgroup of $\mathbf{C}^m$ with a finite set of generators $\Sigma \neq \emptyset$, let W be a subspace of $\mathbf{C}^m$ and let $s \geq 0$ be a real number. For each integer $N \geq 1$, define μ_N as the smallest ratio*

$$\frac{s \dim\big(W/(W\cap V)\big) + \operatorname{rank}(\Gamma) - \operatorname{rank}\big(\Sigma(N) \cap (V + B(0, e^{-N}))\big)}{\dim(\mathbf{C}^m/V)}$$

where V is a subspace of $\mathbf{C}^m$ of dimension $< m$. Then, there exists a constant $c \geq 1$ depending only on m such that (Γ, W, s) admits a sequence of interpolation formulas with exponents $\big([(N/c)^{\mu_N}]\big)_{N\geq 1}$, for the subset Σ of Γ and any basis of W.

In the above statement, the condition that c depends only on m is motivated by the zero estimates of D. Masser in [6]. In practice, it would probably suffice to have $c \geq 1$ independent of N.

In the case where $W = \{0\}$, we introduce the following notation.

Definition 6.2. Let Σ be a non-empty subset of $\mathbf{C}^m$ of finite rank, and let $\rho \geq 0$ be a real number. We denote by $\mu(\Sigma, \rho)$ the smallest ratio

$$\frac{\operatorname{rank}(\Sigma) - \operatorname{rank}\big(\Sigma \cap (V + B(0,\rho))\big)}{\dim(\mathbf{C}^m/V)}$$

where V is a subspace of $\mathbf{C}^m$ of dimension $< m$.

For a finitely generated subgroup Γ of $\mathbf{C}^m$, the quantity $\mu(\Gamma, 0)$ is the Dirichlet exponent of Γ introduced by M. Waldschmidt in [13] and denoted $\mu(\Gamma)$. The above conjecture implies the following version of the Schwarz lemma:

Conjecture 6.3. *Let Σ be a finite non-empty subset of $\mathbf{C}^m$. There exist constants $c, c_1, c_2 \geq 1$ with the following property. For any integer $N \geq 1$, any real numbers r and R with $R \geq r \geq c_1 N$, and any entire function $f\colon \mathbf{C}^m \to \mathbf{C}$ which vanishes at each point of $\Sigma(N)$, we have*

$$|f|_r \leq \left(\frac{c_2 r}{R}\right)^{L_N} |f|_R$$

where $L_N = \big[(N/c)^{\mu_N}\big]$ and $\mu_N = \mu(\Sigma(N), e^{-N})$.

Note that the above inequality follows from the maximum modulus principle when $r \leq R \leq c_2 r$. Therefore, one may assume without loss of generality that $R > c_2 r$.

By Corollary 5.5, both Conjectures 6.1 and 6.3 hold in the case $m = 1$. Moreover, when $m = 1$ or $\operatorname{rank}(\Sigma) \leq m$, Lemma 7.1.3 and Proposition 7.2.1 of [12] imply that Conjecture 6.3 holds with the larger values $\mu_N = \mu(\Sigma(N), 0)$. Equivalently, one can take $\mu_N = \mu(\Gamma)$, where Γ is the subgroup of $\mathbf{C}^m$ generated by Σ, since $\mu(\Sigma(N), 0) = \mu(\Gamma)$ for all large values of N. In view of this, one might hope that Conjecture 6.3 is also true when $\operatorname{rank}(\Sigma) > m > 1$ with the constant sequence $\mu_N = \mu(\Gamma)$. Conjecture 7.1.10 of [12] is slightly less optimistic and proposes that Conjecture 6.3 holds with μ_N replaced by $\mu(\Gamma) - \epsilon$ for any fixed real number $\epsilon > 0$. We will see in the next section that even this is not true, already in the case where $m = 2$ and Σ has rank 3.

7. Two examples of obstruction to the Schwarz lemma.

Our first example shows that Conjecture 6.3 is false if, in the definition of μ_N, we replace the function e^{-N} by the faster decreasing function $e^{-N^{3+\delta}}$ where δ is any fixed positive real number.

Proposition 7.1. *Let ϵ and δ be positive real numbers with $\epsilon \le 1$. There exists a subset Σ of $\mathbf{C}^2$ consisting of three points, with the following property. For infinitely many positive integers N, we have*

$$\mu\Big(\Sigma(N)\,,\,\exp\big(-3N^{3+\delta}\big)\Big) = 3/2$$

and there exists a non-zero polynomial map $f\colon \mathbf{C}^2 \to \mathbf{C}$ vanishing at each point of $\Sigma(N)$ which satisfies

$$|f|_R \le \Big(\frac{8R}{r}\Big)^{5N^{1+\epsilon}} |f|_r$$

for any pair of real numbers r and R with

$$12N \le r \le R \le \exp\Big(\frac{1}{12}N^\delta\Big).$$

In the situation of Proposition 7.1, we have $\mu(\Gamma) = 3/2$ for the subgroup Γ of $\mathbf{C}^2$ generated by Σ. So, for a fixed real number ϵ with $0 < \epsilon < 1/2$, Conjecture 6.3 does not hold if we replace μ_N by $\mu(\Gamma) - \epsilon$ for all $N \ge 1$. However, it might hold with this value of μ_N for infinitely many N. Our second example shows that we cannot even hope this to be the case for some subset Σ of $\mathbf{C}^2$ of rank 5.

Proposition 7.2. *Let ϵ and δ be positive real numbers with $\epsilon \le 1$. There exists a finitely generated subgroup Γ of $\mathbf{C}^2$ of rank 5 with*

$$\mu(\Gamma) = 5/2,$$

a finite set of generators Σ of Γ, and a constant $c > 0$ with the following property. For any sufficiently large positive integer N, there exists a non-zero polynomial map $f\colon \mathbf{C}^2 \to \mathbf{C}$ vanishing at each point of $\Sigma(N)$ which satisfies

$$|f|_R \le \Big(\frac{8R}{r}\Big)^{N^{2+\epsilon}} |f|_r$$

for any pair of real numbers r and R with

$$cN \le r \le R \le \exp\big(N^\delta\big).$$

Note that the above counter-examples cannot extend to arbitrarily large values of r for the following general reason. Let Γ be a subgroup of $\mathbf{C}^2$ generated by a finite set Σ. Theorem 1.1 of [7] combined with Theorem 1 of [6] implies that, for any integer $N \ge 1$, there exists a real number $r_0(N) > 0$ such any entire function $f\colon \mathbf{C}^2 \to \mathbf{C}$ vanishing at each point of $\Sigma(N)$ satisfies

$$|f|_r \le \Big(\frac{2er}{R}\Big)^{N^{\mu(\Gamma)}} |f|_R$$

when $r_0(N) \le r \le R$.

Let $x\colon \mathbf{C}^2 \to \mathbf{C}$ and $y\colon \mathbf{C}^2 \to \mathbf{C}$ be the usual coordinate functions. As in §5, we denote by $|E|$ the cardinality of a finite set E. The following lemma provides the basic construction in the proof of the above two Propositions.

Lemma 7.3. *Let A be a non-empty finite subset of $\mathbf{C}$, and let E be a finite subset of $\mathbf{C}^2$, of cardinality at least equal to that of A, such that the projection $y\colon \mathbf{C}^2 \to \mathbf{C}$ is injective on E. Put $B = y(E)$,*

$$\rho = \max_{p\in E}\Big\{\min_{a\in A}|x(p)-a|\Big\} \quad \textit{and} \quad \Delta = \min_{b\in B}\prod_{\substack{b'\in B\\ b'\neq b}}|b-b'|.$$

Choose also a real number $r_0 \geq 1$ such that $A \subset B_1(0,r_0)$ and $E \subset B_2(0,r_0)$. Then, there exists a polynomial map $f\colon \mathbf{C}^2 \to \mathbf{C}$ which vanishes at each point of E such that, for any real numbers r and R with

$$4r_0 \leq r \leq R \quad \textit{and} \quad \big(3R\big)^{|E|}\rho \leq \Delta, \tag{7.1}$$

we have

$$|f|_R \leq \Big(\frac{8R}{r}\Big)^{|A|}|f|_r.$$

Proof. Consider the polynomial of two variables $f(x,y) = P(x) - Q(y)$ where

$$P(x) = \prod_{a\in A}(x-a) \quad \text{and} \quad Q(y) = \sum_{p\in E}P(x(p))\prod_{\substack{q\in E\\ q\neq p}}\Big(\frac{y-y(q)}{y(p)-y(q)}\Big).$$

By construction, f vanishes at each point of E. Fix a point $p\in E$. Since the projection y induces a bijection from E to B, we have

$$\prod_{\substack{q\in E\\ q\neq p}}|y(p)-y(q)| \geq \Delta.$$

We also have $|x(p)-a| \leq 2r_0$ for any $a\in A$, and $|x(p)-a|\leq\rho$ for at least one choice of $a\in A$. This implies

$$\big|P(x(p))\big| \leq \rho(2r_0)^{|A|-1} \leq \rho(2r_0)^{|A|}.$$

If r and R are real numbers satisfying the conditions (7.1), we deduce

$$|Q|_r \leq |Q|_R \leq \sum_{p\in E}\frac{\rho}{\Delta}(2r_0)^{|A|}(R+r_0)^{|E|} \leq 2^{|E|}\frac{\rho}{\Delta}(2r_0)^{|A|}\Big(\frac{5R}{4}\Big)^{|E|} \leq (2r_0)^{|A|}.$$

Consequently, we get

$$|f|_r \geq |P|_r - |Q|_r \geq (r-r_0)^{|A|} - (2r_0)^{|A|} \geq \Big(\frac{3r}{4}\Big)^{|A|} - \Big(\frac{r}{2}\Big)^{|A|} \geq \Big(\frac{r}{4}\Big)^{|A|},$$

and

$$|f|_R \leq |P|_R + |Q|_R \leq (R+r_0)^{|A|} + (2r_0)^{|A|} \leq \Big(\frac{5R}{4}\Big)^{|A|} + \Big(\frac{R}{2}\Big)^{|A|} \leq (2R)^{|A|}.$$

The conclusion follows. □

Proof of Proposition 7.1.

Put $t = (3+\delta)/\epsilon$ and denote by $\big(\varphi(k)\big)_{k\geq 1}$ the strictly increasing sequence of positive integers defined recursively by

$$\varphi(1) = 1 \quad \text{and} \quad \varphi(k+1) = \big[2^{t\varphi(k)}\big] \quad \text{for } k \geq 1.$$

Define

$$\alpha_k = \sum_{j=1}^{k} 2^{-\varphi(k)}, \quad \text{for } k \geq 1, \quad \text{and} \quad \alpha = \lim_{k\to\infty} \alpha_k.$$

Finally, choose a basis $\{\omega_1, \omega_2, \omega_3\}$ of a cubic extension of $\mathbf{Q}$, with $\max_{1\leq j\leq 3} |\omega_j| \leq 1$. We will show that the set

$$\Sigma = \{\, (0, \omega_1)\,,\, (1, \omega_2)\,,\, (\alpha, \omega_3)\,\}$$

has the required properties for all integers N of the form

$$N = 1 + \Big[2^{\varphi(k)/\epsilon}\Big]$$

where k is a sufficiently large positive integer. Since $\varphi(k+1)/\varphi(k)$ tends to infinity with k, we may already assume $N < 2^{\varphi(k+1)}$.

(i) Put $\rho = \exp\big(-3N^{3+\delta}\big)$. For any non-zero element $p = (\xi, \eta)$ of $\Sigma(N)$, we have $\eta \neq 0$ and Liouville's inequality gives $|\eta| \gg N^{-2}$ (with an implied constant independent of N). So, if k is larger than some constant k_0, we have $\Sigma(N) \cap B_2(0, \rho) = \{0\}$.

Let V be a subspace of $\mathbf{C}^2$ of dimension 1. We will now show that, if k is larger than some constant k_1 independent of V, then the set $\Sigma(N) \cap \big(V + B_2(0, \rho)\big)$ has rank at most 1. Since Σ has rank 3, this will imply $\mu\big(\Sigma(N), \rho\big) = 3/2$ for $k > \max\{k_0, k_1\}$.

Assume on the contrary that $V + B_2(0, \rho)$ contains two $\mathbf{Q}$-linearly independent elements p_1 and p_2 of $\Sigma(N)$. For $j = 1, 2$, write

$$p_j = (\xi_j, \eta_j) = (\, n_{j,2} + n_{j,3}\alpha\,,\; n_{j,1}\omega_1 + n_{j,2}\omega_2 + n_{j,3}\omega_3\,)$$

where $\mathbf{n}_j = (n_{j,1}, n_{j,2}, n_{j,3})$ is a triple of integers with $\|\mathbf{n}_j\| \leq N$. By hypothesis, $\mathbf{n}_1$ and $\mathbf{n}_2$ are $\mathbf{Q}$-linearly independent, so the same is true of η_1 and η_2. Choose an equation of V of the form $ax + by = 0$ with $\max\{|a|, |b|\} = 1$. Since $p_1, p_2 \in V + B_2(0, \rho)$, we find $|a\xi_j + b\eta_j| \leq 2\rho$ for $j = 1, 2$, and therefore

$$\begin{aligned}|\xi_1\eta_2 - \xi_2\eta_1| &= \max\big\{\, \big|(a\xi_1 + b\eta_1)\eta_2 - (a\xi_2 + b\eta_2)\eta_1\big|\,,\, \big|\xi_1(a\xi_2 + b\eta_2) - \xi_2(a\xi_1 + b\eta_1)\big|\,\big\}\\ &\ll N\rho.\end{aligned}$$

Put also $\bar{\xi}_j = n_{j,2} + n_{j,3}\alpha_{k+1}$ for $j = 1, 2$. Since α_{k+1} is a rational number with denominator $2^{\varphi(k+1)} > N$, we have $\bar{\xi}_j = 0$ if and only if $n_{j,2} = n_{j,3} = 0$. Since $\mathbf{n}_1$ and $\mathbf{n}_2$ are linearly independent over $\mathbf{Q}$, this implies that at least one of the numbers $\bar{\xi}_1$ or $\bar{\xi}_2$ is non-zero. Since η_1 and η_2 are also linearly independent over $\mathbf{Q}$, we deduce that the number $\bar{\xi}_1\eta_2 - \bar{\xi}_2\eta_1$ is non-zero and so, using Liouville's inequality, we get

$$\left|\bar{\xi}_1\eta_2 - \bar{\xi}_2\eta_1\right| \gg N^{-4}2^{-3\varphi(k+1)}.$$

Since $|\alpha - \alpha_{k+1}| \leq 2^{-\varphi(k+2)+1}$, we also find

$$\left|(\xi_1 - \bar{\xi}_1)\eta_2 - (\xi_2 - \bar{\xi}_2)\eta_1\right| = \left|\alpha - \alpha_{k+1}\right| \left|n_{1,3}\eta_2 - n_{2,3}\eta_1\right| \ll N^2 2^{-\varphi(k+2)}.$$

Combining these three inequalities, we get

$$N^{-4}2^{-3\varphi(k+1)} \ll N\rho + N^2 2^{-\varphi(k+2)}. \tag{7.1}$$

If k is sufficiently large, we have $\varphi(k+2) \geq 10\varphi(k+1)$. Since $N < 2^{\varphi(k+1)}$, we then get

$$N^2 2^{-\varphi(k+2)} \leq N^2 2^{-10\varphi(k+1)} \leq N^{-5}2^{-3\varphi(k+1)}.$$

Since $\varphi(k+1) \leq 2^{t\varphi(k)} \leq N^{t\epsilon}$, we also have, for sufficiently large k,

$$N\rho = N\exp\left(-3N^{t\epsilon}\right) \leq N\exp\left(-3\varphi(k+1)\right) \leq N^{-5}2^{-3\varphi(k+1)}.$$

So, the inequality (7.1) is satisfied only for finitely many values of k.

(ii) To construct the polynomial f, we apply Lemma 7.3 with $E = \Sigma(N)$, $r_0 = 3N$ and

$$A = \left\{\, n2^{-\varphi(k)}\,;\, n \in \mathbf{Z},\ |n| \leq N2^{\varphi(k)+1} \,\right\}.$$

The condition $|A| \leq |E|$ is satisfied since $\epsilon \leq 1$,

$$|A| = N2^{\varphi(k)+2} + 1 \leq 5N2^{\varphi(k)} \leq 5N^{1+\epsilon} \quad \text{and} \quad |E| = (2N+1)^3.$$

Let ρ, Δ and $f\colon \mathbf{C}^2 \to \mathbf{C}$ be as in Lemma 7.3. It remains to show that, for sufficiently large k, we have $(3R)^{|E|}\rho \leq \Delta$, with $R = \exp(N^\delta/12)$.

For any $p \in \Sigma(N)$, we can write $x(p) = n_2 + n_3\alpha$ with integers n_2, n_3 of absolute values $\leq N$. We have $n_2 + n_3\alpha_k \in A$ since $2^{\varphi(k)}(n_2 + n_3\alpha_k)$ is an integer of absolute value $\leq N2^{\varphi(k)+1}$. Moreover, we find

$$\left|x(p) - (n_2 + n_3\alpha_k)\right| = |n_3|\,|\alpha - \alpha_k| \leq N2^{-\varphi(k+1)+1}.$$

This shows $\rho \leq N2^{-\varphi(k+1)+1}$.

On the other hand, we have $y(E) = \Sigma''(N)$ where $\Sigma'' = \{\omega_1, \omega_2, \omega_3\}$, and Liouville's inequality gives $\Sigma''(N) \cap B_1(0, e^{-N}) = \{0\}$ if N is sufficiently large. Then, Lemma 5.2

gives $\Delta \geq (cN)^{|E|}$ with a constant $c > 0$ depending only on Σ''. So, for sufficiently large values of k, we have $\Delta \geq 2N$, and thus $\Delta/\rho \geq 2^{\varphi(k+1)}$.

Since $\varphi(k+1) \geq 2^{t\varphi(k)} - 1 \geq (N-1)^{t\epsilon} - 1$ and $t\epsilon = 3 + \delta$, we obtain, for sufficiently large k,

$$\begin{aligned}\log\left(\frac{\Delta}{\rho}\right) &\geq \varphi(k+1)\log(2) \geq \big((N-1)^{3+\delta} - 1\big)\log(2)\\ &\geq (2N+1)^3\left(\frac{N^\delta}{12} + \log(3)\right) = |E|\log(3R). \qquad \square\end{aligned}$$

For the proof of Proposition 7.2, we need the following special case of the proposition in the appendix of [8].

Lemma 7.4. *Let $\big(\psi(N)\big)_{N\geq 1}$ be a sequence of positive real numbers. There exist real numbers α, β which are algebraically independent over $\mathbf{Q}$ and have the following additional property. For any integer $N \geq 1$, there exist integers n_1, n_2, n_3 not all zero, with absolute values $\leq N$, such that $|n_1 + n_2\alpha + n_3\beta| \leq \psi(N)$.*

Note that we will simply need that $1, \alpha, \beta$ are linearly independent over $\bar{\mathbf{Q}}$. The original Theorem XIV in Chapter V of [2] does not quite fulfill our needs since it provides the existence of real numbers α, β satisfying the second condition of the lemma with the weaker property that $1, \alpha, \beta$ are linearly independent over $\mathbf{Q}$.

Proof of Proposition 7.2.

Let α and β be real numbers satisfying the conditions of Lemma 7.4 for the sequence $\big(\psi(N)\big)_{N\geq 1}$ given by

$$\psi(N) = \exp\Big(-40\big(100(N+1)\big)^{(5+\delta)/\epsilon}\Big)$$

for any integer $N \geq 1$. We choose $\mathbf{Q}$-linearly independent algebraic numbers $\omega_1, \dots, \omega_5$. Then, the set

$$\Sigma = \big\{\,(1,\omega_1)\,,\,(\alpha,\omega_2)\,,\,(\beta,\omega_3)\,,\,(0,\omega_4)\,,\,(0,\omega_5)\,\big\}.$$

generates a subgroup Γ of $\mathbf{C}^2$ of rank 5. Moreover, the projection $y\colon \mathbf{C}^2 \to \mathbf{C}$ is injective on Γ. We claim that Σ and Γ have all the required properties for a suitable choice of $c > 0$.

Since $\mathrm{rank}(\Gamma) = 5$, we have $\mu(\Gamma) \leq 5/2$. Suppose $\mu(\Gamma) < 5/2$. Then, there exists a subspace V of $\mathbf{C}^2$ of dimension 1 such that $\mathrm{rank}(\Gamma \cap V) \geq 3$. This subspace therefore contains a non-zero element of any subgroup of Γ of rank 3. In particular, V contains a non-zero element of the subgroup of Γ generated by $(1,\omega_1)$, $(0,\omega_4)$ and $(0,\omega_5)$. Thus, V is defined over $\bar{\mathbf{Q}}$. Similarly, V contains a non-zero element (γ,ω) of the subgroup generated by (α,ω_2), (β,ω_3) and $(0,\omega_4)$. Since $\omega \neq 0$ and that V is defined over $\bar{\mathbf{Q}}$, this implies $\gamma \in \bar{\mathbf{Q}}$, thus $\gamma = 0$ because $1, \alpha, \beta$ are linearly independent over $\bar{\mathbf{Q}}$. So, we have $V = \{0\} \times \mathbf{C}$. This is impossible since $\Gamma \cap \{0\} \times \mathbf{C}$ has rank 2. Therefore, we have $\mu(\Gamma) = 5/2$.

Fix an integer $N \geq 100^{1/\epsilon}$ and put $M = \left[100^{-1}N^{\epsilon}\right]$. We consider the sets $E = \Sigma(N)$ and

$$A = \left\{ k_1 + k_2\alpha + k_3\beta\,;\; k_1, k_2, k_3 \in \mathbf{Z},\; \max|k_j| \leq 2N,\; \min|k_j| \leq M \right\}.$$

If N is sufficiently large, the cardinality of A satisfies

$$|A| \leq 3(4N+1)^2(2M+1) \leq 100MN^2 \leq N^{2+\epsilon},$$

which is smaller than $|E| = (2N+1)^5$. We may then apply Lemma 7.3 to this choice of A and E, with $r_0 = cN$ for a suitable constant $c > 0$ independent of N. Let ρ and Δ be the corresponding numbers, as defined in Lemma 7.3, and let $f\colon \mathbf{C}^2 \to \mathbf{C}$ be a polynomial map satisfying the conclusion of this lemma. It remains to show that, for sufficiently large N, we also have $(3R)^{|E|}\rho \leq \Delta$, with $R = \exp\left(N^{\delta}\right)$.

Since $M \geq 1$, we know, by the choice of α and β, that there exist integers m_1, m_2, m_3 not all zero with $\max|m_j| \leq M$ such that

$$|m_1 + m_2\alpha + m_3\beta| \leq \psi(M).$$

Now, choose any point $p \in E$. Since $x(p) = n_1 + n_2\alpha + n_3\beta$ for integers n_1, n_2, n_3 with $\max|n_j| \leq N$, there exists an integer ℓ with $|\ell| \leq N$ such that $x(p) - \ell(m_1 + m_2\alpha + m_3\beta)$ belongs to A. This shows that $\rho \leq N\psi(M)$.

On the other hand, since $y(E) = \Sigma''(N)$ with $\Sigma'' = \{\omega_1, \ldots, \omega_5\}$, Lemma 5.2 together with Liouville's inequality show that, for sufficiently large N, we have $\Delta \geq N$, and we get

$$(3R)^{|E|}\frac{\rho}{\Delta} \leq (3R)^{(2N+1)^5}\psi(M) \leq \exp\left((N^{\delta} + \log 3)(2N+1)^5 - 40N^{5+\delta}\right) \leq 1. \qquad \square$$

8. An application to auxiliary functions

In this section, we analyze two constructions of auxiliary functions coming out of Theorem 3.1 of [13]. For the convenience of the reader, we state below the special case of this theorem for functions of one complex variable and the choice of parameter $R = er$.

Theorem 8.1. (M. Waldschmidt) *Let L be a positive integer, let Δ, r, U be positive real numbers, and let $\varphi_1, \ldots, \varphi_L\colon B_1(0, er) \to \mathbf{C}$ be continuous functions that are holomorphic in the interior of $B_1(0, er)$. Assume*

$$3 \leq U, \quad \Delta \leq U, \quad \sum_{\lambda=1}^{L} |\varphi_\lambda|_{er} \leq e^U \quad \text{and} \quad (8U)^2 \leq \Delta L.$$

Then, there exist integers $p_1, \dots, p_L$ not all zero, in absolute value at most e^{Δ}, such that the function $f = \sum_{\lambda=1}^{L} p_\lambda \varphi_\lambda$ satisfies $|f|_r \le e^{-U}$.

The *height* of a polynomial P, denoted $H(P)$, is defined here as the maximum of the absolute values of its coefficients. Our first result provides an upper bound for the height of polynomials taking small values on points of a subgroup of rank 2 of a one-dimensional analytic subgroup of $\mathbf{G}_{\mathbf{m}}^2(\mathbf{C}) = (\mathbf{C}^\times)^2$.

Proposition 8.2. *Let $a, \lambda, \nu \in \mathbf{C}$ with $a \neq 0$, $\lambda \notin \mathbf{Q}$ and $\nu \notin \mathbf{Q}$. Put*

$$\alpha_{1,1} = e^{a}, \quad \alpha_{1,2} = e^{a\lambda}, \quad \alpha_{2,1} = e^{a\nu}, \quad \alpha_{2,2} = e^{a\lambda\nu}.$$

There exists a constant $c \ge 1$ with the following property. For any integer $N \ge c$ satisfying

$$\begin{aligned} &\min\big\{ |m + n\lambda| \,;\, m, n \in \mathbf{Z},\ 0 < \max\{|m|, |n|\} \le N \big\} > e^{-N}, \\ &\min\big\{ |m + n\nu| \,;\, m, n \in \mathbf{Z},\ 0 < \max\{|m|, |n|\} \le N \big\} > e^{-N}, \end{aligned} \tag{8.1}$$

and for any polynomial $P(z_1, z_2) \in \mathbf{C}[z_1, z_2]$ of degree $\le N/c$, we have

$$H(P) \le c^{N^2} \max\big\{ \big| P(\alpha_{1,1}^{m_1}\alpha_{2,1}^{m_2},\ \alpha_{1,2}^{m_1}\alpha_{2,2}^{m_2}) \big| \,;\, m_1, m_2 \in \mathbf{Z} \text{ and } |m_1|,\ |m_2| \le N \big\}.$$

Proof. We apply Corollary 1.5 with

$$\Sigma = \big\{\, (2\pi i, 0)\,,\ (0, 2\pi i)\,,\ (a, a\lambda)\,,\ (a\nu, a\lambda\nu) \,\big\}.$$

and the exact sequence of linear maps $0 \to \mathbf{C} \xrightarrow{\iota} \mathbf{C}^2 \xrightarrow{\pi} \mathbf{C} \to 0$ given by $\iota(z) = (az, a\lambda z)$ for $z \in \mathbf{C}$ and by $\pi(z_1, z_2) = (z_2 - \lambda z_1)/(2\pi i)$ for $(z_1, z_2) \in \mathbf{C}^2$. We use $\Sigma_1 = \{1, \nu\} = \iota^{-1}(\Sigma)$ and $\Sigma_2 = \{1, -\lambda\} \subset \pi(\Sigma)$. Let $c_1 \ge 1$ be the corresponding constant, and let $N \ge 1$ be an integer satisfying (8.1). In the notation of the corollary, we have $\mu_N = 2$ and so $L_N = N^2$. Thus, for any real numbers r and R with $R \ge r \ge c_1 N$ and any entire function $f\colon \mathbf{C}^2 \to \mathbf{C}$, we have

$$|f|_r \le \left(\frac{c_1 r}{N}\right)^{N^2} \max_{\gamma \in \Sigma(N)} |f(\gamma)| + \left(\frac{c_1 r}{R}\right)^{N^2} |f|_R.$$

Choose $r = c_1 N$, $R = e c_1 r$ and take $f(z_1, z_2) = P(e^{z_1}, e^{z_2})$ where P is a polynomial. Assuming $N \ge c$ and $\deg(P) \le N/c$ with $c = 2ec_1^2$, we get (with the help of Lemma 2.4)

$$|f|_r \ge |P|_{e^{r-\pi}} \ge |P|_1 \quad \text{and} \quad |f|_R \le |P|_{e^R} \le (e^R)^{\deg(P)} |P|_1 \le e^{N^2/2} |P|_1,$$

thus

$$|P|_1 \le c_1^{2N^2} \max_{\gamma \in \Sigma(N)} |f(\gamma)| + e^{-N^2/2} |P|_1.$$

Since $H(P) \le |P|_1$, this gives $H(P) \le c^{N^2} \max_{\gamma \in \Sigma(N)} |f(\gamma)|$. $\square$

Corollary 8.3. *Let $\lambda \in \mathbf{C} \setminus \mathbf{Q}$. There exists a constant $c_0 \geq 1$ with the following properties:*

(i) For any real number c and any integer N with $N \geq c \geq c_0$, there is a non-zero polynomial $P \in \mathbf{Z}[X_1, X_2]$ of degree $\leq N$ and height $\leq \exp(c^{-3/2}N^2)$ such that the entire function $f(z) = P(e^z, e^{\lambda z})$ satisfies $|f|_{c^{-2}N} \leq \exp(-c^{-1}N^2)$.

(ii) For any real number $c \geq c_0$ and any integer $N \geq 1$ with

$$\min\big\{ |m + n\lambda| \,;\, m, n \in \mathbf{Z},\ 0 < \max\{|m|, |n|\} \leq cN + 1 \big\} \geq e^{-cN}$$

and for any non-zero polynomial $P \in \mathbf{Z}[X_1, X_2]$ of degree $\leq N$, the entire function $f(z) = P(e^z, e^{\lambda z})$ satisfies $|f|_{cN} \geq \exp(-c^3N^2)$.

Proof. A quick computation shows that there exists a constant $c_0 \geq 3$ such that, for any real number $c \geq c_0$ and any integer $N \geq c$, all the hypotheses of Theorem 8.1 are satisfied for the $L := N(N+1)/2$ functions $e^{kz}e^{\ell\lambda z}$ with $k, \ell \in \mathbf{N}$ and $k + \ell \leq N$, and the choice of parameters $\Delta = c^{-1/2}U = c^{-3/2}N^2$ and $r = c^{-2}N$. Then, the conclusion of the theorem is precisely the assertion (i) above.

Choose the above constant $c_0 \geq 3$ large enough so that moreover Proposition 8.2 holds with $a = 1/2$, $\nu = i$ and the given λ for any constant $c \geq c_0$. For this choice of c_0, let c, N, P and f as in the assertion (ii) above, and let $M = [cN] + 1$. Then, we have $M \geq c$ and the condition (8.1) of Proposition 8.2 holds with N replaced by M. Since $\deg(P) \leq M/c$, we conclude that there exist $m, n \in \mathbf{Z}$ with $|m|, |n| \leq M$ such that $H(P) \leq c^{M^2}|f((m + ni)/2)|$. Since $H(P) \geq 1$ and $M \leq \sqrt{2}cN$, this implies

$$|f|_{cN} \geq |f((m + ni)/2)| \geq c^{-M^2} \geq \exp(-2c^2 \log(c)N^2) \geq \exp(-c^3N^2). \qquad \square$$

The part (ii) of the above corollary is interesting in relation with the four exponentials conjecture. If λ is the ratio of two non-zero logarithms of algebraic numbers, then lower bounds for linear forms in two logarithms going back to Gel'fond (see the introduction of [1]) show that all sufficiently large integers N satisfy the condition in (ii), independently of the choice of c. This justifies the statement $(*)$ made in the introduction.

The next result deals with a polynomial taking small values on points of a one-dimensional analytic subgroup of $(\mathbf{G_a} \times \mathbf{G_m})(\mathbf{C}) = \mathbf{C} \times \mathbf{C}^\times$, together with some of its derivatives along this analytic subgroup.

Proposition 8.4. *Let $y \in \mathbf{C}^\times$ and let $\alpha = e^y$. There exists a constant $c \geq 1$ with the following property. Let S_0, S_1, T_0, T_1 be positive integers with*

$$\max\Big\{ T_0T_1,\ T_0T_1 \log\Big(\frac{T_0T_1}{S_1}\Big),\ S_1T_1 \Big\} \leq c^{-1}S_0S_1.$$

Then, for any polynomial $P \in \mathbf{C}[X_0, X_1]$ with partial degree $< T_0$ in X_0 and partial degree $< T_1$ in X_1, we have

$$H(P) \le c^{S_0 S_1} \max_{\substack{0 \le k < S_0 \\ 0 \le m < S_1}} \left|(\mathcal{D}^k P)(my, \alpha^m)\right|$$

where $\mathcal{D} = (\partial/\partial X_0) + X_1(\partial/\partial X_1)$.

Proof. We apply Theorem 1 of [11] with $E(z) = P(z, e^z)$. In the notation of this theorem of R. Tijdeman, we have $n = T_1$ and, for $\nu = 0, \dots, n-1$, $m_\nu = T_0$ and $\alpha_\nu = \nu$, thus $m = T_0 T_1$. Similarly, we have $s = S_1$ and, for $\sigma = 0, \dots, s-1$, $r_\sigma = S_0$ and $\beta_\sigma = \sigma y$, thus $r = S_0 S_1$. Using Lemma 5.2, we find that the corresponding quantities a, a_1, a_2 and b, b_1, b_2 of Tijdeman's theorem satisfy

$$a \le T_1,\ a_1 \ge \frac{T_1}{c'},\ a_2 = 1 \quad \text{and} \quad \frac{S_1}{c'} \le b \le c' S_1,\ b_1 \ge \frac{S_1}{c'},\ b_2 \ge \frac{1}{c'},$$

for some constant $c' \ge 1$. In particular, the ratios $a/(a_1 a_2)$ and $b/(b_1 b_2)$ are bounded from above. If c is large enough, the condition $r \ge 2m + 13ab$ is satisfied and the conclusion follows in the form $A \le c^{S_0 S_1} E$, using Tijdeman's notation. ∎

Remark. When $(\mathcal{D}^k P)(my, \alpha^m) = 0$ for $k = 0, \dots, S_0 - 1$ and $m = 0, \dots, S_1 - 1$, the conclusion of the proposition is $P = 0$. Since these represent $S_0 S_1$ linear conditions imposed on the $T_0 T_1$ coefficients of P, this conclusion certainly requires $T_0 T_1 \le S_0 S_1$. Moreover, if α is a d-th root of unity, this conclusion also requires $T_1 \le d S_0$ (look for a polynomial P in X_1 alone). So, in general, the proposition requires at least a condition of the type $\max\{T_0 T_1, S_1 T_1\} \ll S_0 S_1$. However, if α is not "close" to a root of unity, it might be possible to weaken the conditions imposed on the parameters.

Corollary 8.5. *Let s_1, t_0, t_1 and u be positive real numbers.*

(i) If $\max\{1, t_0, s_1 + t_1\} < u < (1 + t_0 + t_1)/2$ then, for any sufficiently large integer N, there exists a non-zero polynomial $P \in \mathbf{Z}[X_0, X_1]$ with $H(P) \le e^N$ and $\deg_{X_j} P < N^{t_j}$ for $j = 1, 2$, such that the function $f(z) = P(z, e^z)$ satisfies $|f|_r \le \exp(-N^u)$ with $r = N^{s_1}$.

(ii) If $\max\{s_1 + t_1,\ t_0 + t_1\} < u$ then, for any sufficiently large integer N and any non-zero polynomial $P \in \mathbf{Z}[X_0, X_1]$ with $\deg_{X_j} P < N^{t_j}$ for $j = 1, 2$, the function $f(z) = P(z, e^z)$ satisfies $|f|_r \ge \exp(-N^u)$ with $r = N^{s_1}$.

Proof. Let N be a positive integer. Put $\Delta = N$, $r = N^{s_1}$, $T_j = [N^{t_j}]$ for $j = 0, 1$, and $U = N^u$. In the case (i), the above choice of parameters and the $L := T_0 T_1$ functions $z^k e^{\ell z}$ with $0 \le k < T_0$ and $0 \le \ell < T_1$ satisfy the hypotheses of Theorem 8.1 for sufficiently large N, and the conclusion follows. In the case (ii), we choose a real number s_0 such that $t_1 < s_0$

and $t_0+t_1 < s_0+s_1 < u$ and we put $S_j = [N^{s_j}]$ for $j = 0, 1$. We also take $y = 1$ and $\alpha = e$. Then, for sufficiently large N, the hypotheses of Proposition 8.4 are satisfied. We deduce that, if N is sufficiently large and if P is any polynomial as in (ii), there exist integers k and ℓ with $0 \le k < S_0$ and $0 \le m < S_1$ such that $H(P) \le \exp(N^u/2)\, |(\mathcal{D}^k P)(m, e^m)|$. We may also assume $k! \le \exp(N^u/2)$. So, for the function $f(z) = P(z, e^z)$, we get

$$|f|_r \ge \frac{1}{k!}\Big|\frac{d^k f}{dz^k}(m)\Big| = \frac{1}{k!}\big|(\mathcal{D}^k P)(m, e^m)\big| \ge \exp(-N^u)H(P) \ge \exp(-N^u). \qquad \square$$

9. An arithmetic criterion for the values of the exponential map of an elliptic curve

Let E be an elliptic curve defined over a number field K. Assume E embedded in $\mathbf{P}^2$ by a Weierstrass model

$$Y_0 Y_2^2 = 4Y_1^3 - g_2 Y_0^2 Y_1 - g_3 Y_0^3$$

with $g_2, g_3 \in K$. Let $\wp(z)$ be the corresponding Weierstrass function, Ω the period lattice of $\wp(z)$, and $\exp_E \colon \mathbf{C} \to E(\mathbf{C})$ the exponential map of $E(\mathbf{C})$ given by

$$\exp_E(z) = \begin{cases} (1 : \wp(z) : \wp'(z)) & \text{if } z \notin \Omega \\ (0 : 0 : 1) & \text{if } z \in \Omega. \end{cases}$$

We choose a constant $c_1 > 1$ and, for $j = 0, 1, 2$, we put

$$\mathcal{V}_j = \big\{(y_0 \colon y_1 \colon y_2) \in E(\mathbf{C})\,;\, \|(y_0, y_1, y_2)\| < c_1 |y_j|\big\}.$$

These three open sets cover $E(\mathbf{C})$. We also define three differential operators on $\mathbf{C}[X, \mathbf{Y}] = \mathbf{C}[X, Y_0, Y_1, Y_2]$:

$$\begin{aligned}
\mathcal{D}_0 &= Y_0 \frac{\partial}{\partial X} + Y_0 Y_2 \frac{\partial}{\partial Y_1} + \Big(6Y_1^2 - \frac{1}{2} g_2 Y_0^2\Big) \frac{\partial}{\partial Y_2},\\
\mathcal{D}_1 &= Y_1 \frac{\partial}{\partial X} - Y_0 Y_2 \frac{\partial}{\partial Y_0} + \Big(\frac{1}{2} Y_2^2 + g_2 Y_0 Y_1 + \frac{3}{2} g_3 Y_0^2\Big) \frac{\partial}{\partial Y_2},\\
\mathcal{D}_2 &= Y_2 \frac{\partial}{\partial X} - \Big(6Y_1^2 - \frac{1}{2} g_2 Y_0^2\Big) \frac{\partial}{\partial Y_0} - \Big(\frac{1}{2} Y_2^2 + g_2 Y_0 Y_1 + \frac{3}{2} g_3 Y_0^2\Big) \frac{\partial}{\partial Y_1}.
\end{aligned}$$

Note that, if $P \in \mathbf{C}[X, \mathbf{Y}]$ is a non-zero polynomial which is homogeneous of degree T in $\mathbf{Y}$, then, for $j = 0, 1, 2$, its derivative $\mathcal{D}_j P$ is zero or homogeneous of degree $T + 1$ in $\mathbf{Y}$. Finally, if $(x, \alpha) \in \mathbf{C} \times \mathbf{P}^2(\mathbf{C})$ and if $P \in \mathbf{C}[X, \mathbf{Y}]$ is homogeneous in $\mathbf{Y}$, we define $|P(x, \alpha)|$ to be $|P(x, \mathbf{y})|$ for any representative $\mathbf{y} \in \mathbf{C}^3$ of α with $\|\mathbf{y}\| = 1$. With this last convention and the notion of height of a polynomial defined in §8, we can now state the main result of this section.

Theorem 9.1. *Let $(x, \alpha) \in \mathbf{C} \times E(\mathbf{C})$, and let s_0, s_1, t_0, t_1, u be positive real numbers satisfying*

$$\max\{1, t_0, 3t_1\} < \min\{s_0, 3s_1\} \quad \textit{and} \quad \max\{s_0, 2s_1 + t_1\} < u < \frac{1}{2}(1 + t_0 + t_1). \tag{9.1}$$

Then the following conditions are equivalent:

(a) there exists an integer $d \geq 1$ such that $d\alpha = \exp_E(dx)$;

(b) for any sufficiently large positive integer N, there exists a polynomial $Q_N \in \mathbf{Z}[X, \mathbf{Y}]$ which is homogeneous in $\mathbf{Y}$, does not vanish identically on $\mathbf{C} \times E(\mathbf{C})$, has partial degree $\leq N^{t_0}$ in X, partial degree $\leq N^{t_1}$ in $\mathbf{Y}$, height $\leq e^N$, and satisfies

$$\left|(\mathcal{D}_j^\tau Q_N)(mx, m\alpha)\right| \leq \exp(-N^u)$$

for any choice of integers j, τ, m with $0 \leq j \leq 2$, $0 \leq \tau \leq N^{s_0}$, $0 \leq m \leq N^{s_1}$, such that $m\alpha \in \mathcal{V}_j$.

Note that, for any point (t_0, t_1) in the interior of the triangle with vertices $(2/3, 1/3)$, $(1, 0)$ and $(3/2, 1/2)$, their exist real numbers s_0, s_1 and u satisfying (9.1). In particular, for any s and u with $1 < s < u < 7/6$, these conditions are fulfilled for the choice of $s_0 = 3s_1 = s$ and $t_0 = 3t_1 = 1$.

The strategy of the proof is the same as that of Theorem 1 of [9]. We will prove each of the implications $(a) \Rightarrow (b)$ and $(b) \Rightarrow (a)$ as separate propositions with conditions on the parameters that are less restrictive than (9.1). We start by introducing further notation and proving two lemmas.

Let $\sigma(z)$ be the Weierstrass σ-function associated to the lattice Ω and let $\varphi\colon \mathbf{C} \to \mathbf{C}^3$ be the entire function given by

$$\varphi(z) = (\varphi_0(z), \varphi_1(z), \varphi_2(z)) = (\sigma(z)^3, \sigma(z)^3\wp(z), \sigma(z)^3\wp'(z))$$

for any $z \in \mathbf{C} \setminus \Omega$. We know that there exists a constant $c_2 > 0$ such that

$$\exp\big(-c_2 \max\{1, |z|\}^2\big) \leq \|\varphi(z)\| \leq \exp\big(c_2 \max\{1, |z|\}^2\big)$$

for any $z \in \mathbf{C}$. For $j = 0, 1, 2$, we define an holomorphic function $\psi_j\colon \varphi^{-1}(\mathcal{V}_j) \to \mathbf{C}^3$ by $\psi_j(z) = \varphi_j(z)^{-1}\varphi(z)$ if $\varphi(z) \in \mathcal{V}_j$.

Lemma 9.2. *Let $(x, \alpha) \in \mathbf{C} \times E(\mathbf{C})$, and let $P \in \mathbf{C}[X, \mathbf{Y}]$ be homogeneous of degree T in $\mathbf{Y}$. Choose $j \in \{0, 1, 2\}$ such that $\alpha \in \mathcal{V}_j$, and $\lambda \in \mathbf{C}$ such that $\exp_E(\lambda) = \alpha$. Then, for any integer $\tau \geq 0$, we have*

$$\left|(\mathcal{D}_j^\tau P)(x, \alpha)\right| \leq \left|\left(\frac{d}{dz}\right)^\tau P(x + z, \psi_j(\lambda + z))\right|_{z=0} \leq c_1^{T+\tau}\left|(\mathcal{D}_j^\tau P)(x, \alpha)\right|.$$

Proof. Since $\alpha \in \mathcal{V}_j$, we have $\exp_E(\lambda+z) \in \mathcal{V}_j$ for any z in a sufficiently small neighborhood of 0 in $\mathbf{C}$. The differential operator $\mathcal{D}_j$ is defined so that, for those values of z, we have

$$\frac{d}{dz} P(x+z, \psi_j(\lambda+z)) = (\mathcal{D}_j P)(x+z, \psi_j(\lambda+z))$$

(see the formulas on page 81, §4 (c) of [15]). Therefore, for any integer $\tau \geq 0$, we find

$$\left| \left(\frac{d}{dz}\right)^{\tau} P(x+z, \psi_j(\lambda+z)) \right|_{z=0} = \left| (\mathcal{D}_j^{\tau} P)(x, \psi_j(\lambda)) \right|.$$

The conclusion follows using the fact that $\mathcal{D}_j^{\tau} P$ is homogeneous of degree $T+\tau$ in $\mathbf{Y}$ and that $1 \leq \|\psi_j(\lambda)\| \leq c_1$ since $\alpha \in \mathcal{V}_j$. ∎

Lemma 9.3. *Let Δ, r, T_0, T_1, U be real numbers ≥ 3. Assume*

$$\log\big((T_0+1)(T_1+1)\big) + \Delta + T_0 \log(er) + c_2(2er)^2 T_1 \leq U \quad \text{and} \quad (8U)^2 \leq \Delta T_0 T_1.$$

Then, for any $a \in B_1(0,r)$, there exists a non-zero polynomial $P \in \mathbf{Z}[X, Y_0, Y_1]$ which has partial degree $\leq T_0$ in X, is homogeneous in (Y_0, Y_1) of degree $\leq T_1$, has height $\leq e^{\Delta}$, and for which the function $f(z) = P(z, \varphi_0(z+a), \varphi_1(z+a))$ satisfies $|f|_r \leq e^{-U}$.

Proof. Put $L = ([T_0]+1)([T_1]+1)$. The result follows from Theorem 8.1 applied to the L functions $z^k \varphi_0(z+a)^{\ell} \varphi_1(z+a)^{[T_1]-\ell}$ with $k = 0, \ldots, [T_0]$ and $\ell = 0, \ldots, [T_1]$. Indeed, for any function θ among these, we have $L|\theta|_{er} \leq L(er)^{T_0} \exp\big(c_2(r+er)^2 T_1\big) \leq e^{U}$. ∎

Proposition 9.4. *Let $(x, \alpha) \in \mathbf{C} \times E(\mathbf{C})$, and let s_0, s_1, t_0, t_1, u be positive real numbers satisfying*

$$\max\{1, s_0, t_0, 2s_1 + t_1\} < u < \frac{1}{2}(1 + t_0 + t_1). \tag{9.2}$$

Assume that there is an integer $d \geq 1$ such that $d\alpha = \exp_E(dx)$. Then, there exists a sequence of polynomials $(Q_N)_{N \geq N_0}$ satisfying the condition (b) of Theorem 9.1 for these values of s_0, s_1, t_0, t_1, u.

Proof. Choose a set $S \subset \Omega$ representing the d^2 classes of Ω modulo $d\Omega$. By (9.2), the conditions of Lemma 9.3 are satisfied for any sufficiently large positive integer N with

$$\Delta = \frac{N}{2d^2}, \quad r = (|x|+1)N^{s_1}, \quad T_0 = \frac{N^{t_0}}{d^2}, \quad T_1 = \frac{N^{t_1}}{d^2} \quad \text{and} \quad U = (d^2+1)N^u.$$

Fix such an integer N, with $S \subset B_1(0, dr)$. For each $\omega \in S$, we choose a polynomial $P_{N,\omega}$ verifying the conclusion of Lemma 9.3 with $a = \omega/d$, and we form the product

$$Q_N = \prod_{\omega \in S} P_{N,\omega}.$$

This is a non-zero element of $\mathbf{Z}[X, Y_0, Y_1]$ which is homogeneous in (Y_0, Y_1). So, it does not vanish identically on $\mathbf{C} \times E(\mathbf{C})$. It verifies also

$$\deg_X(Q_N) \le d^2 T_0 = N^{t_0}, \quad \deg_{\mathbf{Y}}(Q_N) \le d^2 T_1 = N^{t_1},$$

and

$$H(Q_N) \le \Big((T_0+1)(T_1+1)e^{\Delta} \Big)^{d^2} \le e^N$$

provided that N is sufficiently large.

At this point, we note that there is a constant $\delta > 0$ such that, for any $j = 0, 1, 2$, any $u \in \mathbf{C}$ with $\exp_E(u) \in \mathcal{V}_j$, and any $z \in B_1(0, \delta)$, we have $\varphi_j(u+z) \neq 0$ and $\|\psi_j(u+z)\| < 2c_1$. This follows from the fact that, for the complex topology, the closure of $\mathcal{V}_j$ in $E(\mathbf{C})$ is the compact subset $\overline{\mathcal{V}_j}$ of $E(\mathbf{C})$ consisting of all points $(y_0 \colon y_1 \colon y_2)$ of $E(\mathbf{C})$ with $\|(y_0, y_1, y_2)\| \le c_1 |y_j|$ and that this compact set is contained in the open subset $\mathcal{W}_j$ of $E(\mathbf{C})$ consisting of all points $(y_0 \colon y_1 \colon y_2)$ of $E(\mathbf{C})$ with $\|(y_0, y_1, y_2)\| < 2c_1|y_j|$. Therefore, there exists a neighborhood $\mathcal{U}$ of 0 in $E(\mathbf{C})$ such that $\overline{\mathcal{V}_j} + \mathcal{U} \subseteq \mathcal{W}_j$ for $j = 0, 1, 2$. Any $\delta > 0$ with $\exp_E(B_1(0,\delta)) \subseteq \mathcal{U}$ has the required property.

Let j, τ, m be integers with $0 \le \tau \le N^{s_0}$, $0 \le m \le N^{s_1}$ and $m\alpha \in \mathcal{V}_j$. It remains to show that, for sufficiently large N, we have $|(\mathcal{D}_j^\tau Q_N)(mx, m\alpha)| \le \exp(-N^u)$. To this end, we choose $\omega \in \Omega$ such that $m\alpha = \exp_E(mx + \omega/d)$. Using Lemma 9.2 and Cauchy's inequalities, we find

$$\begin{aligned} \big|(\mathcal{D}_j^\tau Q_N)(mx, m\alpha)\big| &\le \Big| \Big(\frac{d}{dz}\Big)^\tau Q_N\Big(mx + z, \psi_j\big(mx + \frac{\omega}{d} + z\big)\Big) \Big|_{z=0} \\ &\le \tau!\, \delta^{-\tau} \sup_{|z| \le \delta} \Big| Q_N\Big(mx + z, \psi_j\big(mx + \frac{\omega}{d} + z\big)\Big) \Big|. \end{aligned}$$

By the choice of δ, we have $\|\psi_j(mx + \omega/d + z)\| < 2c_1$ for any $z \in B_1(0,\delta)$, and so

$$\Big|\varphi_j\Big(mx + \frac{\omega}{d} + z\Big)\Big| \ge \frac{1}{2c_1} \Big\| \varphi\Big(mx + \frac{\omega}{d} + z\Big) \Big\| \ge \Big(2c_1 \exp\big(c_2(2r+\delta)^2\big) \Big)^{-1}.$$

Thus, if N is sufficiently large, we get

$$\begin{aligned} \big|(\mathcal{D}_j^\tau Q_N)(mx, m\alpha)\big| &\le \exp(N^u) \sup_{|z|\le\delta} \Big| Q_N\Big(mx + z, \varphi\big(mx + \frac{\omega}{d} + z\big)\Big) \Big| \\ &\le \exp(N^u) \sup_{|z|\le r} \Big| Q_N\Big(z, \varphi\big(z + \frac{\omega}{d}\big)\Big) \Big| \\ &\le \exp(N^u) \prod_{\omega' \in S} \sup_{|z|\le r} \Big| P_{N,\omega'}\Big(z, \varphi\big(z + \frac{\omega}{d}\big)\Big) \Big|. \end{aligned}$$

For $\omega' \in S$ with $\omega' \neq \omega$, we also find, for large N,

$$\sup_{|z|\le r} \Big| P_{N,\omega'}\Big(z, \varphi\big(z + \frac{\omega}{d}\big)\Big) \Big| \le (T_0+1)(T_1+1)e^{\Delta} r^{T_0} \exp(c_2(2r)^2 T_1) \le \exp(N^u)$$

while, for $\omega' = \omega$, the same quantity is $\le \exp\big(-(d^2+1)N^u\big)$. This gives the required inequality $|(\mathcal{D}_j^\tau Q_N)(mx, m\alpha)| \le \exp(-N^u)$. ∎

For the next proposition, dealing with the implication $(b) \Rightarrow (a)$ in Theorem 9.1, we need a third lemma.

Lemma 9.5. *Let k and m be integers with $1 \le k \le 2m$, let $\omega_1, \dots, \omega_k \in \mathbf{C}^m$ be linearly independent over $\mathbf{R}$ and let $\theta \in \mathbf{C}^m$. We put $\Sigma = \{\omega_1, \dots, \omega_k, \theta\}$. Then, there exists a constant $c > 0$ such that, for infinitely many integers $N \ge 1$, we have $\Sigma(N) \cap B_m(0, c/N) = \{0\}$.*

Proof. If Σ generates a discrete subgroup of $\mathbf{C}^m$, the above property is verified with $c = 1$ for any sufficiently large integer N. Otherwise, we can write $\theta = a_1\omega_1 + \cdots + a_k\omega_k$ with $a_1, \dots, a_k \in \mathbf{R}$ not all rational. Suppose $a_j \notin \mathbf{Q}$. By Lemma 4 of [9], there are infinitely many positive integers N such that

$$\min\Big\{|m + na_j|\,;\, m, n \in \mathbf{Z},\ 0 < \max\{|m|, |n|\} \le N\Big\} \ge \frac{1}{2N+2}.$$

These integers satisfy the required property for a suitable choice of constant $c > 0$. ∎

Proposition 9.6. *Let $(x, \alpha) \in \mathbf{C} \times E(\mathbf{C})$, and let s_0, s_1, t_0, t_1, u be positive real numbers such that*

$$\max\{1, t_0, 3t_1\} < \min\{s_0, 3s_1\} < u.$$

Suppose that $\alpha - \exp_E(x)$ is not a torsion point of $E(\mathbf{C})$. Then there is no sequence of polynomials $(Q_N)_{N \ge N_0}$ satisfying the condition (b) of Theorem 9.1 for these values of s_0, s_1, t_0, t_1, u.

Proof. Assume on the contrary that there exists such a sequence of polynomials $(Q_N)_{N \ge N_0}$. Let $\{\omega_1, \omega_2\}$ be a basis of Ω and let $\lambda \in \mathbf{C}$ such that $\exp_E(\lambda) = \alpha$. We will apply Corollary 1.6 with

$$\Sigma = \big\{\,(0, \omega_1),\ (0, \omega_2),\ (x, \lambda)\,\big\}, \quad \mathbf{w} = (1, 1), \quad s = 3,$$

using the linear map $\pi\colon \mathbf{C}^2 \to \mathbf{C}$ given by $\pi(z_1, z_2) = z_2 - z_1$. Let $c_3 \ge 1$ be the corresponding constant. By hypothesis, the set

$$\Sigma_2 = \pi(\Sigma) = \{\omega_1, \omega_2, \lambda - x\}$$

has rank 3. Moreover, since ω_1 and ω_2 are linearly independent over $\mathbf{R}$, Lemma 9.5 shows that there exists infinitely many integers $N \ge 1$ with

$$\Sigma_2(N) \cap B_1(0, e^{-N}) = \{0\}.$$

Fix such an integer N. In the notation of Corollary 1.6, this implies $\mu_N = 3$ and thus $L_N = N^3$. Letting $r = c_3 N$ and $R = ec_3 r$, we deduce that any entire function $f\colon \mathbf{C}^2 \to \mathbf{C}$ satisfies

$$|f|_r \le c_3^{2N^3} \max_{\substack{\gamma\in\Sigma(N)\\ 0\le\tau\le N^3}} \frac{1}{\tau!}\big|(D_{\mathbf{w}}^\tau f)(\gamma)\big|N^\tau + e^{-N^3}|f|_R. \tag{9.3}$$

Put $\sigma = \min\{s_0/3, s_1\}$ and denote by M the smallest positive integer for which $N \le M^\sigma$. We apply the preceding inequality to the function $f(z_1, z_2) = Q_M(z_1, \varphi(z_2))$. Since $Q_M(X, \mathbf{Y})$ has degree $\le M^{t_0}$ in X, is homogeneous in $\mathbf{Y}$ of degree $\le M^{t_1}$, and has height $\le e^M$, we find

$$\begin{aligned} |f|_R &\le (M^{t_0}+1)(M^{t_1}+1)^2 e^M R^{M^{t_0}} \exp\big(c_2 R^2 M^{t_1}\big) \\ &\le \exp\Big(c_4\big(N^{1/\sigma} + N^{t_0/\sigma}\log(N) + N^{2+t_1/\sigma}\big)\Big) \end{aligned}$$

for some constant $c_4 > 0$ independent of N. Since $\max\{1, t_0, 3t_1\} < 3\sigma$, this implies

$$|f|_R \le \frac{1}{2} e^{N^3/2} \tag{9.4}$$

provided that N is sufficiently large. We claim that, for the same values of N, we also have $|f|_r \ge 2e^{-N^3/2}$. To prove this, denote by I the prime ideal of $\mathbf{C}[X, \mathbf{Y}]$ generated by $Y_0Y_2^2 - 4Y_1^3 + g_2Y_0^2Y_1 + g_3Y_0^3$, denote by T the degree in $\mathbf{Y}$ of Q_M, and consider the polynomial $\overline{Q}_M = Q_M(X, Y_0, Y_1, -Y_2)$. Since $Q_M \notin I$, we have $\overline{Q}_M \notin I$ and thus $Y_0^T Q_M \overline{Q}_M \notin I$. Moreover, since this product contains only even powers of Y_2 with exponents $\le 2T$, we deduce that

$$Y_0^T Q_M \overline{Q}_M \equiv P \bmod I,$$

with a non-zero polynomial $P \in \mathbf{Z}[X, Y_0, Y_1]$, homogeneous of degree $3T$ in Y_0, Y_1. Let $\overline{f}\colon \mathbf{C}^2 \to \mathbf{C}$ be the entire function given by $\overline{f}(z_1, z_2) = \overline{Q}_M(z_1, \varphi(z_2)) = (-1)^{3T} f(z_1, -z_2)$. The above congruence implies

$$\sigma(z_2)^{3T} f(z_1, z_2)\overline{f}(z_1, z_2) = P(z_1, \sigma(z_2)^3, \sigma(z_2)^3\wp(z_2))$$

for any $(z_1, z_2) \in \mathbf{C}^2$, and thus

$$f(z_1, z_2)\overline{f}(z_1, z_2) = P(z_1, \sigma(z_2)^2, \sigma(z_2)^2\wp(z_2))$$

on all $\mathbf{C}^2$. Since the Taylor expansion of $\sigma(z_2)^2$ around $z_2 = 0$ starts out z_2^2, since the one of $\sigma(z_2)^2\wp(z_2)$ starts out 1, and since P has integral coefficients and is homogeneous in (Y_0, Y_1), we deduce that the Taylor expansion of $f\overline{f}$ around the origin has at least one non-zero integral coefficient. According to Cauchy's inequalities, this implies $|f\overline{f}|_1 \ge 1$. Thus, when (9.4) holds, we also have

$$|f|_r \ge |f|_1 \ge |\overline{f}|_1^{-1} = |f|_1^{-1} \ge |f|_R^{-1} \ge 2e^{-N^3/2}. \tag{9.5}$$

Combining (9.3), (9.4) and (9.5), we conclude that, if N is sufficiently large, there is at least one point $\gamma \in \Sigma(N)$ and one integer τ with $0 \le \tau \le N^3$ such that

$$\left|(D^\tau_{\mathbf{w}} f)(\gamma)\right| \ge (ec_3)^{-2N^3}. \tag{9.6}$$

Choose integers m_1, m_2, m of absolute values $\le N$ such that $\gamma = (mx,\ m_1\omega_1 + m_2\omega_2 + m\lambda)$, and choose an index j such that $m\alpha \in \mathcal{V}_j$. We find

$$\left|(D^\tau_{\mathbf{w}} f)(\gamma)\right| = \left|\left(\frac{d}{dz}\right)^\tau f(mx + z, m_1\omega_1 + m_2\omega_2 + m\lambda + z)\right|_{z=0}$$

and, for z in a neighborhood of 0,

$$f(mx + z, m_1\omega_1 + m_2\omega_2 + m\lambda + z) = \varphi_j(m_1\omega_1 + m_2\omega_2 + m\lambda + z)^T Q_M(mx + z, \psi_j(m\lambda + z))$$

because ψ_j is Ω-periodic. Put $c_5 = |\omega_1| + |\omega_2| + |\lambda|$. For $k = 0, \dots, \tau$, Cauchy's inequalities give

$$\begin{aligned}\left|\left(\frac{d}{dz}\right)^{\tau-k} \varphi_j(m_1\omega_1 + m_2\omega_2 + m\lambda + z)^T\right|_{z=0} &\le (\tau - k)!\, |\varphi_j^T|_{1+c_5N} \\ &\le (N^3)! \exp\big(c_2(1 + c_5N)^2 T\big) \\ &\le \exp\big(c_6 M^{3\sigma} \log(M)\big)\end{aligned}$$

for some constant $c_6 > 0$, and Lemma 9.2 gives

$$\begin{aligned}\left|\left(\frac{d}{dz}\right)^{k} Q_M(mx + z, \psi_j(m\lambda + z))\right|_{z=0} &\le c_1^{T+k}\left|(\mathcal{D}_j^k Q_M)(mx, m\alpha)\right| \\ &\le \exp\big(c_7 M^{3\sigma}\big)\left|(\mathcal{D}_j^k Q_M)(mx, m\alpha)\right|\end{aligned}$$

for some constant $c_7 > 0$. Moreover, since $\tau \le M^{3\sigma} \le M^{s_0}$ and $|m| \le M^\sigma \le M^{s_1}$, the hypotheses on Q_M imply $\left|(\mathcal{D}_j^k Q_M)(mx, m\alpha)\right| \le \exp(-M^u)$ for $k = 0, \dots, \tau$. Since $u > 3\sigma$, putting these estimates together provides an upper bound for $|(D^\tau_{\mathbf{w}} f)(\gamma)|$ which contradicts (9.6) when N is large enough. ∎

An elliptic analogue of Schanuel's conjecture is that, if k is the field of endomorphisms of $E(\mathbf{C})$ and if $x_1, \dots, x_\ell \in \mathbf{C}$ are linearly independent over k, then the field $\mathbf{Q}(x_1, \dots, x_\ell, \exp_E(x_1), \dots, \exp_E(x_\ell))$ generated over $\mathbf{Q}$ by $x_1, \dots, x_\ell$ and normalized coordinates of $\exp_E(x_1), \dots, \exp_E(x_\ell)$ has transcendence degree $\ge \ell$ over $\mathbf{Q}$ (see p. 650 of [3]). Using Theorem 9.1 above, we can reformulate this conjecture in the following arithmetic form.

Conjecture 9.7. *Let k be the field of endomorphisms of E, let ℓ be a positive integer, let $x_1, \dots, x_\ell \in \mathbf{C}$ be linearly independent over k and let $\alpha_1, \dots, \alpha_\ell \in E(\mathbf{C})$. Moreover, let s_0, s_1, t_0, t_1, u be positive real numbers satisfying (9.1). Assume that, for any sufficiently large positive integer N, there exists a polynomial $P_N \in \mathbf{Z}[X, \mathbf{Y}]$ which is homogeneous in $\mathbf{Y}$, does not vanish identically on $\mathbf{C} \times E(\mathbf{C})$, has partial degree $\le N^{t_0}$ in X, partial degree $\le N^{t_1}$ in $\mathbf{Y}$, height $\le e^N$, and satisfies*

$$\left| (\mathcal{D}_j^\tau P_N)\Big(\sum_{i=1}^{\ell} m_i x_i, \sum_{i=1}^{\ell} m_i \alpha_i\Big) \right| \le \exp(-N^u),$$

for any integers $j, \tau, m_1, \dots, m_\ell$ with $0 \le j \le 2$, $0 \le \tau \le N^{s_0}$, $\max\{|m_1|, \dots, |m_\ell|\} \le N^{s_1}$, and $\sum_{i=1}^{\ell} m_i \alpha_i \in \mathcal{V}_j$. Then, we have $\mathrm{tr.deg}_{\mathbf{Q}} \mathbf{Q}(x_1, \dots, x_\ell, \alpha_1, \dots, \alpha_\ell) \ge \ell$.

Indeed, if the hypotheses of Conjecture 9.7 are satisfied, Theorem 9.1 implies that there exists an integer $d \ge 1$ such that $d\alpha_j = \exp_E(dx_j)$ for $j = 1, \dots, \ell$. Thus, if the elliptic analogue of Schanuel's conjecture is true, the field $\mathbf{Q}(dx_1, \dots, dx_\ell, d\alpha_1, \dots, d\alpha_\ell)$ has transcendence degree $\ge \ell$ over $\mathbf{Q}$ and therefore the conclusion of Conjecture 9.7 holds. Conversely, assume that Conjecture 9.7 is true. Let $x_1, \dots, x_\ell \in \mathbf{C}$ be linearly independent over k, and define $\alpha_j = \exp_E(x_j)$ for $j = 1, \dots, \ell$. Applying Lemma 9.3 with $\Delta = N$, $r = (1 + \sum |x_j|)N^{s_1}$, $T_0 = N^{t_0}$, $T_1 = N^{t_1}$, $U = 2N^u$ and $a = 0$, produces a sequence of polynomials $(P_N)_{N \ge N_0}$ satisfying all the hypotheses of Conjecture 9.7 (adapt the argument in the proof of Proposition 9.4). This gives $\mathrm{tr.deg}_{\mathbf{Q}} \mathbf{Q}(x_1, \dots, x_\ell, \alpha_1, \dots, \alpha_\ell) \ge \ell$ as claimed by the elliptic analogue of Schanuel's conjecture.

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