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CRITERIA OF ALGEBRAIC INDEPENDENCE WITH MULTIPLICITIES AND APPROXIMATION BY HYPERSURFACES.

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Content

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1. Introduction

The first main result of this paper is a criterion of algebraic independence adapted for the method of interpolation determinants. Recall that this method replaces the classical construction of an auxiliary function, using Dirichlet box principle, by the study of the principal minors of the corresponding interpolation matrix. The estimates obtained in this way have proved to be sharper than those arising from the classical approach with useful applications to lower bounds for linear forms in logarithms (see for example [9]). However, up to recently, this method was restricted to transcendence results. The first proof of algebraic independence using interpolation determinants was given in [22]. It was based on a result of algebraic approximation of E. Wirsing and applied only to specific situations. Other more complete approaches were given in [23], [18] and [10], but they were all concerned with small transcendence degree. Here, we generalize the approach of [10] by providing a criterion of algebraic independence which applies to any transcendence degree. It differs from the previous tools of algebraic independence developed by Yu. V. Nesterenko and P. Philippon in that the new criterion takes into account not only the absolute values of families of polynomials at the point under study, but also the absolute values of their derivatives at that point. The reason for doing this is that the derivatives of the minors of the interpolation matrix can be expressed as linear combinations of minors of smaller order and thus good upper bounds can be given for their absolute values. This is explained in detail in [10], in the case of transcendence degree one, but the general case is similar. Here, we restrict ourselves to the proof of the new criterion, leaving precise applications to interpolation determinants for the future.

The second main result of this paper addresses the problem of finding algebraic subsets of $\mathbf{C}^n$ defined over $\mathbf{Q}$, with bounded degree and bounded height, whose distance to a given point of $\mathbf{C}^n$ is small. The case $n = 1$ has been studied extensively and is well understood (see Thm 3.2 of [23], Prop. 2 of [10] and Thm 2 of [11]). Partial results for $n \leq 3$ can be found in [19]. In general, we have only conjectures (Conj. 1.7 of [24], Conj. 5 and 5' of [8], Problem 10 of [18]). Here, we concentrate on the problem of approximation by hypersurfaces and prove the corresponding conjecture for these. This follows as a corollary of our criterion of algebraic independence with multiplicities, and thus constitutes a first useful application of this criterion.

Our third contribution is a geometric notion of height for projective varieties for which we prove a multi-linearity property as well as an intersection formula. This notion of height depends on a choice of adelic convex bodies in spaces of homogeneous polynomials of fixed degrees, and is reminiscent of the twisted height of J. L. Thunder [26] and the work of S. Zhang [28]. It contains, as a special case, the usual notion of height and, as a limit case, the notion of absolute value of a projective variety at a point, which plays an important role in [13, 14, 15]. We tried to be as general as possible so that, for a specific problem like the one we consider here, one simply has to choose appropriate convex bodies and work with the corresponding height.

In order to handle multiplicities, we use convex bodies which are essentially products of convex bodies in degree one. After this crucial observation, the proof of our criterion follows the same general pattern as in the previous works of Nesterenko and Philippon, using the so called "descent" to construct a sequence of varieties with decreasing dimension. However, our argument differs from the one of [15] even in the case without multiplicities, and leads to a simpler proof from our point of view. Note also that the strategy adopted here is different from that of [10] where, in the case of one variable, we reduced the criterion with multiplicity to Gel'fond's criterion by extracting multiple factors from each polynomial in the given sequence.

The paper is organized as follows. The main results are stated in Chapter 2. Next we define in Chapter 3 the notion of an adelic convex body and the corresponding notion of height for polynomials. The main result of this chapter is Proposition 3.10 which can be viewed as a uniform version of Gel'fond's lemma for the height of a product of polynomials. In Chapter 4, we extend these heights to projective algebraic cycles via their Chow forms and develop the formalism of these heights. The proof of the main criterion (Theorem 2.2) is prepared in Chapter 5 by specializing the general results of Chapter 4 to appropriate families of adelic convex bodies. Finally, Chapter 6 is devoted to the proofs of our three theorems. The paper is then completed by an appendix where we show that the intersection multiplicities defined in terms of Chow forms in §4.2 are those coming from algebraic geometry.

2. The main results

2.1. A first criterion

Through out this paper, we fix a field K of characteristic zero, equipped with a product formula (*cf.* [7], ch. II). This means that the field K comes with a non-empty set $\mathcal{M}_K$ consisting of non-trivial places of K and that, for each place $v \in \mathcal{M}_K$, we have selected an absolute value $|\ |_v$ on K and a real number $\lambda_v > 0$ with the following property. For each nonzero element x of K, we have $|x|_v = 1$ except for finitely many $v \in \mathcal{M}_K$, and

$$\prod_{v\in\mathcal{M}_K} |x|_v^{\lambda_v} = 1.$$

This implies in particular that $\mathcal{M}_K$ contains only finitely many Archimedean places. For each such place v, if any, we define $\epsilon_v = 1$ and we assume that $|\ |_v$ coincides with the usual absolute value on the prime subfield $\mathbf{Q}$ of K. For the remaining places, we define $\epsilon_v = 0$. We also require $\sum_{v\in\mathcal{M}_K} \epsilon_v\lambda_v = 1$ if $\mathcal{M}_K$ contains at least one Archimedean place.

We also fix an integer $m \geq 1$ and, for each field L containing K, we denote by $L[\underline{X}]$ the polynomial ring $L[X_0, \ldots, X_m]$. For each $(m+1)$-tuple $\sigma = (\sigma_0, \ldots, \sigma_m) \in \mathbf{N}^{m+1}$, we put $|\sigma| = \sigma_0 + \cdots + \sigma_m$ and we write $\underline{X}^\sigma$ to denote the monomial $X_0^{\sigma_0} \cdots X_m^{\sigma_m}$. We also denote by $P^{[\sigma]}$ the *divided derivative* of order σ of a polynomial $P \in L[\underline{X}]$ which can be defined either as the coefficient of $T_0^{\sigma_0} \cdots T_m^{\sigma_m}$ in the Taylor expansion of $P(X_0+T_0, \ldots, X_m+T_m)$ viewed as a polynomial in $L[\underline{X}][T_0, \ldots, T_m]$, or as the partial derivative of P of order σ divided by $\sigma_0! \cdots \sigma_m!$, since K is assumed to have characteristic 0.

We choose an algebraic closure $\overline{K}$ of K and, for each place $v \in \mathcal{M}_K$, we choose an absolute value of $\overline{K}$ which extends $|\ |_v$ on K. We denote by $\mathbf{C}_v$ the completion of $\overline{K}$ with respect to this absolute value and we again denote by $|\ |_v$ the absolute value of $\mathbf{C}_v$ which coincides with the chosen absolute value on $\overline{K}$.

Given a place $v \in \mathcal{M}_K$ and a point $\underline{\alpha} = (\alpha_0, \ldots, \alpha_m)$ in $\mathbf{C}_v^{m+1}$, we define the *norm* of $\underline{\alpha}$ by $\|\underline{\alpha}\|_v = \max\{|\alpha_0|_v, \ldots, |\alpha_m|_v\}$. Similarly, we define the *norm* $\|P\|_v$ of a polynomial $P \in \mathbf{C}_v[\underline{X}]$ as the maximum of the absolute values of its coefficients. We also define the *height* of a nonzero polynomial $P \in K[\underline{X}]$ by the formula

$$h(P) = \sum_{v \in \mathcal{M}_K} \lambda_v \log \|P\|_v.$$

Finally, we fix a place $w \in \mathcal{M}_K$, a point $\theta \in \mathbf{P}^m(\mathbf{C}_w)$, and a choice of projective coordinates $\underline{\theta} = (\theta_0, \ldots, \theta_m) \in \mathbf{C}_w^{m+1}$ for θ with norm $\|\underline{\theta}\|_w = 1$. We define the *distance* between θ and a point $\alpha \in \mathbf{P}^m(\mathbf{C}_w)$ by

$$\mathrm{dist}(\theta, \alpha) = \max_{0 \leq i,j \leq m} \frac{|\theta_i \alpha_j - \theta_j \alpha_i|_w}{\|\underline{\theta}\|_w \, \|\underline{\alpha}\|_w}$$

where $\underline{\alpha} = (\alpha_0, \ldots, \alpha_m)$ is any choice of projective coordinates of α in $\mathbf{C}_w^{m+1}$. More generally, we define the *distance* between θ and a subset E of $\mathbf{P}^m(\mathbf{C}_w)$ as the infimum of the distance between θ and a point of E.

We first state a simple criterion of algebraic independence which should nevertheless be sufficient in order to recover most known results in algebraic independence using interpolation determinants as shown in [10] for small transcendence degree.

Theorem 2.1. *Let k be an integer with $0 \leq k \leq m$. Let $(D_n)_{n\geq 1}$ and $(S_n)_{n\geq 1}$ be sequences of positive integers with $S_n \leq D_n$ for each $n \geq 1$, and let $(T_n)_{n\geq 1}$ and $(V_n)_{n\geq 1}$ be sequences of positive real numbers. Suppose that the sequences $(D_n/S_n)_{n\geq 1}$, $(T_n/S_n)_{n\geq 1}$ and $(V_n/S_n)_{n\geq 1}$ are non-decreasing, and that*

$$\limsup_{n\to\infty} \frac{V_n S_n^k}{(D_n + T_n) D_n^k} = \infty.$$

Suppose also that for each $n \geq 2$ there exists a non-empty family $\mathcal{F}_n$ consisting of homogeneous polynomials in $K[\underline{X}]$ which satisfy the following two properties.

(i) For every $P \in \mathcal{F}_n$, we have

$$\deg(P) = D_n, \quad h(P) \leq T_n \quad \textit{and} \quad \max_{|\sigma|<S_n} \left|P^{[\sigma]}(\underline{\theta})\right|_w \leq e^{-V_n} \|P\|_w.$$

(ii) The polynomials of $\mathcal{F}_n$ have no common zero α in $\mathbf{P}^m(\mathbf{C}_w)$ with

$$\mathrm{dist}(\theta, \alpha) \leq \exp\left(-\frac{V_{n-1}}{S_{n-1}}\right).$$

Then, we have $k < m$. Moreover, if i is an index for which $\theta_i \neq 0$, then the transcendence degree over K of the field $K(\theta_0/\theta_i, \ldots, \theta_m/\theta_i)$ is $\geq k+1$.

It is possible to strengthen condition (ii) above in two directions. One is to ask that the polynomials of $\mathcal{F}_n$ have at most finitely many common zeros $\alpha \in \mathbf{P}^m(\mathbf{C}_w)$ with $\mathrm{dist}(\theta, \alpha) \leq \exp(-V_{n-1}/S_{n-1})$. This requires growth conditions on the sequences $(D_n/S_n)_{n\geq 1}$ and $(T_n/S_n)_{n\geq 1}$, and is especially interesting in the case $k = 1$. However, we decided to omit this improvement because, in the case $k = 1$, the generalization of Gel'fond's transcendence criterion that one gets in this way is proved in [10] and [11]. Another direction is to ask that the elements of $\mathcal{F}_n$ have no common zeros $\alpha \in \mathbf{P}^m(\mathbf{C}_w)$ in a smaller region $\mathrm{dist}(\theta, \alpha) \leq \exp(-R_{n-1})$ where $(R_n)_{n\geq 1}$ is a sequence of real numbers with $R_n \geq V_n/S_n$ for each $n \geq 1$. We shall see in §6.2 a strategy to deal with such constraints, using Theorem 2.1 as it stands (see Corollary 6.5).

In the case where $S_n = 1$ for each n, the above theorem essentially implies the criterion of algebraic independence of P. Philippon (Theorem 2.11 of [15]), modulo the above mentioned potential improvements. However, the strategy that we propose in §6.2 in order to deal with smaller zero-free regions provides sharper results.

2.2. A refined quantitative criterion

We now turn to an effective refinement of Theorem 2.1 in the same vein as the statements obtained by Philippon [16], Jabbouri [3] and Jadot [4], when derivatives are not taken into account.

Let $\mathbf{P}^m_K$ denote the projective m-space over K. By a *subvariety* Z of $\mathbf{P}^m_K$, we mean a closed integral subscheme of $\mathbf{P}^m_K$. These objects are in bijection with the homogeneous prime ideals of $K[\underline{X}]$ distinct from $(X_0, \ldots, X_m)$ and the bijection is such that, for any field L containing K, and any subvariety Z of $\mathbf{P}^m_K$, the set of points of Z with values in L, denoted $Z(L)$, is simply the set of common zeros in $\mathbf{P}^m(L)$ of the homogeneous polynomials in the homogeneous prime ideal $\mathfrak{p}$ of $K[\underline{X}]$ corresponding to Z. We denote by $\deg(Z)$ the degree of the homogeneous ideal $\mathfrak{p}$ and by $h(Z) := h(F)$ the *height* of Z, where F stands for the standard Chow form of Z whose definition is recalled in §4.1.

As in the previous section, we fix a place $w \in \mathcal{M}_K$, a point $\theta \in \mathbf{P}^m(\mathbf{C}_w)$ with projective coordinates $\underline{\theta} \in \mathbf{C}_w^{m+1}$ chosen so that $\|\underline{\theta}\|_w = 1$. We also define a function $\Phi\colon \mathbf{N}^* \to \mathbf{R}$ by

$$\Phi(D) = D^{-1} \log \binom{D+m}{m} = D^{-1} \log \frac{(D+m)!}{D!\, m!}.$$

It is easy to show that this function is monotonically decreasing, that it is bounded above by $\log(m+1)$ and that it tends to 0 as D tends to infinity.

Our main result below provides a lower bound for the distance between θ and $Z(\mathbf{C}_w)$ for a k-dimensional subvariety Z of $\mathbf{P}^m_K$, under assumptions similar to those of Theorem 2.1, but with a slightly relaxed condition (i).

Theorem 2.2. *Let Z be a subvariety of $\mathbf{P}^m_K$ of dimension $k \geq 0$ and let p, q be integers with $p < q$. For each $n = p, \dots, q$, let S_n, D_n be positive integers with $S_n \leq D_n$, and let T_n, V_n be positive real numbers. Suppose that the sequences $(D_n/S_n)_{p\leq n\leq q}$, $(T_n/S_n)_{p\leq n\leq q}$ and $(V_n/S_n)_{p\leq n\leq q}$ are non-decreasing. Suppose also that, for each $n = p+1, \dots, q$, there exists a non-empty family $\mathcal{F}_n$ consisting of homogeneous polynomials in $K[\underline{X}]$ which satisfy the following two properties.*

(i) For every $P \in \mathcal{F}_n$ and every $\sigma \in \mathbf{N}^{m+1}$ with $|\sigma| < S_n$ we have

$$\deg(P) = D_n, \quad h(P) \leq T_n \quad \text{and} \quad \left|P^{[\sigma]}(\underline{\theta})\right|_w \leq \|P\|_w \exp\left(-V_n + |\sigma| \frac{V_n}{S_n}\right).$$

(ii) The polynomials of $\mathcal{F}_n$ have no common zero α on $Z(\mathbf{C}_w)$ with

$$\mathrm{dist}(\theta, \alpha) \leq \exp\left(-\frac{V_{n-1}}{S_{n-1}}\right).$$

Let η be a positive real number such that $\Phi(D_n) \leq \eta$ for $n = p+1, \dots, q$. Suppose moreover that

$$\lambda_w S_q^k V_q > (aD_q + bT_q + cS_q) D_q^k \tag{2.1}$$

with

$$\begin{aligned} a &= h(Z) + (k+1)\Big(3\log(m+1) + (k+4)\eta + \epsilon_w \lambda_w \log 4\Big) \deg Z \\ b &= (k+1) \deg Z \\ c &= \lambda_w \left((k+1) \max\left\{ \frac{V_p}{S_p}, \epsilon_w \log 2 \right\} \deg Z + \epsilon_w \log\Big((m+1)(k+1)\deg Z\Big) + \epsilon_w \right). \end{aligned}$$

Then, we have

$$\operatorname{dist}(\theta, Z(\mathbf{C}_w)) \geq \exp\left(-\frac{V_q}{S_q}\right).$$

Remarks.

1) For $p < n \leq q$, the hypothesis $S_n \leq D_n$ is a consequence of condition (i) of the theorem because, for any $P \in \mathcal{F}_n$, we have $\max_{|\sigma|=D_n} |P^{[\sigma]}(\underline{\theta})|_w = \|P\|_w$. The same applies as well to Theorem 2.1.

2) Under the hypotheses of the theorem, we also prove, as Proposition 6.4, a lower bound for the modulus $|Z|_\theta$ of Z at the point θ (see Definition 5.4).

2.3. Application to approximation by hypersurfaces

In this section, we consider the case where K is an algebraic extension of $\mathbf{Q}$ of finite degree d. We choose $\mathcal{M}_K$ to be the set of all non-trivial places of K, and, for each $v \in \mathcal{M}_K$, we choose for $|\ |_v$ the unique absolute value of K which belongs to the place v and extends either the usual absolute value of $\mathbf{Q}$ if v is Archimedean or the usual p-adic absolute value of $\mathbf{Q}$ if v lies above a prime number p. The corresponding exponent in the product formula is given by the quotient $\lambda_v = d_v/d$ where d_v denotes the local degree of K at v.

As in the preceding sections, we fix a non-trivial place w of K, but we choose θ in $\mathbf{P}^m(K_w)$, where K_w denotes the completion of K at w. We also choose a set of projective coordinates $\underline{\theta} \in K_w^{m+1}$ for θ with $\|\underline{\theta}\|_w = 1$. In §6.3, we shall prove the following result as a consequence of Theorem 2.2.

Theorem 2.3. *Let $(D_n)_{n\geq 1}$ be a non-decreasing sequence of positive integers, and let $(T_n)_{n\geq 1}$ be a non-decreasing unbounded sequence of positive real numbers with $T_n \geq D_n$ for each $n \geq 1$. Then, for infinitely many n, there exists a nonzero homogeneous polynomial $P \in K[\underline{X}]$ of degree D_n and height $h(P) \leq T_n$ which admits at least one zero $\alpha \in \mathbf{P}^m(\mathbf{C}_w)$ with*

$$\operatorname{dist}(\theta, \alpha) \leq \exp\Big(-\frac{1}{\gamma} D_{n-1}^m T_{n-1}\Big) \tag{2.2}$$

where $\gamma = 8(d_w/d)(m+1)!$.

Note that, in the case $m = 1$, Theorem 2.3 is not quite as precise as Theorem 2 of [11] because it does not include a lower bound for the degree of the irreducible factors of the polynomial P.

3. Heights attached to adelic convex bodies

3.1. A generalization of John's theorem

Let L be a local field, that is a field which is complete with respect to an absolute value $|\ |$. Let also V be a nonzero finite dimensional vector space over L. Since all norms on V are equivalent, they define the same topology on V. We endow V with this topology. The notion of a *bounded* subset of V is also independent of the choice of a norm. We may therefore introduce the following definition:

Definition 3.1. We say that a subset $\mathcal{C}$ of V is a *convex body* if it contains a neighborhood of zero, if it is bounded, and if it satisfies the condition

$$\lambda x + \mu y \in \mathcal{C}$$

for any choice of points x, y of $\mathcal{C}$ and any choice of elements λ, μ of L with

$$\begin{cases} |\lambda| + |\mu| \le 1 & \text{if } L \text{ is Archimedean,} \\ \\ \max\{|\lambda|, |\mu|\} \le 1 & \text{otherwise.} \end{cases}$$

When $L = \mathbf{R}$ and $V = \mathbf{R}^n$, this is the usual notion of a convex body of $\mathbf{R}^n$. In general, if $L_1, \dots, L_k$ are linear forms from V to L with the origin as their only common zero, and if $\epsilon_1, \dots, \epsilon_k$ are positive real numbers, then the set of points $x \in V$ which satisfy the conditions $|L_i(x)| \le \epsilon_i$ for $i = 1, \dots, k$ is a convex body of V. This is the type of convex body that we shall encounter later in the applications.

For any bounded subset S of V, there is a smallest convex body of V containing S. We call it the *symmetric convex hull* of S. It consists of all points of V of the form $\lambda_1 x_1 + \cdots + \lambda_s x_s$ with $x_1, \dots, x_s \in S$ and $\lambda_1, \dots, \lambda_s \in L$ satisfying $\sum |\lambda_i| \le 1$ if L is Archimedean and $\max |\lambda_i| \le 1$ otherwise.

For each integer $n \ge 1$, we endow the vector space L^n with the maximum norm defined for a point $\underline{x} = (x_1, \dots, x_n) \in L^n$ by $\|\underline{x}\| = \max_{1 \le i \le n} |x_i|$ and we denote by

$$\mathcal{B} = \big\{ \underline{x} \in L^n \,;\, \|\underline{x}\| \le 1 \big\}$$

the *unit ball* of L^n relative to this norm. It is again a convex body.

John's theorem [5] states that, for any compact convex body $\mathcal{C}$ of $\mathbf{R}^n$ there is an ellipsoid E of $\mathbf{R}^n$ such that $E \subseteq \mathcal{C} \subseteq \sqrt{n} E$ (see also Theorem 2A, Chapter IV of [25]). In the present context, we get the following statement:

Proposition 3.2. *Let $\mathcal{C}$ be a convex body of V and let $n \geq 1$ be the dimension of V. Put $\kappa = n$ if L is Archimedean and $\kappa = 1$ otherwise. Then, for any $\rho \in L$ with $|\rho| > \kappa$, there is a linear map $\varphi\colon L^n \to V$ such that*

$$\varphi(\mathcal{B}) \subseteq \mathcal{C} \subseteq \rho\,\varphi(\mathcal{B}). \tag{3.1}$$

When $\mathcal{C}$ is a closed set, this assertion still holds for $|\rho| = \kappa$ provided that L is locally compact or that its valuation group $|L^\times|$ is a discrete subset of $\mathbf{R}_{>0}$.

Proof. Without loss of generality, we may assume that $V = L^n$. Denote by $\mathcal{E}$ the set of all linear maps $\varphi\colon L^n \to L^n$ such that $\varphi(\mathcal{B}) \subseteq \mathcal{C}$. Since $\mathcal{C}$ contains a neighborhood of zero and is bounded, the set $\mathcal{E}$ is not empty and the number

$$\Delta := \sup\big\{|\det\varphi|\,;\ \varphi\in\mathcal{E}\big\}$$

is finite and positive. If $\mathcal{C}$ is closed and L is locally compact, the set $\mathcal{E}$ is compact. It contains therefore an element φ with $|\det(\varphi)| = \Delta$. This is again true if $|L^\times|$ is discrete in $\mathbf{R}_{>0}$. We will show that such a choice of φ satisfies the condition (3.1) for any $\rho \in L$ with $|\rho| \geq \kappa$. When $|\rho| > \kappa$, we will show that an element φ of $\mathcal{E}$ also satisfies this condition as soon as the ratio $\Delta/|\det\varphi|$ is sufficiently close to 1.

Indeed, let $\varphi \in \mathcal{E}$ with $\det\varphi \neq 0$, and assume $\mathcal{C} \not\subseteq \varphi(\rho\mathcal{B})$ for some $\rho \in L$ with $|\rho| \geq \kappa$. Then there exists $\underline{\xi} \in L^n$ with $\varphi(\underline{\xi}) \in \mathcal{C}$ and $\|\underline{\xi}\| > |\rho|$. Since $\varphi(\mathcal{B}) \subseteq \mathcal{C}$, this implies $\varphi(\mathcal{C}') \subseteq \mathcal{C}$, where $\mathcal{C}'$ denotes the symmetric convex hull of $\mathcal{B}\cup\{\underline{\xi}\}$. We show below how to construct an element ψ of $\mathrm{GL}_n(L)$ with $\psi(\mathcal{B}) \subseteq \mathcal{C}'$ and $|\det\psi| > c_L(\rho)$, where $c_L(\rho) \geq 1$ depends only on $|\rho|$ and satisfies $c_L(\rho) > 1$ if $|\rho| > \kappa$. Then, the composite map $\varphi\circ\psi$ belongs to $\mathcal{E}$ and we find:

$$\Delta \geq |\det(\varphi\circ\psi)| > c_L(\rho)|\det\varphi|.$$

We conclude that any choice of $\varphi \in \mathcal{E}$ with $|\det\varphi| \geq \Delta/c_L(\rho)$ satisfies (3.1).

To construct ψ, fix an index i with $|\xi_i| = \|\underline{\xi}\|$, and denote by $\{e_1,\dots,e_n\}$ the canonical basis of L^n. We consider two cases:

When L is not Archimedean, we define ψ as the element of $\mathrm{GL}_n(L)$ which satisfies $\psi(e_i) = \underline{\xi}$ and $\psi(e_j) = e_j$ for $j \neq i$. It is clear that $\psi(\mathcal{B}) \subseteq \mathcal{C}'$ and that $|\det\psi| = |\xi_i| = \|\underline{\xi}\| > |\rho| =: c_L(\rho)$.

When L is Archimedean, we may assume that L is either $\mathbf{R}$ or $\mathbf{C}$. We may also assume, without loss of generality, that $\xi_i = \|\underline{\xi}\|$. For any real t with $0 \leq t \leq 1$, define ψ_t to be the element of $\mathrm{GL}_n(L)$ which satisfies

$$\psi_t(e_i) = te_i + (1-t)\underline{\xi} \quad\text{and}\quad \psi_t(e_j) = te_j \quad\text{for } j \neq i.$$

Then, the map ψ_t satisfies $\psi_t(\mathcal{B}) \subseteq \mathcal{C}'$. Moreover, we find

$$\det \psi_t = v(t) \quad \text{where } v(t) = \big(t + (1-t)\xi_i\big)t^{n-1}.$$

Since $\xi_i > |\rho| \geq n$, the function $v(t)$ achieves its maximum on the interval $[0,1]$ at the point $t_0 = (1-1/n)/(1-1/\xi_i)$ and is strictly monotone decreasing in the interval $[t_0, 1]$. Define $\psi = \psi_{t_0}$ and $c_L(\rho) = v(t_1)$ where $t_1 = 1/(1+1/n-1/|\rho|)$. Since $t_0 < t_1 \leq 1$, we find $\psi(\mathcal{B}) \subseteq \mathcal{C}'$ and $|\det \psi| > c_L(\rho)$ and also $c_L(\rho) \geq v(1) = 1$ with equality if and only if $|\rho| = n$, as required. ∎

3.2. Norm attached to a convex body

Let L and V as in §3.1, and let n be a positive integer. We assume from now on that, if L is Archimedean, then it is isomorphic with the field $\mathbf{C}$ of complex numbers. If it is not Archimedean, we assume instead that its valuation group $|L^\times|$ is dense in $\mathbf{R}_{>0}$. In practice, L will be one of the completions $\mathbf{C}_v$ of $\overline{K}$ defined in §2.1.

To each convex body $\mathcal{C}$ of V, we attach a norm on the vector space of all polynomial maps $F\colon V \to L$, by defining

$$\|F\|_{\mathcal{C}} = \sup\{|F(x)|\,;\, x \in \mathcal{C}\}.$$

When $V = L^n$, this space is naturally isomorphic with the ring $L[u_1, \dots, u_n]$ of polynomials in n indeterminates over L. A more standard norm on $L[u_1, \dots, u_n]$ denoted $\|\ \|$ is given, for a polynomial $F = \sum_{\underline{\tau}} a_{\underline{\tau}} u_1^{\tau_1} \cdots u_n^{\tau_n}$, by the maximum of the absolute values of its coefficients:

$$\|F\| = \max_{\underline{\tau}} |a_{\underline{\tau}}|\,.$$

The next lemma compares these norms when $\mathcal{C}$ is the unit ball $\mathcal{B}$ of L^n for the maximum norm.

Lemma 3.3. *Let $\mathcal{B}$ be the unit ball of L^n for the maximum norm, let $F \in L[u_1, \dots, u_n]$ be a polynomial and let N be the number of nonzero coefficients of F. Then, we have:*
(i) $\|F\| \leq \|F\|_{\mathcal{B}} \leq N\|F\|$ *if L is Archimedean;*
(ii) $\|F\|_{\mathcal{B}} = \|F\|$ *otherwise.*

These are standard inequalities. In the Archimedean case, L can be identified with $\mathbf{C}$ and the relation $\|F\| \leq \|F\|_{\mathcal{B}}$ follows from Cauchy's inequalities. In the ultrametric case, the equality $\|F\|_{\mathcal{B}} = \|F\|$ follows from the hypothesis that $|L^\times|$ is dense in $\mathbf{R}_{>0}$.

Our goal is to show that the norm attached to a convex body is essentially multiplicative. We start by the case of the unit ball of L^n.

Lemma 3.4. *Let $\mathcal{B}$ be as in Lemma 3.3. Assume that L is not Archimedean. Then, for any choice of polynomials $F, G \in L[u_1, \dots, u_n]$, we have:*

$$\|FG\|_{\mathcal{B}} = \|F\|_{\mathcal{B}} \|G\|_{\mathcal{B}}.$$

Proof. Gauss' lemma shows that $\|FG\| = \|F\|\|G\|$. The conclusion then follows from part (ii) of Lemma 3.3. ∎

In the Archimedean case, we proceed in a similar way, by comparison with the *Mahler's measure*. Recall that the Mahler's measure of a nonzero polynomial $F \in \mathbf{C}[u_1, \dots, u_n]$ is given by

$$M(F) = \exp\left\{ \int_0^1 \cdots \int_0^1 \log |F(e(t_1), \cdots, e(t_n))| \, \mathrm{d}t_1 \cdots \mathrm{d}t_n \right\}$$

where $e(t) = \exp(2\pi i t)$ (see [12]). For $F = 0$, we put $M(F) = 0$. This function is multiplicative in the sense that, for any polynomials $F, G \in \mathbf{C}[u_1, \dots, u_n]$, we have

$$M(FG) = M(F)M(G).$$

For the applications to Chow forms, it is crucial to have good inequalities of comparison between $\|F\|_{\mathcal{B}}$ and $M(F)$, especially for the dependence in the number of variables. In [15], P. Philippon proves such precise inequalities of comparison between $\|F\|$ and $M(F)$. Following his approach, we get:

Lemma 3.5. *Let $\mathcal{B}$ be the unit ball of $\mathbf{C}^n$ for the maximum norm. Let $n_1, \dots, n_k$ be positive integers with sum n, and let $u_{j,i}$, $(1 \le j \le k,\ 1 \le i \le n_j)$, be indeterminates. Put $\underline{u}_j = (u_{j,1}, \dots, u_{j,n_j})$ for $j = 1, \dots, k$. If $F \in \mathbf{C}[\underline{u}_1, \dots, \underline{u}_k]$ is a multi-homogeneous polynomial of multi-degree $(d_1, \dots, d_k)$ in the sets of variables $\underline{u}_1, \dots, \underline{u}_k$, then we have:*

$$M(F) \le \|F\|_{\mathcal{B}} \le n_1^{d_1} \cdots n_k^{d_k} M(F).$$

Proof. The inequality $M(F) \le \|F\|_{\mathcal{B}}$ is clear and is valid more generally for any polynomial F of $\mathbf{C}[\underline{u}_1, \dots, \underline{u}_k]$. For the other inequality, write $F = \sum a_{\underline{\tau}_1, \dots, \underline{\tau}_k} \underline{u}_1^{\underline{\tau}_1} \cdots \underline{u}_k^{\underline{\tau}_k}$. Following the proof of Lemma 1.13 of [15], we get that, for any choice of points $\underline{\tau}_1 \in \mathbf{N}^{n_1}, \dots, \underline{\tau}_k \in \mathbf{N}^{n_k}$ with $|\underline{\tau}_1| = d_1, \dots, |\underline{\tau}_k| = d_k$, we have:

$$|a_{\underline{\tau}_1, \dots, \underline{\tau}_k}| \le \frac{d_1!}{\underline{\tau}_1!} \cdots \frac{d_k!}{\underline{\tau}_k!} M(F).$$

This implies

$$\|F\|_{\mathcal{B}} \le \sum |a_{\underline{\tau}_1, \dots, \underline{\tau}_k}| \le \left(\prod_{j=1}^{k} \Big(\sum_{|\underline{\tau}_j| = d_j} \frac{d_j!}{\underline{\tau}_j!} \Big) \right) M(F) = n_1^{d_1} \cdots n_k^{d_k} M(F). \qquad ∎$$

We deduce:

Lemma 3.6. *Let $\mathcal{B}$, $n_1,\dots,n_k$ and $\underline{u}_1,\dots,\underline{u}_k$ be as in Lemma 3.5. If $F_1,\dots,F_s \in \mathbf{C}[\underline{u}_1,\dots,\underline{u}_k]$ are multi-homogeneous polynomials in the sets of variables $\underline{u}_1,\dots,\underline{u}_k$ and if their product $F = F_1\cdots F_s$ has multi-degree $(d_1,\dots,d_k)$, then we have:*

$$\|F\|_{\mathcal{B}} \le \prod_{i=1}^{s} \|F_i\|_{\mathcal{B}} \le n_1^{d_1}\cdots n_k^{d_k}\|F\|_{\mathcal{B}}.$$

Proof. For $i = 1,\dots,s$, denote by $(d_{i,1},\dots,d_{i,k})$ the multi-degree of F_i. Using Lemma 3.5 and the multiplicativity of the Mahler's measure, we find

$$\begin{aligned}\prod_{i=1}^{s} \|F_i\|_{\mathcal{B}} &\le \prod_{i=1}^{s} \big(n_1^{d_{i,1}}\cdots n_k^{d_{i,k}} M(F_i)\big)\\ &= n_1^{d_1}\cdots n_k^{d_k} M(F)\\ &\le n_1^{d_1}\cdots n_k^{d_k}\|F\|_{\mathcal{B}}.\end{aligned}$$

The other inequality is clear. $\square$

Combining Lemmas 3.4 and 3.6 with Proposition 3.2, we find:

Proposition 3.7. *Assume that V is a product $V = V_1\times\cdots\times V_k$ where V_j is a vector space over L of dimension n_j for $j = 1,\dots,k$. Moreover, let $\mathcal{C}$ be a convex body of V in the form of a Cartesian product $\mathcal{C} = \mathcal{C}_1\times\cdots\times\mathcal{C}_k$ where $\mathcal{C}_j$ is a convex body of V_j for $j = 1,\dots,k$. Then, if $F_1,\dots,F_s$ are multi-homogeneous polynomial maps from $V_1\times\cdots\times V_k$ to L and if their product $F = F_1\cdots F_s$ has multi-degree $(d_1,\dots,d_k)$, we have:*

(i) $\|F\|_{\mathcal{C}} \le \prod_{i=1}^{s} \|F_i\|_{\mathcal{C}} \le \big(n_1^{d_1}\cdots n_k^{d_k}\big)^2\|F\|_{\mathcal{C}}$ *if L is Archimedean;*

(i) $\|F\|_{\mathcal{C}} = \prod_{i=1}^{s} \|F_i\|_{\mathcal{C}}$ *otherwise.*

Proof. The lower bound $\prod_{i=1}^{s}\|F_i\|_{\mathcal{C}} \ge \|F\|_{\mathcal{C}}$ is clear and is valid for any convex body $\mathcal{C}$ of V. To prove the upper bound, denote by $\mathcal{B}_j$ the unit ball of L^{n_j} relative to the maximum norm, for $j = 1,\dots,k$. Choose also $\rho_j \in L$ with $|\rho_j| > n_j$ if L is Archimedean, with $|\rho_j| > 1$ otherwise. By Proposition 3.2, there is a linear map $\varphi_j\colon L^{n_j} \to V_j$ such that

$$\varphi_j(\mathcal{B}_j) \subseteq \mathcal{C}_j \subseteq \varphi_j(\rho_j\mathcal{B}_j).$$

For $i = 1,\dots,s$, define $G_i = F_i\circ(\varphi_1,\dots,\varphi_k)$. Put also $\mathcal{B} = \mathcal{B}_1\times\cdots\times\mathcal{B}_k$ and $G = G_1\cdots G_s$, and assume that $n = \dim V$. Then, $\mathcal{B}$ is the unit ball of L^n for the maximum norm and we have $G = F \circ (\varphi_1, \dots, \varphi_k)$. Using the above inclusions together with Lemmas 3.4 and 3.6, we find:

$$\begin{aligned}\prod_{i=1}^{s} \|F_i\|_{\mathcal{C}} &\le \prod_{i=1}^{s} \|G_i\|_{(\rho_1\mathcal{B}_1)\times\cdots\times(\rho_k\mathcal{B}_k)}\\ &= |\rho_1|^{d_1}\cdots|\rho_k|^{d_k} \prod_{i=1}^{s} \|G_i\|_{\mathcal{B}}\\ &\le \big(|\rho_1|^{d_1}\cdots|\rho_k|^{d_k}\big)^2 \|G\|_{\mathcal{B}}\\ &\le \big(|\rho_1|^{d_1}\cdots|\rho_k|^{d_k}\big)^2 \|F\|_{\mathcal{C}}.\end{aligned}$$

The conclusion then follows from the fact that $|L^\times|$ is dense in $\mathbf{R}_{>0}$. ∎

3.3. Heights

We use the notation of §2.1. Let V be a finite dimensional vector space over K. For each place v of K, we view V as a K-subspace of $\mathbf{C}_v \otimes_K V$ under the map which sends an element x of V to $1 \otimes x$. Given a polynomial map $F: V \to K$, we also denote by the same letter F the unique polynomial map from $\mathbf{C}_v \otimes_K V$ to $\mathbf{C}_v$ which coincides with F on V.

Definition 3.8. Let V be a vector space over K of finite dimension $n > 0$. An *adelic convex body* for V is a product

$$\mathcal{C} = \prod_{v\in\mathcal{M}_K} \mathcal{C}_v \subseteq \prod_{v\in\mathcal{M}_K} \mathbf{C}_v \otimes_K V$$

which satisfies the following conditions:

(i) for each $v \in \mathcal{M}_K$, the set $\mathcal{C}_v$ is a convex body of $\mathbf{C}_v \otimes_K V$;

(ii) there exists an invertible linear map $\varphi: K^n \to V$ such that, except for finitely many places $v \in \mathcal{M}_K$, the set $\mathcal{C}_v$ is the image of the unit ball $\mathcal{B}_v$ of $\mathbf{C}_v^n$ under the linear map $\varphi_v: \mathbf{C}_v^n \to \mathbf{C}_v \otimes_K V$ which extends φ.

We define the *height* of a nonzero polynomial map $F: V \to K$ relative to such a convex $\mathcal{C}$ by

$$h_{\mathcal{C}}(F) = \sum_{v\in\mathcal{M}_K} \lambda_v \log \|F\|_{\mathcal{C}_v}.$$

In the above condition (ii), the choice of the linear map $\varphi: K^n \to V$ plays no particular role: if this condition holds for at least one invertible linear map, it holds for all of them. This follows from the fact that an automorphism of K^n induces by linearity an automorphism of $\mathbf{C}_v^n$ which maps $\mathcal{B}_v$ to itself except for at most finitely many places v.

On the other hand, assuming both conditions (i) and (ii), we find $\|F\|_{\mathcal{C}_v} > 0$ for each $v \in \mathcal{M}_K$, since $\mathcal{C}_v$ is a neighborhood of 0 in $\mathbf{C}_v \otimes_K V$. Moreover, we get

$\|F\|_{\mathcal{C}_v} = \|F \circ \varphi_v\|_{\mathcal{B}_v} = 1$ except for a finite set of places $v \in \mathcal{M}_K$. Thus, the series defining $h_{\mathcal{C}}(F)$ reduces to a finite sum. Note also that, by the product formula, we have $h_{\mathcal{C}}(aF) = h_{\mathcal{C}}(F)$, for any $a \in K^\times$.

We mention that this definition of height is closely related to the notion of twisted height introduced by J. L. Thunder in [26] and extended to polynomials in §1 of [21]. This follows from the approximation property of convex bodies given by Proposition 3.2.

When $V = K^n$, the product of the unit balls $\mathcal{B}_v$ of $\mathbf{C}_v^n$ for each $v \in \mathcal{M}_K$ defines an adelic convex body for K^n:

$$\mathcal{B} = \prod_{v \in \mathcal{M}_K} \mathcal{B}_v \subseteq \prod_{v \in \mathcal{M}_K} \mathbf{C}_v^n.$$

On the other hand, any $F \in K[u_1, \ldots, u_n]$ defines a polynomial map from K^n to K. We denote again by $h_{\mathcal{B}}(F)$ the height of this map relative to $\mathcal{B}$. The next lemma compares this height with the more standard one defined in §2.1.

Lemma 3.9. *For any nonzero polynomial $F \in K[u_1, \ldots, u_n]$ and for the above choice of $\mathcal{B}$, we have*

$$h(F) \le h_{\mathcal{B}}(F) \le h(F) + \log N,$$

where N denotes the number of nonzero coefficients of F. In particular, we have $h_{\mathcal{B}}(F) \ge 0$.

Proof. For each place $v \in \mathcal{M}_K$, Lemma 3.3 provides inequalities relating $\|F\|_v$ and $\|F\|_{\mathcal{B}_v}$. The conclusion follows by multiplying the logarithm of these inequalities by λ_v and summing over $v \in \mathcal{M}_K$. ∎

Similarly, Proposition 3.7 implies the following version of Gel'fond's lemma:

Proposition 3.10. *Let k and s be positive integers. For $j = 1, \ldots, k$, let V_j be a vector space over K of finite dimension $n_j > 0$, and let $\mathcal{C}_j$ be an adelic convex body for V_j. Put $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_k$ and $V = V_1 \times \cdots \times V_k$. Moreover, for $i = 1, \ldots, s$, let F_i be a multi-homogeneous polynomial map from V to K. Denote by $F = F_1 \cdots F_s$ the product of these maps and by $(d_1, \ldots, d_k)$ the multi-degree of F. Then, $\mathcal{C}$ is an adelic convex body for V, and we have:*

$$h_{\mathcal{C}}(F) \le \sum_{i=1}^{s} h_{\mathcal{C}}(F_i) \le h_{\mathcal{C}}(F) + 2\sum_{j=1}^{k} d_j \log n_j\,.$$

4. Heights of projective cycles

We introduce in this section the main ingredients needed in order to prove our criterion of algebraic independence. Our first goal is to define a notion of height of projective varieties associated to a product of adelic convex bodies. This type of height will be used in the next section for some specific choice of convex bodies. Here, we establish general properties of it modeled on the formalism of arithmetic intersection theory. The key tool in this study is our generalized version of Gel'fond's lemma from Section 3 about the height of a product of polynomials.

Through out the section, we use the notation of §2.1 and §2.2. For each integer $D \geq 0$, we denote by $K[\underline{X}]_D$ the homogeneous part of degree D of $K[\underline{X}] = K[X_0, \ldots, X_m]$. Moreover, we fix an integer t with $0 \leq t \leq m$ and a sequence $\underline{D} = (D_0, \ldots, D_t)$ of $t+1$ positive integers. In order to simplify notations, we also define

$$\eta_j = \Phi(D_j) = D_j^{-1} \log \binom{D_j + m}{m} \qquad \text{for} \qquad j = 0, \ldots, t. \tag{4.1}$$

4.1. Cycles and Chow forms

Let Z be a subvariety of $\mathbf{P}^m_K$ of dimension t. We define a *Chow form of Z in degree $\underline{D}$* to be a polynomial map

$$F \colon K[\underline{X}]_{D_0} \times \cdots \times K[\underline{X}]_{D_t} \longrightarrow K$$

which satisfies the following conditions:

(i) F is *irreducible* over K (i.e. F cannot be written as the product of two non-constant such maps);

(ii) for any algebraically closed field L containing K, the zeros of the polynomial map from $L[\underline{X}]_{D_0} \times \cdots \times L[\underline{X}]_{D_t}$ to L which extends F (also denoted F in the sequel) are the $(t+1)$-tuples of polynomials $(P_0, \ldots, P_t)$ which have at least one common zero on $Z(L)$.

The existence of such a map F follows from Propositions 1.3 (ii), 1.4 and 1.5 (iii) of [15]. Moreover, the above conditions (i) and (ii) imply that it is unique up to multiplication by an element of $K^\times$. Conversely, they also show that Z is uniquely determined by F. More formally, one can think of F as an element of

$$\mathrm{Sym}(K[\underline{X}]^*_{D_0}) \otimes \cdots \otimes \mathrm{Sym}(K[\underline{X}]^*_{D_t}),$$

where, given a vector space V over K, the notation $\mathrm{Sym}(V^*)$ stands for the symmetric algebra over the dual V^* of V.

The *standard Chow form of Z* mentioned in §2.2 is the map $F : (K[\underline{X}]_1)^{t+1} \longrightarrow K$ corresponding to the special case $D_0 = \ldots = D_t = 1$. We view it as a polynomial over K in $(t+1)(m+1)$ independent variables by identifying any linear form $a_0X_0+\cdots+a_mX_m$ with the $(m+1)$-tuple $(a_0,\ldots,a_m)$ of its coefficients. Then, we have a height $h(F)$ associated to F, as defined in §2.1, and it makes sense to define the (standard) height of Z by $h(Z) = h(F)$, as in §2.2.

We also define a *cycle* of dimension t in $\mathbf{P}^m_K$ to be a formal linear combination

$$Z = m_1Z_1 + \cdots + m_sZ_s$$

with positive integer coefficients $m_1,\ldots,m_s \geq 1$ of subvarieties $Z_1,\ldots,Z_s$ of $\mathbf{P}^m_K$ of dimension t. The subvarieties $Z_1,\ldots,Z_s$ are called the *components* of Z. We define the *degree* of such a cycle Z by

$$\deg(Z) = \sum_{i=1}^{s} m_i \deg(Z_i).$$

Finally, we define a *Chow form of Z in degree $\underline{D}$* to be a polynomial map F from $K[\underline{X}]_{D_0} \times \cdots \times K[\underline{X}]_{D_t}$ to K of the form

$$F = \zeta F_1^{m_1} \cdots F_s^{m_s}$$

where $\zeta \in K^\times$ is a constant and where F_i is a Chow form of Z_i in degree $\underline{D}$ for $i = 1,\ldots,s$. We also set the convention that there exists exactly one cycle of $\mathbf{P}^m_K$ of dimension -1. We denote it by 0 and we define a Chow form of 0 to be any constant in $K^\times$. The following lemma groups together general properties of the Chow forms.

Lemma 4.1. *Let Z be a cycle of $\mathbf{P}^m_K$ of dimension t, and let F be a Chow form of Z in degree $\underline{D}$.*

(i) The map F is multi-homogeneous of multi-degree $(N_0,\ldots,N_t)$ where

$$N_j = D_j^{-1}\big(D_0\cdots D_t\big)\deg(Z) \quad \text{for } j = 0,\ldots,t. \tag{4.2}$$

(ii) Let σ be a permutation of $\{0,\ldots,t\}$ and let F_σ be a Chow form of Z in degree $(D_{\sigma(0)},\ldots,D_{\sigma(t)})$. Then, there exists $\xi \in K^\times$ such that

$$F_\sigma(P_{\sigma(0)},\ldots,P_{\sigma(t)}) = \xi F(P_0,\ldots,P_t) \tag{4.3}$$

for any field L containing K and any choice of polynomials $P_j \in L[\underline{X}]_{D_j}$ for $j = 0,\ldots,t$.

(iii) *Let I be a finite set of indices and let $\left(D^{(i)}\right)_{i\in I}$ be a sequence of positive integers indexed by I such that $D_t = \sum_{i\in I} D^{(i)}$. For each $i \in I$, let F_i be a Chow form of Z in degree $(D_0, \dots, D_{t-1}, D^{(i)})$. Then, there exists $\zeta \in K^\times$ such that*

$$F\Big(P_0, \dots, P_{t-1}, \prod_{i\in I} Q_i\Big) = \zeta \prod_{i\in I} F_i(P_0, \dots, P_{t-1}, Q_i) \tag{4.4}$$

for any field L containing K and every choice of polynomials $Q_i \in L[\underline{X}]_{D^{(i)}}$, $i \in I$, and $P_j \in L[\underline{X}]_{D_j}$, $j = 0, \dots, t-1$.

Proof. By virtue of the definitions, it suffices to prove the lemma in the case where Z is a subvariety of $\mathbf{P}^m_K$. Then, (i) follows from Lemma 1.8 of [15] together with Remark 1) on page 15 of [15]. The properties (ii) and (iii) are also well known, at least in the case where $Z = \mathbf{P}^m_K$ (see §5.7 and §5.8 of [6]). The proof in general is similar. To sketch it, view both sides of (4.3) as defining functions on $\prod_{j=0}^t K[\underline{X}]_{D_j}$. The defining properties of Chow forms show that these polynomial maps are irreducible over K and that they have the same zeros over the algebraically closed field $\overline{K} \supseteq K$. So, they differ by a constant factor in $K^\times$. Similarly, view both sides of (4.4) as defining functions on the product $\left(\prod_{j=0}^{t-1} K[\underline{X}]_{D_j}\right) \times \left(\prod_{i\in I} K[\underline{X}]_{D^{(i)}}\right)$. Again these polynomial maps have the same zeros over $\overline{K}$, so they have the same irreducible factors over K. Moreover, all factors on the right hand side of (4.4) are distinct and irreducible. So the right hand side divides the left hand side. Finally, part (i) of the lemma shows that both sides are multi-homogeneous of the same multi-degree, so they differ by a factor in $K^\times$. ∎

Note that Lemma 4.1 (i) is not valid in general for a field K of positive characteristic p. In this case, the actual value of N_j may differ from the right hand side of (4.2) by a power of p, contrary to the affirmation in Remark 1) on page 15 of [15]. For example, let r and s be indeterminates over the field $\mathbf{F}_p$ of p elements, and let $K = \mathbf{F}_p(r, s)$. The ideal of $K[X_0, X_1, X_2]$ generated by $X_1^p - rX_0^p$ and $X_2^p - sX_0^p$ determines a subvariety Z of $\mathbf{P}^2_K$ of degree p^2. However, Z has only one purely inseparable point $(1 : r^{1/p} : s^{1/p})$ over an algebraic closure of K, and its Chow form in degree 1 sends a linear form $a_0X_0 + a_1X_1 + a_2X_2$ to $a_0^p + ra_1^p + sa_2^p$. Thus, this Chow form has degree $p < \deg(Z)$. This is the reason why we restricted to characteristic 0. One way to circumvent the problem could be to use the "resulting" forms developped by P. Philippon in Exposé no. 4 of [17] and generalized by G. Rémond in §3.2 of [20]. In our example, a resulting form turns out to be the p-th power of a Chow form, so that we recover the expected degree p^2, but we loose the irreducibility over K.

4.2. Intersections

Given a non-constant homogeneous polynomial $Q \in K[\underline{X}]$, we denote by $\mathrm{div}(Q)$ the *divisor* of Q in $\mathbf{P}^m_K$. If Q is irreducible, this is just the subvariety of dimension $m-1$ of $\mathbf{P}^m_K$ defined by Q. In general, if Q is written as a product of powers of irreducible polynomials $Q = \zeta Q_1^{e_1} \cdots Q_k^{e_k}$ with $\zeta \in K^\times$, then $\mathrm{div}(Q)$ is the cycle $e_1\mathrm{div}(Q_1) + \cdots + e_k\mathrm{div}(Q_k)$.

Let Z be a cycle of $\mathbf{P}^m_K$ of dimension t with components $Z_1, \ldots, Z_s$, let Q be a non-constant homogeneous polynomial in $K[\underline{X}]$ as above, and let L be an algebraically closed field containing K. We define the *support* of Z in $\mathbf{P}^m(L)$ to be the union $Z(L) = \bigcup_{i=1}^s Z_i(L)$. In particular, $\mathrm{div}(Q)(L)$ is the set of zeros of Q in $\mathbf{P}^m(L)$. We also put $0(L) = \emptyset$ for the cycle 0 of dimension -1. Moreover, asking that Q not vanish identically on any of the sets $Z_1(L), \ldots, Z_s(L)$ is clearly equivalent to asking that Q does not belong to any of the prime ideals of $K[\underline{X}]$ defining $Z_1, \ldots, Z_s$. So, this condition is independent of L. When it holds, we say that Q *does not vanish identically on any of the components of* Z. We can now state:

Lemma 4.2. *Let F and Z be as in Lemma 4.1, let $Q \in K[\underline{X}]_{D_t}$, and let F' be the polynomial map from $K[\underline{X}]_{D_0} \times \cdots \times K[\underline{X}]_{D_{t-1}}$ to K given by*

$$F'(P_0, \ldots, P_{t-1}) = F(P_0, \ldots, P_{t-1}, Q)$$

for any $P_j \in K[\underline{X}]_{D_j}$ with $j = 0, \ldots, t-1$. Then, F' is nonzero if and only if Q does not vanish identically on any of the components of Z. Moreover, if this is the case, then F' is a Chow form in degree $(D_0, \ldots, D_{t-1})$ of a cycle Z' of $\mathbf{P}^m_K$ of dimension $t-1$. This cycle depends only on Z and Q, not on the integers $D_0, \ldots, D_{t-1}$. It satisfies $Z'(L) = Z(L) \cap \mathrm{div}(Q)(L)$ for any algebraically closed field L containing K. Moreover, if $t \geq 1$, its degree is $D_t \deg(Z)$.

When the conditions of the lemma are satisfied, we say that the cycle Z' is the *intersection product* of Z and $\mathrm{div}(Q)$ and we denote this cycle as $Z \cdot \mathrm{div}(Q)$. This is consistent with the geometric notion of intersection of cycles from intersection theory, as we explain in the appendix.

Proof. Again it suffices to prove the lemma in the case where Z is a subvariety of $\mathbf{P}^m_K$. We may also assume that Q does not vanish identically on Z otherwise $P_0, \ldots, P_{t-1}$ and Q have at least one common zero on Z for any choice of polynomials $P_j \in K[\underline{X}]_{D_j}$ for $j = 0, \ldots, t-1$, and thus F' is identically zero. We may also assume $t \geq 1$ otherwise the lemma is satisfied with $Z' = 0$. Under these restrictions, Lemma 2.4 of [15] shows the existence of a cycle Z' of $\mathbf{P}^m_K$ of dimension $t-1$ such that F' is a Chow form of Z' and such that $Z'(L) = Z(L) \cap \mathrm{div}(Q)(L)$ for any algebraically closed field $L \supseteq K$. In particular, F' is nonzero. Moreover, applying Lemma 4.1 (i) to Z shows that F' is homogeneous of degree $D_1 \cdots D_t \deg Z$ on the factor $K[\underline{X}]_{D_0}$. Applying it to Z' shows that this degree is also given by $D_1 \cdots D_{t-1} \deg Z'$. By comparison, we deduce $\deg Z' = D_t \deg Z$.

It remains only to show that Z' is independent of the integers $D_0, \dots, D_{t-1}$. For this purpose, put $N = D_0 \cdots D_{t-1}$. Fix a Chow form E of Z in degree $(1, \dots, 1, D_t)$ and a Chow form E' of Z' in degree $(1, \dots, 1)$. Lemma 4.1 (iii) applied to Z and Z' shows that there are constants $\zeta, \zeta' \in K^\times$ such that

$$F\big(L_0^{D_0}, \dots, L_{t-1}^{D_{t-1}}, Q\big) = \zeta E(L_0, \dots, L_{t-1}, Q)^N,$$
$$F'\big(L_0^{D_0}, \dots, L_{t-1}^{D_{t-1}}\big) = \zeta' E'(L_0, \dots, L_{t-1})^N$$

for any choice of linear forms $L_0, \dots, L_{t-1} \in K[\underline{X}]_1$. Comparing both expressions, we deduce from the unique factorization property of polynomials that there exists $\xi \in K^\times$ such that

$$E'(L_0, \dots, L_{t-1}) = \xi E(L_0, \dots, L_{t-1}, Q)$$

for any $L_0, \dots, L_{t-1} \in K[\underline{X}]_1$. Thus, E' is independent of $D_0, \dots, D_{t-1}$ and so, Z' is also independent of these integers. $\square$

For any field L containing K, any integer $D \geq 0$, and any point $\underline{\alpha} \in L^{m+1}$, we denote by $\mathcal{L}_{\underline{\alpha}}$ the linear map from $L[\underline{X}]_D$ to L which sends a polynomial $P \in L[\underline{X}]_D$ to its value $P(\underline{\alpha})$ at $\underline{\alpha}$. With this notation, we are ready to prove:

Lemma 4.3. *Let F and Z be as in Lemma 4.1, and let L be an algebraically closed field containing K. Choose $Q_j \in K[\underline{X}]_{D_j}$ for $j = 1, \dots, t$, and put $N_0 = D_1 \cdots D_t \deg(Z)$. Then, there exist nonzero points $\underline{\alpha}_1, \dots, \underline{\alpha}_{N_0}$ of L^{m+1} which are independent of D_0 and represent (not necessarily distinct) elements of $Z(L)$, and there exists $\xi \in L$ such that*

$$F(P, Q_1, \dots, Q_t) = \xi \prod_{k=1}^{N_0} P(\underline{\alpha}_k),$$

for any $P \in L[\underline{X}]_{D_0}$.

Proof. Denote by G the polynomial map from $K[\underline{X}]_{D_0}$ to K given by $G(P) = F(P, Q_1, \dots, Q_t)$ for any $P \in K[\underline{X}]_{D_0}$. Lemma 4.2 shows, by induction on t, that either G is identically zero or it is a Chow form in degree D_0 of a cycle Z' of $\mathbf{P}^m_K$ of dimension 0 and degree N_0. Moreover, it shows that whether the first or the second case happens is independent of the choice of D_0, and that, when the second case happens, the cycle Z' is independent of D_0 and satisfies $Z'(L) \subseteq Z(L)$. So, if G is identically zero, we can choose any set of points $\underline{\alpha}_1, \dots, \underline{\alpha}_{N_0}$ of L^{m+1} representing elements of $Z(L)$ and put $\xi = 0$. Otherwise, write $Z' = m_1 Z'_1 + \cdots + m_s Z'_s$ where $Z'_1, \dots, Z'_s$ are the distinct components of Z'. For each $i = 1, \dots, s$, denote by n_i the degree of Z'_i and let G_i be a Chow form of

Z'_i in degree D_0. Since the field K has characteristic zero, each set $Z'_i(L)$ consists of n_i distinct points which are conjugate over K. Choose representatives $\underline{\alpha}_{i,1}, \dots, \underline{\alpha}_{i,n_i}$ for these points in L^{m+1}. Since G_i is homogeneous of degree n_i and since its set of zeros in $L[\underline{X}]_{D_0}$ is the union of the sets of zeros of the maps $\mathcal{L}_{\underline{\alpha}_{i,j}}\colon L[\underline{X}]_{D_0} \to L$ for $j = 1, \dots, n_i$, we get $G_i = \xi_i \prod_{j=1}^{n_i} \mathcal{L}_{\underline{\alpha}_{i,j}}$ for some $\xi_i \in L^\times$. The conclusion follows because $G = \zeta \prod_{i=1}^{s} G_i^{m_i}$ for some $\zeta \in K^\times$. $\square$

4.3. Heights of cycles

We define an *adelic convex body in degree $\underline{D}$* to be an adelic convex body for the K-vector space $K[\underline{X}]_{D_0} \times \cdots \times K[\underline{X}]_{D_t}$ which has the form of a Cartesian product $\mathcal{C} = \mathcal{C}_0 \times \cdots \times \mathcal{C}_t$ where $\mathcal{C}_j$ is an adelic convex body for $K[\underline{X}]_{D_j}$, $j = 0, \dots, t$. Fix such an adelic convex body $\mathcal{C}$. When Z is an irreducible subvariety of $\mathbf{P}^m_K$ of dimension t, we define its *height* relative to $\mathcal{C}$ by the formula

$$h_{\mathcal{C}}(Z) = h_{\mathcal{C}}(F),$$

where F denotes any Chow form of Z in degree $\underline{D}$. This is independent of the choice of F (see §3.3). For a cycle $Z = \sum_{i=1}^{s} m_i Z_i$ of dimension t, we extend the definition of height by linearity :

$$h_{\mathcal{C}}(Z) = \sum_{i=1}^{s} m_i h_{\mathcal{C}}(F_i).$$

We also define the *normalized height* of Z by

$$\tilde{h}_{\mathcal{C}}(Z) = (D_0 \cdots D_t \deg Z)^{-1} h_{\mathcal{C}}(Z).$$

By convention, we define $\tilde{h}_{\mathcal{C}}(0) = h_{\mathcal{C}}(0) = 0$ for the cycle 0 of dimension -1 (recall that a Chow form of 0 is any element of $K^\times$). The next lemma groups together useful properties of the normalized height.

Lemma 4.4. *Let $\mathcal{C}$ be as above, and let Z be a cycle of $\mathbf{P}^m_K$ of dimension t.*

(i) If σ is a permutation of $\{0, \dots, t\}$, and if we put $\mathcal{C}_\sigma = \mathcal{C}_{\sigma(0)} \times \cdots \times \mathcal{C}_{\sigma(t)}$, then we have $\tilde{h}_{\mathcal{C}_\sigma}(Z) = \tilde{h}_{\mathcal{C}}(Z)$.

(ii) There is at least one component Z' of Z such that $\tilde{h}_{\mathcal{C}}(Z') \le \tilde{h}_{\mathcal{C}}(Z)$.

(iii) If F is a Chow form of Z in degree $\underline{D}$ then

$$0 \le \tilde{h}_{\mathcal{C}}(Z) - (D_0 \cdots D_t \deg Z)^{-1} h_{\mathcal{C}}(F) \le 2 \sum_{j=0}^{t} \eta_j \,,$$

where $\eta_0, \dots, \eta_t$ are defined by (4.1).

Proof. The assertion (i) follows immediately from Lemma 4.1 (ii). For the remaining assertions, write $Z = \sum_{i=1}^{s} m_i Z_i$ where $Z_1, \dots, Z_s$ are the distinct components of Z. By definition, we have $\tilde{h}_{\mathcal{C}}(Z) = \sum_{i=1}^{s} \lambda_i \tilde{h}_{\mathcal{C}}(Z_i)$ where $\lambda_i = m_i \deg(Z_i)/\deg(Z)$ for $i = 1, \dots, s$. Since $\sum_{i=1}^{s} \lambda_i = 1$, this implies $\tilde{h}_{\mathcal{C}}(Z) \geq \min_{1\leq i\leq s} \tilde{h}_{\mathcal{C}}(Z_i)$ and thus proves (ii). Finally, let F_i be a Chow form of Z_i in degree $\underline{D}$ for $i = 1, \dots, s$. By definition, we can write $F = \zeta \prod_{i=1}^{s} F_i^{m_i}$ for some $\zeta \in K^\times$ and so, Proposition 3.10 together with Lemma 4.1 (i) give

$$h_{\mathcal{C}}(F) \leq \sum_{i=1}^{s} m_i h_{\mathcal{C}}(F_i) \leq h_{\mathcal{C}}(F) + 2 \sum_{j=0}^{t} N_j \log \binom{D_j + m_j}{m_j}$$

where $N_j = D_j^{-1}(D_0 \cdots D_t) \deg Z$ for $j = 0, \dots, t$. The assertion (iii) follows since $\sum_{i=1}^{s} m_i h_{\mathcal{C}}(F_i) = \sum_{i=1}^{s} m_i h_{\mathcal{C}}(Z_i) = h_{\mathcal{C}}(Z)$. $\square$

4.4. Heights of polynomials and local norms

Let v be a place of K, let D be a positive integer, and let $\mathcal{C}_v$ be a convex body of $\mathbf{C}_v[\underline{X}]_D$. For any point $\underline{\alpha} \in \mathbf{C}_v^{m+1}$, we denote by $\mathcal{L}_{\underline{\alpha}}\colon \mathbf{C}_v[\underline{X}]_D \to \mathbf{C}_v$ the linear map of evaluation at $\underline{\alpha}$, as in §4.2. According to the definitions from §3.2, we have

$$\|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}_v} = \sup_{Q\in\mathcal{C}_v} |\mathcal{L}_{\underline{\alpha}}(Q)|_v = \sup_{Q\in\mathcal{C}_v} |Q(\underline{\alpha})|_v.$$

Note that this number is strictly positive if $\underline{\alpha} \neq 0$. Now, let $P \in \mathbf{C}_v[\underline{X}]_D$. We define the *norm of P relative to $\mathcal{C}_v$* by

$$\|P\|_{\mathcal{C}_v} = \inf\big\{|\rho|_v\,;\ \rho \in \mathbf{C}_v \text{ and } P \in \rho\,\mathcal{C}_v\big\}.$$

For a point $\alpha \in \mathbf{P}^m(\mathbf{C}_v)$, we define the *local norm of P at α relative to $\mathcal{C}_v$* by

$$\|P\|_{\alpha,\mathcal{C}_v} = \frac{|P(\underline{\alpha})|_v}{\|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}_v}},$$

where $\underline{\alpha}$ denotes any representative of α in $\mathbf{C}_v^{m+1}$. This is clearly independent of the choice of the representative $\underline{\alpha}$, and we have

$$\|P\|_{\alpha,\mathcal{C}_v} \leq \|P\|_{\mathcal{C}_v}. \tag{4.5}$$

We also formally define $\|\mathcal{L}_\alpha\|_{\mathcal{C}_v} := \|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}_v}$ for any choice of projective coordinates $\underline{\alpha}$ of α with $\|\underline{\alpha}\|_v = 1$ (although the symbol $\mathcal{L}_\alpha$ on its own has no meaning).

We adelize the preceding notions in the following way. Let $\mathcal{C} = \prod_v \mathcal{C}_v$ be an adelic convex body for $K[\underline{X}]_D$ and let $P \in K[\underline{X}]_D$. If P is nonzero, we define its *height relative to $\mathcal{C}$* by

$$h_{\mathcal{C}}(P) = \sum_{v\in\mathcal{M}_K} \lambda_v \log \|P\|_{\mathcal{C}_v}$$

(see §2.1 for the definition of the numbers λ_v). This is a finite sum since $\|P\|_{\mathcal{C}_v} = 1$ except for finitely many places $v \in \mathcal{M}_K$. We also define an *adelic projective point* of $\mathbf{P}^m_K$ to be an element $\alpha = (\alpha_v)_{v\in\mathcal{M}_k}$ of the product $\prod_{v\in\mathcal{M}_K} \mathbf{P}^m(\mathbf{C}_v)$. For such a point α, we define the *local norm of P at α relative to $\mathcal{C}$* by

$$\|P\|_{\alpha,\mathcal{C}} = \prod_{v\in\mathcal{M}_K} \|P\|^{\lambda_v}_{\alpha_v,\mathcal{C}_v}.$$

The upper bound (4.5) then implies

$$\|P\|_{\alpha,\mathcal{C}} \le e^{h_{\mathcal{C}}(P)}.$$

Note also that, given any projective adelic point α, the symbol $\|\mathcal{L}_{\alpha_v}\|_{\mathcal{C}_v}$ is equal to 1 except for finitely many places. So, the product

$$\|\mathcal{L}_\alpha\|_{\mathcal{C}} := \prod_{v\in\mathcal{M}_K} \|\mathcal{L}_{\alpha_v}\|^{\lambda_v}_{\mathcal{C}_v}$$

is convergent. Finally, we define the *set of adelic points* of a cycle Z of $\mathbf{P}^m_K$ to be the product $Z_{\mathbf{A}} = \prod_{v\in\mathcal{M}_K} Z(\mathbf{C}_v)$, and we say that an element of $Z_{\mathbf{A}}$ is an *adelic point* of Z.

4.5. Multi-linearity of the height

We show in this section that the height $h_{\mathcal{C}_0\times\cdots\times\mathcal{C}_t}(Z)$ behaves like an intersection product $Z\cdot\mathcal{C}_0\cdot\ldots\cdot\mathcal{C}_t$ in the sense that it is quasi-linear in each of $Z,\mathcal{C}_0,\ldots,\mathcal{C}_t$. By definition, we already know that the height is linear in Z. The quasi-linearity in terms of each of the adelic convex bodies $\mathcal{C}_0,\ldots,\mathcal{C}_t$ relates to the following notion of product:

Definition. Let $D', D'' \ge 0$ be integers and let $D = D' + D''$. Moreover, let $\mathcal{C}' = \prod_v \mathcal{C}'_v$ and $\mathcal{C}'' = \prod_v \mathcal{C}''_v$ be adelic convex bodies for $K[\underline{X}]_{D'}$ and $K[\underline{X}]_{D''}$ respectively. We define their *product* $\mathcal{C}'\mathcal{C}''$ as the adelic convex body $\mathcal{C} = \prod_v \mathcal{C}_v$ for $K[\underline{X}]_D$ whose v-component $\mathcal{C}_v$ is the symmetric convex hull of the set of products $Q'Q''$ with $Q' \in \mathcal{C}'_v$ and $Q'' \in \mathcal{C}''_v$, for each place $v \in \mathcal{M}_K$.

It is easily seen using definition 3.8 that this construction gives indeed an adelic convex body for $K[\underline{X}]_D$. We will need the following lemma.

Lemma 4.5. *Let $v \in \mathcal{M}_K$ be any place and let $\underline{\alpha} \in \mathbf{C}_v^{m+1}$ be any point. With the notations of the above definition, we have*

$$\|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}_v} = \|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}'_v}\|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}''_v}.$$

Proof. Every element $Q \in \mathcal{C}_v$ can be written as a finite sum $Q = \sum_{i=1}^s \lambda_i Q'_i Q''_i$ where $Q'_i \in \mathcal{C}'_v$ and $Q''_i \in \mathcal{C}''_v$ for $i = 1,\ldots,s$, and where the coefficients $\lambda_1,\ldots,\lambda_s \in \mathbf{C}_v$ satisfy

$\sum_i |\lambda_i|_v \le 1$ if v is Archimedean and $\max_i |\lambda_i|_v \le 1$ otherwise. In both cases, we get $|Q(\underline{\alpha})|_v \le \|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}'_v} \|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}''_v}$ and thus $\|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}_v} \le \|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}'_v} \|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}''_v}$. To establish the reverse inequality, we choose $Q = Q'Q''$ with $Q' \in \mathcal{C}'_v$ and $Q'' \in \mathcal{C}''_v$, and we take the supremum of both sides of the equality $|Q'(\underline{\alpha})|_v |Q''(\underline{\alpha})|_v = |Q(\underline{\alpha})|_v$ over the set of all such polynomials Q. $\square$

Proposition 4.6. *Let $\mathcal{C} = \mathcal{C}_0 \times \cdots \times \mathcal{C}_t$ be an adelic convex body in degree $\underline{D}$ and let s be an integer with $0 \le s \le t$. Suppose that D_s is written as a finite sum of positive integers $D_s = \sum_{i \in I} D^{(i)}$ and that we have a corresponding decomposition of $\mathcal{C}_s$ into a product*

$$\mathcal{C}_s = \prod_{i \in I} \mathcal{C}^{(i)} ,$$

where $\mathcal{C}^{(i)}$ is a convex body associated to $K[\underline{X}]_{D^{(i)}}$ for each $i \in I$. Suppose also that η' is some positive real number such that $\Phi(D^{(i)}) \le \eta'$ for any $i \in I$. For each $i \in I$, denote by $\mathcal{E}^{(i)}$ the adelic convex body which, as a Cartesian product, has the same factors as $\mathcal{C}$ except that the s-th factor $\mathcal{C}_s$ is replaced by $\mathcal{C}^{(i)}$. Then, for any cycle Z of dimension t, we have

$$-2\Big(\sum_{\substack{j=0\\ j\neq s}}^{t} \eta_j\Big)\Big(\prod_{j=0}^{t} D_j\Big) \deg Z \;\le\; h_{\mathcal{C}}(Z) - \sum_{i \in I} h_{\mathcal{E}^{(i)}}(Z) \;\le\; 2\eta'\Big(\prod_{j=0}^{t} D_j\Big) \deg Z ,$$

where $\eta_0, \dots, \eta_t$ are defined by (4.1).

Proof. Since, by Lemma 4.4 (i), the height of Z relative to a Cartesian product of adelic convex bodies does not change under a permutation of its factors, we may assume, without loss of generality, that $s = t$. By linearity, we may also assume that Z is a subvariety of $\mathbf{P}^m_K$.

Let F be a Chow form of Z in degree $(D_0, \dots, D_t)$ and, for $i \in I$, let F_i be a Chow form of Z in degree $(D_0, \dots, D_{t-1}, D^{(i)})$. Lemma 4.1 (iii) shows that there exists $\zeta \in K^\times$ such that

$$F\Big(P_0, \dots, P_{t-1}, \prod_{i \in I} Q_i\Big) = \zeta \prod_{i \in I} F_i(P_0, \dots, P_{t-1}, Q_i) \tag{4.6}$$

for any field L containing K and any choice of polynomials $Q_i \in L[\underline{X}]_{D^{(i)}}$ for $i \in I$ and $P_j \in L[\underline{X}]_{D_j}$ for $j = 0, \dots, t-1$. Define $N_0, \dots, N_t$ as in Lemma 4.1 (i).

For the upper bound, fix a place $v \in \mathcal{M}_K$ and a choice of polynomials $P_j \in \mathcal{C}_{j,v}$ for $j = 0, \dots, t-1$. Define polynomial maps $G\colon \mathbf{C}_v[\underline{X}]_{D_t} \to \mathbf{C}_v$ and $G_i\colon \mathbf{C}_v[\underline{X}]_{D^{(i)}} \to \mathbf{C}_v$ for $i \in I$ by putting

$$G(Q) = F(P_0, \dots, P_{t-1}, Q) \quad \text{and} \quad G_i(Q_i) = F_i(P_0, \dots, P_{t-1}, Q_i)$$

for any $Q \in \mathbf{C}_v[\underline{X}]_{D_t}$ and any $Q_i \in \mathbf{C}_v[\underline{X}]_{D^{(i)}}$ with $i \in I$. Lemma 4.3 shows that there exist nonzero elements $\underline{\alpha}_1, \dots, \underline{\alpha}_{N_t}$ of $\mathbf{C}_v^{m+1}$ and constants $\xi \in \mathbf{C}_v$ and $\xi_i \in \mathbf{C}_v$ for $i \in I$ such that

$$G(Q) = \xi \prod_{k=1}^{N_t} Q(\underline{\alpha}_k) \quad \text{and} \quad G_i(Q_i) = \xi_i \prod_{k=1}^{N_t} Q_i(\underline{\alpha}_k)$$

for any choice of polynomials Q and Q_i with $i \in I$, as above. By virtue of (4.6), we have $G(Q) = \zeta \prod_{i\in I} G_i(Q_i)$ whenever $Q = \prod_{i\in I} Q_i$ and therefore $\xi = \zeta \prod_{i\in I} \xi_i$. Applying Proposition 3.7 to the above factorizations of the maps G and G_i into products of linear forms, we find

$$\|G\|_{\mathcal{C}_{t,v}} \le |\xi|_v \prod_{k=1}^{N_t} \|\mathcal{L}_{\underline{\alpha}_k}\|_{\mathcal{C}_{t,v}} \quad \text{and} \quad |\xi_i|_v \prod_{k=1}^{N_t} \|\mathcal{L}_{\underline{\alpha}_k}\|_{\mathcal{C}_v^{(i)}} \le \binom{D^{(i)}+m}{m}^{2N_t\epsilon_v} \|G_i\|_{\mathcal{C}_v^{(i)}}.$$

On the other hand, Lemma 4.5 shows that, for any $\underline{\alpha} \in \mathbf{C}_v^{m+1}$, we have $\|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}_{t,v}} = \prod_{i\in I} \|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}_v^{(i)}}$. Combining this with the previous inequalities and using the relation $\xi = \zeta \prod_{i\in I} \xi_i$, we get

$$\|G\|_{\mathcal{C}_{t,v}} \le |\zeta|_v \prod_{i\in I} \left(\binom{D^{(i)}+m}{m}^{2N_t\epsilon_v} \|G_i\|_{\mathcal{C}_v^{(i)}} \right).$$

Taking the supremum of both sides of this inequality over all choices of $P_0, \dots, P_{t-1}$, we deduce

$$\|F\|_{\mathcal{C}_v} \le |\zeta|_v \prod_{i\in I} \left(\binom{D^{(i)}+m}{m}^{2N_t\epsilon_v} \|F_i\|_{\mathcal{E}_v^{(i)}} \right).$$

Finally, taking the logarithms of both sides, multiplying them by λ_v, summing over all $v \in \mathcal{M}_K$ and taking into account the product formula for ζ, we get

$$h_{\mathcal{C}}(Z) \le 2N_t \sum_{i\in I} \log \binom{D^{(i)}+m}{m} + \sum_{i\in I} h_{\mathcal{E}^{(i)}}(Z).$$

For the lower bound, fix a place $v \in \mathcal{M}_K$ and a choice of polynomials $Q_i \in \mathcal{C}_v^{(i)}$ for $i \in I$. Put $Q = \prod_{i\in I} Q_i$ and define $\mathbf{C}_v$-valued polynomial maps E and E_i for $i \in I$ on the product space $\prod_{j=0}^{t-1} \mathbf{C}_v[\underline{X}]_{D_j}$ by putting

$$E(P_0, \dots, P_{t-1}) = F(P_0, \dots, P_{t-1}, Q) \quad \text{and} \quad E_i(P_0, \dots, P_{t-1}) = F_i(P_0, \dots, P_{t-1}, Q_i)$$

for any choice of $P_j \in \mathbf{C}_v[\underline{X}]_{D_j}$ for $j = 0, \dots, t-1$. By virtue of (4.6), we have $E = \zeta \prod_{i\in I} E_i$. Since E is multi-homogeneous of multi-degree $(N_0, \dots, N_{t-1})$, Proposition 3.7 gives

$$|\zeta|_v \prod_{i\in I} \|E_i\|_{\mathcal{C}_{0,v}\times\cdots\times\mathcal{C}_{t-1,v}} \le \left(\prod_{j=0}^{t-1} \binom{D_j+m}{m}^{2N_j\epsilon_v} \right) \|E\|_{\mathcal{C}_{0,v}\times\cdots\times\mathcal{C}_{t-1,v}}.$$

Taking the supremum of both sides over all choices of polynomials $Q_i \in \mathcal{C}_v^{(i)}$ with $i \in I$ and using the fact that $Q = \prod_{i\in I} Q_i$ then belongs to $\mathcal{C}_{t,v}$, we get

$$|\zeta|_v \prod_{i\in I} \|F_i\|_{\mathcal{E}_v^{(i)}} \le \left(\prod_{j=0}^{t-1} \binom{D_j+m}{m}^{2N_j\epsilon_v} \right) \|F\|_{\mathcal{C}_v}.$$

Arguing as in the previous situation, we deduce

$$\sum_{i\in I} h_{\mathcal{E}^{(i)}}(Z) \le h_{\mathcal{C}}(Z) + 2\sum_{j=0}^{t-1} N_j \log \binom{D_j+m}{m}.$$

The conclusion follows in both cases from the choice of η', the definition of $\eta_0, \dots, \eta_t$ in (4.1) and the definition of $N_0, \dots, N_t$ in (4.2). ∎

4.6. Comparison between heights

We start with a very simple observation:

Lemma 4.7. *Let $\mathcal{C} = \mathcal{C}_0 \times \cdots \times \mathcal{C}_t$ and $\mathcal{C}' = \mathcal{C}'_0 \times \cdots \times \mathcal{C}'_t$ be adelic convex bodies in degree $\underline{D}$, and let Z be a cycle of $\mathbf{P}_K^m$ of dimension t. Then, there exists a finite subset $\mathcal{S}$ of $\{0, \dots, t\} \times \mathcal{M}_K$ such that $\mathcal{C}'_{j,v} = \mathcal{C}_{j,v}$ for any $(j,v) \notin \mathcal{S}$. Moreover, for each $(j,v) \in \mathcal{S}$, there exists $\rho_{j,v} \in \mathbf{C}_v^\times$ such that $\mathcal{C}'_{j,v} \subseteq \rho_{j,v}\mathcal{C}_{j,v}$. For such a choice of set $\mathcal{S}$ and numbers $\rho_{j,v}$ with $(j,v) \in \mathcal{S}$, we have*

$$\tilde{h}_{\mathcal{C}'}(Z) \le \tilde{h}_{\mathcal{C}}(Z) + \sum_{(j,v)\in\mathcal{S}} \lambda_v D_j^{-1} \log |\rho_{j,v}|_v.$$

Proof. The existence of $\mathcal{S}$ and $(\rho_{j,v})_{(j,v)\in\mathcal{S}}$ follows from the definition of adelic convex bodies (see Definition 3.8). To prove the main assertion of the lemma, we may assume, by induction on the cardinality of $\mathcal{S}$, that $\mathcal{S}$ consists of only one point (ℓ, w). Moreover, by permuting the factors of $\mathcal{C}$ and $\mathcal{C}'$ if necessary, we may assume, thanks to Lemma 4.4 (i), that $\ell = 0$. We may also assume that Z is a subvariety of $\mathbf{P}_K^m$. This being so, let F be a Chow form of Z in degree $\underline{D}$. Since, by Lemma 4.1 (i), the map F is homogeneous of degree $N_0 = D_1 \cdots D_t \deg Z$ on the factor $K[\underline{X}]_{D_0}$, we find

$$\|F\|_{\mathcal{C}'_w} \le \|F\|_{\mathcal{C}_w} |\rho_{0,w}|_w^{N_0}.$$

For the other places $v \neq w$, we have $\|F\|_{\mathcal{C}'_v} = \|F\|_{\mathcal{C}_v}$ since $\mathcal{C}'_v = \mathcal{C}_v$. Hence, we get

$$h_{\mathcal{C}'}(Z) \le h_{\mathcal{C}}(Z) + \lambda_w N_0 \log |\rho_{0,w}|_w.$$

The required upper bound follows by dividing both sides by $D_0 N_0$. ∎

The next lemma refines the above result. It uses an idea of P. Philippon from Proposition 2.5 of [15], which exploits the factorization property of the Chow forms stated in Lemma 4.3.

Lemma 4.8. *Let $\mathcal{C}$, $\mathcal{C}'$ and Z be as in Lemma 4.7, and let $\mathcal{S}$ be a finite subset of $\{0,\dots,t\}\times\mathcal{M}_K$ such that $\mathcal{C}'_{j,v}=\mathcal{C}_{j,v}$ for any $(j,v)\notin\mathcal{S}$. Then, for each $(j,v)\in\mathcal{S}$, there is a positive real number $A_{j,v}$ which satisfies*

$$\|\mathcal{L}_\alpha\|_{\mathcal{C}'_{j,v}} \le A_{j,v}\|\mathcal{L}_\alpha\|_{\mathcal{C}_{j,v}} \tag{4.7}$$

for every $\alpha\in Z(\mathbf{C}_v)$. For such a choice of numbers $A_{j,v}$ with $(j,v)\in\mathcal{S}$, we have

$$\tilde{h}_{\mathcal{C}'}(Z) \le \tilde{h}_{\mathcal{C}}(Z) + \sum_{(j,v)\in\mathcal{S}} \lambda_v\left(D_j^{-1}\log A_{j,v} + 2\eta_j\epsilon_v\right),$$

where $\eta_0,\dots,\eta_t$ are given by (4.1) and where, for $v\in\mathcal{M}_K$, ϵ_v is 1 or 0 depending on whether v is Archimedean or not (see §2.1).

Proof. Lemma 4.7 shows that, for any $(j,v)\in\mathcal{S}$, there exists $\rho_{j,v}\in\mathbf{C}_v^\times$ such that $\mathcal{C}'_{j,v}\subseteq\rho_{j,v}\mathcal{C}_{j,v}$. Then, (4.7) holds with $A_{j,v}=|\rho_{j,v}|_v$, for any $\alpha\in Z(\mathbf{C}_v)$. This shows the existence of the numbers $A_{j,v}$.

To prove the second part of the lemma, it suffices simply, as in the proof of Lemma 4.7, to consider the case where $\mathcal{S}$ consists of one point of the form $(0,w)$, and where Z is a subvariety of $\mathbf{P}^m_K$. Let F be a Chow form of Z in degree $\underline{D}$ and let $N_0=D_1\cdots D_t\deg Z$. Fix a choice of polynomials $Q_j\in\mathcal{C}_{j,w}=\mathcal{C}'_{j,w}$ for $j=1,\dots,t$ and consider the polynomial map $E\colon\mathbf{C}_w[\underline{X}]_{D_0}\to\mathbf{C}_w$ given by

$$E(P)=F(P,Q_1,\dots,Q_t)$$

for any $P\in\mathbf{C}_w[\underline{X}]_{D_0}$. If $E\neq0$, Lemma 4.3 shows that there exist $\xi\in\mathbf{C}_w^\times$ and elements $\underline{\alpha}_1,\dots,\underline{\alpha}_{N_0}$ of $\mathbf{C}_w^{m+1}$ representing points of $Z(\mathbf{C}_w)$ such that

$$E(P)=\xi\prod_{k=1}^{N_0}P(\underline{\alpha}_k)$$

for any $P\in\mathbf{C}_w[\underline{X}]_{D_0}$. Applying Proposition 3.7 to this factorization of E gives

$$\|E\|_{\mathcal{C}'_{0,w}} \le |\xi|_w\prod_{k=1}^{N_0}\|\mathcal{L}_{\underline{\alpha}_k}\|_{\mathcal{C}'_{0,w}} \quad\text{and}\quad |\xi|_w\prod_{k=1}^{N_0}\|\mathcal{L}_{\underline{\alpha}_k}\|_{\mathcal{C}_{0,w}} \le \binom{D_0+m}{m}^{2N_0\epsilon_w}\|E\|_{\mathcal{C}_{0,w}}.$$

On the other hand, the hypotheses imply $\|\mathcal{L}_{\underline{\alpha}_k}\|_{\mathcal{C}'_{0,w}}\le A_{0,w}\|\mathcal{L}_{\underline{\alpha}_k}\|_{\mathcal{C}_{0,w}}$ for $k=1,\dots,N_0$. Combining this with the previous inequalities, we get

$$\|E\|_{\mathcal{C}'_{0,w}} \le A_{0,w}^{N_0}\binom{D_0+m}{m}^{2N_0\epsilon_w}\|E\|_{\mathcal{C}_{0,w}}.$$

Note that this last inequality still holds even if $E=0$. Taking the supremum of both sides over all possible choices of $(Q_1,\dots,Q_t)\in\mathcal{C}_{1,w}\times\cdots\times\mathcal{C}_{t,w}$, it becomes

$$\|F\|_{\mathcal{C}'_w}\le A_{0,w}^{N_0}\binom{D_0+m}{m}^{2N_0\epsilon_w}\|F\|_{\mathcal{C}_w}.$$

For the other places $v\neq w$, we have $\mathcal{C}'_v=\mathcal{C}_v$ and therefore $\|F\|_{\mathcal{C}'_v}=\|F\|_{\mathcal{C}_v}$. Hence, we deduce

$$h_{\mathcal{C}'}(Z)\le h_{\mathcal{C}}(Z)+\lambda_w N_0\left(\log(A_{0,w})+2\epsilon_w\log\binom{D_0+m}{m}\right).$$

Finally, dividing both sides by D_0N_0, we get the required upper bound

$$\tilde h_{\mathcal{C}'}(Z)\le\tilde h_{\mathcal{C}}(Z)+\lambda_w\Big(D_0^{-1}\log(A_{0,w})+2\epsilon_w\eta_0\Big).\qquad\square$$

4.7. The height of a section by an hypersurface

Through out this section, we fix an adelic convex body $\mathcal{C}=\mathcal{C}_0\times\cdots\times\mathcal{C}_t$ in degree $\underline{D}=(D_0,\dots,D_t)$. We also extend the notation $h_{\mathcal{C}}(Z)$ to any cycle Z of $\mathbf{P}^m_K$ of dimension $s\le t$ by putting $h_{\mathcal{C}}(Z)=h_{\mathcal{C}_0\times\cdots\times\mathcal{C}_s}(Z)$ when $s\ge 0$ and $h_{\mathcal{C}}(0)=0$ when $s=-1$. We define similarly $\tilde h_{\mathcal{C}}(Z)$ and also $h_{\mathcal{C}}(F)$ when F is a Chow form of Z in degree $(D_0,\dots,D_s)$. We start with the following version of an arithmetic Bézout's inequality.

Proposition 4.9. *Let Z be a cycle of dimension t and let $P\in K[\underline{X}]_{D_t}$. Assume that P does not vanish identically on any component of Z, and let Z' be the intersection product $Z\cdot\mathrm{div}(P)$ defined after Lemma 4.2. Then, we have*

$$\tilde h_{\mathcal{C}}(Z')\le\tilde h_{\mathcal{C}}(Z)+(D_t)^{-1}h_{\mathcal{C}_t}(P)+2\sum_{j=0}^{t-1}\eta_j.$$

Moreover, when $t\ge 1$, there exists at least one component Z'' of Z' which satisfies the above inequality with Z' replaced by Z''.

Proof. Let F be a Chow form of Z in degree $\underline{D}$. Write $N_t:=(D_0\cdots D_{t-1})\deg Z$ and consider the polynomial map F' from $K[\underline{X}]_{D_0}\times\cdots\times K[\underline{X}]_{D_{t-1}}$ to K given by

$$F'(P_0,\dots,P_{t-1}):=F(P_0,\dots,P_{t-1},P)$$

for any choice of $P_j\in K[\underline{X}]_{D_j}$ for $j=0,\dots,t-1$. Lemma 4.2 shows that F' is a Chow form of the cycle Z'. Since F is homogeneous of degree N_t on the factor $K[\underline{X}]_{D_t}$, we have the upper bound

$$\|F'\|_{\mathcal{C}_{0,v}\times\cdots\times\mathcal{C}_{t-1,v}}\le\|P\|_{\mathcal{C}_{t,v}}^{N_t}\|F\|_{\mathcal{C}_v}$$

for any place $v \in \mathcal{M}_K$, and thus, using Lemma 4.4 (iii), we find

$$(D_0 \cdots D_t \deg Z)^{-1} h_{\mathcal{C}}(F') \le \tilde{h}_{\mathcal{C}}(Z) + (D_t)^{-1} h_{\mathcal{C}_t}(P).$$

If $t = 0$, the map F' is a constant in $K^\times$ and the product formula shows that the left hand side of this inequality reduces to 0. The proposition then follows since, in this case, we have $Z' = 0$ and thus $\tilde{h}_{\mathcal{C}}(Z') = 0$ according to our convention. Otherwise, the cycle Z' has dimension ≥ 0 and degree $D_t \deg Z$, and Lemma 4.4 (iii) gives

$$\tilde{h}_{\mathcal{C}}(Z') \le (D_0 \cdots D_t \deg Z)^{-1} h_{\mathcal{C}}(F') + 2 \sum_{j=0}^{t-1} \eta_j.$$

This completes the proof of the first part of the proposition. The second part follows from Lemma 4.4 (ii). ∎

We also have the following finer arithmetic intersection formula involving the values of the local norms of the polynomial P at the adelic points of Z, whose set $Z_{\mathbf{A}}$ is defined at the end of §4.4.

Proposition 4.10. *With the notations of Proposition 4.9, we have*

$$\tilde{h}_{\mathcal{C}}(Z') \le \tilde{h}_{\mathcal{C}}(Z) + (D_t)^{-1} \log \sup_{\alpha \in Z_{\mathbf{A}}} \|P\|_{\alpha, \mathcal{C}_t} + 2 \sum_{j=0}^{t} \eta_j.$$

Moreover, when $t \ge 1$, there exists at least one irreducible component Z'' of Z' which satisfies the above inequality with Z' replaced by Z''.

Proof. Let F and F' be as in the proof of Proposition 4.9. We first observe that, if $\mathcal{C}'_t = \prod_v \mathcal{C}'_{t,v}$ is an adelic convex body for $K[\underline{X}]_{D_t}$ such that $P \in \mathcal{C}'_{t,v}$ for each $v \in \mathcal{M}_K$, then the Cartesian product $\mathcal{C}' = \mathcal{C}_0 \times \cdots \times \mathcal{C}_{t-1} \times \mathcal{C}'_t$ satisfies

$$h_{\mathcal{C}}(F') = h_{\mathcal{C}'}(F') \le h_{\mathcal{C}'}(F) \le h_{\mathcal{C}'}(Z). \tag{4.8}$$

Fix an arbitrarily small positive number δ. By (4.5), the number

$$A_v = \sup_{\alpha \in Z(\mathbf{C}_v)} \|P\|_{\alpha, \mathcal{C}_{t,v}}$$

is finite for each $v \in \mathcal{M}_K$. Denote by $\mathcal{S}_t$ the finite set consisting of all places v of K for which $A_v \ne 1$. We claim that, for any given place $v \in \mathcal{M}_K$, there exists a convex body $\mathcal{C}'_{t,v}$ of $\mathbf{C}_v[\underline{X}]_{D_t}$ which contains P and satisfies

$$\|\mathcal{L}_\alpha\|_{\mathcal{C}'_{t,v}} \le (A_v + \delta) \|\mathcal{L}_\alpha\|_{\mathcal{C}_{t,v}}$$

for any $\alpha \in Z(\mathbf{C}_v)$. To see this, observe that, for each real $\delta_v > 0$ there is a convex body $\mathcal{C}'_{t,v}$ of $\mathbf{C}_v[\underline{X}]_{D_t}$ which contains P and satisfies $\|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}'_{t,v}} \leq |P(\underline{\alpha})|_v + \delta_v$ for any $\underline{\alpha} \in \mathbf{C}_v^{m+1}$ with $\|\underline{\alpha}\|_v = 1$. Moreover, the definition of A_v implies $|P(\underline{\alpha})|_v \leq A_v \|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}_{t,v}}$ for any such $\underline{\alpha}$ which represents a point of $Z(\mathbf{C}_v)$. Thus, $\mathcal{C}'_{t,v}$ has the required properties if we choose δ_v such that $\delta_v \leq \delta \|\mathcal{L}_{\underline{\alpha}}\|_{\mathcal{C}_{t,v}}$ for any $\underline{\alpha} \in \mathbf{C}_v^{m+1}$ with $\|\underline{\alpha}\|_v = 1$.

Choose a convex body $\mathcal{C}'_{t,v}$ as above for each $v \in \mathcal{S}_t$. For the other places $v \notin \mathcal{S}_t$, put $\mathcal{C}'_{t,v} = \mathcal{C}_{t,v}$. Define $\mathcal{C}'_t = \prod_v \mathcal{C}'_{t,v}$ and $\mathcal{C}' = \mathcal{C}_0 \times \cdots \times \mathcal{C}_{t-1} \times \mathcal{C}'_t$. Then, Lemma 4.8 applies to the present situation with $\mathcal{S} = \{t\} \times \mathcal{S}_t$ and $A_{t,v} = A_v + \delta$ for any $v \in \mathcal{S}_t$. This gives

$$\tilde{h}_{\mathcal{C}'}(Z) \leq \tilde{h}_{\mathcal{C}}(Z) + \sum_{v \in \mathcal{S}_t} \lambda_v \big(D_t^{-1} \log(A_v + \delta) + 2\eta_t \epsilon_v \big).$$

Combining this with (4.8) and letting $\delta > 0$ tend to zero, we deduce

$$(D_0 \cdots D_t \deg Z)^{-1} h_{\mathcal{C}}(F') \leq \tilde{h}_{\mathcal{C}}(Z) + D_t^{-1} \log \sup_{\alpha \in Z_{\mathbf{A}}} \|P\|_{\alpha, \mathcal{C}_t} + 2\eta_t.$$

The conclusion follows using Lemma 4.4, as in the proof of Proposition 4.9. ∎

Note that, since $\|P\|_{\alpha,\mathcal{C}_t} \leq \exp\big(h_{\mathcal{C}_t}(P)\big)$ for any projective adelic point α, Proposition 4.9 is a consequence of Proposition 4.10, up to the value of the constants. Later, we shall regard these propositions as lower bounds for the height of a cycle Z in terms of the height of one of its sections. Here is a first application of this principle.

Proposition 4.11. *For any cycle Z of $\mathbf{P}_K^m$ of dimension t we have*

$$\tilde{h}_{\mathcal{C}}(Z) \geq \sum_{j=0}^{t} D_j^{-1} \log \Big(\inf_{\alpha \in Z_{\mathbf{A}}} \|\mathcal{L}_\alpha\|_{\mathcal{C}_j} \Big) - 2 \sum_{j=0}^{t} (t - j + 1)\eta_j \,.$$

Proof. Suppose that Z_s is a subvariety of $\mathbf{P}_K^m$ of dimension s, with $0 \leq s \leq t$, which is contained in at least one component of Z. Select a coordinate index k such that Z_s is not contained in the hyperplane $X_k = 0$. If $s \geq 1$, Proposition 4.10 shows that there is a component Z_{s-1} of $Z_s \cdot \mathrm{div}(X_k^{D_s})$ which satisfies

$$\tilde{h}_{\mathcal{C}}(Z_{s-1}) \leq \tilde{h}_{\mathcal{C}}(Z_s) + D_s^{-1} \log \Big(\sup_{\alpha \in (Z_s)_{\mathbf{A}}} \|X_k^{D_s}\|_{\alpha, \mathcal{C}_s} \Big) + 2 \sum_{j=0}^{s} \eta_j \,.$$

If $s = 0$, it shows that the same inequality holds by taking $Z_{-1} = 0$. In both cases, since $(Z_s)_{\mathbf{A}}$ is contained in $Z_{\mathbf{A}}$ and since $\|X_k^{D_s}\|_{\alpha,\mathcal{C}_s} \leq \|\mathcal{L}_\alpha\|_{\mathcal{C}_s}^{-1}$ for any adelic projective point α, we deduce

$$\tilde{h}_{\mathcal{C}}(Z_{s-1}) \leq \tilde{h}_{\mathcal{C}}(Z_s) - D_s^{-1} \log \Big(\inf_{\alpha \in Z_{\mathbf{A}}} \|\mathcal{L}_\alpha\|_{\mathcal{C}_s} \Big) + 2 \sum_{j=0}^{s} \eta_j \,. \tag{4.9}$$

Starting with a component Z_t of Z such that $\tilde{h}_{\mathcal{C}}(Z_t) \le \tilde{h}_{\mathcal{C}}(Z)$ (see Lemma 4.4 (ii)), this process generates a sequence of subvarieties $Z_t \supseteq \cdots \supseteq Z_0$ of $\mathbf{P}^m_K$ together with the cycle $Z_{-1} = 0$ so that (4.9) holds for $s = 0, \dots, t$. The conclusion follows, taking into account that $\tilde{h}_{\mathcal{C}}(Z_{-1}) = 0$. $\square$

5. Estimates for a choice of adelic convex bodies

We now turn to families of adelic convex bodies which are appropriate to the context of our criterion of algebraic independence with multiplicities. Their definition is given in the next section §5.1. It involves various conditions including upper bounds for the absolute values of the divided derivatives of the polynomials at the point $\underline{\theta}$. We show that these convex bodies can be essentially decomposed into products of more elementary convex bodies in degree 1. In §5.2, we use this fact to deduce upper estimates for the height of a cycle of $\mathbf{P}^m_K$ relative to a Cartesian product of such convex bodies. Finally, §5.3 provides an explicit version of the arithmetic Bézout's inequality of §4.7 in this context.

5.1. Comparison between convex bodies

The notation being as in §2.1, we fix a place $w \in \mathcal{M}_K$, a point $\underline{\theta} = (\theta_0, \dots, \theta_m)$ in $\mathbf{C}_w^{m+1}$ with norm $\|\underline{\theta}\|_w = 1$, and we denote by θ the point of $\mathbf{P}^m(\mathbf{C}_w)$ with projective coordinates $\underline{\theta}$.

Let D, S be positive integers with $S \le D$ and let V be a positive real number. We attach to these parameters two adelic convex bodies $\mathcal{B}$ and $\mathcal{C}$ for $K[\underline{X}]_D$ and two adelic convex bodies $\mathcal{D}$ and $\mathcal{E}$ for $K[\underline{X}]_1$. They are determined by the following formulas which describe their components at each place $v \in \mathcal{M}_K$:

$$\mathcal{B}_v = \big\{ P \in \mathbf{C}_v[\underline{X}]_D \ ; \ \|P\|_v \le 1 \big\},$$

$$\mathcal{C}_v = \begin{cases} \mathcal{B}_v & \text{if } v \ne w, \\ \Big\{ P \in \mathcal{B}_w \ ; \ |P^{[\sigma]}(\underline{\theta})|_w \le \exp\Big(-V + |\sigma| \dfrac{V}{S} \Big) \text{ for } |\sigma| < S \Big\} & \text{if } v = w, \end{cases}$$

$$\mathcal{D}_v = \big\{ L \in \mathbf{C}_v[\underline{X}]_1 \ ; \ \|L\|_v \le 1 \big\},$$

$$\mathcal{E}_v = \begin{cases} \mathcal{D}_v & \text{if } v \ne w, \\ \Big\{ L \in \mathcal{D}_w \ ; \ |L(\underline{\theta})|_w \le \exp\Big(-\dfrac{V}{S} \Big) \Big\} & \text{if } v = w. \end{cases}$$

We first prove:

Lemma 5.1. *Let v be any place of K. We have the inclusions*

$$\begin{aligned}(m+1)^{-D\epsilon_v}\,\mathcal{D}_v^D \quad &\subseteq \quad \mathcal{B}_v \quad \subseteq \quad \binom{D+m}{m}^{\epsilon_v}\mathcal{D}_v^D\,,\\ \big(2(m+1)\big)^{-D\epsilon_w}\,\mathcal{D}_w^{D-S}\mathcal{E}_w^S \quad &\subseteq \quad \mathcal{C}_w \quad \subseteq \quad \left(\binom{D+m}{m}2^D\right)^{\epsilon_w}\mathcal{D}_w^{D-S}\mathcal{E}_w^S\,.\end{aligned}$$

Note that these inclusions become equalities for non-Archimedean places, since $\epsilon_v = 0$ for such places v.

Proof. The inclusions relative to $\mathcal{B}_v$ express standard properties of the maximum norm for polynomials. So, we concentrate on the inclusions relative to $\mathcal{C}_w$. Without loss of generality, we may assume that $|\theta_0|_w = \|\underline{\theta}\|_w = 1$.

To prove the right inclusion, fix $P \in \mathcal{C}_w$. Taylor's formula applied to the dehomogeneized polynomial P/X_0^D at the point $\theta_0^{-1}\underline{\theta}$ reads

$$P(\underline{X}) = \sum_{\substack{|\sigma|\le D\\ \sigma_0=0}} P^{[\sigma]}(\underline{\theta})L_0(\underline{X})^{D-|\sigma|}\prod_{j=1}^m L_j(\underline{X})^{\sigma_j}\,, \tag{5.1}$$

where the sum extends to all $\sigma = (\sigma_0,\dots,\sigma_m) \in \mathbf{N}^{m+1}$ with $|\sigma| \le D$ and $\sigma_0 = 0$, and where

$$L_0(\underline{X}) = \frac{1}{\theta_0}X_0 \quad \text{and} \quad L_j(\underline{X}) = X_j - \frac{\theta_j}{\theta_0}X_0 \quad (1\le j\le m).$$

Observe that $L_0 \in \mathcal{D}_w$ and that $L_1,\dots,L_m \in \mathcal{E}_w$. Thus, for $|\sigma| \ge S$, the product $L_0^{D-|\sigma|}\prod_{j=1}^m L_j^{\sigma_j}$ belongs to $\mathcal{D}_w^{D-S}\mathcal{E}_w^S$. On the other hand, for $|\sigma| < S$, we have $P^{[\sigma]}(\underline{\theta})L_0^{S-|\sigma|} \in \mathcal{E}_w^{S-|\sigma|}$ since $|P^{[\sigma]}(\underline{\theta})|_w \le \big(e^{-V/S}\big)^{S-|\sigma|}$ and since $\mathbf{C}_w$, being algebraically closed, contains a root of $P^{[\sigma]}(\underline{\theta})$ of order $S-|\sigma|$. We conclude that, for any σ, we have

$$P^{[\sigma]}(\underline{\theta})L_0^{D-|\sigma|}\prod_{j=1}^m L_j^{\sigma_j} \in p_\sigma\mathcal{D}_w^{D-S}\mathcal{E}_w^S$$

where $p_\sigma = 1$ when $|\sigma| < S$ and $p_\sigma = P^{[\sigma]}(\underline{\theta})$ otherwise. If w is ultrametric, we find $|p_\sigma|_w \le 1$ for any σ and therefore $P \in \mathcal{D}_w^{D-S}\mathcal{E}_w^S$. Otherwise, we get $P \in c\mathcal{D}_w^{D-S}\mathcal{E}_w^S$ where

$$\begin{aligned} c \;=\; \sum_{\substack{|\sigma|\le D\\ \sigma_0=0}} |p_\sigma|_w &\le \sum_{\substack{|\sigma|\le D\\ \sigma_0=0}}\ \sum_{\substack{|\tau|\le D\\ \tau\ge\sigma\\ \tau_0=0}} \binom{\tau_1}{\sigma_1}\cdots\binom{\tau_m}{\sigma_m}\\ &= \sum_{\substack{|\tau|\le D\\ \tau_0=0}}\ \sum_{\substack{\sigma\le\tau\\ \sigma_0=0}} \binom{\tau_1}{\sigma_1}\cdots\binom{\tau_m}{\sigma_m} = \sum_{\substack{|\tau|\le D\\ \tau_0=0}} 2^{|\tau|} \le \binom{D+m}{m}2^D. \end{aligned} \tag{5.2}$$

To prove the left inclusion, choose $M_1, \dots, M_S \in \mathcal{E}_w$ and $M_{S+1}, \dots, M_D \in \mathcal{D}_w$, and put $P = M_1 \cdots M_D$. Then P belongs to $(m+1)^{D\epsilon_w}\mathcal{B}_w$. Let $\sigma \in \mathbf{N}^{m+1}$ with $|\sigma| < S$. Leibniz' formula gives

$$P^{[\sigma]} = \sum_{\tau_1+\cdots+\tau_D=\sigma} M_1^{[\tau_1]} \cdots M_D^{[\tau_D]}$$

where the sum extends to all D-tuples $(\tau_1, \dots, \tau_D)$ of elements of $\mathbf{N}^{m+1}$ with $\tau_1+\cdots+\tau_D = \sigma$. Moreover, the term of the sum corresponding to such a D-tuple is zero unless there is a subset J of $\{1, \dots, D\}$ of cardinality $|\sigma|$ such that $|\tau_j| = 1$ if $j \in J$ and $\tau_j = 0$ otherwise. Assume that we are in the latter case for some fixed D-tuple $(\tau_1, \dots, \tau_D)$. For each $j \in J$, the polynomial $M_j^{[\tau_j]}$ reduces to one of the coefficients of M_j. It is therefore an element of $\mathbf{C}_w$ of absolute value ≤ 1. On the other hand, we have $M_j^{[\tau_j]} = M_j$ for $j \notin J$. Hence $|M_j^{[\tau_j]}(\underline{\theta})|_w$ is $\le (m+1)^{\epsilon_w}$ if $j \notin J$ and it is $\le e^{-V/S}$ if moreover $j \in \{1, \dots, S\}$. Since there are at least $S - |\sigma|$ elements of $\{1, \dots, S\}$ not belonging to J, we deduce

$$\left| M_1^{[\tau_1]}(\underline{\theta}) \cdots M_D^{[\tau_D]}(\underline{\theta}) \right|_w \le (m+1)^{(D-S)\epsilon_w} \left(e^{-V/S}\right)^{S-|\sigma|}.$$

Finally, since there are at most $\binom{D}{|\sigma|}(m+1)^{|\sigma|} \le 2^D(m+1)^S$ families $(\tau_1, \dots, \tau_D)$ giving rise to a nonzero product, we conclude that $|P^{[\sigma]}(\underline{\theta})|_w$ is $\le \big(2(m+1)\big)^{D\epsilon_w} \left(e^{-V/S}\right)^{S-|\sigma|}$. Hence, P belongs to $\big(2(m+1)\big)^{D\epsilon_w}\mathcal{C}_w$. □

The inclusions of Lemma 5.1 imply $\|\mathcal{L}_\alpha\|_{\mathcal{C}_w} \gg\ll \|\mathcal{L}_\alpha\|_{\mathcal{D}_w}^{D-S} \|\mathcal{L}_\alpha\|_{\mathcal{E}_w}^S$ for any $\alpha \in \mathbf{P}^m(\mathbf{C}_w)$, where the constants involved in the symbol $\gg\ll$ are independent of V and α. Thus, estimating the local norms $\|\mathcal{L}_\alpha\|_{\mathcal{C}_w}$ reduces to estimating $\|\mathcal{L}_\alpha\|_{\mathcal{D}_w}$ and $\|\mathcal{L}_\alpha\|_{\mathcal{E}_w}$. Below, we estimate separately $\|\mathcal{L}_\alpha\|_{\mathcal{E}_w}$ and $\|\mathcal{L}_\alpha\|_{\mathcal{C}_w}$ in order to achieve better constants.

Lemma 5.2. *Let $\alpha \in \mathbf{P}^m(\mathbf{C}_w)$ and denote by $\rho_\alpha = \mathrm{dist}(\theta, \alpha)$ the projective distance between θ and α. Then, we have*

$$\max\left\{e^{-V/S}, \frac{\rho_\alpha}{2^{\epsilon_w}}\right\} \le \|\mathcal{L}_\alpha\|_{\mathcal{E}_w} \le (m+1)^{\epsilon_w} \max\left\{e^{-V/S}, \rho_\alpha\right\},$$
$$\max\left\{\frac{e^{-V}}{2^{D\epsilon_w}}, \left(\frac{\rho_\alpha}{2^{\epsilon_w}}\right)^S\right\} \le \|\mathcal{L}_\alpha\|_{\mathcal{C}_w} \le \binom{D+m}{m}^{\epsilon_w} \max\left\{e^{-V}, 2^{D\epsilon_w}\rho_\alpha^S\right\}.$$

In particular, when w is not Archimedean, we have $\|\mathcal{L}_\alpha\|_{\mathcal{C}_w} = \|\mathcal{L}_\alpha\|_{\mathcal{E}_w}^S = \max\left\{e^{-V}, \rho_\alpha^S\right\}$.

Proof. For simplicity, we restrict to the case where w is Archimedean, the computations being essentially the same in the ultrametric case. Again, we may assume without loss of generality that $|\theta_0|_w = \|\underline{\theta}\|_w = 1$.

Select projective coordinates $\underline{\alpha}$ of the point α with norm $\|\underline{\alpha}\|_w = 1$. Let i and j be indices for which $\rho_\alpha = |\theta_j\alpha_i - \theta_i\alpha_j|_w$ and let k be an index with $|\alpha_k|_w = 1$. The convex body $\mathcal{E}_w$ contains the linear forms $e^{-V/S}X_k$ and $2^{-1}(\theta_j X_i - \theta_i X_j)$. This gives the lower bound

$$\|\mathcal{L}_\alpha\|_{\mathcal{E}_w} = \sup_{L\in\mathcal{E}_w} |L(\underline{\alpha})|_w \geq \max\left\{e^{-V/S}, 2^{-1}\rho_\alpha\right\}.$$

On the other hand, for any linear form $L = a_0X_0 + \cdots + a_mX_m \in \mathbf{C}_w[\underline{X}]_1$, we have the decomposition $\theta_0 L(\underline{\alpha}) = \alpha_0 L(\underline{\theta}) + \sum_{\ell=1}^m a_\ell(\theta_0\alpha_\ell - \theta_\ell\alpha_0)$. For $L \in \mathcal{E}_w$, we deduce

$$|L(\underline{\alpha})|_w \leq e^{-V/S} + m\rho_\alpha \leq (m+1)\max\{e^{-V/S}, \rho_\alpha\},$$

since $|\theta_0|_w = 1$ and $|\alpha_0|_w \leq 1$. This proves the upper bound for $\|\mathcal{L}_\alpha\|_{\mathcal{E}_w}$.

Similarly, since $\mathcal{C}_w$ contains $2^{-D}e^{-V}X_k^D$ and $2^{-S}(\theta_jX_i - \theta_iX_j)^S X_k^{D-S}$, we get

$$\|\mathcal{L}_\alpha\|_{\mathcal{C}_w} = \sup_{Q\in\mathcal{C}_w} |Q(\underline{\alpha})|_w \geq \max\left\{2^{-D}e^{-V}, 2^{-S}\rho_\alpha^S\right\}.$$

For the upper bound, choose $P \in \mathcal{C}_w$ and substitute $\underline{X} = \underline{\alpha}$ in the formula (5.1) from the proof of Lemma 5.1. This gives

$$|P(\underline{\alpha})|_w \leq \sum_{\substack{|\sigma|\leq D\\ \sigma_0=0}} |P^{[\sigma]}(\underline{\theta})|_w \rho_\alpha^{|\sigma|}$$

For $|\sigma| < S$, we have

$$|P^{[\sigma]}(\underline{\theta})|_w\rho_\alpha^{|\sigma|} \leq e^{-V}\left(e^{V/S}\rho_\alpha\right)^{|\sigma|} \leq e^{-V}\max\left\{1, e^{V/S}\rho_\alpha\right\}^S = \max\left\{e^{-V}, \rho_\alpha^S\right\}.$$

If $\rho_\alpha \leq 1$, we deduce

$$\begin{aligned}|P(\underline{\alpha})|_w &\leq \max\left\{e^{-V}, \rho_\alpha^S\right\}\Bigg(\sum_{\substack{|\sigma|<S\\ \sigma_0=0}} 1\Bigg) \;+\; \rho_\alpha^S\Bigg(\sum_{\substack{S\leq|\sigma|\leq D\\ \sigma_0=0}} |P^{[\sigma]}(\underline{\theta})|_w\Bigg)\\ &\leq \binom{D+m}{m}\max\{e^{-V}, 2^D\rho_\alpha^S\},\end{aligned}$$

using the estimate (5.2) from the proof of Lemma 5.1. If $\rho_\alpha > 1$, this inequality still holds because the constraint $\|\underline{\alpha}\|_w = 1$ implies $|P(\underline{\alpha})|_w \leq \binom{D+m}{m}$. This proves the upper bound for $\|\mathcal{L}_\alpha\|_{\mathcal{C}_w}$. ∎

5.2. Upper bounds for the heights

Through out this section, we fix an integer t with $0 \leq t \leq m$. For $j = 0, \ldots, t$, we choose positive integers D_j and S_j with $S_j \leq D_j$, and a positive real number V_j. We denote by $\mathcal{B}_j$, $\mathcal{C}_j$, $\mathcal{D}_j$ and $\mathcal{E}_j$ the corresponding adelic convex bodies defined as in §5.1 for the choice of $D = D_j$, $S = S_j$ and $V = V_j$. We put

$$\mathcal{B} = \mathcal{B}_0\times\cdots\times\mathcal{B}_t\,,\quad \mathcal{C} = \mathcal{C}_0\times\cdots\times\mathcal{C}_t\,,\quad \mathcal{D} = \mathcal{D}_0\times\cdots\times\mathcal{D}_t\,,\quad \mathcal{E} = \mathcal{E}_0\times\cdots\times\mathcal{E}_t,$$

so that $\mathcal{B}$ and $\mathcal{C}$ are now adelic convex bodies in degree $\underline{D} = (D_0, \dots, D_t)$ while $\mathcal{D}$ and $\mathcal{E}$ are adelic convex bodies in degree $(1, \dots, 1)$. We also choose a positive real number η with

$$\eta \geq \max_{0 \leq j \leq m} \{\Phi(D_j)\} \quad \text{where} \quad \Phi(D_j) = D_j^{-1} \log \binom{D_j + m}{m}.$$

The next proposition provides upper bounds for the heights of a cycle Z relative to $\mathcal{B}$ and $\mathcal{C}$ in terms of its heights relative to $\mathcal{D}$ and $\mathcal{E}$ (compare with Proposition 2.8 of [15]).

Proposition 5.3. *Let Z be a cycle of dimension t of $\mathbf{P}^m_K$. We have the upper bounds*

$$\tilde{h}_{\mathcal{B}}(Z) \leq \tilde{h}_{\mathcal{D}}(Z) + a \quad \text{and} \quad \tilde{h}_{\mathcal{C}}(Z) \leq \tilde{h}_{\mathcal{D}}(Z) + \left(\prod_{j=0}^{t} \frac{S_j}{D_j} \right) \left(\tilde{h}_{\mathcal{E}}(Z) - \tilde{h}_{\mathcal{D}}(Z) \right) + b$$

with $a = (t+1)\big(2\log(m+1) + \eta\big)$ and $b = a + (t+1)\epsilon_w \lambda_w \log 2$.

Proof. We first establish the upper bound concerning $\tilde{h}_{\mathcal{C}}(Z)$. Consider the adelic convex body in degree $\underline{D}$ given by $\mathcal{C}' = \mathcal{C}'_0 \times \cdots \times \mathcal{C}'_t$ where $\mathcal{C}'_j$ denotes the product $\mathcal{D}_j^{D_j - S_j} \mathcal{E}_j^{S_j}$ for $j = 0, \dots, t$. Lemma 5.1 shows that, for any $j = 0, \dots, t$ and any place $v \in \mathcal{M}_K$, we have $\mathcal{C}_{j,v} \subseteq \rho_{j,v} \mathcal{C}'_{j,v}$ where

$$\rho_{j,v} = \binom{D_j + m}{m}^{\epsilon_v} \text{if } v \neq w, \quad \text{and} \quad \rho_{j,w} = \left(\binom{D_j + m}{m} 2^{D_j} \right)^{\epsilon_w}.$$

Using Lemma 4.7, we deduce

$$\tilde{h}_{\mathcal{C}}(Z) \leq \tilde{h}_{\mathcal{C}'}(Z) + \sum_{j=0}^{t} \sum_{v \in \mathcal{M}_K} \frac{\lambda_v}{D_j} \log \rho_{j,v} \leq \tilde{h}_{\mathcal{C}'}(Z) + (t+1)\left(\eta + \epsilon_w \lambda_w \log 2\right)$$

since $\Phi(D_j) \leq \eta$ for $j = 0, \dots, t$. For each subset J of $\{0, \dots, t\}$, define an adelic convex body $\mathcal{E}^{(J)} = \mathcal{E}_0^{(J)} \times \cdots \times \mathcal{E}_t^{(J)}$ in degree $(1, \dots, 1)$ by putting $\mathcal{E}_j^{(J)} = \mathcal{E}_j$ if $j \in J$ and $\mathcal{E}_j^{(J)} = \mathcal{D}_j$ otherwise. Note that $\mathcal{E}^{(J)} = \mathcal{E}$ when $J = \{0, \dots, t\}$ and that $\mathcal{E}^{(J)} \subseteq \mathcal{D}$ for any subset J of $\{0, \dots, t\}$. Applying recursively Proposition 4.6 to each factor of $\mathcal{C}'$, using $\eta' = \log(m+1)$, we find

$$h_{\mathcal{C}'}(Z) \leq \left(\sum_{J \subseteq \{0,\dots,t\}} \Big(\prod_{j \notin J} (D_j - S_j) \Big) \Big(\prod_{j \in J} S_j \Big) h_{\mathcal{E}^{(J)}}(Z) \right) + (t+1) 2\eta' \Big(\prod_{j=0}^{t} D_j \Big) \deg Z.$$

For a proper subset $J \subset \{0, \dots, t\}$, we use the upper bound $h_{\mathcal{E}^{(J)}}(Z) \leq h_{\mathcal{D}}(Z)$ which comes from the inclusion $\mathcal{E}^{(J)} \subseteq \mathcal{D}$. Since

$$\sum_{J \subset \{0,\dots,t\}} \Big(\prod_{j \notin J} (D_j - S_j) \Big) \Big(\prod_{j \in J} S_j \Big) = \Big(\prod_{j=0}^{t} D_j \Big) - \Big(\prod_{j=0}^{t} S_j \Big),$$

this gives

$$h_{\mathcal{C}'}(Z) \le \Big(\prod_{j=0}^{t} D_j\Big) h_{\mathcal{D}}(Z) + \Big(\prod_{j=0}^{t} S_j\Big)\Big(h_{\mathcal{E}}(Z) - h_{\mathcal{D}}(Z)\Big) + (t+1)2\eta'\Big(\prod_{j=0}^{t} D_j\Big) \deg Z\,.$$

The upper bound for $\tilde{h}_{\mathcal{C}}(Z)$ follows by dividing both sides of this inequality by $\big(\prod_{j=0}^{t} D_j\big) \deg Z$, and by combining the result with the upper bound for $\tilde{h}_{\mathcal{C}}(Z)$ in terms of $\tilde{h}_{\mathcal{C}'}(Z)$.

The proof of the upper bound for $\tilde{h}_{\mathcal{B}}(Z)$ is similar and simpler. In this case, we compare the height relative to $\mathcal{B}$ with the height relative to $\mathcal{B}' = \mathcal{B}'_0 \times \cdots \times \mathcal{B}'_t$ where $\mathcal{B}'_j = \mathcal{D}_j^{D_j}$ for $j = 0, \ldots, t$. In this situation, Lemma 5.1 gives $\mathcal{B}_{j,v} \subseteq \rho_{j,v}\mathcal{B}'_{j,v}$ with $\rho_{j,v} = \binom{D_j+m}{m}^{\epsilon_v}$ for any $j = 0, \ldots, t$ and any $v \in \mathcal{M}_K$. Applying Lemma 4.7, we deduce $\tilde{h}_{\mathcal{B}}(Z) \le \tilde{h}_{\mathcal{B}'}(Z) + (t+1)\eta$. On the other hand, Proposition 4.6 leads to $\tilde{h}_{\mathcal{B}'}(Z) \le \tilde{h}_{\mathcal{D}}(Z) + 2(t+1)\eta'$. The result follows. ∎

Remark. It is also possible to show

$$\tilde{h}_{\mathcal{B}}(Z) \ge \tilde{h}_{\mathcal{D}}(Z) - a' \quad \text{and} \quad \tilde{h}_{\mathcal{C}}(Z) \ge \tilde{h}_{\mathcal{D}}(Z) + \left(\prod_{j=0}^{t} \frac{S_j}{D_j}\right) \Big(\tilde{h}_{\mathcal{E}}(Z) - \tilde{h}_{\mathcal{D}}(Z)\Big) - b'$$

with $a' = (t+1)\big(\log(m+1) + t\eta\big)$ and $b' = a' + (t+1)\epsilon_w\lambda_w \log 2$. The argument is similar except that, in the proof of the lower bound for $\tilde{h}_{\mathcal{C}}(Z)$, one needs the inequality $\tilde{h}_{\mathcal{E}^{(J)}}(Z) \ge \tilde{h}_{\mathcal{D}}(Z) - (t+1)\epsilon_w\lambda_w \log 2$ for any proper subset J of $\{0, \ldots, t\}$. A proof of this last inequality can be given along the lines of the remark on page 33 of [15]. The details are left to the reader.

In order to estimate the height $h_{\mathcal{E}}(Z)$, we need the following notion which, in a different form, is originally due to Yu. V. Nesterenko (see §1 of [13]).

Definition 5.4. Let Z be a cycle of dimension t in $\mathbf{P}^m_K$ and let F be the standard Chow form of the cycle Z in degree $(1, \ldots, 1)$. We define *the modulus of Z at the point $\theta \in \mathbf{P}^m(\mathbf{C}_w)$* to be

$$|Z|_\theta = \frac{\|F\|_\theta}{\|F\|_{\mathcal{D}_w}}$$

where $\|F\|_\theta$ denotes the supremum of the numbers $|F(L_0, \ldots, L_t)|_w$ for all choices of linear forms $L_0, \ldots, L_t$ in $\mathbf{C}_w[\underline{X}]_1$ with $\|L_j\|_w \le 1$ and $L_j(\theta) = 0$ for $j = 0, \ldots, t$.

Note that $|Z|_\theta$ is independent of the choice of F. Moreover, it is clear that $|Z|_\theta \le 1$ and that $|Z|_\theta = 0$ exactly when $\theta \in Z(\mathbf{C}_w)$.

We leave the reader check that the original symbol $|Z(\underline{\theta})|$ introduced by Nesterenko in §1 of [13] satisfies

$$|Z|_\theta \le |Z(\underline{\theta})| \le \big(m(m+1)\big)^{\epsilon_w (t+1)\deg(Z)} |Z|_\theta.$$

Similar inequalities also hold between $|Z|_\theta$ and the symbol $\mathrm{Dist}_{w,1}(Z,\theta)$ of Philippon (see Definition 1.15 of [15]).

Proposition 5.5. *Let Z be a cycle in $\mathbf{P}^m_K$ of dimension $t \ge 0$. Denote by ρ the distance between θ and $Z(\mathbf{C}_w)$ and put $U = \min_{0\le j\le t}\{V_j/S_j\}$.*
(i) If $\rho < e^{-U}$, we have

$$h_{\mathcal{E}}(Z) - h_{\mathcal{D}}(Z) \le -\lambda_w U + \epsilon_w \lambda_w \Big(1 + \log\Big((m+1)(t+1)\deg Z\Big)\Big).$$

(ii) If $\rho \ge e^{-U}$, we have

$$h_{\mathcal{E}}(Z) - h_{\mathcal{D}}(Z) \le \lambda_w \log|Z|_\theta + 3\epsilon_w\lambda_w(t+1)\log(m+1)\deg Z.$$

Proof. For any point $\alpha \in \mathbf{P}^m(\mathbf{C}_w)$, we define $\rho_\alpha = \mathrm{dist}(\theta,\alpha)$.

(i) Suppose first $\rho < e^{-U}$. Let $N = (t+1)\deg Z$, and let F be the Chow form of Z in degree $(1,\dots,1)$. Since

$$h_{\mathcal{E}}(Z) - h_{\mathcal{D}}(Z) = \lambda_w \log\frac{\|F\|_{\mathcal{E}_w}}{\|F\|_{\mathcal{D}_w}},$$

we simply need to show

$$|F(L_0,\dots,L_t)|_w \le \|F\|_{\mathcal{D}_w} e^{-U}\Big(e(m+1)N\Big)^{\epsilon_w}, \tag{5.3}$$

for any choice of linear forms $L_0 \in \mathcal{E}_{0,w},\dots,L_t \in \mathcal{E}_{t,w}$.

The assumption $\rho < e^{-U}$ means that there exists a point $\alpha \in Z(\mathbf{C}_w)$ such that $\rho_\alpha < e^{-U}$. Fix such a point α and a representative $\underline{\alpha} \in \mathbf{C}_w^{m+1}$ with $\|\underline{\alpha}\|_w = 1$. Choose also an index ℓ with $|\alpha_\ell|_w = 1$ and define

$$M_j(\underline{X}) = \frac{L_j(\underline{\alpha})}{\alpha_\ell} X_\ell \qquad \text{for} \qquad 0 \le j \le t,$$

so that all linear forms $L_j - M_j$ vanish at the point $\underline{\alpha}$. Since $\alpha \in Z(\mathbf{C}_w)$, we get

$$F(L_0 - M_0,\dots,L_t - M_t) = 0.$$

Let R be any real number > 1. We define rational functions φ and f on $\mathbf{C}_w$ by

$$\varphi(z) = \frac{z-1}{(1-R^{-2}z)^{\epsilon_w}} \quad \text{and} \quad f(z) = F\big(L_0 + \varphi(z)M_0, \dots, L_t + \varphi(z)M_t\big).$$

Since $f(0) = 0$, Schwarz' lemma provides the upper bound

$$|F(L_0, \dots, L_t)|_w = |f(1)|_w \le \frac{|f|_R}{R},$$

where $|f|_R$ denotes the maximum of $|f(z)|_w$ on the disk $|z|_w \le R$. Note that, with the same notation, the Blaschke function $\varphi(z)$ satisfies $|\varphi|_R = R$. Moreover, using Lemma 5.2, we find

$$\|M_j\|_w = |L_j(\underline{\alpha})|_w \le \|\mathcal{L}_\alpha\|_{\mathcal{E}_{j,w}} \le (m+1)^{\epsilon_w} e^{-U} \qquad \text{for} \qquad 0 \le j \le t.$$

Since the polynomial F is homogeneous of degree N in the whole set of coefficients of $L_0, \dots, L_t$, we deduce

$$|f|_R \le \begin{cases} \big(1 + R(m+1)e^{-U}\big)^N \|F\|_{\mathcal{D}_w} & \text{if } w \text{ is Archimedean,} \\ \max\{1, Re^{-U}\}^N \|F\|_{\mathcal{D}_w} & \text{else.} \end{cases}$$

In the ultrametric case, the inequality (5.3) follows by choosing $R = e^U$. Suppose now that w is an Archimedean place. If $e^U \le (m+1)N$, the inequality (5.3) is obvious since $\mathcal{E}_{j,w} \subseteq \mathcal{D}_{j,w}$ for $j = 0, \dots, t$. Otherwise, we may choose $R = e^U/\big((m+1)N\big) > 1$. Then (5.3) follows from the estimates

$$|f(1)|_w \le (m+1)Ne^{-U}\left(1 + \frac{1}{N}\right)^N \|F\|_{\mathcal{D}_w} \le (m+1)Ne^{-U+1}\|F\|_{\mathcal{D}_w}.$$

(ii) Suppose now $\rho \ge e^{-U}$. Fix a real $\delta > 0$ and consider the new adelic convex body $\mathcal{F} = \mathcal{F}_0 \times \cdots \times \mathcal{F}_t$ in degree $(1, \dots, 1)$ with components

$$\mathcal{F}_{j,v} = \begin{cases} \mathcal{D}_{j,v} & \text{if } v \ne w, \\ \big\{L \in \mathcal{D}_{j,w}\,;\, |L(\underline{\theta})|_w \le \delta\big\} & \text{if } v = w. \end{cases}$$

It is clear that

$$\lim_{\delta \to 0} h_{\mathcal{F}}(Z) = h_{\mathcal{D}}(Z) + \lambda_w \log |Z|_\theta.$$

On the other hand, for any point $\alpha \in Z(\mathbf{C}_w)$ and any $j = 0, \dots, t$, Lemma 5.2 gives

$$\|\mathcal{L}_\alpha\|_{\mathcal{E}_{j,w}} \le (m+1)^{\epsilon_w} \max\{e^{-V_j/S_j}, \rho_\alpha\} \le (m+1)^{\epsilon_w} \rho_\alpha$$

since $\rho_\alpha \geq \rho \geq e^{-U} \geq e^{-V_j/S_j}$. Moreover, since $\mathcal{F}_{j,w}$ contains the linear forms $\theta_i X_\ell - \theta_\ell X_i$ with $0 \leq i, \ell \leq m$, we also have $\|\mathcal{L}_\alpha\|_{\mathcal{F}_{j,w}} \geq \rho_\alpha$. We deduce

$$\|\mathcal{L}_\alpha\|_{\mathcal{E}_{j,w}} \leq (m+1)^{\epsilon_w} \|\mathcal{L}_\alpha\|_{\mathcal{F}_{j,w}}.$$

Since $\mathcal{F}_{j,v} = \mathcal{E}_{j,v}$ for $v \neq w$, Lemma 4.8 applies to this situation with $\mathcal{C}' = \mathcal{E}$, $\mathcal{C} = \mathcal{F}$, $\mathcal{S} = \{0, \dots, t\} \times \{w\}$ and $A_{j,w} = (m+1)^{\epsilon_w}$ for $j = 0, \dots, t$. It gives

$$\tilde{h}_{\mathcal{E}}(Z) \leq \tilde{h}_{\mathcal{F}}(Z) + \sum_{j=0}^{t} \lambda_w \big(3\epsilon_w \log(m+1)\big)$$

using $\eta_j = \log(m+1)$ for $j = 0, \dots, t$. Multiplying both sides by $\deg Z$ and taking the limit as $\delta \to 0$, we get the required upper bound for $h_{\mathcal{E}}(Z) - h_{\mathcal{D}}(Z)$. □

Remark. As $\mathcal{F} \subseteq \mathcal{E}$ for sufficiently small δ, we also have the lower bound

$$h_{\mathcal{E}}(Z) \geq h_{\mathcal{F}}(Z) \geq h_{\mathcal{D}}(Z) + \lambda_w \log |Z|_\theta,$$

which compares to the upper bound in the case $\rho \geq e^{-U}$.

Finally, we mention the following result as a corollary of Propositions 5.3 and 5.5.

Corollary 5.6. *Let Z be a cycle of dimension t of $\mathbf{P}^m_K$, and let F be a Chow form of Z in degree $\underline{D}$. Assume $\theta \in Z(\mathbf{C}_w)$. For $j = 0, \dots, t$, let P_j be a homogeneous polynomial in $\mathbf{C}_w[\underline{X}]_{D_j}$ vanishing with multiplicity $\geq S_j$ at the point θ. Then F vanishes at the point $(P_0, \dots, P_t)$ with multiplicity $\geq S := \min_{0 \leq j \leq t} \Big\{ \prod_{\substack{i=0 \\ i \neq j}}^{t} S_i \Big\}$.*

Proof. Fix a choice of polynomial $Q_j \in \mathbf{C}[\underline{X}]_{D_j}$ for $j = 0, \dots, t$. To prove the corollary, it suffices to show

$$\log \big| F(P_0 + zQ_0, \dots, P_t + zQ_t) \big|_w \leq S \log |z|_w + \mathcal{O}(1)$$

for any $z \in \mathbf{C}_w$ with $0 < |z|_w < 1$, where the symbol $\mathcal{O}(1)$ stands for a real number independent of z. For this purpose, we may assume, without loss of generality, that

$$\|P_j\|_w \leq 1/2\,, \qquad \|Q_j\|_w \leq 1/2 \quad \text{and} \quad \max_{|\sigma| < S_j} |Q_j^{[\sigma]}(\theta)|_w \leq 1\,, \quad (0 \leq j \leq t).$$

For given $z \in \mathbf{C}_w$ with $0 < |z|_w < 1$, we apply the preceding results of this section with the choice of parameters

$$V_0 = \cdots = V_t = -\log |z|_w\,.$$

For $j = 0, \ldots, t$, the hypotheses on P_j and Q_j imply that the polynomial $P_j + zQ_j$ belongs to $\mathcal{C}_{j,w}$. Thus we have the upper bound

$$|F(P_0 + zQ_0, \ldots, P_t + zQ_t)|_w \le \|F\|_{\mathcal{C}_w} .$$

On the other hand, since the components $\mathcal{C}_{j,v}$ are independent of z whenever $v \neq w$ while $\mathcal{C}_{j,w}$ depends upon z through $V_j = -\log |z|_w$, we can write

$$\lambda_w \log \|F\|_{\mathcal{C}_w} \le h_{\mathcal{C}}(Z) + \mathcal{O}(1) .$$

Note also that the first case of Proposition 5.5 applies to the present situation since $\theta \in Z(\mathbf{C}_w)$. Using this result together with Proposition 5.3, we obtain

$$\lambda_w^{-1} h_{\mathcal{C}}(Z) \le -S_0 \cdots S_t \min_{0 \le j \le t} \big\{ V_j / S_j \big\} + \mathcal{O}(1) = S \log |z|_w + \mathcal{O}(1).$$

The combination of the three above upper bounds gives the desired estimate. ∎

Note that, in the case where $Z = \mathbf{P}^m_K$, Corollary 5.6 can also be viewed as a consequence of Proposition 6.2.1 of [6]. In the general case, we could not find a reference for it.

5.3. The height of a section by an hypersurface

We keep the notation of §5.2 and apply the results and conventions of §4.7 to our specific choice of adelic convex body $\mathcal{C}$. The next proposition provides a situation where intersecting a subvariety with the divisor of a certain type of polynomial produces subvarieties of lower dimension with a good control on the normalized height relative to $\mathcal{C}$. In our setting, it plays a similar role as Lemma 4 of [14] or Proposition 2.5 of [15].

Proposition 5.7. *Let Z be a subvariety of $\mathbf{P}^m_K$ of dimension t, let D', S' be positive integers with $S' \le D'$, $D'/S' \le D_t/S_t$ and $\Phi(D') \le \eta$, and let V' be a positive real number. Suppose that there exists a homogeneous polynomial $Q \in K[\underline{X}]$ of degree D' which does not vanish identically on Z and satisfies*

$$|Q^{[\sigma]}(\underline{\theta})|_w \le \|Q\|_w \exp\Big(-V' + |\sigma| \frac{V'}{S'} \Big)$$

for each $\sigma \in \mathbf{N}^{m+1}$ with $|\sigma| < S'$. Denote by ρ the distance between the point θ and $Z(\mathbf{C}_w)$. Suppose moreover that

$$e^{-V'/S'} \le \max\Big\{ \rho, e^{-V_t/S_t} \Big\} .$$

If $t \geq 1$, then there exists a subvariety Z' of $\mathbf{P}^m_K$ of dimension $t-1$ contained in Z such that

$$\tilde{h}_{\mathcal{C}}(Z') \leq \tilde{h}_{\mathcal{C}}(Z) + \frac{S_t}{D_t S'} h(Q) + (2t+3)\eta + \epsilon_w \lambda_w \left(1 + \frac{S_t}{D_t}\right) \log 2.$$

If $t = 0$, the same inequality holds with $Z' = 0$.

Remark. When $D' = D_t$, $S' = S_t$ and $V' = V_t$, the conclusion follows directly from Proposition 4.9 since $h_{\mathcal{C}_t}(Q) \leq h_{\mathcal{B}_t}(Q) \leq h(Q) + \eta D_t$. In the general case, we shall deduce the above statement from the refined Bézout's inequality given by Proposition 4.10.

Proof. Denote by $\mathfrak{p} \subseteq K[\underline{X}]$ the homogeneous prime ideal defining Z. We may assume without loss of generality that $X_0 \notin \mathfrak{p}$. Let k be an arbitrary large positive integer and define $\ell = [kS_t/S']$. Since $\ell D' \leq kS_t D'/S' \leq kD_t$, the product $P = X_0^{kD_t - \ell D'} Q^\ell$ is a homogeneous polynomial of degree kD_t which does not belong to $\mathfrak{p}$. Put $\mathcal{C}^{(k)} = \mathcal{C}_0 \times \cdots \times \mathcal{C}_{t-1} \times \mathcal{C}_t^k$, so that $\mathcal{C}^{(k)}$ is an adelic convex body in degree $(D_0, \dots, D_{t-1}, kD_t)$. By virtue of Proposition 4.6, we have

$$h_{\mathcal{C}^{(k)}}(Z) \leq k h_{\mathcal{C}}(Z) + 2k\eta \Big(\prod_{j=0}^{t} D_j \Big) \deg Z$$

and thus $\tilde{h}_{\mathcal{C}^{(k)}}(Z) \leq \tilde{h}_{\mathcal{C}}(Z) + 2\eta$. Denote by $Z'' = Z \cdot \mathrm{div}(P)$ the intersection product of Z and $\mathrm{div}(P)$. Applying Proposition 4.10 to $\mathcal{C}^{(k)}$ and P, and using the above upper bound for $\tilde{h}_{\mathcal{C}^{(k)}}(Z)$, we deduce

$$\tilde{h}_{\mathcal{C}}(Z'') \leq \big(\tilde{h}_{\mathcal{C}}(Z) + 2\eta\big) + (kD_t)^{-1} \log \sup_{\alpha \in Z_{\mathbf{A}}} \|P\|_{\alpha, \mathcal{C}_t^k} + 2t\eta + 2\Phi(kD_t).$$

We now estimate the local norm $\|P\|_{\alpha, \mathcal{C}_t^k}$ for an adelic point $\alpha = (\alpha_v)_v$ in $Z_{\mathbf{A}}$. To this end, we choose projective coordinates $\underline{\alpha}_v \in \mathbf{C}_v^{m+1}$ for α_v with $\|\underline{\alpha}_v\|_v = 1$, for each $v \in \mathcal{M}_K$. We begin with the contribution of the place w. Let $\mathcal{C}'_w \subseteq \mathbf{C}_w[\underline{X}]_{D'}$ be the convex body defined in the same way as $\mathcal{C}_w$ in §5.1, but for the choice of parameters D', S' and V' instead of D, S and V. Our assumptions on Q translate into the norm inequality $\|Q\|_{\mathcal{C}'_w} \leq \|Q\|_w$ so that

$$|Q(\underline{\alpha}_w)|_w \leq \|Q\|_w \|\mathcal{L}_{\alpha_w}\|_{\mathcal{C}'_w} \quad \text{and} \quad |P(\underline{\alpha}_w)|_w \leq \big(\|Q\|_w \|\mathcal{L}_{\alpha_w}\|_{\mathcal{C}'_w}\big)^\ell,$$

and thus

$$\|P\|_{\alpha_w, \mathcal{C}_{t,w}^k} = \frac{|P(\underline{\alpha}_w)|_w}{\|\mathcal{L}_{\alpha_w}\|_{\mathcal{C}_{t,w}^k}} = \frac{|P(\underline{\alpha}_w)|_w}{\big(\|\mathcal{L}_{\alpha_w}\|_{\mathcal{C}_{t,w}}\big)^k} \leq \|Q\|_w^\ell \frac{\big(\|\mathcal{L}_{\alpha_w}\|_{\mathcal{C}'_w}\big)^\ell}{\big(\|\mathcal{L}_{\alpha_w}\|_{\mathcal{C}_{t,w}}\big)^k}.$$

Now we use Lemma 5.2 to bound the last ratio. Applied to the convex body $\mathcal{C}'_w$, it provides the upper bound

$$\begin{aligned}\|\mathcal{L}_{\alpha_w}\|_{\mathcal{C}'_w} &\le \binom{D'+m}{m}^{\epsilon_w} \max\{e^{-V'}, 2^{D'\epsilon_w}\rho_{\alpha_w}^{S'}\}\\ &\le e^{\eta D'\epsilon_w}\max\{e^{-V_t/S_t}, 2^{(D_t/S_t)\epsilon_w}\rho_{\alpha_w}\}^{S'}\end{aligned}$$

since $D'/S' \le D_t/S_t$ and $e^{-V'/S'} \le \max\{e^{-V_t/S_t}, \rho\} \le \max\{e^{-V_t/S_t}, \rho_{\alpha_w}\}$ according to the hypotheses. On the other hand, Lemma 5.2 also provides the lower bound

$$\begin{aligned}\|\mathcal{L}_{\alpha_w}\|_{\mathcal{C}_{t,w}} &\ge \max\{2^{-D_t\epsilon_w}e^{-V_t}, 2^{-S_t\epsilon_w}\rho_{\alpha_w}^{S_t}\}\\ &\ge 2^{-(S_t+D_t)\epsilon_w}\max\{e^{-V_t/S_t}, 2^{(D_t/S_t)\epsilon_w}\rho_{\alpha_w}\}^{S_t}.\end{aligned}$$

Combining the above inequalities, we get

$$\begin{aligned}\|P\|_{\alpha_w,\mathcal{C}^k_{t,w}} &\le \|Q\|_w^\ell \big(2^{S_t+D_t}e^{\eta D_t}\big)^{k\epsilon_w}\max\{e^{-V_t/S_t}, 2^{(D_t/S_t)\epsilon_w}\rho_{\alpha_w}\}^{\ell S'-kS_t}\\ &\le \|Q\|_w^\ell \big(2^{S_t+D_t}e^{\eta D_t}\big)^{k\epsilon_w} e^{S'V_t/S_t}\end{aligned}$$

since $\ell D' \le kD_t$ and $-S' < \ell S' - kS_t \le 0$. For the remaining places $v \ne w$, we have $\mathcal{C}_{t,v} = \mathcal{B}_{t,v}$ and $\|\mathcal{L}_{\alpha_v}\|_{\mathcal{C}_{t,v}} \ge \|\underline{\alpha}_v\|_v = 1$. Thus, we obtain

$$\|P\|_{\alpha_v,\mathcal{C}^k_{t,v}} \le |P(\underline{\alpha}_v)|_v \le \left(\binom{D'+m}{m}^{\epsilon_v}\|Q\|_v\right)^\ell \le e^{\eta D_t k\epsilon_v}\|Q\|_v^\ell.$$

Combining the last inequalities, we find

$$\begin{aligned}(kD_t)^{-1}\log\|P\|_{\alpha,\mathcal{C}^k_t} &\le \frac{\ell}{kD_t}h(Q) + \eta + \epsilon_w\lambda_w\left(1+\frac{S_t}{D_t}\right)\log 2 + \lambda_w\frac{S'V_t}{kS_tD_t}\\ &\le \frac{S_t}{S'D_t}h(Q) + \eta + \epsilon_w\lambda_w\left(1+\frac{S_t}{D_t}\right)\log 2 + o(1),\end{aligned}$$

where $o(1)$ denotes a function of k which tends to 0 as k tends to infinity. We substitute this estimation in the upper bound for $\tilde{h}_{\mathcal{C}}(Z'')$. Since $\Phi(kD_t)$ tends to 0 as k tends to infinity, we find

$$\tilde{h}_{\mathcal{C}}(Z'') \le \tilde{h}_{\mathcal{C}}(Z) + \frac{S_t}{D_tS'}h(Q) + (2t+3)\eta + \epsilon_w\lambda_w\left(1+\frac{S_t}{D_t}\right)\log 2 + o(1).$$

For $t = 0$ the left hand side of this inequality is 0 and we get the conclusion by letting k tend to infinity. If $t \ge 1$, Lemma 4.4 shows that there exists at least one component of Z' of Z'' which satisfies the above inequality with Z'' replaced by Z'. This component Z' depends a priori on the integer k. However, it belongs to a finite set of irreducible subvarieties of $\mathbf{P}^m_K$ since Z'' has the same components as $Z\cdot\mathrm{div}(X_0Q)$. For infinitely many integers k we thus obtain the same component Z' and, for such Z', the above upper bound remains valid without the term $o(1)$. ∎

Finally, Proposition 4.11 gives:

Proposition 5.8. *Let Z be a cycle of $\mathbf{P}^m_K$ of dimension t. Denote by ρ the distance between θ and $Z(\mathbf{C}_w)$. Then*

$$\tilde{h}_{\mathcal{C}}(Z) \geq \lambda_w \Big(\sum_{j=0}^{t} \frac{S_j}{D_j} \Big) \log \Big(\frac{\rho}{2^{\epsilon_w}} \Big) - (t+1)(t+2)\eta.$$

Proof. For any index $j = 0, \dots, t$, any place $v \in \mathcal{M}_K$ with $v \neq w$ and any point $\alpha_v \in Z(\mathbf{C}_v)$, we have $\|\mathcal{L}_{\alpha_v}\|_{\mathcal{C}_{j,v}} \geq 1$ since $\mathcal{C}_{j,v} = \mathcal{B}_{j,v}$. On the other hand, for any $\alpha_w \in Z(\mathbf{C}_w)$, Lemma 5.2 gives the lower bound

$$\|\mathcal{L}_{\alpha_w}\|_{\mathcal{C}_{j,w}} \geq \Big(\frac{\rho_{\alpha_w}}{2^{\epsilon_w}} \Big)^{S_j} \geq \Big(\frac{\rho}{2^{\epsilon_w}} \Big)^{S_j}$$

where $\rho_{\alpha_w} = \mathrm{dist}(\theta, \alpha_w)$. These inequalities imply that, for any adelic point $\alpha = (\alpha_v)_{v \in \mathcal{M}_K}$ in $Z_{\mathbf{A}}$, we have

$$D_j^{-1} \log \|\mathcal{L}_\alpha\|_{\mathcal{C}_j} \geq \lambda_w \frac{S_j}{D_j} \log \Big(\frac{\rho}{2^{\epsilon_w}} \Big), \qquad (0 \leq j \leq t)\,.$$

Proposition 4.11 then provides the desired lower bound for $\tilde{h}_{\mathcal{C}}(Z)$. □

6. Proofs

Since Theorem 2.1 is a consequence of Theorem 2.2, we begin by proving the latter in §6.1. We also establish a lower bound for $|Z|_\theta$ under the hypotheses of Theorem 2.2. In §6.2, we prove Theorem 2.1 and a corollary of it which extends this result to a situation with smaller zero-free regions (in condition (ii)). Finally, §6.3 is devoted to the proof of Theorem 2.3.

6.1. Proof of Theorem 2.2

Let the notation be as in Theorem 2.2. For simplicity, we set

$$D = D_q\,, \quad S = S_q\,, \quad T = T_q\,, \quad V = V_q$$

and we denote by $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ the adelic convex bodies defined as in §5.1 for the present choice of parameters D, S and V. Thus, $\mathcal{C}$ is an adelic convex body for $K[\underline{X}]_D$ while $\mathcal{D}$ and $\mathcal{E}$ are adelic convex bodies for $K[\underline{X}]_1$. Given a cycle Z_t of $\mathbf{P}^m_K$ of dimension t, we denote by $h_{\mathcal{C}}(Z_t)$ its height relative to the Cartesian product $\mathcal{C} \times \cdots \times \mathcal{C}$ of $t+1$ copies of $\mathcal{C}$. We define similarly $h_{\mathcal{D}}(Z_t)$ and $h_{\mathcal{E}}(Z_t)$. The corresponding normalized heights are thus given by

$$\tilde{h}_{\mathcal{C}}(Z_t) = \frac{h_{\mathcal{C}}(Z_t)}{D^{t+1} \deg Z_t}, \quad \tilde{h}_{\mathcal{D}}(Z_t) = \frac{h_{\mathcal{D}}(Z_t)}{\deg Z_t} \quad \text{and} \quad \tilde{h}_{\mathcal{E}}(Z_t) = \frac{h_{\mathcal{E}}(Z_t)}{\deg Z_t}\,.$$

Note that the expressions D_n, S_n and V_n do not have the same meaning here as in §5.

The proof of Theorem 2.2 is achieved in three steps. We first get a lower bound for the normalized height $\tilde{h}_{\mathcal{C}}(Z)$ of the given subvariety Z of $\mathbf{P}^n_K$. This lower bound is obtained through iterated applications of our arithmetic Bézout's inequality in the form of Proposition 5.7. Secondly, we deduce a lower bound for the difference $h_{\mathcal{E}}(Z) - h_{\mathcal{D}}(Z)$ using Proposition 5.3. Finally, we conclude using Proposition 5.5. This produces not only the expected lower bound $\rho \geq e^{-V/S}$ for the projective distance ρ between θ and $Z(\mathbf{C}_w)$, but also a lower bound for the modulus $|Z|_\theta$. We prove this as Proposition 6.4 below.

In order to carry out this program, we need more notation. For each integer $t = -1, 0, \dots, k$, we define

$$\begin{aligned} a_t &= (t+1)(t+3)\eta + (t+1)\epsilon_w\lambda_w \log 2\,, \\ b_t &= t+1\,, \\ c_t &= (t+1)\max\left\{\lambda_w \frac{V_p}{S_p}\,,\ \epsilon_w\lambda_w\log 2\right\}\,. \end{aligned}$$

We start with a preliminary lemma:

Lemma 6.1. *Let t be an integer with $0 \leq t \leq k$ and let Z_t be any subvariety of $\mathbf{P}^m_K$ of dimension t. Denote by ρ_t the projective distance between θ and $Z_t(\mathbf{C}_w)$, and suppose that*

$$\tilde{h}_{\mathcal{C}}(Z_t) \leq -a_t - b_t\frac{T}{D} - c_t\frac{S}{D}.$$

Then, we have $\rho_t < \exp\big(-V_p/S_p\big)$.

Proof. We apply Proposition 5.8 to the cycle Z_t. We find

$$\begin{aligned} \lambda_w(t+1)\frac{S}{D}\log\rho_t &\leq \tilde{h}_{\mathcal{C}}(Z_t) + (t+1)\epsilon_w\lambda_w\frac{S}{D}\log 2 + (t+1)(t+2)\eta \\ &\leq -a_t - b_t\frac{T}{D} - c_t\frac{S}{D} + (t+1)\epsilon_w\lambda_w\log 2 + (t+1)(t+2)\eta \\ &= -(t+1)\eta - b_t\frac{T}{D} - c_t\frac{S}{D} \\ &< -(t+1)\lambda_w\frac{V_p}{S_p}\frac{S}{D}, \end{aligned}$$

and thus $\log\rho_t < -V_p/S_p$. $\square$

The next two results can be viewed as higher dimensional versions of Liouville's inequality.

Lemma 6.2. *Under the hypotheses of Theorem 2.2, we have*

$$\tilde{h}_{\mathcal{C}}(Z) \geq -a_k - b_k\frac{T}{D} - c_k\frac{S}{D}.$$

Proof. We proceed by contradiction. Suppose on the contrary that the above inequality does not hold. Let t be the smallest integer with $0 \le t \le k$ for which there exists a subvariety Z_t of $\mathbf{P}^m_K$ of dimension t which is contained in Z and satisfies

$$\tilde{h}_{\mathcal{C}}(Z_t) < -a_t - b_t\frac{T}{D} - c_t\frac{S}{D}.$$

Let ρ_t be the distance between θ and $Z_t(\mathbf{C}_w)$. By virtue of Lemma 6.1, we have $\rho_t < \exp(-V_p/S_p)$. Thus, there exists a largest integer n in the interval $p < n \le q$ such that

$$\rho_t < \exp\left(-\frac{V_{n-1}}{S_{n-1}}\right).$$

Because of the assumption (ii) of the theorem, the points of $Z_t(\mathbf{C}_w)$ whose distance to θ is $< \exp(-V_{n-1}/S_{n-1})$ cannot be common zeros of the family of polynomials $\mathcal{F}_n$. Thus, there exists at least one polynomial of this family, say Q, which does not vanish identically on Z_t. By virtue of the choice of n, we also have

$$\exp\left(-\frac{V_n}{S_n}\right) \le \max\left\{\rho_t\,,\, \exp\left(-\frac{V}{S}\right)\right\}.$$

Thus, the polynomial Q satisfies all the hypotheses of Proposition 5.7 for the choice of parameters $D' = D_n$, $S' = S_n$ and $V' = V_n$. If $t \ge 1$, we conclude that there is a subvariety Z_{t-1} of $\mathbf{P}^m_K$ of dimension $t-1$ which is contained in Z_t and satisfies

$$\tilde{h}_{\mathcal{C}}(Z_{t-1}) \le \tilde{h}_{\mathcal{C}}(Z_t) + \frac{S}{DS_n}h(Q) + (2t+3)\eta + \epsilon_w\lambda_w\left(1+\frac{S}{D}\right)\log 2.$$

If $t = 0$, Proposition 5.7 shows that the same inequality holds with $Z_{t-1} = 0$. Since $h(Q)/S_n \le T_n/S_n \le T/S$, this implies

$$\begin{aligned}\tilde{h}_{\mathcal{C}}(Z_{t-1}) &< -a_t - b_t\frac{T}{D} - c_t\frac{S}{D} + \frac{T}{D} + (2t+3)\eta + \epsilon_w\lambda_w\left(1+\frac{S}{D}\right)\log 2\\ &\le -a_{t-1} - b_{t-1}\frac{T}{D} - c_{t-1}\frac{S}{D}.\end{aligned}$$

If $t \ge 1$, this contradicts the choice of t. If $t = 0$, this is still a contradiction because $\tilde{h}_{\mathcal{C}}(Z_{-1}) = 0$ and $a_{-1} = b_{-1} = c_{-1} = 0$. ∎

Lemma 6.3. *Under the hypotheses of Theorem 2.2, we have*

$$\left(\frac{S}{D}\right)^{k+1}\Big(h_{\mathcal{E}}(Z) - h_{\mathcal{D}}(Z)\Big) \ge -a - b\frac{T}{D} - c_k\deg(Z)\frac{S}{D},$$

where a and b are as in the statement of Theorem 2.2.

Proof. By Proposition 5.3, we have

$$\left(\frac{S}{D}\right)^{k+1}\Big(\tilde{h}_{\mathcal{E}}(Z)-\tilde{h}_{\mathcal{D}}(Z)\Big) \ge \tilde{h}_{\mathcal{C}}(Z)-\tilde{h}_{\mathcal{D}}(Z)-a' \tag{6.1}$$

where

$$a' = (k+1)\big(2\log(m+1)+\eta+\epsilon_w\lambda_w\log 2\big).$$

On the other hand, since a Chow form of Z in degree $(1,\dots,1)$ is a multi-homogeneous polynomial map from the $(k+1)$-fold Cartesian product $\big(K[\underline{X}]_1\big)^{k+1}$ to K which has degree $\deg Z$ on each factor, we deduce from Lemma 3.9 the upper bound

$$h_{\mathcal{D}}(Z) \le (k+1)\log\binom{\deg(Z)+m}{m} + h(Z) \le (k+1)(\log(m+1))\deg(Z)+h(Z).$$

The conclusion follows by multiplying both sides of (6.1) by $\deg Z$ and by combining the result with the lower bound for $\tilde{h}_{\mathcal{C}}(Z)$ given by Lemma 6.2 and also with the above upper bound for $h_{\mathcal{D}}(Z)$. This simply uses the relations

$$a = \big(a_k + a' + (k+1)\log(m+1)\big)\deg(Z) + h(Z) \quad\text{and}\quad b = b_k\deg(Z). \qquad \square$$

Proof of Theorem 2.2. Once again we proceed by contradiction. Suppose on the contrary that $\rho = \operatorname{dist}(\theta, Z(\mathbf{C}_w)) < e^{-V/S}$. Then Proposition 5.5 gives the upper bound

$$h_{\mathcal{E}}(Z) - h_{\mathcal{D}}(Z) \le -\lambda_w\frac{V}{S} + c'$$

where

$$c' = \epsilon_w\lambda_w\Big(1+\log\big((m+1)(k+1)\deg Z\big)\Big).$$

Combining this inequality with Lemma 6.3, we find

$$-a - b\frac{T}{D} - c_k\deg(Z)\frac{S}{D} \le \left(\frac{S}{D}\right)^{k+1}\left(-\lambda_w\frac{V}{S}+c'\right) \le -\lambda_w\frac{S^kV}{D^{k+1}} + c'\frac{S}{D}.$$

This contradicts (2.1) since, in the notation of Theorem 2.2, we have $c = c' + c_k\deg Z$. $\square$

Proposition 6.4. *Under the hypotheses of Theorem 2.2, we also have*

$$\lambda_w\log|Z|_\theta \ge -\left(a\frac{D}{S}+b\frac{T}{S}+c''\right)\left(\frac{D}{S}\right)^k$$

where

$$c'' = \lambda_w(k+1)\left(\max\left\{\frac{V_p}{S_p}, \epsilon_w \log 2\right\} + 3\epsilon_w \log(m+1)\right) \deg Z.$$

Proof. By Theorem 2.2, we already know that $\rho \geq e^{-V/S}$. Then, Proposition 5.5 reads

$$h_{\mathcal{E}}(Z) - h_{\mathcal{D}}(Z) \leq \lambda_w \log |Z|_\theta + c''',$$

where $c''' = 3\epsilon_w\lambda_w(k+1)\log(m+1)\deg(Z)$. The conclusion follows, as in the proof of Theorem 2.2, by comparing this upper bound with the lower bound provided by Lemma 6.3 and using the relation $c'' = c''' + c_k \deg Z$. □

6.2. Proof of Theorem 2.1

Suppose on the contrary that the transcendence degree of the field $K(\theta_0/\theta_i, \dots, \theta_m/\theta_i)$ were $\leq k$ for some index i with $\theta_i \neq 0$. Then the point θ would belong to $Z(\mathbf{C}_w)$ for some subvariety Z of $\mathbf{P}^m_K$ of dimension k. Since the sequence $V_n S_n^k / \big((D_n+T_n)D_n^k\big)$ is unbounded, we could then select an index $n = q$ for which the inequality (2.1) of Theorem 2.2 is satisfied with say $p = 1$. The conditions (i) and (ii) of Theorem 2.2 would also be satisfied because of the stronger corresponding conditions of Theorem 2.1. Since the point θ belongs to $Z(\mathbf{C}_w)$, its distance to $Z(\mathbf{C}_w)$ is equal to 0, contradicting the conclusion of Theorem 2.2. □

In the case of smaller zero-free regions, Theorem 2.1 gives:

Corollary 6.5. *Let $(E_n)_{n\geq 1}$ be a non-decreasing sequence of positive integers. Suppose that all the hypotheses of Theorem 2.1 are satisfied except that, instead of condition (ii), we make the weaker hypothesis that, for each sufficiently large n, the polynomials of $\mathcal{F}_n$ have no common zero α in $\mathbf{P}^m(\mathbf{C}_w)$ with*

$$\mathrm{dist}(\theta, \alpha) \leq \exp\left(-\frac{E_{n-1}V_{n-1}}{S_{n-1}}\right).$$

Then, the conclusion of Theorem 2.1 still holds provided that

$$\lim_{n\to\infty} \frac{V_n S_n^k}{(D_n + T_n) D_n^k E_n^k} = \infty. \tag{6.2}$$

Proof. Define new sequences of positive integers $(D'_n)_{n\geq 1}$ and $(S'_n)_{n\geq 1}$, and new sequences of positive real numbers $(T'_n)_{n\geq 1}$ and $(V'_n)_{n\geq 1}$ by putting, for each $n \geq 1$,

$$D'_n = 2E_nD_n, \quad S'_n = S_n, \quad T'_n = 2E_n(T_n + D_n\log(m+1)) \quad \text{and} \quad V'_n = E_nV_n.$$

Assuming (6.2), these sequences satisfy all the hypotheses of Theorem 2.1. Define also

$$\mathcal{F}'_n = \left\{ P^{2E_n}\,;\, P \in \mathcal{F}_n \right\}$$

for each $n \geq 2$. Then $\mathcal{F}'_n$ consists of homogeneous polynomials of $K[\underline{X}]$ of degree D'_n and height $\leq T'_n$. Moreover, for each sufficiently large n, the elements of $\mathcal{F}'_n$ have no common zero $\alpha \in \mathbf{P}^m(\mathbf{C}_w)$ with $\mathrm{dist}(\theta, \alpha) \leq \exp\big(-V'_{n-1}/S'_{n-1}\big)$. To conclude by using Theorem 2.1, we only have to show that, for sufficiently large n, any element Q of $\mathcal{F}'_n$ satisfies

$$\max_{|\sigma|<S'_n} \left| Q^{[\sigma]}(\underline{\theta}) \right|_w \leq \|Q\|_w e^{-V'_n}.$$

To this end, fix a choice of n, a choice of $\sigma \in \mathbf{N}^{m+1}$ with $|\sigma| < S_n$ and a choice of $Q = P^{2E_n}$ with $P \in \mathcal{F}_n$. We find

$$\begin{aligned}
\left| Q^{[\sigma]}(\underline{\theta}) \right| &\leq \sum_{\sigma_1+\cdots+\sigma_{2E_n}=\sigma} \left| P^{[\sigma_1]}(\underline{\theta}) \cdots P^{[\sigma_{2E_n}]}(\underline{\theta}) \right| \\
&\leq (m+1)^{2E_nD_n} \Big(\|P\|_w e^{-V_n} \Big)^{2E_n} \\
&\leq \|Q\|_w \exp\Big(-2E_nV_n + 6E_nD_n \log(m+1) \Big).
\end{aligned}$$

This gives the required estimate for sufficiently large n since condition (6.2) implies that V_n/D_n tends to infinity with n. ∎

6.3. Proof of Theorem 2.3

We now assume that the field K is an algebraic extension of $\mathbf{Q}$ as in §2.3. In order to prove Theorem 2.3, we require a few intermediate lemmas. The first one is a direct application of Dirichlet box principle. It constructs polynomials having small absolute value at the point $\underline{\theta}$.

Lemma 6.6. *Let c_1 be a real number $> (d_w/d)m!$. For each positive integer D and each positive real number T greater than some constant depending only on the choice of c_1, there exists a nonzero homogeneous polynomial $P \in K[\underline{X}]$ which satisfies*

$$\deg(P) = D, \quad h(P) \leq T \quad \text{and} \quad |P(\underline{\theta})|_w \leq \|P\|_w \exp\Big(-\frac{1}{c_1} D^m T \Big).$$

Proof. Let D be a positive integer and let T be a positive real number. Define

$$\delta = \exp\Big(-\frac{1}{c_1} D^m T + \epsilon_w T - \frac{d}{d_w} T \Big), \quad N = \binom{D+m}{m} \quad \text{and} \quad R = \Big(N \frac{e^T}{2} \Big)^{\epsilon_w}.$$

Denote also by E the set of all elements P of $K[\underline{X}]$ which satisfy $\|P\|_v \le (e^T/2)^{\epsilon_v}$ for each place $v \in \mathcal{M}_K$. The dimension of $K[\underline{X}]_D$ over K being N, we find that the cardinality of E is $\ge \exp\big(dN(T-c_2)\big)$ where c_2 denotes a positive constant which depends only on K. On the other hand, we have $|P(\underline{\theta})|_w \le R$ for any $P \in E$. Moreover, it is possible to cover the closed disk of K_w of center 0 and radius R by a union of at most $(c_3R/\delta)^{d_w}$ closed disks of K_w of radius $\delta/2^{\epsilon_w}$, where c_3 is a positive constant depending only on the place w. One verifies that, for sufficiently large T, this number of disks is strictly less than the cardinality of E. Hence, for these values of T, there exist at least two distinct elements P_1, P_2 of E with $|P_1(\underline{\theta}) - P_2(\underline{\theta})|_w \le \delta$. The polynomial $P = P_1 - P_2$ then fulfills all conditions of the lemma. It is nonzero and homogeneous of degree D. Its height $h(P)$ is $\le T$ since $\|P\|_v \le e^{T\epsilon_v}$ for any place $v \in \mathcal{M}_K$. Finally, it satisfies the last condition of the lemma because we have $|P(\underline{\theta})|_w \le \delta$ and because the inequality $\prod_v \|P\|_v^{d_v/d} \ge 1$ implies $\|P\|_w \ge \exp\big(\epsilon_w T - (d/d_w)T\big)$. $\square$

The next lemma refines Lemma 1.1 on page 424 of [1] and extends it to ultrametric domains as well. It is valid for any integer $m \ge 0$ and any point $\underline{\theta} \in \mathbf{C}_w^{m+1}$. It shows that, if a polynomial P has a small absolute value at $\underline{\theta}$ and if no zero of P lies close to $\underline{\theta}$, then the divided derivatives of P also have small absolute value at $\underline{\theta}$.

Lemma 6.7. *Let $D \ge 0$ be an integer, let $P \in \mathbf{C}_w[\underline{X}]$ be an arbitrary polynomial of degree $\le D$ with $P(\underline{\theta}) \ne 0$, and let δ be a positive real number such that any zero $\underline{\alpha}$ of P in $\mathbf{C}_w^{m+1}$ verifies $\|\underline{\theta} - \underline{\alpha}\|_w \ge \delta$. Then, for any $\sigma \in \mathbf{N}^{m+1}$, we have*

$$\big|P^{[\sigma]}(\underline{\theta})\big|_w \le \binom{D}{|\sigma|}^{\epsilon_w} \delta^{-|\sigma|} |P(\underline{\theta})|_w.$$

Proof. Let $\underline{u} \in \mathbf{C}_w^{m+1}$ with $\|\underline{u}\|_w = 1$. Define $Q(T) = P(\underline{\theta} + T\underline{u}) \in \mathbf{C}_w[T]$ and denote by ℓ the degree of Q. Since $Q(0) \ne 0$ and since $\mathbf{C}_w$ is algebraically closed, we can write

$$Q(T) = b(T-\beta_1)\cdots(T-\beta_\ell)$$

with $b, \beta_1, \dots, \beta_\ell \in \mathbf{C}_w^\times$. By construction, the roots of Q verify $|\beta_j|_w \ge \delta$ for $j = 1, \dots, \ell$. Let s be any integer with $0 \le s \le \ell$. By Leibniz' formula, the s-th divided derivative of Q is given by

$$Q^{[s]}(T) = \sum_{\substack{E \subseteq \{1,\dots,\ell\} \\ |E| = \ell - s}} b \prod_{j \in E} (T - \beta_j)$$

where the sum extends to all subsets E of $\{1, \dots, \ell\}$ with cardinality $\ell - s$. Since there are at most $\binom{D}{s}$ such subsets E, and since for each of these we have $\big|b\prod_{j\in E}\beta_j\big|_w \le \delta^{-s}\big|Q(0)\big|_w$, we deduce

$$\big|Q^{[s]}(0)\big|_w \le \binom{D}{s}^{\epsilon_w} \delta^{-s} \big|Q(0)\big|_w.$$

Note that this inequality also holds for $s > \ell$ because, in this case, its left hand side is 0. Moreover, we have $Q(0) = P(\underline{\theta})$ and the chain rule gives $Q^{[s]}(0) = \sum_{|\sigma|=s} P^{[\sigma]}(\underline{\theta})\underline{u}^{\sigma}$. So, the preceding observations show that, for any fixed integer $s \geq 0$, we have

$$\sup_{\|\underline{u}\|_w=1} \left| \sum_{|\sigma|=s} P^{[\sigma]}(\underline{\theta})\underline{u}^{\sigma} \right|_w \leq \binom{D}{s}^{\epsilon_w} \delta^{-s} \big|P(\underline{\theta})\big|_w.$$

The conclusion follows because, by Cauchy's inequalities, the left hand side of this inequality is an upper bound for each of the numbers $\big|P^{[\sigma]}(\underline{\theta})\big|_w$ with $|\sigma| = s$. □

The last lemma puts the preceding result in a projective context.

Lemma 6.8. *Let $P \in \mathbf{C}_w[\underline{X}]$ be a homogeneous polynomial of positive degree D, and let ρ be the distance between θ and the set of zeros of P in $\mathbf{P}^m(\mathbf{C}_w)$. Suppose $\rho > 0$. Then, for any $\sigma \in \mathbf{N}^{m+1}$, we have*

$$\big|P^{[\sigma]}(\underline{\theta})\big|_w \leq \binom{D}{|\sigma|}^{\epsilon_w} \left(\frac{\rho}{2^{\epsilon_w}}\right)^{-|\sigma|} |P(\underline{\theta})|_w.$$

Proof. By virtue of Lemma 6.7, it suffices to show that any zero $\underline{\alpha}$ of P in $\mathbf{C}_w^{m+1}$ verifies $\|\underline{\theta} - \underline{\alpha}\|_w \geq 2^{-\epsilon_w}\rho$. To see this, write $\underline{\alpha} = (\alpha_0, \dots, \alpha_m)$. If $\underline{\alpha} \neq 0$, the relations

$$\big|\theta_i\alpha_j - \theta_j\alpha_i\big|_w = \big|\alpha_j(\theta_i - \alpha_i) + \alpha_i(\alpha_j - \theta_j)\big|_w \leq 2^{\epsilon_w}\|\underline{\alpha}\|_w\|\underline{\theta} - \underline{\alpha}\|_w$$

for $i, j = 0, \dots, m$ imply $\mathrm{dist}(\theta, \alpha) \leq 2^{\epsilon_w}\|\underline{\theta} - \underline{\alpha}\|_w$ where α denotes the point of $\mathbf{P}^m(\mathbf{C}_w)$ with projective coordinates $\underline{\alpha}$. Thus, in this case, we get $\rho \leq 2^{\epsilon_w}\|\underline{\theta} - \underline{\alpha}\|_w$. If $\underline{\alpha} = 0$, this inequality still holds since $\rho \leq 2^{\epsilon_w}$ and $\|\underline{\theta}\|_w = 1$. □

Proof of Theorem 2.3. Let $c_1 = 1.01(d_w/d)m!$. For each sufficiently large integer n, Lemma 6.6 shows the existence of a nonzero homogeneous polynomial $P_n \in K[\underline{X}]$ with

$$\deg(P_n) = D_n, \quad h(P_n) \leq T_n \quad \text{and} \quad |P_n(\underline{\theta})|_w \leq \|P_n\|_w \exp\Big(-\frac{1}{c_1}D_n^m T_n\Big).$$

For each of these n, fix such a polynomial P_n. We claim that, for infinitely many n, there is at least one zero α of P_n in $\mathbf{P}^m(\mathbf{C}_w)$ which verifies the condition (2.2) of the Theorem.

To show this, we proceed by contradiction. We assume that there exists an index $n_0 \geq 2$ such that, for any $n \geq n_0$, none of the zeros of P_n satisfies this condition. Define

$$V_n = \frac{1}{c_1}D_n^m T_n \quad \text{and} \quad S_n = S = 7(m+1)$$

for each $n \geq n_0 - 1$. For $n \geq n_0$, the distance from θ to the set of zeros of P_n in $\mathbf{P}^m(\mathbf{C}_w)$ is bounded below by

$$\exp\Big(-\frac{1}{\gamma} D_{n-1}^m T_{n-1} \Big) = \exp\Big(-\frac{7.07}{8} \frac{V_{n-1}}{S_{n-1}} \Big) \geq \exp\Big(-\frac{7.07}{8} \frac{V_n}{S_n} \Big).$$

By Lemma 6.8, we deduce that, for each $\sigma \in \mathbf{N}^{m+1}$ with $|\sigma| < S_n$, we have

$$\big| P_n^{[\sigma]}(\underline{\theta}) \big|_w \leq \|P_n\|_w \exp\Big(-V_n + \frac{7.07}{8} |\sigma| \frac{V_n}{S_n} + (D_n + S_n) \log 2 \Big).$$

Since $D_n + S_n = o(V_n/S_n)$, we conclude that there exists an integer $p \geq n_0$ such that, for any integer $n \geq p$, the condition (i) of Theorem 2.2 is satisfied by the family $\mathcal{F}_n = \{P_n\}$. According to the first remark after Theorem 2.2, this implies in particular $D_n \geq S_n$ for each $n \geq p$ and thus $\Phi(D_n) \leq \eta$ where $\eta = \Phi(S)$. Moreover, the condition (ii) of this theorem is also verified with $Z = \mathbf{P}^m_K$. Let a, b, c be the constants defined in Theorem 2.2 for the present choice of Z and η. Since $\theta \in Z(\mathbf{C}_w)$, it remains only to show

$$\frac{d_w}{d} S_n^m V_n \geq (a D_n + b T_n + c S_n) D_n^m$$

for sufficiently large n to get the required contradiction. Note that $D_n \leq T_n$ and $S_n = S = o(T_n)$. So, it is enough to show

$$S^m > 1.01(a+b)m!\,.$$

Moreover, according to its definition in §4.1, the standard Chow form of Z is simply the determinant. So, we have $\deg(Z) = 1$ and $h(Z) = 0$, and thus

$$a + b = (m+1)\Big(1 + 3\log(m+1) + (m+4)\eta + \epsilon_w \frac{d_w}{d} \log 4\Big).$$

Since

$$\binom{S+m}{m} \leq \frac{(S+m)^m}{m!} \leq \frac{8^m (m+1)^m}{m!} \leq 8^m e^{m+1} \leq (8e)^{m+1},$$

we get $\eta = \Phi(S) \leq (1/7)\log(8e) \leq 0.44$. We also have $3\log(m+1) \leq (m+1)\log 3$. Since $m \geq 1$, we deduce

$$a + b \leq (m+1)\big(1 + (m+1)\log 3 + 0.44 \cdot (m+4) + \log 4\big) \leq 3.4(m+1)^2.$$

and thus $1.01(a+b)m! \leq 3.5(m+1)^2 m! \leq 7(m+1)^m \leq S^m$, as announced. ∎

7. Appendix on intersection multiplicities and Chow forms

The goal of this appendix is to show that the specific notion of intersection of cycles which was considered in §4.2 coincides with the classical one from algebraic geometry.

Denote by K any ground field and recall first some facts concerning intersection theory on the smooth variety $\mathbf{P}^m_K$. Let Z and Z' be two cycles in $\mathbf{P}^m_K$ with respective dimensions $t \geq 0$ and $t' \geq 0$. We suppose that $t + t' \geq m$ and that the cycles Z and Z' have a *proper intersection* in the sense that the set theoretic intersection $Z(\overline{K}) \cap Z'(\overline{K})$ is an algebraic set of dimension $t + t' - m$. Note that the cycles Z and Z' intersect properly exactly when any component of Z intersects properly any component of Z' and that for subvarieties $Z = Z(\mathfrak{p})$ and $Z' = Z(\mathfrak{p}')$ corresponding to homogeneous prime ideals $\mathfrak{p}$ and $\mathfrak{p}'$ in $K[\underline{X}]$, the condition of proper intersection means algebraically that all the minimal primes $\mathfrak{q}$ associated to the ideal $\mathfrak{p} + \mathfrak{p}'$ have rank equal to $\operatorname{rank}(\mathfrak{p}) + \operatorname{rank}(\mathfrak{p}') = 2m - t - t'$. We assign to cycles Z and Z' intersecting properly an *intersection cycle*

$$Z \cdot Z' = \sum_{\dim Z'' = t + t' - m} m'' Z''$$

where Z'' ranges along the set of subvarieties in $\mathbf{P}^m_K$ of dimension $t + t' - m$ and where the *intersection multiplicities* m'' are non-negative integers. The coefficient m'' is ≥ 1 exactly when $Z'' = Z(\mathfrak{q})$ for some minimal prime $\mathfrak{q}$ as above. We refer to Chapters 7 and 8 of [2] for a general definition of these multiplicities.

We shall be concerned with the special case when one of the two cycles, say Z', is the cycle of a complete intersection. Let k be an integer with $1 \leq k \leq t$ and let $P_1, \ldots, P_k$ be homogeneous polynomials in $K[\underline{X}]$ with positive degrees $D_1, \ldots, D_k$. We suppose that the intersection $Z(\overline{K}) \cap \operatorname{div}(P_1)(\overline{K}) \cap \ldots \cap \operatorname{div}(P_k)(\overline{K})$ has dimension $t - k$. This condition implies that we can successively intersect the cycle Z with $\operatorname{div}(P_1), \ldots, \operatorname{div}(P_k)$ in any order because the process of proper intersection is known to be associative and commutative. As the intersection pairing is bi-additive, we may suppose that $Z = Z(\mathfrak{p})$ is irreducible and we set

$$Z \cdot \operatorname{div}(P_1) \cdot \ldots \cdot \operatorname{div}(P_k) = \sum_{\mathfrak{q}} m_{\mathfrak{q}} Z(\mathfrak{q})$$

where $\mathfrak{q}$ ranges along the set of minimal prime ideals containing $\mathfrak{p} + (P_1) + \cdots + (P_k)$. The multiplicity $m_{\mathfrak{q}}$ is then given by Serre's formula (*cf.* §7.1.2 from [2])

$$m_{\mathfrak{q}} = \sum_{i=0}^{k} (-1)^i \operatorname{length} H_i \Big(C_\bullet \Big(P_1, \ldots, P_k \,;\, (K[\underline{X}]/\mathfrak{p})_{\mathfrak{q}} \Big) \Big)$$

where the subscript $\mathfrak{q}$ indicates localization at $\mathfrak{q}$, $C_\bullet$ denotes the Koszul's complex associated to the sequence $P_1, \ldots, P_k$ in the local ring $(K[\underline{X}]/\mathfrak{p})_{\mathfrak{q}}$ and H_i is the i-th homology module of the complex $C_\bullet$. As an example, suppose that $k = 1$ and that the polynomial $P = P_1$ does not belong to $\mathfrak{p}$. The intersection formula then reduces to the simple expression

$$Z \cdot \mathrm{div}(P) = \sum_{\mathfrak{q}} \mathrm{length}\left(\frac{K[\underline{X}]}{\mathfrak{p} + (P)}\right)_{\mathfrak{q}} Z(\mathfrak{q}) \ .$$

We are interested in the Chow forms of the cycle $Z \cdot \mathrm{div}(P_1) \cdot \ldots \cdot \mathrm{div}(P_k)$ in any degree. The following proposition shows that they are obtained by specializing at $P_1, \ldots, P_k$ the Chow forms of Z. In Section 4.2 our approach was inverse in the sense that we defined $Z \cdot \mathrm{div}(P_1) \cdot \ldots \cdot \mathrm{div}(P_k)$ as being the cycle whose Chow forms are precisely these specialized polynomials. Thus the two notions of intersection coincide, since any cycle is determined by its Chow form.

Proposition 7.1. *let Z be a cycle of dimension $t \geq 0$ in $\mathbf{P}^m_K$ and let F be a Chow form of Z in degree $(D_0, \ldots, D_t)$. Let $P_1, \ldots, P_k$ be homogeneous polynomials in $K[\underline{X}]$ with respective degrees $D_{t-k+1}, \ldots, D_t$. Suppose that the set theoretic intersection $Z(\overline{K}) \cap \mathrm{div}(P_1)(\overline{K}) \cap \ldots \cap \mathrm{div}(P_k)(\overline{K})$ has dimension $t-k$. Then the polynomial function*

$$F'(U_0, \ldots, U_{t-k}) := F(U_0, \ldots, U_{t-k}, P_1, \ldots, P_k)$$

is a Chow form of the cycle $Z \cdot \mathrm{div}(P_1) \cdot \ldots \cdot \mathrm{div}(P_k)$ in degree $(D_0, \ldots, D_{t-k})$. In other words, if we decompose

$$Z \cdot \mathrm{div}(P_1) \cdot \ldots \cdot \mathrm{div}(P_k) = \sum_{\ell} m_\ell Z_\ell$$

as a sum of irreducible cycles Z_ℓ with multiplicities $m_\ell \geq 1$ and if we choose for each index ℓ a Chow form F_ℓ of the cycle Z_ℓ in degree $(D_0, \ldots, D_{t-k})$, there exists some element $\xi \in K^\times$ such that

$$F' = \xi \prod_{\ell} F_\ell^{m_\ell}.$$

Proof. For simplicity set $Z' = Z \cdot \mathrm{div}(P_1) \cdot \ldots \cdot \mathrm{div}(P_k)$. Without loss of generality, we may suppose that the field K is algebraically closed. By multi-linearity, we can also assume that Z is a subvariety and that the polynomials $P_1, \ldots, P_k$ are irreducible in $K[\underline{X}]$.

We first verify Proposition 7.1 for $k = t$, in which case the cycle Z' is zero-dimensional. Fix a one parameter family

$$P_{j,x} = P_j + xQ_j \qquad (1 \leq j \leq t)$$

of polynomials with $Q_j \in K[\underline{X}]_{D_j}$ such that, for each $x \in K^\times$, with at most finitely many exceptions, the intersection $Z'_x(K) := Z(K) \cap \mathrm{div}(P_{1,x})(K) \cap \ldots \cap \mathrm{div}(P_t, x)(K)$ is transverse. This means that any point belonging to $Z'_x(K)$ is smooth on each algebraic subset $Z(K)$, $\mathrm{div}(P_{1,x})(K), \dots, \mathrm{div}(P_{t,x})(K)$ and that their respective tangent spaces at this point intersect properly. Then $Z'_x(K)$ contains exactly $N_0 = (\prod_{j=1}^t D_j) \deg Z$ distinct points $\{\alpha_{i,x}\}$ with multiplicity one. We can also select projective coordinates $\underline{\alpha}_{i,x}$ of the points $\alpha_{i,x}$ which are algebraic functions of the variable x regular at the origin $x = 0$. Arguing as in Lemma 4.2, we obtain for $x \neq 0$ a product expansion

$$F(U_0, P_{1,x}, \dots, P_{t,x}) = \xi_x \prod_{i=1}^{N_0} U_0(\underline{\alpha}_{i,x}),$$

where ξ_x is also an algebraic function of x regular at $x = 0$. Now by the dynamical definition of the intersection multiplicity, we know that for each point $\alpha_\ell \in Z'(K)$ the intersection multiplicity m_ℓ at the point α_ℓ is equal to the number of points $\alpha_{i,x}$ tending to α_ℓ when $x \to 0$. See Proposition 10.1 from [2] and its corollary for a purely algebraic formulation of this argument of continuity involving the notion of specialization of a cycle in an algebraic family. We substitute $x = 0$ in the above product expansion to find the expected expression

$$F'(U_0) = \xi \prod_\ell U_0(\underline{\alpha}_\ell)^{m_\ell}$$

where $Z' = \sum_\ell m_\ell(\alpha_\ell)$ and $\xi = \xi_0$.

Suppose now $k < t$ and write $Z' = \sum_\ell m_\ell Z_\ell$. We must verify that F' is a constant multiple of $\prod_\ell F_\ell^{m_\ell}$. Since the polynomial functions F_ℓ are irreducible, we know from the characteristic property of the Chow forms that $F' = \xi \prod_\ell F_\ell^{\mu_\ell}$ for some exponents $\mu_\ell \geq 1$. Now we select homogeneous polynomials $Q_1, \dots, Q_{t-k}$ of degrees $D_1, \dots, D_{t-k}$ in $K[\underline{X}]$ such that for each index ℓ the 0-cycle

$$Z_\ell \cdot \mathrm{div}(Q_1) \cdot \ldots \cdot \mathrm{div}(Q_{t-k}) = \sum_i (\alpha_{\ell,i})$$

is a sum of simple points which are all together pairwise distinct. Using the rules of commutativity and associativity of proper intersections, we observe that

$$\begin{aligned} Z'' :=& Z \cdot \mathrm{div}(Q_1) \cdot \ldots \cdot \mathrm{div}(Q_{t-k}) \cdot \mathrm{div}(P_1) \cdot \ldots \cdot \mathrm{div}(P_k) \\ =& Z' \cdot \mathrm{div}(Q_1) \cdot \ldots \cdot \mathrm{div}(Q_{t-k}) = \sum_{\ell,i} m_\ell(\alpha_{\ell,i}). \end{aligned}$$

From the first step, the Chow form F'' in degree D_0 of the 0-dimensional cycle Z'' is obtained by specializing in F the generic homogeneous polynomials $U_1, \dots, U_t$ at $Q_1, \dots, Q_{t-k}, P_1, \dots, P_k$. We find

$$\begin{aligned} F''(U_0) =& F(U_0, Q_1, \dots, Q_{t-k}, P_1, \dots, P_k) = F'(U_0, Q_1, \dots, Q_{t-k}) \\ =& \xi \prod_\ell F_\ell(U_0, Q_1, \dots, Q_{t-k})^{\mu_\ell} = \xi' \prod_{\ell,i} U_0(\underline{\alpha}_{\ell,i})^{\mu_\ell} \end{aligned}$$

since, for each index ℓ, the specialized polynomial $F_\ell(U_0, Q_1, \ldots, Q_{t-k})$ is a Chow form in degree D_0 of the cycle $Z_\ell \cdot \mathrm{div}(Q_1) \cdot \ldots \cdot \mathrm{div}(Q_{t-k})$. On the other hand, the product $\prod_{\ell,i} U_0(\underline{\alpha}_{\ell,i})^{m_\ell}$ is a Chow form in degree D_0 of the cycle $Z'' = \sum_{\ell,i} m_\ell(\alpha_{\ell,i})$. Comparing both product expansions, we find that $m_\ell = \mu_\ell$ for each index ℓ. $\square$

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