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On simultaneous approximation to a real number, its square, and its cube, II

by

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Abstract. In a previous paper with the same title, we gave an upper bound for the exponent of uniform rational approximation to a quadruple of $\mathbb{Q}$-linearly independent real numbers in geometric progression. Here, we explain why this upper bound is not optimal.

1. Introduction. For each positive integer n and each real number ξ, we follow Bugeaud and Laurent [2] and denote by $\widehat{\lambda}_n(\xi)$ the exponent of uniform rational approximation to the geometric progression $(1, \xi, \dots, \xi^n)$ of length $n+1$ and ratio ξ. This is defined as the supremum of all $\lambda \in \mathbb{R}$ for which the inequalities

$$|x_0| \le X, \qquad \max_{1\le i\le n} |x_0\xi^i - x_i| \le X^{-\lambda}$$

admit a non-zero solution $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{Z}^{n+1}$ for each large enough real number X. In their 1969 seminal paper [3], Davenport and Schmidt established upper bounds for $\widehat{\lambda}_n(\xi)$ that are independent of ξ when $[\mathbb{Q}(\xi) : \mathbb{Q}] > n$, that is, when $1, \xi, \dots, \xi^n$ are linearly independent over $\mathbb{Q}$. Then, using an argument of geometry of numbers, they deduced a result on approximation to such ξ by algebraic integers of degree at most $n+1$. For $n = 1$ and $n = 2$, both estimates are best possible. For $n = 1$, we have $\widehat{\lambda}_1(\xi) = 1$ for each $\xi \in \mathbb{R} \setminus \mathbb{Q}$, and the corresponding result on approximation by algebraic integers of degree at most 2 is best possible as explained in [3, §1]. For $n = 2$, it is shown in [3, Theorem 1a] that, for each $\xi \in \mathbb{R}$ with $[\mathbb{Q}(\xi) : \mathbb{Q}] > 2$, we have $\widehat{\lambda}_2(\xi) \le 1/\gamma \cong 0.618$, where $\gamma = (1+\sqrt{5})/2$ stands for the golden ratio. In [6], we showed that this upper bound is best possible, and in [7] that the corresponding result on approximation by algebraic integers of degree at

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most 3 is also best possible. For $n > 2$, refined upper bounds for $\widehat{\lambda}_n(\xi)$ have been established in [1, 4, 5, 8–10] but the least upper bound is unknown. This paper deals with the case $n = 3$.

Let $\lambda_3 \cong 0.4245$ denote the smallest positive root of $T^2 - \gamma^3 T + \gamma$, where $\gamma = (1 + \sqrt{5})/2$, as above. In the previous paper [8] with the same title, I proved the following statement.

THEOREM 1.1. *Let $\xi \in \mathbb{R}$ with $[\mathbb{Q}(\xi) : \mathbb{Q}] > 3$, and let c and λ be positive real numbers. Suppose that, for any sufficiently large value of X, the inequalities*

$$|x_0| \le X, \qquad \max_{1 \le i \le 3} |x_0 \xi^i - x_i| \le c X^{-\lambda} \tag{1.1}$$

admit a non-zero solution $\mathbf{x} = (x_0, x_1, x_2, x_3) \in \mathbb{Z}^4$. Then $\lambda \le \lambda_3$. Moreover, if $\lambda = \lambda_3$, then c is bounded below by a positive constant depending only on ξ.

In particular, any $\xi \in \mathbb{R}$ with $[\mathbb{Q}(\xi) : \mathbb{Q}] > 3$ has $\widehat{\lambda}_3(\xi) \le \lambda_3$.

For several years, before the publication of [8], I thought that the upper bound λ_3 for λ in Theorem 1.1 could be optimal until I realized that it is not. However, I did not include the proof of this as it was only leading to a small improvement over λ_3. The goal of this paper is to present that argument in the hope that it will help finding the least upper bound. In fact, we will prove the following result.

THEOREM 1.2. *Under the hypotheses of Theorem 1.1, we have $\lambda < \lambda_3$.*

Using the same method, it is possible to compute an explicit $\epsilon > 0$ such that $\widehat{\lambda}_3(\xi) \le \lambda_3 - \epsilon$. I refrain from doing that here in order to keep the presentation as simple as possible. In a further paper, I plan to provide more tools to make progress on this problem.

In the next two sections, we recall most of the results of [8] with some precision added, including the notion of minimal points and the definition of the important polynomial map $C : (\mathbb{R}^4)^2 \to \mathbb{R}^2$ that was already implicit in [3]. In Section 4, we introduce a new pair of polynomial maps Ψ_- and Ψ_+ from $(\mathbb{R}^4)^3$ to $\mathbb{R}^4$ and we elaborate on their analytic and algebraic properties. In Sections 5 to 7 we use these tools to study the behavior of the minimal points assuming that the hypothesis of Theorem 1.1 holds with $\lambda = \lambda_3$. In each section, we get new algebraic relations that link the minimal points. In Section 6, they involve the polynomial map C, and in Section 7, the maps $\Psi_\pm$. In the process, we isolate a very rigid structure among the subspaces spanned by consecutive minimal points. Using this, we end up with a contradiction in Section 8, and this proves Theorem 1.2. For some of the main results that we establish along the way, we indicate weaker conditions on λ for which they hold, but we omit the proofs to keep the paper reasonably short.

In an addendum, we provide a further algebraic relation involving another polynomial map with interesting algebraic properties.

2. Notation and preliminaries. The notation is as in [8]. We fix a real number ξ with $[\mathbb{Q}(\xi)\colon \mathbb{Q}] > 3$ and a real number $\lambda > 0$ which fulfills the hypothesis of Theorem 1.1 for some constant $c > 0$. For brevity, we use the symbols $\ll$ and $\gg$ to denote inequalities involving multiplicative constants that depend only on ξ and λ. We also denote by $\asymp$ their conjunction. As we are not interested in the dependence on c, we consider that $c \asymp 1$, contrary to what is done in [8].

For each integer $n \geq 1$ and each point $\mathbf{x} = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$, we define

$$\mathbf{x}^- = (x_0, \dots, x_{n-1}), \qquad \mathbf{x}^+ = (x_1, \dots, x_n), \qquad \Delta\mathbf{x} = \mathbf{x}^+ - \xi\mathbf{x}^-, \tag{2.1}$$

$$\|\mathbf{x}\| = \max_{0\le i\le n} |x_i| \quad \text{and} \quad L(\mathbf{x}) = \max_{1\le i\le n} |x_0\xi^i - x_i|. \tag{2.2}$$

For each $p = 1, \dots, n+1$, we identify $\bigwedge^p \mathbb{R}^{n+1}$ with $\mathbb{R}^{\binom{n+1}{p}}$ via an ordering of the Grassmann coordinates as in [11, Chap. I, §5]. If $n \in \{1, 2, 3\}$ and if $\mathbf{x}$ is a non-zero point of $\mathbb{Z}^{n+1}$, then $L(\mathbf{x}) \neq 0$ and we have

$$L(\mathbf{x}) \asymp \|\Delta\mathbf{x}\| \asymp \|\mathbf{x} \wedge (1, \xi, \dots, \xi^n)\|. \tag{2.3}$$

As in [8, §2], we fix a sequence $(\mathbf{x}_i)_{i\ge1}$ of non-zero points of $\mathbb{Z}^4$ with the following properties:

(a) the positive integers $X_i := \|\mathbf{x}_i\|$ form a strictly increasing sequence;
(b) the positive real numbers $L_i := L(\mathbf{x}_i)$ form a strictly decreasing sequence;
(c) if some non-zero point $\mathbf{x} \in \mathbb{Z}^4$ satisfies $L(\mathbf{x}) < L_i$ for some $i \geq 1$, then $\|\mathbf{x}\| \geq X_{i+1}$.

This is slightly different than the construction of Davenport and Schmidt in [3, §4], but it plays the same role. In particular, using (2.3), our hypothesis translates into the basic estimate

$$L_i \asymp \|\Delta\mathbf{x}_i\| \ll X_{i+1}^{-\lambda}. \tag{2.4}$$

We say that $(\mathbf{x}_i)_{i\ge1}$ is a sequence of *minimal points* for ξ in $\mathbb{Z}^4$.

For any integer $n \geq 1$, we define the *height* of a non-zero vector subspace V of $\mathbb{R}^n$ defined over $\mathbb{Q}$ to be

$$H(V) = \|\mathbf{y}_1 \wedge \cdots \wedge \mathbf{y}_p\|,$$

where $(\mathbf{y}_1, \dots, \mathbf{y}_p)$ is any basis of $V \cap \mathbb{Z}^n$ over $\mathbb{Z}$. We also set $H(\{0\}) = 1$. We now recall some definitions and results from [8, §3] relative to the subspaces $\langle \mathbf{x}_i, \dots, \mathbf{x}_j\rangle_{\mathbb{R}}$ of $\mathbb{R}^4$ spanned by consecutive minimal points $\mathbf{x}_i, \dots, \mathbf{x}_j$. They use the well-known fact that, since $[\mathbb{Q}(\xi):\mathbb{Q}]>3$, any proper subspace of $\mathbb{R}^4$ contains finitely many minimal points (see [8, Lemma 2.4]).

We first recall that each $\mathbf{x}_i$ is a *primitive* point of $\mathbb{Z}^4$, that is, a non-zero point of $\mathbb{Z}^4$ whose gcd of the coordinates is 1. Thus, $\langle \mathbf{x}_i\rangle_{\mathbb{R}}$ has height X_i for each $i\ge 1$, and so $\langle \mathbf{x}_i\rangle_{\mathbb{R}} \neq \langle \mathbf{x}_j\rangle_{\mathbb{R}}$ for distinct integers $i,j\ge 1$. For each $i\ge 2$, we define

$$W_i = \langle \mathbf{x}_{i-1}, \mathbf{x}_i\rangle_{\mathbb{R}}.$$

Then W_i has dimension 2 and the set I of integers $i\ge 2$ for which $W_i\neq W_{i+1}$ is infinite. For each $i\in I$, we define the *successor* of i in I to be the smallest element j of I with $j>i$. We also say that elements $i<j$ of I are *consecutive* in I if j is the successor of i in I. For such i and j, we have

$$W_i \neq W_{i+1} = \cdots = W_j \neq W_{j+1},$$

thus $W_{i+1}=W_j=\langle \mathbf{x}_i,\dots,\mathbf{x}_j\rangle_{\mathbb{R}} = \langle \mathbf{x}_i,\mathbf{x}_j\rangle_{\mathbb{R}}$. For each $i\in I$, we also define

$$U_i = W_i + W_{i+1} = \langle \mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}\rangle_{\mathbb{R}}.$$

Then U_i has dimension 3. Finally, we define J to be the set of all $i\in I$ for which $U_i\neq U_j$ where j is the successor of i in I. This is an infinite subset of I. For each triple of consecutive elements $h<i<j$ of I, we have

$$U_i = W_{h+1}+W_{i+1} = \langle \mathbf{x}_h,\mathbf{x}_i\rangle_{\mathbb{R}} + \langle \mathbf{x}_i,\mathbf{x}_j\rangle_{\mathbb{R}} = \langle \mathbf{x}_h,\mathbf{x}_i,\mathbf{x}_j\rangle_{\mathbb{R}},$$

thus $(\mathbf{x}_h,\mathbf{x}_i,\mathbf{x}_j)$ is a basis of U_i. We also note that $U_i = \langle \mathbf{x}_h,\dots,\mathbf{x}_j\rangle_{\mathbb{R}}$. Moreover, we have $\mathbf{x}_{j+1}\notin U_i$ if and only if $i\in J$. The heights of these subspaces of $\mathbb{R}^4$ can be estimated as follows.

PROPOSITION 2.1.

(i) *For each $i\ge 2$, the pair $(\mathbf{x}_{i-1},\mathbf{x}_i)$ is a basis of $W_i\cap\mathbb{Z}^4$ and we have*

$$H(W_i) \asymp X_i L_{i-1} \ll X_i^{1-\lambda}.$$

(ii) *For each $i\in I$, we have*

$$X_i H(U_i) \ll H(W_i)H(W_{i+1}).$$

(iii) *For each pair of consecutive elements $i<j$ of I with $i\in J$, we have*

$$H(W_j) \ll H(U_i)H(U_j).$$

Part (i) is [8, Lemma 3.1]. Part (ii) follows from a general inequality of Schmidt from [11, Chap. I, Lemma 8A] on the basis that W_i and W_{i+1} have sum U_i and intersection $\langle \mathbf{x}_i\rangle_{\mathbb{R}}$. Part (iii) follows from the same formula upon noting that the sum of U_i and U_j is $\mathbb{R}^4$ with height 1 and that their intersection is W_j.

Determinants play a crucial role in this theory. For each integer $n\ge 0$ and each choice of $\mathbf{y}_i=(y_{i,0},\dots,y_{i,n})\in\mathbb{R}^{n+1}$ for $i=0,\dots,n$, we denote by $\det(\mathbf{y}_0,\dots,\mathbf{y}_n)$ the determinant of the matrix $(y_{i,j})$ whose rows are $\mathbf{y}_0,\dots,\mathbf{y}_n$. We will need the following formula.

LEMMA 2.2. *Suppose that $n \geq 1$. Then, for $\mathbf{y}_0, \dots, \mathbf{y}_n$ as above, we have*

$$\det(\mathbf{y}_0, \dots, \mathbf{y}_n) = \sum_{i=0}^{n} (-1)^i y_{i,0} \det(\Delta\mathbf{y}_0, \dots, \widehat{\Delta\mathbf{y}_i}, \dots, \Delta\mathbf{y}_n)$$

where the hat on $\Delta\mathbf{y}_i$ on the right hand side indicates that this point is omitted from the list.

Proof. The linear map $\varphi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ sending each $\mathbf{y} = (y_0, \dots, y_n)$ of $\mathbb{R}^{n+1}$ to

$$\varphi(\mathbf{y}) = (y_0, y_1 - \xi y_0, \dots, y_n - \xi y_{n-1}) = (y_0, \Delta\mathbf{y})$$

has determinant 1. Thus, the square matrix with rows $\mathbf{y}_0, \dots, \mathbf{y}_n$ has the same determinant as that with rows $\varphi(\mathbf{y}_0), \dots, \varphi(\mathbf{y}_n)$. The result follows by expanding the determinant of the latter matrix along its first column. ∎

The formula of Lemma 2.2 yields the standard estimate

$$(2.5) \qquad |\det(\mathbf{y}_0, \dots, \mathbf{y}_n)| \ll \sum_{i=0}^{n} \|\mathbf{y}_i\| L(\mathbf{y}_0) \cdots \widehat{L(\mathbf{y}_i)} \cdots L(\mathbf{y}_n)$$

for any choice of $\mathbf{y}_0, \dots, \mathbf{y}_n \in \mathbb{R}^{n+1}$ with $n \leq 3$. We add the condition $n \leq 3$ so that the implicit constant in (2.5) is independent of n. In this paper, we will need finer estimates of the following form.

COROLLARY 2.3. *Let $n \in \{1, 2, 3\}$ and let $\mathbf{y}_0, \dots, \mathbf{y}_n$ be linearly independent elements of $\mathbb{Z}^{n+1}$. Then*

$$|\det(\mathbf{y}_0, \dots, \mathbf{y}_n)| \asymp \|\mathbf{y}_n\| \, |\det(\Delta\mathbf{y}_0, \dots, \Delta\mathbf{y}_{n-1})|$$

if $L(\mathbf{y}_n) < 1$ and if the n products $\|\mathbf{y}_i\| L(\mathbf{y}_0) \cdots \widehat{L(\mathbf{y}_i)} \cdots L(\mathbf{y}_n)$ with index $i = 0, \dots, n-1$ are smaller than some positive function δ of ξ.

Proof. Put $d = \det(\mathbf{y}_0, \dots, \mathbf{y}_n)$. Lemma 2.2 yields

$$\left| d - (-1)^n y_{n,0} \det(\Delta\mathbf{y}_0, \dots, \Delta\mathbf{y}_{n-1}) \right| \leq c \sum_{i=0}^{n-1} \|\mathbf{y}_i\| L(\mathbf{y}_0) \cdots \widehat{L(\mathbf{y}_i)} \cdots L(\mathbf{y}_n)$$

for some $c = c(\xi) > 0$. Since d is a non-zero integer, we have $|d| \geq 1$. So, if the conditions of the corollary are fulfilled with $\delta = 1/(2nc)$, we obtain

$$\left| d - (-1)^n y_{n,0} \det(\Delta\mathbf{y}_0, \dots, \Delta\mathbf{y}_{n-1}) \right| \leq 1/2 \leq |d|/2,$$

and the result follows since the condition $L(\mathbf{y}_n) < 1$ implies that $y_{n,0} \neq 0$ and $\|\mathbf{y}_n\| \asymp |y_{n,0}|$. ∎

We also recall that (2.5) generalizes to

$$(2.6) \qquad \|\mathbf{y}_0 \wedge \cdots \wedge \mathbf{y}_p\| \ll \sum_{i=0}^{p} \|\mathbf{y}_i\| L(\mathbf{y}_0) \cdots \widehat{L(\mathbf{y}_i)} \cdots L(\mathbf{y}_p)$$

for any choice of $\mathbf{y}_0, \dots, \mathbf{y}_p \in \mathbb{R}^{n+1}$ with $0 \le p \le n \le 3$. We conclude with the following estimates from [8, Lemma 2.1].

LEMMA 2.4. *Let $C \in \mathbb{Z}^2$ and $\mathbf{x} \in \mathbb{Z}^{n+1}$ with $n \in \{1,2,3\}$. Then the point $\mathbf{y} = C^+\mathbf{x}^- - C^-\mathbf{x}^+ \in \mathbb{Z}^n$ satisfies*

$$\|\mathbf{y}\| \ll \|\mathbf{x}\|L(C) + \|C\|L(\mathbf{x}) \quad \textit{and} \quad L(\mathbf{y}) \ll \|C\|L(\mathbf{x}).$$

3. The maps C and E. For each non-zero point $\mathbf{x}$ of $\mathbb{R}^4$, we define

$$V(\mathbf{x}) = \langle \mathbf{x}^-, \mathbf{x}^+ \rangle_{\mathbb{R}} \subseteq \mathbb{R}^3.$$

We also define a polynomial map $C\colon \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^2$ by

$$C(\mathbf{x}, \mathbf{y}) = (\det(\mathbf{x}^-, \mathbf{x}^+, \mathbf{y}^-), \det(\mathbf{x}^-, \mathbf{x}^+, \mathbf{y}^+))$$

and note that, for a given $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^4 \times \mathbb{R}^4$, we have

$$C(\mathbf{x}, \mathbf{y}) \neq 0 \iff (\dim V(\mathbf{x}) = 2 \text{ and } V(\mathbf{y}) \not\subseteq V(\mathbf{x})). \tag{3.1}$$

Since C is quadratic in its first argument, there is a unique tri-linear map $E\colon (\mathbb{R}^4)^3 \to \mathbb{R}^2$ such that

$$E(\mathbf{w}, \mathbf{x}, \mathbf{y}) = E(\mathbf{x}, \mathbf{w}, \mathbf{y}) \quad \text{and} \quad E(\mathbf{x}, \mathbf{x}, \mathbf{y}) = 2C(\mathbf{x}, \mathbf{y}) \tag{3.2}$$

for each choice of $\mathbf{w}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^4$. It is given by

$$\begin{aligned} E(\mathbf{w}, \mathbf{x}, \mathbf{y}) = (&\det(\mathbf{w}^-, \mathbf{x}^+, \mathbf{y}^-) - \det(\mathbf{w}^+, \mathbf{x}^-, \mathbf{y}^-), \\ &\det(\mathbf{w}^-, \mathbf{x}^+, \mathbf{y}^+) - \det(\mathbf{w}^+, \mathbf{x}^-, \mathbf{y}^+)). \end{aligned}$$

Besides (3.2), we note that this map satisfies

$$E(\mathbf{x}, \mathbf{y}, \mathbf{y}) = E(\mathbf{y}, \mathbf{x}, \mathbf{y}) = -C(\mathbf{y}, \mathbf{x}) \tag{3.3}$$

for each $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^4 \times \mathbb{R}^4$.

The following result uses the operator $\varDelta$ defined in (2.1). We write $\varDelta^2$ to denote its double iteration. Thus, for a point $\mathbf{x} \in \mathbb{R}^4$, we have $\varDelta^2\mathbf{x} = \varDelta(\varDelta\mathbf{x})$. We also denote by $\varDelta\mathbf{x}^-$ the vector $\varDelta(\mathbf{x}^-) = (\varDelta\mathbf{x})^-$, omitting parentheses. Similarly, $\varDelta\mathbf{x}^+$ stands for $\varDelta(\mathbf{x}^+) = (\varDelta\mathbf{x})^+$.

LEMMA 3.1. *For any $\mathbf{x} = (x_0, \dots, x_3)$ and $\mathbf{y} = (y_0, \dots, y_3)$ in $\mathbb{R}^4$, we have*

$$\begin{aligned} C(\mathbf{x}, \mathbf{y})^- &= x_0 \det(\varDelta^2\mathbf{x}, \varDelta\mathbf{y}^-) + y_0 \det(\varDelta\mathbf{x}^-, \varDelta^2\mathbf{x}) + \mathcal{O}(L(\mathbf{x})^2 L(\mathbf{y})), \\ C(\mathbf{x}, \mathbf{y})^+ &= x_0 \det(\varDelta^2\mathbf{x}, \varDelta\mathbf{y}^+) + y_0\xi \det(\varDelta\mathbf{x}^-, \varDelta^2\mathbf{x}) + \mathcal{O}(L(\mathbf{x})^2 L(\mathbf{y})), \\ \varDelta C(\mathbf{x}, \mathbf{y}) &= x_0 \det(\varDelta^2\mathbf{x}, \varDelta^2\mathbf{y}) + \mathcal{O}(L(\mathbf{x})^2 L(\mathbf{y})). \end{aligned}$$

Proof. For any choice of sign ϵ, we have

$$C(\mathbf{x}, \mathbf{y})^\epsilon = \det(\mathbf{x}^-, \mathbf{x}^+, \mathbf{y}^\epsilon) = \det(\mathbf{x}^-, \varDelta\mathbf{x}, \mathbf{y}^\epsilon).$$

Thus, Lemma 2.2 gives

$$C(\mathbf{x}, \mathbf{y})^\epsilon = x_0 \det(\varDelta^2\mathbf{x}, \varDelta\mathbf{y}^\epsilon) + (\mathbf{y}^\epsilon)_0 \det(\varDelta\mathbf{x}^-, \varDelta^2\mathbf{x}) + \mathcal{O}(L(\mathbf{x})^2 L(\mathbf{y})),$$

where $(\mathbf{y}^-)_0 = y_0$ and $(\mathbf{y}^+)_0 = y_1 = y_0\xi + \mathcal{O}(L(\mathbf{y}))$. This explains the first two formulas. The last one follows from them by the definition of Δ. ∎

The above estimates have the following immediate consequence.

COROLLARY 3.2. *For any* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^4$, *we have*

$$\|C(\mathbf{x},\mathbf{y})\| \ll \|\mathbf{x}\|L(\mathbf{x})L(\mathbf{y}) + \|\mathbf{y}\|L(\mathbf{x})^2, \qquad L(C(\mathbf{x},\mathbf{y})) \ll \|\mathbf{x}\|L(\mathbf{x})L(\mathbf{y}).$$

For brevity, we write

$$V_i = V(\mathbf{x}_i) \quad \text{and} \quad C_{i,j} = C(\mathbf{x}_i, \mathbf{x}_j) \tag{3.4}$$

for each pair of positive integers i and j. Then we have the following non-vanishing result.

LEMMA 3.3. *Suppose that* $\lambda > \sqrt{2} - 1 \cong 0.4142$. *There is an integer* $i_0 \geq 1$ *with the following properties:*

- (i) $\dim V_i = 2$ *and* $V_i \neq V_{i+1}$ *for any integer* $i \geq i_0$.
- (ii) *For any integer* $i \geq i_0$ *and any non-zero* $\mathbf{y} \in \mathbb{Z}^3$, *there is a choice of signs* ϵ *and* η *for which the integer* $\det(\mathbf{x}_i^\epsilon, \mathbf{x}_{i+1}^\eta, \mathbf{y})$ *is non-zero.*
- (iii) *For any pair of consecutive elements* $i < j$ *of* I *with* $i \geq i_0$, *the four points* $C_{i,i+1}$, $C_{i,j}$, $C_{j,j-1}$ *and* $C_{j,i}$ *are all non-zero, and* $C_{i,j} = bC_{i,i+1}$ *for some non-zero integer* b *with* $|b| \asymp X_j/X_{i+1}$.

Proof. For each sufficiently large integer $i \geq 1$, we have $\dim V_i = 2$ by [8, Lemma 2.3] and $V_i \neq V_{i+1}$ by [8, Proposition 5.2]. Thus property (i) holds for some integer $i_0 \geq 1$. We now show that (ii) and (iii) also hold for such i_0.

To prove (ii), fix an integer $i \geq i_0$ and a non-zero point $\mathbf{y} \in \mathbb{Z}^3$. If $\mathbf{y} \in V_i$, we can write $V_i = \langle \mathbf{x}_i^\epsilon, \mathbf{y}\rangle_{\mathbb{R}}$ for a choice of sign ϵ, and then we obtain $\mathbb{R}^3 = V_i + V_{i+1} = \langle \mathbf{x}_i^\epsilon, \mathbf{x}_{i+1}^\eta, \mathbf{y}\rangle_{\mathbb{R}}$ for a choice of sign η. If $\mathbf{y} \notin V_i$, then $\langle \mathbf{x}_i^\epsilon, \mathbf{y}\rangle_{\mathbb{R}}$ is a subspace of $\mathbb{R}^3$ of dimension 2 for any choice of sign ϵ. Choosing ϵ such that $\mathbf{x}_i^\epsilon \notin V_{i+1}$, we find again that $\mathbb{R}^3 = \langle \mathbf{x}_i^\epsilon, \mathbf{y}\rangle_{\mathbb{R}} + V_{i+1} = \langle \mathbf{x}_i^\epsilon, \mathbf{x}_{i+1}^\eta, \mathbf{y}\rangle_{\mathbb{R}}$ for a choice of sign η. In both cases the triple $(\mathbf{x}_i^\epsilon, \mathbf{x}_{i+1}^\eta, \mathbf{y})$ is linearly independent, so its determinant is a non-zero integer.

To prove (iii), fix a pair of consecutive elements $i < j$ of I with $i \geq i_0$. In view of (3.1), we have $C_{i,i+1} \neq 0$ and $C_{j-1,j} \neq 0$. Moreover [8, Lemma 4.2] gives $C_{i,j} = bC_{i,i+1}$ for some non-zero integer b with $|b| \asymp X_j/X_{i+1}$. Thus, we have $C_{i,j} \neq 0$. By (3.1), this means that $V_i \neq V_j$ and so $C_{j,i} \neq 0$. ∎

We conclude with two growth estimates for the sequence $(X_i)_{i\geq 1}$:

LEMMA 3.4. *Suppose that* $\lambda > \sqrt{2} - 1$. *Then, for each pair of consecutive elements* $i < j$ *of* I, *we have*

$$X_{j+1} \ll X_{i+1}^\theta \quad \textit{where} \quad \theta = \frac{1-\lambda}{\lambda}. \tag{3.5}$$

If moreover $i \in J$, then we also have

$$X_i \ll X_j^{\theta^2-1}. \tag{3.6}$$

Proof. Let $i < j$ be consecutive elements of I. If i is large enough, we have $C_{j,j-1} \neq 0$ by Lemma 3.3. Since $C_{j,j-1} \in \mathbb{Z}^2$, this implies that

$$1 \leq \|C_{j,j-1}\| \ll X_j L_{j-1} L_j,$$

where the second estimate comes from Corollary 3.2. As Proposition 2.1 gives $X_j L_{j-1} \asymp H(W_j) = H(W_{i+1}) \asymp X_{i+1} L_i$, we deduce that

$$1 \ll X_{i+1} L_i L_j \ll X_{i+1}^{1-\lambda} X_{j+1}^{-\lambda}$$

and (3.5) follows. For $i \in J$, the estimate (3.6) follows from [8, Corollary 5.3, equation (11)]. ∎

4. The maps Ψ_- and Ψ_+. For each choice of sign ϵ among $\{-,+\}$, we define a polynomial map $\Psi_\epsilon \colon (\mathbb{R}^4)^3 \to \mathbb{R}^4$ by the formula

$$\Psi_\epsilon(\mathbf{x},\mathbf{y},\mathbf{z}) = C(\mathbf{y},\mathbf{z})^\epsilon \mathbf{x} + E(\mathbf{y},\mathbf{z},\mathbf{x})^\epsilon \mathbf{y} - C(\mathbf{y},\mathbf{x})^\epsilon \mathbf{z}. \tag{4.1}$$

We first note the following identities.

LEMMA 4.1. *For any choice of $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}^4$, we have*

$$\Psi_-(\mathbf{x},\mathbf{y},\mathbf{z})^- = \det(\mathbf{x}^-,\mathbf{y}^-,\mathbf{z}^+)\mathbf{y}^- - \det(\mathbf{x}^-,\mathbf{y}^-,\mathbf{z}^-)\mathbf{y}^+,$$
$$\Psi_+(\mathbf{x},\mathbf{y},\mathbf{z})^+ = \det(\mathbf{x}^+,\mathbf{y}^+,\mathbf{z}^+)\mathbf{y}^- - \det(\mathbf{x}^+,\mathbf{y}^+,\mathbf{z}^-)\mathbf{y}^+.$$

Proof. For any choice of $\mathbf{y}_1,\dots,\mathbf{y}_4 \in \mathbb{R}^3$, we have

$$\sum_{i=1}^{4} (-1)^{i-1} \det(\mathbf{y}_1,\dots,\widehat{\mathbf{y}}_i,\dots,\mathbf{y}_4)\mathbf{y}_i = 0,$$

where $(\mathbf{y}_1,\dots,\widehat{\mathbf{y}}_i,\dots,\mathbf{y}_4)$ denotes the sequence obtained by removing $\mathbf{y}_i$ from $(\mathbf{y}_1,\dots,\mathbf{y}_4)$. The first formula follows from this identity applied to the points $\mathbf{x}^-,\mathbf{y}^-,\mathbf{y}^+,\mathbf{z}^- \in \mathbb{R}^3$. We obtain the second formula by applying it to $\mathbf{x}^+,\mathbf{y}^-,\mathbf{y}^+,\mathbf{z}^+ \in \mathbb{R}^3$. ∎

PROPOSITION 4.2. *For any choice of $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}^4 \setminus \{0\}$ with*

$$\frac{L(\mathbf{x})}{\|\mathbf{x}\|} \geq \frac{L(\mathbf{y})}{\|\mathbf{y}\|} \geq \frac{L(\mathbf{z})}{\|\mathbf{z}\|}, \tag{4.2}$$

and for any choice of sign ϵ, we have

$$\|\Psi_\epsilon(\mathbf{x},\mathbf{y},\mathbf{z})\| \ll \|\mathbf{y}\|^2 L(\mathbf{x})L(\mathbf{z}) + \|\mathbf{z}\| L(\mathbf{x})L(\mathbf{y})^2, \tag{4.3}$$

$$L(\Psi_\epsilon(\mathbf{x},\mathbf{y},\mathbf{z})) \ll \|\mathbf{z}\| L(\mathbf{x})L(\mathbf{y})^2. \tag{4.4}$$

Proof. Fix $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}^4 \setminus \{0\}$ with property (4.2) and set $\psi_\epsilon = \Psi_\epsilon(\mathbf{x},\mathbf{y},\mathbf{z})$ for some sign ϵ among $\{-,+\}$. We first note that $\Delta\psi_\epsilon$ is a sum of four terms of the form

$$\mathbf{v} = \pm \det(\mathbf{y}_1^\pm,\mathbf{y}_2^\pm,\mathbf{y}_3^\pm)\Delta\mathbf{y}_4,$$

where $(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4)$ is a permutation of $(\mathbf{x}, \mathbf{y}, \mathbf{y}, \mathbf{z})$ with

$$\frac{L(\mathbf{y}_1)}{\|\mathbf{y}_1\|} \ge \frac{L(\mathbf{y}_2)}{\|\mathbf{y}_2\|} \ge \frac{L(\mathbf{y}_3)}{\|\mathbf{y}_3\|}.$$

By the general estimate (2.5), we find that

$$\|\mathbf{v}\| \ll \|\mathbf{y}_3\| L(\mathbf{y}_1) L(\mathbf{y}_2) L(\mathbf{y}_4) \le \|\mathbf{z}\| L(\mathbf{x}) L(\mathbf{y})^2,$$

and (4.4) follows.

Substituting $\mathbf{y}^+ = \xi \mathbf{y}^- + \Delta \mathbf{y}$ and $\mathbf{z}^+ = \xi \mathbf{z}^- + \Delta \mathbf{z}$ in the formulas of Lemma 4.1, we find

$$\psi_\epsilon^\epsilon = \det(\mathbf{x}^\epsilon, \mathbf{y}^\epsilon, \Delta \mathbf{z})\mathbf{y}^- - \det(\mathbf{x}^\epsilon, \mathbf{y}^\epsilon, \mathbf{z}^-)\Delta \mathbf{y}.$$

Using (2.5) and (4.2), this gives

$$\|\psi_\epsilon^\epsilon\| \ll \|\mathbf{y}\|^2 L(\mathbf{x}) L(\mathbf{z}) + \|\mathbf{z}\| L(\mathbf{x}) L(\mathbf{y})^2,$$

and (4.3) follows because $\|\psi_\epsilon\| \ll \|\psi_\epsilon^\epsilon\| + L(\psi_\epsilon)$. ∎

Corollary 4.3. *For any non-zero* $\mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^4 \setminus \{0\}$ *with*

$$\frac{L(\mathbf{v})}{\|\mathbf{v}\|} \ge \frac{L(\mathbf{w})}{\|\mathbf{w}\|} \ge \frac{L(\mathbf{x})}{\|\mathbf{x}\|} \ge \frac{L(\mathbf{y})}{\|\mathbf{y}\|} \ge \frac{L(\mathbf{z})}{\|\mathbf{z}\|}, \tag{4.5}$$

and for any choice of sign ϵ, *the integer*

$$d_\epsilon = \det(\mathbf{v}, \mathbf{w}, \mathbf{x}, \Psi_\epsilon(\mathbf{x}, \mathbf{y}, \mathbf{z}))$$

satisfies

$$|d_\epsilon| \ll \big(\|\mathbf{y}\|^2 L(\mathbf{x}) L(\mathbf{z}) + \|\mathbf{x}\|\,\|\mathbf{z}\| L(\mathbf{y})^2\big) L(\mathbf{v}) L(\mathbf{w}) L(\mathbf{x}). \tag{4.6}$$

Proof. In view of (4.5), the estimate (2.5) gives

$$|d_\epsilon| \ll \|\Psi_\epsilon(\mathbf{x}, \mathbf{y}, \mathbf{z})\| L(\mathbf{v}) L(\mathbf{w}) L(\mathbf{x}) + \|\mathbf{x}\| L(\mathbf{v}) L(\mathbf{w}) L(\Psi_\epsilon(\mathbf{x}, \mathbf{y}, \mathbf{z})).$$

Then (4.6) follows from the estimates of the proposition. ∎

When the right hand side of (4.6) is sufficiently small, the integers d_- and d_+ must both be 0. The next proposition analyses the outcome of such a vanishing in a context that we will encounter later.

Proposition 4.4. *Let* $(\mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y})$ *be a basis of* $\mathbb{R}^4$ *with* $\mathbf{x}^- \wedge \mathbf{x}^+ \ne 0$, *and let*

$$\mathbf{z} = a\mathbf{y} + b\mathbf{x} + c\mathbf{w} \tag{4.7}$$

for some $a, b, c \in \mathbb{R}$. *Suppose that*

$$\det(\mathbf{v}, \mathbf{w}, \mathbf{x}, \Psi_\epsilon(\mathbf{x}, \mathbf{y}, \mathbf{z})) = 0 \tag{4.8}$$

for any choice of sign ϵ. *Then there exists* $t \in \mathbb{R}$ *such that*

(i) $C(\mathbf{y}, \mathbf{z}) = tC(\mathbf{x}, \mathbf{y})$,
(ii) $C(\mathbf{z}, \mathbf{y}) = ctC(\mathbf{x}, \mathbf{w})$,
(iii) $\det(C(\mathbf{z}, \mathbf{x}), C(\mathbf{x}, \mathbf{w})) = c^2 \det(C(\mathbf{w}, \mathbf{x}), C(\mathbf{x}, \mathbf{w}))$.

Proof. We will use the tri-linearity of the map E as well as its properties (3.2) and (3.3). We first substitute formula (4.1) for $\Psi_\epsilon(\mathbf{x},\mathbf{y},\mathbf{z})$ into (4.8). This gives

$$\det(\mathbf{v},\mathbf{w},\mathbf{x},\mathbf{y})E(\mathbf{y},\mathbf{z},\mathbf{x})^\epsilon - \det(\mathbf{v},\mathbf{w},\mathbf{x},\mathbf{z})C(\mathbf{y},\mathbf{x})^\epsilon = 0$$

for any choice of sign ϵ. In view of (4.7), we also have

$$\det(\mathbf{v},\mathbf{w},\mathbf{x},\mathbf{z}) = a\det(\mathbf{v},\mathbf{w},\mathbf{x},\mathbf{y}).$$

Since $\det(\mathbf{v},\mathbf{w},\mathbf{x},\mathbf{y}) \neq 0$, the former formula simplifies to

$$E(\mathbf{y},\mathbf{z},\mathbf{x})^\epsilon - aC(\mathbf{y},\mathbf{x})^\epsilon = 0,$$

which can also be rewritten as

$$(\mathbf{y}^- \wedge \mathbf{z}^+ - \mathbf{y}^+ \wedge \mathbf{z}^- - a\mathbf{y}^- \wedge \mathbf{y}^+) \wedge \mathbf{x}^\epsilon = 0.$$

As $\mathbf{x}^- \wedge \mathbf{x}^+ \neq 0$, we therefore have

$$\mathbf{y}^- \wedge \mathbf{z}^+ - \mathbf{y}^+ \wedge \mathbf{z}^- - a\mathbf{y}^- \wedge \mathbf{y}^+ = -t\mathbf{x}^- \wedge \mathbf{x}^+$$

for some $t \in \mathbb{R}$. This in turn implies that

$$E(\mathbf{y},\mathbf{z},\mathbf{u}) = aC(\mathbf{y},\mathbf{u}) - tC(\mathbf{x},\mathbf{u}) \tag{4.9}$$

for any $\mathbf{u} \in \mathbb{R}^4$.

For the choice of $\mathbf{u} = \mathbf{x}$, formula (4.9) reduces to

$$E(\mathbf{y},\mathbf{z},\mathbf{x}) = aC(\mathbf{y},\mathbf{x}). \tag{4.10}$$

For $\mathbf{u} = \mathbf{y}$, (4.9) yields (i) since $E(\mathbf{y},\mathbf{z},\mathbf{y}) = -C(\mathbf{y},\mathbf{z})$. For $\mathbf{u} = \mathbf{z}$, it gives

$$\begin{aligned}
C(\mathbf{z},\mathbf{y}) &= -aC(\mathbf{y},\mathbf{z}) + tC(\mathbf{x},\mathbf{z}) \\
&= -atC(\mathbf{x},\mathbf{y}) + tC(\mathbf{x},\mathbf{z}) \qquad \text{by (i)} \\
&= tC(\mathbf{x},\mathbf{z} - a\mathbf{y}) \\
&= tC(\mathbf{x}, b\mathbf{x} + c\mathbf{w}) \qquad \text{by (4.7)} \\
&= ctC(\mathbf{x},\mathbf{w}),
\end{aligned}$$

which is (ii). Upon substituting formula (4.7) for $\mathbf{z}$ into (4.10), we find

$$\begin{aligned}
0 &= E(\mathbf{y}, a\mathbf{y} + b\mathbf{x} + c\mathbf{w}, \mathbf{x}) - aC(\mathbf{y},\mathbf{x}) \\
&= aC(\mathbf{y},\mathbf{x}) - bC(\mathbf{x},\mathbf{y}) + cE(\mathbf{w},\mathbf{y},\mathbf{x}).
\end{aligned}$$

Using this relation, we obtain

$$\begin{aligned}
C(\mathbf{z},\mathbf{x}) &= (1/2)E(a\mathbf{y} + b\mathbf{x} + c\mathbf{w}, a\mathbf{y} + b\mathbf{x} + c\mathbf{w}, \mathbf{x}) \\
&= a^2C(\mathbf{y},\mathbf{x}) - abC(\mathbf{x},\mathbf{y}) + acE(\mathbf{w},\mathbf{y},\mathbf{x}) - bcC(\mathbf{x},\mathbf{w}) + c^2C(\mathbf{w},\mathbf{x}) \\
&= -bcC(\mathbf{x},\mathbf{w}) + c^2C(\mathbf{w},\mathbf{x}),
\end{aligned}$$

which yields (iii). ∎

5. First step. By [8, Corollary 6.3], the complement $I \setminus J$ of J in I is infinite if $\lambda > \lambda_2$, where $\lambda_2 \cong 0.4241$ denotes the positive root of the polynomial $P_2(T) = 3T^4 - 4T^3 + 2T^2 + 2T - 1$. In fact, we can show that $I \setminus J$ is infinite as long as $\lambda > (3-\sqrt{3})/3 \cong 0.4226$ but we will not go into this here as the proof is relatively elaborate.

Below, we recall the proof that $\lambda \le \lambda_3$ where $\lambda_3 \cong 0.4245$ is as in the introduction and we study in some detail the limit case where $\lambda = \lambda_3$. We start with a lemma which uses the notation $\theta = (1-\lambda)/\lambda$ from (3.5).

Lemma 5.1. *For each pair of consecutive elements $k < l$ of I, we have*

$$H(U_l)^{1/\lambda} \ll X_{l+1}^{\theta} X_{k+1}^{-1}. \tag{5.1}$$

When this is optimal in the sense that $H(U_l)^{1/\lambda} \gg X_{l+1}^{\theta} X_{k+1}^{-1}$ for another implicit constant depending only on ξ, we also have

$$X_{k+1} \asymp X_l, \qquad L_k \asymp X_{k+1}^{-\lambda}, \qquad L_l \asymp X_{l+1}^{-\lambda}. \tag{5.2}$$

Proof. Using the estimates of Proposition 2.1, we find

$$\begin{aligned} H(U_l) &\ll X_l^{-1} H(W_l) H(W_{l+1}) \le X_{k+1}^{-1} H(W_l) H(W_{l+1}),\\ H(W_l) &= H(W_{k+1}) \asymp X_{k+1} L_k \ll X_{k+1}^{1-\lambda},\\ H(W_{l+1}) &\asymp X_{l+1} L_l \ll X_{l+1}^{1-\lambda}, \end{aligned}$$

thus $H(U_l) \ll X_{l+1}^{1-\lambda} X_{k+1}^{-\lambda}$, which is equivalent to (5.1). If this is optimal for a set of pairs $k < l$, then all the above estimates are optimal for those pairs and this yields (5.2). ∎

Proposition 5.2. *Suppose that $\lambda \ge \lambda_3$. Then $\lambda = \lambda_3$ and there are infinitely many sequences of consecutive elements $g < h < i < j$ of I with $h \notin J$ and $i \in J$. For each of them, we have*

$$\begin{matrix} X_{g+1} \asymp X_h, & X_{h+1} \asymp X_i \asymp X_h^{\theta}, & X_{i+1} \asymp X_j \asymp X_i^{\gamma/\theta}, & X_{j+1} \asymp X_j^{\theta},\\ L_g \asymp X_{g+1}^{-\lambda}, & L_h \asymp X_{h+1}^{-\lambda}, & L_i \asymp X_{i+1}^{-\lambda}, & L_j \asymp X_{j+1}^{-\lambda}, \end{matrix} \tag{5.3}$$

and

$$H(U_h) \asymp X_{h+1}^{\lambda/\gamma}. \tag{5.4}$$

If h is large enough, we also have $g \in J$ and $X_g \ll X_h^{\theta/\gamma}$.

Note that (5.3) yields $X_{i+1} \asymp X_j \asymp X_h^{\gamma}$ and $X_{j+1} \asymp X_h^{\gamma\theta}$.

Proof of Proposition 5.2. Since $\lambda > \lambda_2$, we know by [8, Corollary 6.3] that $I \setminus J$ is infinite. Since J is infinite as well, there are arbitrarily large consecutive sequences of elements $g < h < i < j$ of I with $h \notin J$ and $i \in J$. Consider any such sequence, and set

$$U = W_h + W_{h+1} = W_i + W_{i+1}.$$

Then $U \neq U_j = W_j + W_{j+1}$ and Proposition 2.1 gives

$$H(W_j) \ll H(U)H(U_j), \tag{5.5}$$

$$X_j H(U_j) \ll H(W_j)H(W_{j+1}), \tag{5.6}$$

$$H(W_{j+1}) \asymp X_{j+1}L_j \ll X_{j+1}^{1-\lambda}. \tag{5.7}$$

Thus, we obtain

$$H(U) \gg \frac{H(W_j)}{H(U_j)} \gg \frac{X_j}{H(W_{j+1})} \gg \frac{X_j}{X_{j+1}^{1-\lambda}} \geq \frac{X_{i+1}}{X_{j+1}^{1-\lambda}}. \tag{5.8}$$

Using $1/\lambda = 1 + \theta$ and the estimate $X_{j+1} \ll X_{i+1}^{\theta}$ from Lemma 3.4, this gives

$$H(U)^{1/\lambda} \gg \frac{X_{i+1}^{1+\theta}}{X_{j+1}^{\theta}} \gg X_{i+1}^{1+\theta-\theta^2}. \tag{5.9}$$

Applying Lemma 5.1 to $U = U_h = U_i$, we also find

$$H(U)^{1/\lambda} \ll X_{i+1}^{\theta} X_{h+1}^{-1}, \tag{5.10}$$

$$H(U)^{1/\lambda} \ll X_{h+1}^{\theta} X_{g+1}^{-1} \ll X_{h+1}^{\theta - 1/\theta}, \tag{5.11}$$

where the second inequality in (5.11) uses the estimate $X_{h+1} \ll X_{g+1}^{\theta}$ from Lemma 3.4. Combining (5.9) and (5.10), we obtain

$$X_{h+1} \ll X_{i+1}^{\theta^2-1}, \tag{5.12}$$

and so (5.9) and (5.11) yield

$$X_{i+1}^{1+\theta-\theta^2} \ll H(U)^{1/\lambda} \ll X_{h+1}^{\theta-1/\theta} \ll X_{i+1}^{(\theta-1/\theta)(\theta^2-1)}. \tag{5.13}$$

As h can be chosen arbitrarily large, we conclude that

$$1 + \theta - \theta^2 \leq (\theta - 1/\theta)(\theta^2 - 1) = \theta(\theta - 1/\theta)^2,$$

which can be rewritten as

$$1 \leq (\theta - 1/\theta) + (\theta - 1/\theta)^2.$$

This gives $\theta - 1/\theta \geq 1/\gamma$ and so $\lambda^2 - \gamma^3\lambda + \gamma \geq 0$, which in turn implies that $\lambda \leq \lambda_3$. Since $\lambda \geq \lambda_3$, we conclude that $\lambda = \lambda_3$, thus $\theta - 1/\theta = 1/\gamma$ and the inequalities (5.13) are optimal. Going backwards, we deduce that all estimates (5.5) to (5.13) are optimal.

Since (5.10) and (5.11) are optimal, Lemma 5.1 gives

$$X_{g+1} \asymp X_h, \quad X_{h+1} \asymp X_i, \quad L_g \asymp X_{g+1}^{-\lambda}, \quad L_h \asymp X_{h+1}^{-\lambda}, \quad L_i \asymp X_{i+1}^{-\lambda}.$$

Optimality in (5.7) and (5.8) also yields $X_{i+1} \asymp X_j$ and $L_j \asymp X_{i+1}^{-\lambda}$. Finally, (5.9), (5.11) and (5.12) being optimal, we have

$$X_{j+1} \asymp X_{i+1}^{\theta}, \quad X_{h+1} \asymp X_{g+1}^{\theta}, \quad X_{h+1} \asymp X_{i+1}^{\theta^2-1} = X_{i+1}^{\theta/\gamma}$$

and $H(U) \asymp X_{h+1}^{(\theta-1/\theta)\lambda} = X_{h+1}^{\lambda/\gamma}$. This proves (5.3) and (5.4). Finally, using Lemma 5.1 as in (5.11), we find that

$$H(U_g) \ll X_{g+1}^{(\theta-1/\theta)\lambda} = X_{g+1}^{\lambda/\gamma}.$$

So, if h is large enough, we have $U_g \neq U = U_h$ and thus $g \in J$. Then Lemma 3.4 gives $X_g \ll X_h^{\theta^2-1} = X_h^{\theta/\gamma}$. ∎

We conclude this section with three consequences of the above estimates in the limit case where $\lambda = \lambda_3$.

Corollary 5.3. *Suppose that $\lambda = \lambda_3$. Then any pair of large enough consecutive elements of I contains at least one element of J.*

Proof. Otherwise, since J is infinite, there would exist arbitrarily large triples of consecutive elements $g < h < i$ of I with $g \notin J$, $h \notin J$ and $i \in J$, contrary to the last assertion of the proposition. ∎

More precise estimates based on similar arguments show that Corollary 5.3 holds for $\lambda \geq 0.42094$.

Corollary 5.4. *Suppose that $\lambda = \lambda_3$, and let $h < i < j$ be consecutive elements of I with $h \notin J$. If h is large enough, then $(\mathbf{x}_h, \mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_{j+1})$ is a basis of $\mathbb{R}^4$ with*

$$1 \asymp |\det(\mathbf{x}_h, \mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_{j+1})| \asymp X_{j+1}|\det(\Delta\mathbf{x}_h, \Delta\mathbf{x}_i, \Delta\mathbf{x}_j)| \asymp X_{j+1}L_hL_iL_j.$$

Proof. For h large enough, Corollary 5.3 gives $i \in J$, and then

$$\mathbb{R}^4 = U_i + U_j = \langle \mathbf{x}_h, \mathbf{x}_i, \mathbf{x}_j \rangle_{\mathbb{R}} + \langle \mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_{j+1} \rangle_{\mathbb{R}} = \langle \mathbf{x}_h, \mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_{j+1} \rangle_{\mathbb{R}},$$

thus $(\mathbf{x}_h, \mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_{j+1})$ is a basis of $\mathbb{R}^4$. As this basis is made up of integer points, its determinant d is a non-zero integer. Since

$$X_jL_hL_iL_{j+1} \leq X_jL_hL_iL_j \asymp X_h^{\gamma-\lambda\theta-\lambda\gamma-\lambda\gamma\theta} \ll X_h^{-0.575},$$

Corollary 2.3 gives

$$|d| \asymp X_{j+1}|\det(\Delta\mathbf{x}_h, \Delta\mathbf{x}_i, \Delta\mathbf{x}_j)| \ll X_{j+1}L_hL_iL_j,$$

and the conclusion follows from the computation

$$X_{j+1}L_hL_iL_j \asymp X_h^{\gamma\theta-\lambda\theta-\lambda\gamma-\lambda\theta\gamma} = 1,$$

since $\gamma\theta - \lambda\theta - \lambda\gamma - \lambda\theta\gamma = \lambda\gamma(\theta^2 - 1) - \lambda\theta = 0$. ∎

Proposition 5.5. *Suppose that $\lambda = \lambda_3$, and let $g < h < i < j$ be consecutive elements of I with $h \notin J$. If h is large enough, then*

(i) $|\det(\Delta^2\mathbf{x}_g, \Delta^2\mathbf{x}_h)| \asymp L_gL_h$ *and* $|\det(\Delta^2\mathbf{x}_i, \Delta^2\mathbf{x}_j)| \asymp L_iL_j$,

(ii) $1 \asymp \|C_{h,g}\| \asymp L(C_{h,g})$ *and* $1 \asymp \|C_{j,i}\| \asymp L(C_{j,i})$,

(iii) $L(C_{g,h}) \asymp X_g/X_h$ *and* $L(C_{i,j}) \asymp X_i/X_j$.

Proof. Since $g < h$ and $i < j$ are pairs of consecutive elements of I, Lemma 3.3(iii) shows that $C_{h,g}$ and $C_{j,i}$ are non-zero points of $\mathbb{Z}^2$ if h is large enough. As Corollary 3.2 gives

$$\|C_{h,g}\| \ll X_h L_g L_h \asymp X_h^{1-\lambda-\theta\lambda} = 1,$$

we deduce that $\|C_{h,g}\| \asymp 1$ for large enough h, and thus $L(C_{h,g}) \asymp 1$ since $\xi \notin \mathbb{Q}$. Since $L_g L_h^2$ tends to 0 as $h \to \infty$, Lemma 3.1 yields

$$1 \asymp L(C_{h,g}) \asymp X_h |\det(\Delta^2 \mathbf{x}_g, \Delta^2 \mathbf{x}_h)| \ll X_h L_g L_h \asymp 1,$$

and then

$$L(C_{g,h}) \asymp X_g |\det(\Delta^2 \mathbf{x}_g, \Delta^2 \mathbf{x}_h)| \asymp X_g L_g L_h,$$

because L_g/X_g tends to 0 as $h \to \infty$. This proves the first parts of (i)–(iii). The second parts are proved in the same way. ■

6. A new set of algebraic relations. From now on, we assume that $\lambda = \lambda_3 \cong 0.4245$ and so the estimates of Proposition 5.2 apply. To alleviate the notation, we also set

$$C_i := C_{i,i+1} = C(\mathbf{x}_i, \mathbf{x}_{i+1})$$

for each $i \in I$. By Lemma 3.3(iii), this is a non-zero point of $\mathbb{Z}^2$ for each large enough i. In this section, we show that $\det(C_j, C_k) = 0$ for any triple of consecutive elements $i < j < k$ of I with $i \in J$ large enough, and we deduce that J contains finitely many triples of consecutive elements of I. By a finer analysis that we avoid here, one can show that this finiteness property holds whenever $\lambda > \lambda_2$, where $\lambda_2 \cong 0.4241$ is defined at the beginning of Section 5.

LEMMA 6.1. *Let $h < i < j$ be consecutive elements of I with $h \notin J$. Then*

$$\|C_h\| \ll X_h^{\theta(1-2\lambda)}, \quad L(C_h) \ll X_h^{-\lambda/\gamma}, \quad \|C_i\| \ll X_h^{\gamma(1-2\lambda)}, \quad L(C_i) \ll X_h^{-\lambda/\gamma}.$$

Moreover, $\det(C_h, C_i) = 0$ *if h is large enough.*

Proof. The estimates of Corollary 3.2 and Proposition 5.2 yield

$$\|C_h\| \ll X_{h+1} L_h^2 \asymp X_h^{\theta(1-2\lambda)} \quad \text{and} \quad \|C_i\| \ll X_{i+1} L_i^2 \asymp X_h^{\gamma(1-2\lambda)}.$$

If h is large enough, Lemma 3.3(iii) gives $C_{h,i} = bC_h$ for some non-zero integer b. Then, using Corollary 3.2, we find

$$L(C_h) \le L(C_{h,i}) \ll X_h L_h L_i \asymp X_h^{1-\lambda\theta-\lambda\gamma}.$$

Similarly, if h is large enough, Lemma 3.3(iii) gives $C_{i,j} = b'C_i$ for some non-zero integer b' and, using Corollary 3.2, we find

$$L(C_i) \le L(C_{i,j}) \ll X_i L_i L_j \asymp X_h^{\theta-\lambda\gamma-\lambda\theta\gamma}.$$

This proves the remaining estimates since

$$1 - \lambda\theta - \lambda\gamma = \lambda - \lambda\gamma = -\lambda/\gamma \quad \text{and} \quad \theta - \lambda\gamma - \lambda\theta\gamma = \theta - \gamma = -\lambda/\gamma.$$

Finally, using these estimates, Lemma 2.4 gives

$$|\det(C_h, C_i)| \ll \|C_h\| L(C_i) + \|C_i\| L(C_h) \ll X_h^a,$$

where $a = \gamma(1 - 2\lambda) - \lambda/\gamma < -0.018$. As $\det(C_h, C_i)$ is an integer, it must be 0 if h is large enough. ∎

LEMMA 6.2. *Let $i < j < k$ be consecutive elements of I with $i \in J$. If i is large enough, then* $\det(C_j, C_k) = 0$.

Proof. If $j \notin J$, this follows from Lemma 6.1. So, we may assume that $j \in J$. Then we have $\{i, j\} \subset J$ and [8, Lemma 6.1] gives

$$L(C_j) \ll X_{k+1}^{\alpha} \quad \text{where} \quad \alpha = \frac{-\lambda^4 + \lambda^3 + \lambda^2 - 3\lambda + 1}{\lambda(\lambda^2 - \lambda + 1)} \cong -0.1536. \tag{6.1}$$

If $k \in J$, we also have $\{j, k\} \subset J$ and the same result gives $L(C_k) \ll X_{l+1}^{\alpha}$, where l is the successor of k in I, and so, a fortiori,

$$L(C_k) \ll X_{j+1}^{\alpha}. \tag{6.2}$$

If $k \notin J$, the last estimate still holds as Lemma 6.1 gives $L(C_k) \ll X_k^{-\lambda/\gamma} \le X_{j+1}^{-\lambda/\gamma}$ where $-\lambda/\gamma \cong -0.2623 < \alpha$. Using (6.1) and (6.2) together with the estimates for $\|C_j\|$ and $\|C_k\|$ coming from Corollary 3.2, Lemma 2.4 gives

$$\begin{aligned} |\det(C_j, C_k)| &\ll \|C_j\| L(C_k) + \|C_k\| L(C_j) \\ &\ll X_{j+1}^{1-2\lambda+\alpha} + X_{k+1}^{1-2\lambda+\alpha} \ll X_{j+1}^{-0.0026}. \end{aligned}$$

Thus $\det(C_j, C_k) = 0$ if i is large enough. ∎

PROPOSITION 6.3. *The set J contains finitely many triples of consecutive elements of I.*

Proof. Suppose on the contrary that J contains infinitely many such triples. Then there are infinitely many maximal sequences of consecutive elements $i < j < \cdots < r$ of I contained in J, with cardinality at least 3. If i is large enough, such a sequence extends to a sequence

$$h < i < j < \cdots < r < h' < i'$$

of consecutive elements of I with $h \notin J$ and $h' \notin J$, and by Lemma 6.2, the integer points $C_j, \dots, C_r, C_{h'}, C_{i'}$ are all integral multiples of a single primitive point C of $\mathbb{Z}^2$. Using Corollary 3.2 and Lemma 6.1, we find that

$$\|C\| \le \|C_j\| \ll X_{j+1}^{1-2\lambda} \quad \text{and} \quad L(C) \le L(C_{i'}) \ll X_{h'}^{-\lambda/\gamma}.$$

As $r \in J$, Lemma 3.4 gives $X_r \ll X_{h'}^{\theta^2-1} = X_{h'}^{\theta/\gamma}$. As $r > j$, we also have $X_r \ge X_{j+1} \asymp X_j^{\theta}$ using the estimates of Proposition 5.2. Thus, we obtain

$$L(C) \ll X_r^{-\lambda/\theta} \ll X_j^{-\lambda}.$$

We form the point

$$\mathbf{y} = C^-\mathbf{x}_j^+ - C^+\mathbf{x}_j^- \in \mathbb{Z}^3.$$

If h is large enough, then $V_j = \langle \mathbf{x}_j^-, \mathbf{x}_j^+ \rangle_{\mathbb{R}}$ has dimension 2 by Lemma 3.3(i), and so $\mathbf{y}$ is non-zero. Using Lemma 2.4, we find

$$L(\mathbf{y}) \ll \|C\| L_j \ll X_{j+1}^{1-3\lambda}, \qquad \|\mathbf{y}\| \ll X_j L(C) + \|C\| L_j \ll X_j^{1-\lambda}$$

since $1 - 3\lambda < 0$. So, for any choice of signs ϵ and η, we obtain, using the general estimate (2.5),

$$\begin{aligned} |\det(\mathbf{x}_{h-1}^{\epsilon}, \mathbf{x}_h^{\eta}, \mathbf{y})| &\ll X_h L_{h-1} L(\mathbf{y}) + \|\mathbf{y}\| L_{h-1} L_h \\ &\ll X_h^{1-\lambda+\gamma\theta(1-3\lambda)} + X_h^{\gamma(1-\lambda)-\lambda-\theta\lambda} \ll X_h^{-0.024}. \end{aligned}$$

By Lemma 3.3(ii), this is impossible if h is large enough. ∎

7. Another set of algebraic relations. As in the preceding section, we assume that $\lambda = \lambda_3 \cong 0.4245$. We start with the following observation.

LEMMA 7.1. *Let $g < h < i < j$ be consecutive elements of I with $h \notin J$. Then we have*

$$p\mathbf{x}_j = q\mathbf{x}_i + r\mathbf{x}_h + s\mathbf{x}_g$$

for some integers p, q, r, s with $1 \le |p| \ll 1$ and $1 \le |s| \ll 1$. Moreover, if h is large enough, then $(\mathbf{x}_{g-1}, \mathbf{x}_g, \mathbf{x}_h, \mathbf{x}_i)$ is a basis of $\mathbb{R}^4$.

Proof. Set $U = U_h = U_i$. Then $(\mathbf{x}_g, \mathbf{x}_h, \mathbf{x}_i)$ and $(\mathbf{x}_h, \mathbf{x}_i, \mathbf{x}_j)$ are bases of U as a vector space over $\mathbb{R}$, while $(\mathbf{x}_h, \mathbf{x}_i)$ is a basis of $W_{h+1} = W_i$ over $\mathbb{R}$.

By Proposition 2.1(i), the pair $(\mathbf{x}_h, \mathbf{x}_{h+1})$ is a basis of $W_{h+1} \cap \mathbb{Z}^4$ over $\mathbb{Z}$. Thus, it can be extended to a basis $(\mathbf{x}_h, \mathbf{x}_{h+1}, \mathbf{y})$ of $U \cap \mathbb{Z}^4$ over $\mathbb{Z}$. By the above, we can write

$$\begin{aligned} \mathbf{x}_i &= a\mathbf{x}_h + b\mathbf{x}_{h+1}, \\ \mathbf{x}_g &= a'\mathbf{x}_h + b'\mathbf{x}_{h+1} + c'\mathbf{y}, \\ \mathbf{x}_j &= a''\mathbf{x}_h + b''\mathbf{x}_{h+1} + c''\mathbf{y} \end{aligned}$$

for a unique choice of integers $a, a', a'', b, b', b'', c', c''$ with $b \neq 0$, $c' \neq 0$ and $c'' \neq 0$. For these integers, we find that

$$bc'\mathbf{x}_j - bc''\mathbf{x}_g \in \langle \mathbf{x}_h, \mathbf{x}_i \rangle_{\mathbb{Z}}. \tag{7.1}$$

We claim that $|bc'| \ll 1$ and $|bc''| \ll 1$. To prove this, we note that $\mathbf{x}_h \wedge \mathbf{x}_i = b\mathbf{x}_h \wedge \mathbf{x}_{h+1}$, thus

$$\|\mathbf{x}_g \wedge \mathbf{x}_h \wedge \mathbf{x}_i\| = \|b\mathbf{x}_g \wedge \mathbf{x}_h \wedge \mathbf{x}_{h+1}\| = \|bc'\mathbf{y} \wedge \mathbf{x}_h \wedge \mathbf{x}_{h+1}\| = |bc'| H(U).$$

Similarly, we find that

$$\|\mathbf{x}_j \wedge \mathbf{x}_h \wedge \mathbf{x}_i\| = |bc''| H(U).$$

The claim is then a consequence of the following computations based on the general estimate (2.6) and the estimates of Proposition 5.2:

$$\|\mathbf{x}_g \wedge \mathbf{x}_h \wedge \mathbf{x}_i\| \ll X_i L_g L_h \asymp X_h^{\theta-\lambda-\lambda\theta} = X_h^{\lambda\theta/\gamma} \asymp H(U),$$
$$\|\mathbf{x}_j \wedge \mathbf{x}_h \wedge \mathbf{x}_i\| \ll X_j L_h L_i \asymp X_h^{\gamma-\lambda\theta-\lambda\gamma} = X_h^{\lambda\theta/\gamma} \asymp H(U),$$

as $\theta - \lambda - \lambda\theta = \lambda(\theta^2 - 1) = \lambda\theta/\gamma$ and $\gamma - \lambda\theta - \lambda\gamma = (1-\lambda)(\gamma - 1) = \lambda\theta/\gamma$. This claim together with (7.1) proves the first assertion of the lemma.

Finally, if h is large enough, Proposition 5.2 gives $g \in J$, and thus we have $U_g + U_h = \mathbb{R}^4$. Since $(\mathbf{x}_{g-1}, \mathbf{x}_g, \mathbf{x}_h)$ is a basis of U_g while $(\mathbf{x}_g, \mathbf{x}_h, \mathbf{x}_i)$ is a basis of U_h, it follows that $(\mathbf{x}_{g-1}, \mathbf{x}_g, \mathbf{x}_h, \mathbf{x}_i)$ is a basis of $\mathbb{R}^4$. ∎

The next result plays a crucial role and holds whenever $\lambda \geq 0.42094$. Here, we only prove it under our current hypothesis that $\lambda = \lambda_3$.

Proposition 7.2. *Let $g < h < i < j$ be consecutive elements of I with $h \notin J$, and let ϵ be a sign among $\{-,+\}$. If h is large enough, then*

$$\det(\mathbf{x}_{g-1}, \mathbf{x}_g, \mathbf{x}_h, \Psi_\epsilon(\mathbf{x}_h, \mathbf{x}_i, \mathbf{x}_j)) = 0. \tag{7.2}$$

Proof. The conditions (4.5) of Corollary 4.3 are fulfilled for the sequence $(\mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{x}_{g-1}, \mathbf{x}_g, \mathbf{x}_h, \mathbf{x}_i, \mathbf{x}_j)$. So, upon denoting by d_ϵ the determinant on the left hand side of (7.2), we obtain

$$|d_\epsilon| \ll (X_i^2 L_h L_j + X_h X_j L_i^2) L_{g-1} L_g L_h.$$

Using the estimates (5.3) of Proposition 5.2, we find

$$X_i^2 L_h L_j \asymp X_h^{2\theta-\lambda\theta-\lambda\gamma\theta} \leq X_h^{1.2047} \quad \text{and} \quad X_h X_j L_i^2 \asymp X_h^{1+\gamma-2\lambda\gamma} \leq X_h^{1.2444}.$$

Since Lemma 3.4 gives $X_{g+1} \ll X_g^\theta$, we also have $L_{g-1} \ll X_g^{-\lambda} \ll X_{g+1}^{-\lambda/\theta}$, thus

$$L_{g-1} L_g L_h \ll X_h^{-\lambda/\theta-\lambda-\lambda\theta} \leq X_h^{-1.3131},$$

and so $|d_\epsilon| \ll X_h^{-0.687}$. As d_ϵ is an integer, we conclude that $d_\epsilon = 0$ if h is large enough. ∎

Corollary 7.3. *Let $g < h < i < j$ be consecutive elements of I with $h \notin J$. If h is large enough, there are non-zero rational numbers c and t whose numerators and denominators are bounded only in terms of ξ, such that*

(i) $C_{i,j} = tC_{h,i}$,
(ii) $C_{j,i} = ctC_{h,g}$,
(iii) $\det(C_{j,h}, C_{h,g}) = c^2 \det(C_{g,h}, C_{h,g})$.

Proof. Lemma 7.1 and Proposition 7.2 show that the hypotheses of Proposition 4.4 are fulfilled with $(\mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{x}_{g-1}, \mathbf{x}_g, \mathbf{x}_h, \mathbf{x}_i, \mathbf{x}_j)$ and $c = s/p$ for bounded non-zero integers p and s, if h is large enough. Then (i)–(iii) hold for some $t \in \mathbb{R}$. If h is large enough, Proposition 5.5(ii) also gives $\|C_{j,i}\| \asymp \|C_{h,g}\| \asymp 1$. Then (ii) implies that ct is a non-zero rational number with bounded numerator and denominator. Since c has the same property, this applies to t as well. ∎

The third identity of the corollary has the following consequence.

LEMMA 7.4. *Let $g < h < i < j$ be consecutive elements of I with $h \notin J$. If h is sufficiently large, then*

$$\|C_g\| \asymp |\det(C_{j,h}, C_{h,g})| \ll \|C_{j,h}\| \ll X_h^{\lambda^2/\gamma}.$$

Since $\lambda^2/\gamma \cong 0.111$, this is a significant improvement on the generic upper bound $\|C_g\| \ll X_{g+1}^{1-2\lambda} \asymp X_h^{1-2\lambda}$ coming from Corollary 3.2, where $1 - 2\lambda \cong 0.151$.

Proof of Lemma 7.4. If h is large enough, we have $1 \asymp \|C_{h,g}\| \asymp L(C_{h,g})$ by Proposition 5.5(ii), and Lemma 3.3(iii) gives $C_{g,h} = bC_g$ for some non-zero integer b with $|b| \asymp X_h/X_{g+1} \asymp 1$. Thus, if h is sufficiently large, Corollary 7.3(iii) yields

$$|\det(C_g, C_{h,g})| \asymp |\det(C_{j,h}, C_{h,g})| \ll \|C_{j,h}\|.$$

Using Corollary 3.2, we also find

$$\|C_{j,h}\| \ll X_j L_j L_h \asymp X_h^{\gamma-\lambda\gamma\theta-\lambda\theta} = X_h^{\lambda^2/\gamma}$$

since $\gamma - \lambda\gamma\theta - \lambda\theta = -1 + \gamma^2\lambda = \lambda^2/\gamma$. On the other hand, we note that

$$L(C_g) = |b|^{-1} L(C_{g,h}) \asymp X_g/X_h \ll X_h^{\theta/\gamma-1} \le X_h^{-0.162}$$

using Proposition 5.5(iii) and the estimate $X_g \ll X_h^{\theta/\gamma}$ of Proposition 5.2. In particular, this means that $\|C_g \wedge (1,\xi)\| \asymp L(C_g)$ tends to 0 as $h \to \infty$. As $\|C_{h,g} \wedge (1,\xi)\| \asymp L(C_{h,g}) \asymp 1$, we conclude that the angle between C_g and $C_{h,g}$ is bounded away from 0 as $h \to \infty$ and so

$$|\det(C_g, C_{h,g})| \asymp \|C_g\|\,\|C_{h,g}\| \asymp \|C_g\|. \ ∎$$

PROPOSITION 7.5. *Any pair of sufficiently large consecutive elements of I contains exactly one element of J.*

Proof. By Corollary 5.3, any pair of sufficiently large consecutive elements of I contains at least one element of J. So, it remains to show that J contains finitely many pairs of consecutive elements of I.

Suppose on the contrary that J contains infinitely many such pairs. Then it follows from Proposition 6.3 and Corollary 5.3 that there exist arbitrarily long sequences of consecutive elements $g < h < i < j < k < l$ of I with

$$g \in J, \quad h \notin J, \quad i \in J, \quad j \in J, \quad k \notin J, \quad l \in J.$$

Since $k \notin J$, Lemma 7.4 gives

$$\|C_j\| \ll X_k^{\lambda^2/\gamma}. \tag{7.3}$$

On the other hand, if h is large enough, Lemma 3.3(iii) gives $C_{j,k} = bC_j$ for some non-zero $b \in \mathbb{Z}$ with $|b| \asymp X_k/X_{j+1} \asymp 1$. In view of this, Proposition 5.5(iii) gives

$$L(C_j) \asymp L(C_{j,k}) \asymp X_j/X_k \asymp X_k^{1/\theta-1} = X_k^{-\lambda/\gamma}, \tag{7.4}$$

using the fact that $X_k \asymp X_{j+1} \asymp X_j^\theta$ since $h \notin J$ and $k \notin J$. Combining (7.3) and (7.4), we obtain $L(C_j) \ll \|C_j\|^{-1/\lambda}$. By [8, Lemma 2.2], this implies that $L(C_j) \asymp \|C_j\|^{-1/\lambda}$, but we will not need that. We will get the desired contradiction by considering the sequence $e < f < g < h$ of four consecutive elements of I ending with h, and by forming the point

$$\mathbf{y} = C_j^- \mathbf{x}_e^+ - C_j^+ \mathbf{x}_e^- \in \mathbb{Z}^3.$$

If h is large enough, Lemma 3.3(i) shows that the points $\mathbf{x}_e^-$ and $\mathbf{x}_e^+$ are linearly independent and thus $\mathbf{y}$ is non-zero. By Lemma 2.4, we have

$$\|\mathbf{y}\| \ll \|C_j\| L_e + X_e L(C_j). \tag{7.5}$$

If $f \notin J$, Proposition 5.2 gives, for h large enough,

$$X_e \ll X_f^{\theta/\gamma}, \quad L_e \asymp X_f^{-\lambda}, \quad X_f^\gamma \asymp X_h. \tag{7.6}$$

If $f \in J$ and h is large enough, Proposition 6.3 tells us that $e \notin J$ because $f, g \in J$. Then Proposition 5.2 shows that the estimates (7.6) still hold. In fact, it even gives the stronger estimate $X_e \asymp X_f^{1/\theta}$ with exponent $1/\theta < \theta/\gamma$. Combining (7.3)–(7.6) and using the estimate $X_k \asymp X_{j+1} \asymp X_h^{\gamma\theta}$ coming from Proposition 5.2, we find that

$$\|\mathbf{y}\| \ll X_k^{\lambda^2/\gamma} X_h^{-\lambda/\gamma} + X_h^{\theta/\gamma^2} X_k^{-\lambda/\gamma} \ll X_h^{\lambda^2\theta-\lambda/\gamma} + X_h^{\theta/\gamma^2-\lambda\theta} \ll X_h^{-0.018}.$$

For h large enough, this is impossible as $\mathbf{y} \neq 0$. ∎

8. Final contradiction. In this section, we assume that our fixed real number ξ of Section 2 satisfies the hypotheses of Theorem 1.1 for $\lambda = \lambda_3 \cong 0.4245$ and we prove Theorem 1.2 by reaching a contradiction.

More precisely, we will show that if $f < g < h < i < j < k < l$ are consecutive elements of I with $h \notin J$ large enough, then the points $C_{f,h}$ and $C_{k,l}$ are linearly dependent with

$$\|C_{f,h}\| < \|C_{k,l}\| \quad \text{and} \quad L(C_{f,h}) > L(C_{k,l}), \tag{8.1}$$

which is impossible. To show this, we will need sharp estimates on the above quantities.

Proposition 7.5 greatly simplifies the problem by showing that large consecutive elements of I alternate between J and $I \setminus J$. By Proposition 5.2, this provides sharp estimates on the minimal points. Explicitly, if $h < i < j$ are large consecutive elements of I with $h \notin J$, then Proposition 7.5 shows that $i \in J$ and $j \notin J$, and Proposition 5.2 gives

$$(8.2) \qquad X_{h+1} \asymp X_i \asymp X_h^{\theta}, \quad X_{i+1} \asymp X_j \asymp X_i^{\gamma/\theta}, \quad L_h \asymp X_{h+1}^{-\lambda}, \quad L_i \asymp X_{i+1}^{-\lambda}.$$

In particular, this gives $X_i^{\gamma/\theta} \ll X_{i+1} \ll X_i^{\theta}$ for each $i \in I$. Corollary 7.3(ii) also has the following consequence.

LEMMA 8.1. *There is a primitive point $(a,b) \in \mathbb{Z}^2$ such that*

$$C_{h,g} \in \langle (a,b) \rangle_{\mathbb{Z}}$$

for each pair of large enough consecutive elements $g < h$ of I with $h \notin J$.

Proof. For each pair of large enough consecutive elements $g < h$ of I with $h \notin J$, the next pair of consecutive elements $i < j$ of I has $j \notin J$, and Corollary 7.3(ii) shows that $C_{h,g}$ and $C_{j,i}$ are linearly dependent. As the latter are non-zero points of $\mathbb{Z}^2$, they are integer multiples of the same primitive point (a,b) of $\mathbb{Z}^2$. The result follows by induction on h. ∎

For each integer $i \geq 1$, we define

$$\widetilde{\Delta \mathbf{x}}_i = \frac{\Delta \mathbf{x}_i}{\|\Delta \mathbf{x}_i\|} \quad \text{and} \quad \widetilde{\Delta^2 \mathbf{x}}_i = \frac{\Delta^2 \mathbf{x}_i}{\|\Delta \mathbf{x}_i\|} = \Delta(\widetilde{\Delta \mathbf{x}}_i).$$

Since $\|\Delta \mathbf{x}_i\| \asymp L_i$, Corollary 5.4 and Proposition 5.5(i) have the following immediate consequences.

LEMMA 8.2. *For any large enough consecutive elements $g < h < i < j$ of I with $h \notin I$, we have*

$$|\det(\widetilde{\Delta \mathbf{x}}_h, \widetilde{\Delta \mathbf{x}}_i, \widetilde{\Delta \mathbf{x}}_j)| \asymp 1 \quad \textit{and} \quad |\det(\widetilde{\Delta^2 \mathbf{x}}_g, \widetilde{\Delta^2 \mathbf{x}}_h)| \asymp 1.$$

The next lemma asks for precise estimates for $|\det(\widetilde{\Delta \mathbf{x}}_i^-, \widetilde{\Delta \mathbf{x}}_i^+)|$ as i goes to infinity in I.

LEMMA 8.3. *For any large enough integers $i < j$ with $i \in I$, we have*

$$1 \leq \|C_{i,j}\| \asymp \frac{X_j}{X_{i+1}} \|C_i\| \asymp X_j L_i^2 |\det(\widetilde{\Delta \mathbf{x}}_i^-, \widetilde{\Delta \mathbf{x}}_i^+)|.$$

Proof. For integers $1 \leq i < j$, Lemma 3.1 gives

$$\|C_{i,j}\| = c|x_{j,0}|\,|\det(\Delta \mathbf{x}_i^-, \Delta^2 \mathbf{x}_i)| + \mathcal{O}(X_i L_i L_j),$$

where $c = \max\{1, |\xi|\}$ and where $x_{j,0}$ is the first coordinate of $\mathbf{x}_j$. If $i \in I$, we also have $X_{i+1} \gg X_i^{\gamma/\theta}$ by the remark below (8.2), thus

$$X_i L_i L_j \leq X_i L_i^2 \ll X_i^{1-2\lambda\gamma/\theta} \ll X_i^{-0.0133}.$$

As $\Delta^2 \mathbf{x}_i = \Delta \mathbf{x}_i^+ - \xi \Delta \mathbf{x}_i^-$, we deduce that

$$\|C_{i,j}\| = c|x_{j,0}|\,|\det(\Delta \mathbf{x}_i^-, \Delta \mathbf{x}_i^+)| + \mathcal{O}(X_i^{-0.0133}).$$

Moreover, if i is large enough, Lemma 3.3 shows that $C_i = C_{i,i+1}$ is a non-zero point of $\mathbb{Z}^2$. Then the above estimate with $j = i+1$ yields

$$1 \le \|C_i\| \asymp X_{i+1}|\det(\Delta\mathbf{x}_i^-, \Delta\mathbf{x}_i^+)|,$$

and the conclusion follows. ∎

We now exploit the various estimates of Corollary 7.3 and their consequences developed in Lemmas 7.4 and 8.1.

Proposition 8.4. *Let (a,b) be as in Lemma 8.1. For any large enough consecutive elements $g < h < i < j$ of I with $h \notin J$, we have*

(i) $|\det(\widetilde{\Delta\mathbf{x}_h^-}, \widetilde{\Delta\mathbf{x}_h^+})| \asymp X_h^\sigma |\det(\widetilde{\Delta\mathbf{x}_i^-}, \widetilde{\Delta\mathbf{x}_i^+})|$,

(ii) $|\det(\widetilde{\Delta^2\mathbf{x}_h}, a\widetilde{\Delta\mathbf{x}_g^+} - b\widetilde{\Delta\mathbf{x}_g^-})| \ll X_h^{\gamma\lambda-1} \ll X_h^{-0.3131}$,

(iii) $|\det(\widetilde{\Delta\mathbf{x}_g^-}, \widetilde{\Delta\mathbf{x}_g^+})| \asymp X_h^{-\sigma}|\det(\widetilde{\Delta^2\mathbf{x}_j}, a\widetilde{\Delta\mathbf{x}_h^+} - b\widetilde{\Delta\mathbf{x}_h^-})|$,

where $\sigma = 2 - (3+\gamma)\lambda \cong 0.0396$.

Proof. By Corollary 7.3(i), we have $\|C_{h,i}\| \asymp \|C_{i,j}\|$, and thus

$$X_i L_h^2 |\det(\widetilde{\Delta\mathbf{x}_h^-}, \widetilde{\Delta\mathbf{x}_h^+})| \asymp X_j L_i^2 |\det(\widetilde{\Delta\mathbf{x}_i^-}, \widetilde{\Delta\mathbf{x}_i^+})|$$

by the previous lemma. As we find that $X_i L_h^2 \asymp X_i^{1-2\lambda} \asymp X_h^{\theta(1-2\lambda)}$ and that $X_j L_i^2 \asymp X_j^{1-2\lambda} \asymp X_h^{\gamma(1-2\lambda)}$, this yields the estimate of part (i) with $\sigma = (\gamma-\theta)(1-2\lambda) = (\lambda/\gamma)(1-2\lambda) = 2-(3+\gamma)\lambda$.

Lemma 8.1 implies that $aC_{h,g}^+ - bC_{h,g}^- = 0$. By the formulas of Lemma 3.1, this gives

$$X_h|\det(\Delta^2\mathbf{x}_h, a\Delta\mathbf{x}_g^+ - b\Delta\mathbf{x}_g^-)| \ll X_g L_h^2,$$

and part (ii) follows as $X_g L_h^2/(X_h L_g L_h) = X_g L_h/(X_h L_g) \asymp X_h^{\theta/\gamma-\lambda\theta-1+\lambda} = X_h^{\gamma\lambda-1}$.

Finally, Lemma 7.4 gives $\|C_g\| \asymp |\det(C_{j,h}, C_{h,g})|$. As $C_{h,g}$ is a non-zero multiple of (a,b) by Lemma 8.1, and as it has bounded norm by Proposition 5.5(ii), it is a bounded non-zero multiple of (a,b). We deduce that

$$\begin{aligned}1 \le \|C_g\| \asymp |aC_{j,h}^+ - bC_{j,h}^-| &= |\det(\mathbf{x}_j^-, \Delta\mathbf{x}_j, a\mathbf{x}_h^+ - b\mathbf{x}_h^-)| \\ &\asymp X_j|\det(\Delta^2\mathbf{x}_j, a\Delta\mathbf{x}_h^+ - b\Delta\mathbf{x}_h^-)|,\end{aligned}$$

using Lemma 2.2 to expand the determinant, and noting that $X_h L_j^2$ tends to 0 as $h \to \infty$. Since Lemma 8.3 gives $\|C_g\| \asymp X_{g+1}|\det(\Delta\mathbf{x}_g^-, \Delta\mathbf{x}_g^+)|$, we obtain the estimate of part (iii) by observing that $X_j L_h L_j/(X_{g+1}L_g^2) \asymp X_h^{\gamma-\theta\lambda-\gamma\theta\lambda-1+2\lambda} = X_h^{-\sigma}$. ∎

In a first step, we deduce upper bound estimates for the quantities $|\det(\widetilde{\Delta\mathbf{x}_i^-}, \widetilde{\Delta\mathbf{x}_i^+})|$ with $i \in I$. We will show later that they are best possible up to multiplicative constants.

Corollary 8.5. *Let σ be as in Proposition* 8.4. *For any pair of consecutive elements $g < h$ of I with $h \notin J$, we have*

(i) $|\det(\widetilde{\Delta\mathbf{x}}_g^-, \widetilde{\Delta\mathbf{x}}_g^+)| \ll X_h^{-\sigma}$,

(ii) $|\det(\widetilde{\Delta\mathbf{x}}_h^-, \widetilde{\Delta\mathbf{x}}_h^+)| \ll X_h^{-\sigma/\gamma}$.

Proof. We may assume that $g < h$ are large enough so that Proposition 8.4 applies to the sequence of four consecutive elements $g < h < i < j$ of I starting with g. Then part (i) follows immediately from Proposition 8.4(iii). For part (ii), we may assume that h is large enough so that $j \notin J$ and thus the estimate of part (i) holds with the pair $g < h$ replaced by $i < j$. Then Proposition 8.4(i) gives

$$|\det(\widetilde{\Delta\mathbf{x}}_h^-, \widetilde{\Delta\mathbf{x}}_h^+)| \ll X_h^{\sigma} X_j^{-\sigma} \asymp X_h^{\sigma-\gamma\sigma} = X_h^{-\sigma/\gamma}. \blacksquare$$

Corollary 8.6. *Let σ be as in Proposition* 8.4. *For any pair of consecutive elements $g < h$ of I with $h \notin J$, there are points (s_g, t_g) and (s_h, t_h) of norm* 1 *in $\mathbb{R}^2$ such that*

$$\widetilde{\Delta\mathbf{x}}_g = \pm(s_g^2, s_g t_g, t_g^2) + \mathcal{O}(X_h^{-\sigma}) \quad \textit{and} \quad \widetilde{\Delta\mathbf{x}}_h = \pm(s_h^2, s_h t_h, t_h^2) + \mathcal{O}(X_h^{-\sigma/\gamma}).$$

As $\widetilde{\Delta\mathbf{x}}_g$ and $\widetilde{\Delta\mathbf{x}}_h$ are points of norm 1 in $\mathbb{R}^3$, this is a direct consequence of Corollary 8.5 and of the following simple observation.

Lemma 8.7. *Let $\mathbf{y} \in \mathbb{R}^3$ with $\|\mathbf{y}\| = 1$, and let $\delta = |\det(\mathbf{y}^-, \mathbf{y}^+)|$. There exists a point $(r, s) \in \mathbb{R}^2$ with $\|(r, s)\| = 1$ such that*

$$\|\mathbf{y} \pm (r^2, rs, s^2)\| \le 2\delta.$$

Proof. We may assume that $\delta < 1$, otherwise any point (r, s) of norm 1 has the required property. Writing $\mathbf{y} = (a, b, c)$, we have $\delta = |ac - b^2|$. By permuting a and c, and by multiplying $\mathbf{y}$ by -1 if necessary, we may assume that $a = |a| \ge |c|$. We set $(r, s) = (1, b)$. Then we have $\|(r, s)\| = 1$ since $|b| \le \|\mathbf{y}\| \le 1$ and we find

$$\|\mathbf{y} - (1, b, b^2)\| = \max\{1 - a, |b^2 - c|\}. \tag{8.3}$$

If $a < 1$, we have $|c| < 1$, thus $|b| = 1$ since $\|\mathbf{y}\| = 1$, and then $\delta = 1 - ac$. As $\delta < 1$, this implies that $c > 0$, thus the right hand side of (8.3) becomes $\max\{1 - a, 1 - c\} \le \delta$. If $a = 1$, it reduces to $|b^2 - c| = \delta$. In both cases, we are done. $\blacksquare$

From now on, we fix a pair of points (s_g, t_g) and (s_h, t_h) as in Corollary 8.6 for each pair of consecutive elements $g < h$ of I with $g \in J$ and $h \notin J$. This yields a unique point (s_i, t_i) for each large enough $i \in I$.

Proposition 8.8. *For any large enough consecutive elements $g < h < i < j$ of I with $h \notin J$, we have*

(i) $1 \asymp |t_g - \xi s_g| \asymp |t_h - \xi s_h| \asymp |s_g t_h - s_h t_g|$,
(ii) $1 \asymp |s_h t_i - s_i t_h| \asymp |s_h t_j - s_j t_h| \asymp |s_i t_j - s_j t_i|$.

Proof. Using the formulas of Corollary 8.6, the estimates of Lemma 8.2 become

$$1 \asymp |\det(\widetilde{\Delta^2\mathbf{x}}_g, \widetilde{\Delta^2\mathbf{x}}_h)| = \left|(t_g - \xi s_g)(t_h - \xi s_h)\det\begin{pmatrix} s_g & t_g \\ s_h & t_h\end{pmatrix}\right| + \mathcal{O}(X_h^{-\sigma/\gamma}),$$

$$1 \asymp |\det(\widetilde{\Delta\mathbf{x}}_h, \widetilde{\Delta\mathbf{x}}_i, \widetilde{\Delta\mathbf{x}}_j)| = \left|\det\begin{pmatrix} s_h^2 & s_h t_h & t_h^2 \\ s_i^2 & s_i t_i & t_i^2 \\ s_j^2 & s_j t_j & t_j^2\end{pmatrix}\right| + \mathcal{O}(X_h^{-\sigma/\gamma})$$
$$= |(s_h t_i - s_i t_h)(s_h t_j - s_j t_h)(s_i t_j - s_j t_i)| + \mathcal{O}(X_h^{-\sigma/\gamma}).$$

The conclusion follows since all the factors involved have bounded absolute values. ∎

In particular, Proposition 8.8(i) implies that $|t_i - \xi s_i| \asymp 1$ for each large enough $i \in I$. Analyzing in the same way the estimate of Proposition 8.4(ii), we find the following relation.

Proposition 8.9. *Let (a,b) be as in Lemma 8.1 and let $\kappa = \|(a,b)\|$. For each pair of consecutive elements $g < h$ of I with $h \notin J$, we have*

$$(s_g, t_g) = \pm\kappa^{-1}(a,b) + \mathcal{O}(X_h^{-\sigma}), \tag{8.4}$$

where σ is as in Proposition 8.4. If h is large enough, then we also have $|at_h - bs_h| \asymp 1$.

Proof. We may assume that the pair $g < h$ comes from a sequence of consecutive elements $g < h < i < j$ of I with $h \notin J$ large enough so that Proposition 8.4 applies. Using the formula of Corollary 8.6 for $\widetilde{\Delta\mathbf{x}}_g$, we find that

$$X_h^{-0.3131} \gg |\det(\widetilde{\Delta^2\mathbf{x}}_h, a\widetilde{\Delta\mathbf{x}}_g^{+} - b\widetilde{\Delta\mathbf{x}}_g^{-})|$$
$$= |(at_g - bs_g)\det(\widetilde{\Delta^2\mathbf{x}}_h, (s_g, t_g))| + \mathcal{O}(X_h^{-\sigma}).$$

Using the formula of Corollary 8.6 for $\widetilde{\Delta\mathbf{x}}_h$ and Proposition 8.8(i), we also note that

$$|\det(\widetilde{\Delta^2\mathbf{x}}_h, (s_g, t_g))| = |(t_h - \xi s_h)(s_g t_h - s_h t_g)| + \mathcal{O}(X_h^{-\sigma/\gamma}) \asymp 1.$$

So, we conclude that

$$|at_g - bs_g| \ll X_h^{-\sigma}.$$

If $|b| \le |a|$, we have $1 \le |a| = \kappa \ll 1$ and this gives $t_g = (b/a)s_g + \mathcal{O}(X_h^{-\sigma})$, thus

$$(s_g, t_g) = s_g\big(1, b/a\big) + \mathcal{O}(X_h^{-\sigma}).$$

Since $\|(s_g,t_g)\|=1$, this implies that $s_g=\pm1+\mathcal{O}(X_h^{-\sigma})$ and (8.4) follows. The case where $|a|\le|b|$ is similar and also yields (8.4). Using this formula for (s_g,t_g) and assuming h large enough, we conclude from Proposition 8.8(i) that

$$|at_h-bs_h|=\kappa|s_gt_h-t_gs_h|+\mathcal{O}(X_h^{-\sigma})\asymp 1. \quad\blacksquare$$

We deduce the following strengthening of Corollary 8.5.

Corollary 8.10. *Let σ be as in Proposition* 8.4. *For any large enough consecutive elements $g<h$ of I with $h\notin J$, we have*

(i) $|\det(\widetilde{\Delta\mathbf{x}}_g^-,\widetilde{\Delta\mathbf{x}}_g^+)|\asymp X_h^{-\sigma}$,
(ii) $|\det(\widetilde{\Delta\mathbf{x}}_h^-,\widetilde{\Delta\mathbf{x}}_h^+)|\asymp X_h^{-\sigma/\gamma}$.

Proof. For large enough consecutive elements $g<h<i<j$ of I with $h\notin J$, we have $j\notin J$ and we find

$$\begin{aligned}|\det(\widetilde{\Delta^2\mathbf{x}}_j,a\widetilde{\Delta\mathbf{x}}_h^+-b\widetilde{\Delta\mathbf{x}}_h^-)|&\\ &=|(t_j-\xi s_j)(at_h-bs_h)(s_jt_h-s_ht_j)|+\mathcal{O}(X_h^{-\sigma/\gamma})\asymp 1\end{aligned}$$

using the formulas of Corollary 8.6, the estimates of Proposition 8.8, and the last estimate of Proposition 8.9. This gives (i) as a consequence of Proposition 8.4(iii). Finally, (ii) follows from (i) with g replaced by i, together with Proposition 8.4(i), similarly to the proof of Corollary 8.5(ii). $\blacksquare$

Proposition 8.11. *Let σ be as in Proposition* 8.4. *For any large enough consecutive elements $g<h<i<j$ of I with $h\notin J$, we have*

$$\|C_{g,h}\|\asymp X_h^{\gamma^2\lambda-1},\ L(C_{g,h})\asymp X_h^{-\lambda/\gamma^2},\ \|C_{h,j}\|\asymp X_h^{\gamma(3\lambda-1)},\ L(C_{h,j})\asymp X_h^{\gamma^2\lambda-\gamma}.$$

Proof. Using Lemma 8.3 and the estimates of the previous corollary, we find that

$$\begin{aligned}\|C_{g,h}\|&\asymp X_hL_g^2|\det(\widetilde{\Delta\mathbf{x}}_g^-,\widetilde{\Delta\mathbf{x}}_g^+)|\asymp X_h^{1-2\lambda-\sigma}=X_h^{\gamma^2\lambda-1},\\ \|C_{h,j}\|&\asymp X_jL_h^2|\det(\widetilde{\Delta\mathbf{x}}_h^-,\widetilde{\Delta\mathbf{x}}_h^+)|\asymp X_h^{\gamma-2\theta\lambda-\sigma/\gamma}=X_h^{\gamma(3\lambda-1)}.\end{aligned}$$

By Proposition 5.5(iii), we also have

$$L(C_{g,h})\asymp X_g/X_h\asymp X_h^{\theta/\gamma-1}=X_h^{-\lambda/\gamma^2}.$$

Finally, Lemma 3.1 gives

$$\Delta C_{h,j}=x_{h,0}\det(\Delta^2\mathbf{x}_h,\Delta^2\mathbf{x}_j)+\mathcal{O}(L_h^2L_j),$$

where $x_{h,0}$ is the first coordinate of $\mathbf{x}_h$. Using the formulas of Corollary 8.6 together with the estimates of Proposition 8.8, we find that

$$|\det(\widetilde{\Delta^2\mathbf{x}}_h,\widetilde{\Delta^2\mathbf{x}}_j)|=|(t_h-\xi s_h)(t_j-\xi s_j)(s_ht_j-s_jt_h)|+\mathcal{O}(X_h^{-\sigma/\gamma})\asymp 1,$$

and so

$$L(C_{h,j}) = |\Delta C_{h,j}| \asymp X_h L_h L_j \asymp X_h^{1-\theta\lambda-\gamma\theta\lambda} = X_h^{\gamma^2\lambda-\gamma}. \ \blacksquare$$

Final contradiction. Let $f < g < h < i < j < k < l$ be consecutive elements of I with $h \notin J$. If h is large enough, we have

$$\{f,h,j,l\} \subset I \setminus J, \quad \{g,i,k\} \subset J, \quad X_f \asymp X_h^{1/\gamma}, \quad X_l \asymp X_h^{\gamma^2},$$

and Proposition 8.11 gives

$$\begin{aligned} \|C_{k,l}\| &\asymp X_h^{\gamma^4\lambda-\gamma^2} = X_h^{0.2915\ldots}, & L(C_{k,l}) &\asymp X_h^{-\lambda} = X_h^{-0.4245\ldots}, \\ \|C_{f,h}\| &\asymp X_h^{3\lambda-1} = X_h^{0.2735\ldots}, & L(C_{f,h}) &\asymp X_h^{\gamma\lambda-1} = X_h^{-0.3131\ldots}. \end{aligned}$$

Using the standard estimate (2.5) for determinants, we deduce that

$$|\det(C_{f,h}, C_{k,l})| \ll \|C_{f,h}\| L(C_{k,l}) + \|C_{k,l}\| L(C_{f,h}) \ll X_h^{-0.021}.$$

As this determinant is an integer, it vanishes if h is large enough, and we conclude that $C_{f,h} = \rho C_{k,l}$ for some non-zero $\rho \in \mathbb{Q}$ that depends on h. If h is large enough, we also note that $\|C_{f,h}\| < \|C_{k,l}\|$ and $L(C_{f,h}) > L(C_{k,l})$, as claimed in (8.1). This is impossible since the first inequality implies that $|\rho| < 1$, while the second yields $|\rho| > 1$. This contradiction completes the proof of Theorem 1.2.

9. Addendum. Although the above shows that the hypotheses of Theorem 1.1 are not satisfied for $\lambda = \lambda_3$, it is nevertheless useful to search for further polynomial relations satisfied by the sequence $(\mathbf{x}_i)_{i\in I}$, assuming that $\lambda = \lambda_3$, because these relations may continue to hold for smaller values of λ. They may also suggest new constructions that will eventually produce some $\xi \in \mathbb{R}$ with $[\mathbb{Q}(\xi) : \mathbb{Q}] > 3$ whose exponent $\widehat{\lambda}_3(\xi)$ is largest possible, in a similar way to what is done in [6] for the exponent $\widehat{\lambda}_2(\xi)$.

I found several such relations. For brevity, I will simply indicate one of them. It is linked with the polynomial map $\Xi\colon (\mathbb{R}^4)^3 \to \mathbb{R}^4$ given by

$$\begin{aligned} \Xi(\mathbf{x},\mathbf{y},\mathbf{z}) &= C(\mathbf{z},\mathbf{x})^{-}\Psi_{+}(\mathbf{y},\mathbf{x},\mathbf{z}) - C(\mathbf{z},\mathbf{x})^{+}\Psi_{-}(\mathbf{y},\mathbf{x},\mathbf{z}) \\ &= -\det(E(\mathbf{x},\mathbf{z},\mathbf{y}), C(\mathbf{z},\mathbf{x}))\mathbf{x} - \det(C(\mathbf{x},\mathbf{z}), C(\mathbf{z},\mathbf{x}))\mathbf{y} \\ &\quad + \det(C(\mathbf{x},\mathbf{y}), C(\mathbf{z},\mathbf{x}))\mathbf{z}. \end{aligned}$$

This polynomial map has algebraic properties that are similar to the map from $(\mathbb{R}^3)^2$ to $\mathbb{R}^3$ that plays a central role in [6, Corollary 5.2] and sends a pair $(\mathbf{x},\mathbf{y})$ to $[\mathbf{x},\mathbf{x},\mathbf{y}]$ in the notation of [6, §2]. The present map sends $(\mathbb{Z}^4)^3$ to $\mathbb{Z}^4$, and it can be shown (or checked on a computer) that, for any

$\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^4$, the point $\mathbf{w} = \Xi(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathbb{R}^4$ satisfies

$$\begin{aligned} C(\mathbf{w}, \mathbf{x}) &= \det(C(\mathbf{z}, \mathbf{x}), C(\mathbf{z}, \mathbf{y})) \det(C(\mathbf{x}, \mathbf{y}), C(\mathbf{x}, \mathbf{z})) C(\mathbf{x}, \mathbf{z}), \\ C(\mathbf{x}, \mathbf{w}) &= \det(C(\mathbf{x}, \mathbf{y}), C(\mathbf{x}, \mathbf{z})) C(\mathbf{z}, \mathbf{x}), \\ \Xi(\mathbf{x}, \mathbf{z}, \mathbf{w}) &= \det(C(\mathbf{w}, \mathbf{x}), C(\mathbf{x}, \mathbf{w}))\, \mathbf{z} \\ &= \det(C(\mathbf{z}, \mathbf{x}), C(\mathbf{z}, \mathbf{y})) \det(C(\mathbf{x}, \mathbf{y}), C(\mathbf{x}, \mathbf{z}))^2 \\ &\quad \cdot \det(C(\mathbf{x}, \mathbf{z}), C(\mathbf{z}, \mathbf{x})) \mathbf{z}. \end{aligned}$$

It can also be shown that, for $\mathbf{x}, \mathbf{y}, \mathbf{z}$ as in Proposition 4.2, the point $\mathbf{w}$ has

$$\begin{aligned} L(\mathbf{w}) &\ll \|\mathbf{z}\|^2 L(\mathbf{x})^3 L(\mathbf{y}) L(\mathbf{z}), \\ \|\mathbf{w}\| &\ll \|\mathbf{z}\|^2 L(\mathbf{x})^3 L(\mathbf{y}) L(\mathbf{z}) + \|\mathbf{x}\|^2 \|\mathbf{z}\| L(\mathbf{x}) L(\mathbf{y}) L(\mathbf{z})^2. \end{aligned}$$

Suppose that $\lambda = \lambda_3$, and let $j_1 < j_2 < j_3 < \cdots$ denote the elements of I in increasing order. Without loss of generality, by dropping the first element of I if necessary, we may assume that $j_{2i-1} \in J$ and $j_{2i} \notin J$ for each large enough i. Then, upon setting $\mathbf{y}_i = \mathbf{x}_{j_i}$ for each $i \geq 1$, one finds using the above estimates that, when i is large enough,

$$\det(\mathbf{y}_{2i-6}, \mathbf{y}_{2i-5}, \mathbf{y}_{2i-4}, \Xi(\mathbf{y}_{2i}, \mathbf{y}_{2i+1}, \mathbf{y}_{2i+2})) = 0.$$

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