EXTREMAL NUMBERS AND MULTI-PARAMETRIC GEOMETRY OF NUMBERS

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In memory of Bertrand Russell, mathematician and philosopher, for his commitment to peace.

Abstract. We study weighted simultaneous rational approximation to points of the form $(1, \xi, \xi^2)$, for a class of extremal real numbers ξ, within the framework of multi-parametric geometry of numbers.

1. Introduction

Fix an integer $n \geq 2$. For each $A = (a_{i,j}) \in \mathrm{GL}_n(\mathbb{R})$ and each $\mathbf{q} = (q_1, \dots, q_n) \in \mathbb{R}^n$, we denote by $\mathcal{C}_A(\mathbf{q})$ the parallelepiped of $\mathbb{R}^n$ made of the points $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ satisfying

$$|a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n| \leq \exp(-q_i) \quad \text{for } i = 1, \dots, n. \tag{1.1}$$

For each $j = 1, \dots, n$, we also denote by $L_{A,j}(\mathbf{q})$ the logarithm of its j-th minimum with respect to $\mathbb{Z}^n$, namely the smallest $t \in \mathbb{R}$ such that the product $e^t\mathcal{C}_A(\mathbf{q})$ defined by

$$|a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n| \leq \exp(t - q_i) \quad (1 \leq i \leq n)$$

contains at least j linearly independent points $\mathbf{x} = (x_1, \dots, x_n)$ of $\mathbb{Z}^n$. The map

$$\begin{aligned} \mathbf{L}_A : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ \mathbf{q} &\longmapsto (L_{A,1}(\mathbf{q}), L_{A,2}(\mathbf{q}), \dots, L_{A,n}(\mathbf{q})). \end{aligned}$$

carries much information about Diophantine approximation to the matrix A and little is lost in estimating $\mathbf{L}_A$ up to bounded error on $\mathbb{R}^n$.

Geometry of numbers imposes constraints on the components of $\mathbf{L}_A$, as shown in [11, 12, 13] in one parameter setting, and in [2] in the general case. For example, since the logarithm of the volume of $\mathcal{C}_A(\mathbf{q})$ is $-(q_1 + \cdots + q_n) + \mathcal{O}_A(1)$, Minkowski's second convex body theorem gives

$$L_{A,1}(\mathbf{q}) + \cdots + L_{A,n}(\mathbf{q}) = q_1 + \cdots + q_n + \mathcal{O}_A(1), \tag{1.2}$$

where $\mathcal{O}_A(1)$ stands for a function of $\mathbf{q}$ whose absolute value is bounded above by a constant depending only of A. The main open problem is whether or not, for given $n \geq 3$, this and other such conditions suffice to characterize the set of all maps $\mathbf{L}_A$ with $A \in \mathrm{GL}_n(\mathbb{R})$ modulo the additive group of bounded functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ (cf. [9]). Even a characterization of the maps $\mathbf{L}_A$ modulo the additive group of functions $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ with $\|\mathbf{f}(\mathbf{q})\|/\|\mathbf{q}\| \to 0$ for

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$\|\mathbf{q}\| \to \infty$, using say the maximum norm, would be useful in questions related to spectra of exponents of Diophantine approximation. We do not address this problem here. Our goal instead is to estimate the map $\mathbf{L}_A$ for a specific type of matrices $A \in \mathrm{GL}_3(\mathbb{R})$.

We first restrict to matrices of the form

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \xi_1 & -1 & 0 \\ \xi_2 & 0 & -1 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{R})$$

attached to points $\boldsymbol{\xi} = (1, \xi_1, \xi_2) \in \mathbb{R}^3$. For convenience, we adapt the notation as follows. For each $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2$, we set $\mathcal{C}_{\boldsymbol{\xi}}(\mathbf{q}) = \mathcal{C}_A(0, q_1, q_2)$, and, for each $j = 1, 2, 3$, we set $L_{\boldsymbol{\xi},j}(\mathbf{q}) = L_{A,j}(0, q_1, q_2)$, that is the smallest $t \in \mathbb{R}$ for which the inequalities

$$|x_0| \le \exp(t), \quad |x_0\xi_1 - x_1| \le \exp(t - q_1), \quad |x_0\xi_2 - x_2| \le \exp(t - q_2) \tag{1.3}$$

admit at least j linearly independent solutions $\mathbf{x} = (x_0, x_1, x_2)$ in $\mathbb{Z}^3$. Finally, we define

$$\begin{array}{rccl} \mathbf{L}_{\boldsymbol{\xi}} : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ & \mathbf{q} & \longmapsto & (L_{\boldsymbol{\xi},1}(\mathbf{q}), L_{\boldsymbol{\xi},2}(\mathbf{q}), L_{\boldsymbol{\xi},3}(\mathbf{q})). \end{array} \tag{1.4}$$

No information is lost in the process as we can recover $\mathbf{L}_A$ from $\mathbf{L}_{\boldsymbol{\xi}}$ using

$$L_{A,j}(q_0, q_1, q_2) = q_0 + L_{\boldsymbol{\xi},j}(q_1 - q_0, q_2 - q_0) \quad (1 \le j \le 3)$$

for any $(q_0, q_1, q_2) \in \mathbb{R}^3$. Moreover, the estimate (1.2) becomes

$$L_{\boldsymbol{\xi},1}(\mathbf{q}) + L_{\boldsymbol{\xi},2}(\mathbf{q}) + L_{\boldsymbol{\xi},3}(\mathbf{q}) = q_1 + q_2 + \mathcal{O}_{\boldsymbol{\xi}}(1) \tag{1.5}$$

for each $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2$. As for the maps $\mathbf{L}_A$ with $A \in \mathrm{GL}_3(\mathbb{R})$, one may ask for a characterization of the set of all maps $\mathbf{L}_{\boldsymbol{\xi}} \colon \mathbb{R}^2 \to \mathbb{R}^3$ with $\boldsymbol{\xi} = (1, \xi_1, \xi_2) \in \mathbb{R}^3$ modulo bounded functions on $\mathbb{R}^2$. In the next section, we show that this can easily be done outside of the critical sector

$$\mathcal{D} = \{(q_1, q_2) \in \mathbb{R}^2 \,;\, 0 \le q_1/2 \le q_2 \le 2q_1\} \tag{1.6}$$

if we restrict to points $\boldsymbol{\xi}$ for which ξ_1 and ξ_2 are badly approximable.

In this paper, we further restrict to points $\boldsymbol{\xi} = (1, \xi, \xi^2)$ where ξ belongs to a countably infinite set of extremal real numbers defined in [5, Theorem 3.1] and shown there to have the strongest possible measure of approximation by cubic algebraic integers (cf. [1, Theorem 1]). For each such $\boldsymbol{\xi}$, we construct an explicit approximation of $\mathbf{L}_{\boldsymbol{\xi}}$ with bounded difference on the set of points (q_1, q_2) of $\mathbb{R}^2$ with $q_1 \ge 0$ and $q_2 \le q_1$, an angular sector which covers the lower half of $\mathcal{D}$. An outline of the main result and of its proof is given in section 3. As an application, we use this to compute exponents of weighted rational approximation to these points in section 15. Numerical experiments, which we do not include, suggest a relatively chaotic behaviour for the functions $\mathbf{L}_{\boldsymbol{\xi}}$ on the upper half of $\mathcal{D}$.

We believe that these are the first examples of points $\boldsymbol{\xi}$ in $\mathbb{R}^3$ with $\mathbb{Q}$-linearly independent coordinates, for which $\mathbf{L}_{\boldsymbol{\xi}}$ is estimated up to bounded difference in an angular sector of $\mathbb{R}^2$ with positive angle contained in $\mathcal{D}$.

2. Basic tool and preliminary observations

For each integer $n \geq 1$ and each point $\mathbf{x}$ of $\mathbb{R}^n$, we denote by $\|\mathbf{x}\|$ the maximum norm of $\mathbf{x}$. As in [9, Section 1], we set

$$\Delta_3 = \{(p_1, p_2, p_3) \in \mathbb{R}^3 \,;\, p_1 \leq p_2 \leq p_3\}, \tag{2.1}$$

and denote by $\Phi\colon \mathbb{R}^3 \to \Delta_3$ the continuous map which lists the coordinates of a point in non-decreasing order. Clearly, the functions $\mathbf{L}_{\boldsymbol{\xi}}$ defined by (1.4) take values in Δ_3 and so $\Phi \circ \mathbf{L}_{\boldsymbol{\xi}} = \mathbf{L}_{\boldsymbol{\xi}}$. We also note that Φ is 1-Lipschitz, namely that

$$\|\Phi(\mathbf{p}) - \Phi(\mathbf{p}')\| \leq \|\mathbf{p} - \mathbf{p}'\| \quad \text{for any } \mathbf{p}, \mathbf{p}' \in \mathbb{R}^3. \tag{2.2}$$

Fix a point $\boldsymbol{\xi} = (1, \xi_1, \xi_2) \in \mathbb{R}^3$. We define the *trajectory* of a non-zero integer point $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3$ as the map $L_{\mathbf{x}}\colon \mathbb{R}^2 \to \mathbb{R}$ whose value at a point $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2$ is the smallest $t \in \mathbb{R}$ for which $\mathbf{x} \in e^t \mathcal{C}_{\boldsymbol{\xi}}(\mathbf{q})$ or, equivalently, for which (1.3) holds. Thus,

$$L_{\mathbf{x}}(\mathbf{q}) = \max\{\log|x_0|, q_1 + \log|x_0\xi_1 - x_1|, q_2 + \log|x_0\xi_2 - x_2|\}, \tag{2.3}$$

with the convention that we omit a term in the maximum when it involves $\log 0 = -\infty$. Aside when this happens, the trajectory $L_{\mathbf{x}}$ of $\mathbf{x}$ is the maximum of three affine maps and Figure 1 shows the angular sectors of $\mathbb{R}^2$ where they realize this maximum.

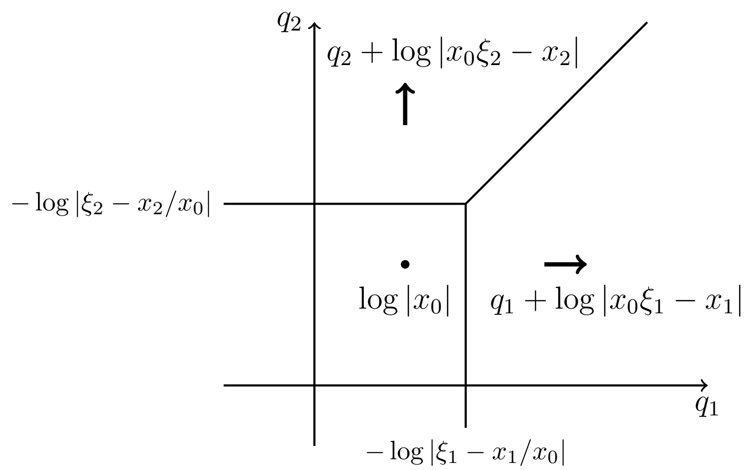

Figure 1. The map $L_{\mathbf{x}}$ in the generic case.

It follows from (2.3) that

$$|L_{\mathbf{x}}(\mathbf{q}) - L_{\mathbf{x}}(\mathbf{q}')| \leq \|\mathbf{q} - \mathbf{q}'\| \quad \text{for any } \mathbf{q}, \mathbf{q}' \in \mathbb{R}^2. \tag{2.4}$$

We now show that $\mathbf{L}_{\boldsymbol{\xi}}$ is also 1-Lipschitz.

Lemma 2.1. *For any $\mathbf{q}, \mathbf{q}' \in \mathbb{R}^2$, we have $\|\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) - \mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}')\| \leq \|\mathbf{q} - \mathbf{q}'\|$.*

Proof. Let $\mathbf{q}, \mathbf{q}' \in \mathbb{R}^2$. Choose linearly independent integer points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{Z}^3$ such that

$$\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) = (L_{\mathbf{x}_1}(\mathbf{q}), L_{\mathbf{x}_2}(\mathbf{q}), L_{\mathbf{x}_3}(\mathbf{q})).$$

For each $j = 1,2,3$, the points $\mathbf{x}_1,\dots,\mathbf{x}_j$ are linearly independent, and so, using (2.4) for each of these points, we find that

$$L_{\boldsymbol{\xi},j}(\mathbf{q}') \le \max_{1\le i\le j} L_{\mathbf{x}_i}(\mathbf{q}') \le \max_{1\le i\le j}\big(L_{\mathbf{x}_i}(\mathbf{q}) + \|\mathbf{q}-\mathbf{q}'\|\big) = L_{\boldsymbol{\xi},j}(\mathbf{q}) + \|\mathbf{q}-\mathbf{q}'\|.$$

The result follows as we may permute the roles of $\mathbf{q}$ and $\mathbf{q}'$ in the above estimates. □

The main tool for estimating $\mathbf{L}_{\boldsymbol{\xi}}$ is the following result, similar to [8, Lemma 4.1].

Lemma 2.2. *Let $\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3 \in \mathbb{Z}^3$ be linearly independent integer points. Let $\mathbf{q} = (q_1,q_2) \in \mathbb{R}^2$, $\mathbf{p} = (p_1,p_2,p_3) \in \mathbb{R}^3$ and $\delta \ge 0$ such that*

$$p_1 + p_2 + p_3 = q_1 + q_2 \quad \textit{and} \quad L_{\mathbf{x}_j}(\mathbf{q}) \le p_j + \delta \quad \textit{for } j = 1,2,3. \tag{2.5}$$

Then, we have $\big\|\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) - \Phi(\mathbf{p})\big\| \le 5\delta + c_1$ for a constant c_1 depending only on $\boldsymbol{\xi}$.

Proof. Set $\mathbf{p}' = (L_{\mathbf{x}_1}(\mathbf{q}), L_{\mathbf{x}_2}(\mathbf{q}), L_{\mathbf{x}_3}(\mathbf{q}))$. By definition of $\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q})$, the difference $\Phi(\mathbf{p}')-\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q})$ has non-negative coordinates. Thus, its norm is bounded above by the sum of its coordinates. Using (1.5) and the first part of (2.5), this gives

$$\begin{aligned}\|\,\Phi(\mathbf{p}') - \mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q})\,\| &\le \sum_{j=1}^{3} L_{\mathbf{x}_j}(\mathbf{q}) - \sum_{j=1}^{3} L_{\boldsymbol{\xi},j}(\mathbf{q})\\ &\le \sum_{j=1}^{3} L_{\mathbf{x}_j}(\mathbf{q}) - (q_1+q_2) + c = \sum_{j=1}^{3}(L_{\mathbf{x}_j}(\mathbf{q}) - p_j) + c,\end{aligned}$$

for a constant $c = c(\boldsymbol{\xi}) \ge 0$. By the second part of (2.5), this implies that

$$\|\,\Phi(\mathbf{p}') - \mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q})\,\| \le 3\delta + c \quad \text{and} \quad -2\delta - c \le L_{\mathbf{x}_j}(\mathbf{q}) - p_j \le \delta \quad \text{for } j = 1,2,3.$$

Thus, $\|\,\Phi(\mathbf{p}) - \Phi(\mathbf{p}')\,\| \le \|\,\mathbf{p}-\mathbf{p}'\,\| \le 2\delta + c$, and so $\big\|\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) - \Phi(\mathbf{p})\big\| \le 5\delta + 2c$. □

The next statement provides an estimate for $\mathbf{L}_{\boldsymbol{\xi}}$ outside the angular sector $\mathcal{D}$ defined by (1.6) when ξ_1 and ξ_2 are badly approximable, as illustrated in Figure 2. Recall that a real number ξ is called *badly approximable* when there exists a constant $c = c(\xi) > 0$ such that $|x_0\xi - x_1| \ge c^{-1}|x_0|^{-1}$ for any $(x_0,x_1) \in \mathbb{Z}^2$ with $x_0 \ne 0$.

Lemma 2.3. *Let $\mathbf{q} = (q_1,q_2) \in \mathbb{R}^2$.*

(i) *If $q_1 \le 0$ and $q_2 \le 0$, then* $\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) = \Phi(0,q_1,q_2) + \mathcal{O}_{\boldsymbol{\xi}}(1)$.

(ii) *Suppose that ξ_1 is badly approximable. If $q_1 \ge \max\{0, 2q_2\}$, then*
$$\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) = (q_2, q_1/2, q_1/2) + \mathcal{O}_{\boldsymbol{\xi}}(1).$$

(iii) *Suppose that ξ_2 is badly approximable. If $q_2 \ge \max\{0, 2q_1\}$, then*
$$\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) = (q_1, q_2/2, q_2/2) + \mathcal{O}_{\boldsymbol{\xi}}(1).$$

Proof. For the basis $\{\mathbf{e}_1 = (1,0,0),\ \mathbf{e}_2 = (0,1,0),\ \mathbf{e}_3 = (0,0,1)\}$ of $\mathbb{Z}^3$, we find

$$L_{\mathbf{e}_1}(\mathbf{q}) = \max\{0, q_1 + \log|\xi_1|, q_2 + \log|\xi_2|\}, \quad L_{\mathbf{e}_2}(\mathbf{q}) = q_1, \quad L_{\mathbf{e}_3}(\mathbf{q}) = q_2.$$

If $q_1 \le 0$ and $q_2 \le 0$, we have $L_{\mathbf{e}_1}(\mathbf{q}) = \mathcal{O}_{\boldsymbol{\xi}}(1)$, and Lemma 2.2 yields

$$\|\,\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) - \Phi(0,q_1,q_2)\,\| \le 5|L_{\mathbf{e}_1}(\mathbf{q})| + c_1 = \mathcal{O}_{\boldsymbol{\xi}}(1),$$

which proves (i).

Suppose that ξ_1 is badly approximable and that $q_1 \ge \max\{0, 2q_2\}$. Set

$$(t_1, t_2) = \mathbf{L}_A(0, q_1) \quad \text{where} \quad A = \begin{pmatrix} 1 & 0 \\ \xi_1 & -1 \end{pmatrix}. \tag{2.6}$$

Then, choose linearly independent points $(x_{1,0}, x_{1,1})$, $(x_{2,0}, x_{2,1})$ in $\mathbb{Z}^2$ such that, for $j = 1, 2$,

$$|x_{j,0}| \le \exp(t_j) \quad \text{and} \quad |x_{j,0}\xi_1 - x_{j,1}| \le \exp(t_j - q_1).$$

Since ξ_1 is badly approximable, these estimates imply that $t_j \ge q_1/2 + \mathcal{O}_{\xi_1}(1)$. As Minkowski's inequality (1.2) applied to (2.6) gives $t_1 + t_2 = q_1 + \mathcal{O}_{\xi_1}(1)$, we deduce that

$$t_j = q_1/2 + \mathcal{O}_{\xi_1}(1) \quad \text{for } j = 1, 2.$$

Finally, for $j = 1, 2$, set $\mathbf{x}_j = (x_{j,0}, x_{j,1}, x_{j,2})$ for an integer $x_{j,2}$ such that

$$|x_{j,0}\xi_2 - x_{j,2}| \le 1 \le \exp(q_1/2 - q_2),$$

and set $\mathbf{x}_3 = (0, 0, 1)$. Then $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly independent points of $\mathbb{Z}^3$ with

$$L_{\mathbf{x}_j}(\mathbf{q}) \le \max\{t_j, q_1/2\} \le q_1/2 + \mathcal{O}_{\xi_1}(1) \quad \text{for } j = 1, 2.$$

As $L_{\mathbf{x}_3}(\mathbf{q}) = q_2$, Lemma 2.2 yields $\|\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) - \Phi(q_1/2, q_1/2, q_2)\| = \mathcal{O}_{\boldsymbol{\xi}}(1)$. This proves (ii). The proof of (iii) is similar. □

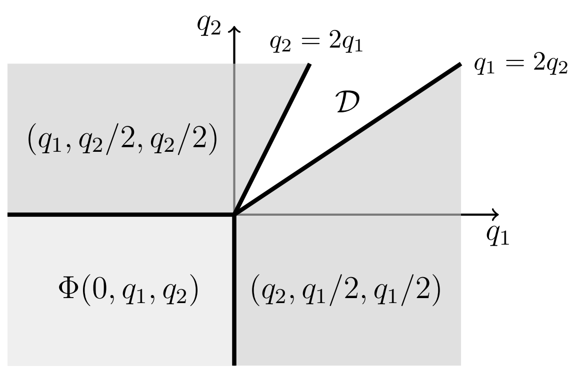

FIGURE 2. The map $\mathbf{L}_{\boldsymbol{\xi}}$ up to bounded difference outside of $\mathcal{D}$, when ξ_1 and ξ_2 are badly approximable

Littlewood's conjecture states that, for any choice of $\xi_1, \xi_2 \in \mathbb{R}$ and any $\epsilon > 0$, there exists $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3$ with $x_0 \neq 0$ such that

$$|x_0|\,|x_0\xi_1 - x_1|\,|x_0\xi_2 - x_2| \le \epsilon.$$

We leave the reader check that, in the present setting, an equivalent formulation of the conjecture is that there is no pair of badly approximable numbers $\xi_1, \xi_2 \in \mathbb{R}$ such that the point $\boldsymbol{\xi} = (1, \xi_1, \xi_2) \in \mathbb{R}^3$ satisfies

$$\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) = \frac{q_1 + q_2}{3}(1, 1, 1) + \mathcal{O}_{\boldsymbol{\xi}}(1) \quad \text{for each } \mathbf{q} = (q_1, q_2) \in \mathcal{D}.$$

Although, the construction that we present in the next section is far from producing a counterexample, it could eventually inspire one that does so. In this respect, we stress that, for each integer $n \geq 2$, there are matrices $A \in \mathrm{GL}_n(\mathbb{R})$ such that

$$\mathbf{L}_A(\mathbf{q}) = \frac{q_1 + \cdots + q_n}{n}(1, \dots, 1) + \mathcal{O}_A(1) \quad \text{for each } \mathbf{q} = (q_1, \dots, q_n) \in \mathbb{R}^n,$$

and so the problem disappears for matrices (this follows for example from [3, Section 4.2, Theorem 1]).

3. Notation and main results

3.1. **Monoid of words.** Let E be a non-empty set. We denote by E^* the monoid of words on the alphabet E, namely the set of finite (possibly empty) sequences of elements of E, with product given by concatenation of words. Its neutral element is the empty word ϵ. Given words $u, w \in E^*$, we say that u is a *prefix* of w and write $u \leq w$ if $w = uv$ for some $v \in E^*$. More generally, we say that u is a *factor* of w if $w = vuv'$ for some $v, v' \in E^*$. We denote by $[u, w]$ the set of words $v \in E^*$ with $u \leq v \leq w$. We write $u < w$ if $u \leq w$ and $u \neq w$. Using standard convention, we also denote by $]u, w]$, $[u, w[$ and $]u, w[$ the sets obtained by removing respectively u, w and $\{u, w\}$ from $[u, w]$. The reverse of a non-empty word $a_1 a_2 \cdots a_n$ of E^* is defined as the word $a_n \cdots a_2 a_1$ with letters written in reverse order. The reverse of ϵ is itself. Thus ϵ is an example of palindrome in E^*, that is a word of E^* which coincides with its reverse.

3.2. **Fibonacci sequences.** We say that a sequence $(x_i)_{i \geq 1}$ in a monoid $\mathcal{M}$ is a *Fibonacci sequence* if it satisfies $x_{i+2} = x_{i+1} x_i$ for each $i \geq 1$. Such a sequence is uniquely determined by its first two elements x_1 and x_2. Moreover, if $\varphi \colon \mathcal{M} \to \mathcal{N}$ is a morphism of monoids, then a Fibonacci sequence $(x_i)_{i \geq 1}$ in $\mathcal{M}$ yields a Fibonacci sequence $(\varphi(x_i))_{i \geq 1}$ in $\mathcal{N}$.

The set $\mathbb{N} = \{0, 1, 2, \dots\}$ of non-negative integers is a monoid under addition. We denote by $(F_i)_{i \geq -1}$, the usual Fibonacci sequence starting with $F_{-1} = 0$, $F_0 = 1$ and obeying $F_{i+2} = F_{i+1} + F_i$ for each $i \geq -1$. It is given in closed form by Binet's formula

$$F_i = (\gamma^{i+1} - (-\gamma)^{-i-1})/\sqrt{5} \quad \text{where} \quad \gamma = (1 + \sqrt{5})/2. \tag{3.1}$$

Let $E = \{a, b\}$ be an alphabet of two letters. The Fibonacci sequence $(w_i)_{i \geq 1}$ in E^* starting with $w_1 = a$ and $w_2 = ab$, has $w_3 = aba$, $w_4 = abaab$, etc. Since any word in that sequence is a prefix of the next word, this sequence converges pointwise to an infinite word

$$f_{a,b} := w_\infty := \lim_{i \to \infty} w_i = abaababaabaab \dots \tag{3.2}$$

From now on, we reserve the notation $(w_i)_{i \geq 1}$ for the above Fibonacci sequence in E^* and w_∞ for its limit. We say that a word u in E^* is a prefix of w_∞ and write $u < w_\infty$, if $u \leq w_i$ for some $i \geq 1$. Then we denote by $[u, w_\infty[$ (resp. $]u, w_\infty[$) the set of words $v \in E^*$ with $u \leq v < w_\infty$ (resp. $u < v < w_\infty$). We say that u is a factor of w_∞ if it is a factor of w_i for some $i \geq 1$.

We denote by $\theta \colon E^* \to E^*$ the morphism of monoids determined by the conditions $\theta(a) = ab$ and $\theta(b) = a$. It satisfies

$$\theta(w_i) = w_{i+1} \quad \text{for each } i \geq 1, \tag{3.3}$$

and preserves the partial order on E^*. Thus it sends to itself the set $[\epsilon, w_\infty[$ of prefixes of w_∞.

The map which sends a word w to its length $|w|$ yields a morphism of monoids from E^* to $\mathbb{N}$ which restricts to an order preserving bijection from $[\epsilon, w_\infty[$ to $\mathbb{N}$. It satisfies

$$|w_i| = F_i \quad \text{for each } i \geq 1. \tag{3.4}$$

In general, given any words u, v on any alphabet, we denote by $f_{u,v} = uvuuv\dots$ the limit of the Fibonacci sequence starting with $(u, uv, uvu, \dots)$, and we call it the *infinite Fibonacci word on* (u, v).

3.3. The sets $\mathcal{V}_\ell$.

Let

$$\mathcal{F} = \{\epsilon, w_1, w_2, w_3, \dots\} = \{\epsilon, a, ab, aba, abaab, \dots\} \tag{3.5}$$

denote the set made of ϵ and all Fibonacci words w_i with $i \geq 1$. We define a map ι from $[\epsilon, w_\infty[$ to itself in the following way. If a prefix v of w_∞ belongs to $\mathcal{F}$, we set $\iota(v) = v$. Otherwise, we have $v \in]w_i, w_{i+1}[$ for some integer $i \geq 3$ and we define $\iota(v)$ to be the prefix of w_∞ of length $|v| - 2F_{i-2}$. This makes sense since $|v| - 2F_{i-2} > F_i - 2F_{i-2} = F_{i-3} \geq 1$. In all cases, we have $\iota(v) \leq v$ and so, the sequence $(\iota^k(v))_{k\geq 1}$ of iterates of ι at v is eventually constant, equal to some element of $\mathcal{F}$. We thus obtain a map $\alpha : [\epsilon, w_\infty[\to \mathcal{F}$ by sending v to this word:

$$\alpha(v) = \lim_{k\to\infty} \iota^k(v). \tag{3.6}$$

The following table provides the values of the maps ι and α on the first 8 prefixes of w_∞.

(3.7)

v	ϵ	$w_1 = a$	$w_2 = ab$	$w_3 = aba$	$abaa$	$w_4 = abaab$	$abaaba$	$abaabab$
$\iota(v)$	ϵ	a	ab	aba	ab	$abaab$	ab	aba
$\alpha(v)$	ϵ	a	ab	aba	ab	$abaab$	ab	aba

For each integer $\ell \geq 1$, we define

$$\mathcal{V}_\ell = \{v \in [\epsilon, w_\infty[\ ;\ \alpha(v) \geq w_\ell\}. \tag{3.8}$$

This is an infinite set as it contains w_i for each $i \geq \ell$. We say that elements $v_1 < v_2 < \cdots < v_k$ of $\mathcal{V}_\ell$ are *consecutive* in $\mathcal{V}_\ell$ if, for each index i with $1 \leq i < k$, there is no element v of $\mathcal{V}_\ell$ with $v_i < v < v_{i+1}$. Sections 4 and 5 are devoted to the combinatorial properties of the sets $\mathcal{V}_\ell$. In particular, Corollary 4.8 shows that, for any consecutive elements $u < v$ of $\mathcal{V}_\ell$ with $\ell \geq 3$, the difference $|v| - |u|$ is $F_{\ell-2}$ or $F_{\ell-1}$. Proposition 5.7 shows that $\mathcal{V}_{\ell+1} = \theta(\mathcal{V}_\ell)$ for each $\ell \geq 4$.

3.4. The functions P_v.

For each $v \in [\epsilon, w_\infty[$, we define a function $P_v \colon \mathbb{R}^2 \to \mathbb{R}$ by

$$P_v(q_1, q_2) = \max\{q_1 - |v|, q_2 - |\alpha(v)|, |v|\}, \tag{3.9}$$

and we denote by $\mathcal{A}(v)$, $\mathcal{B}(v)$ and $\mathcal{C}(v)$ the closed sets made of the points $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2$ where $P_v(\mathbf{q})$ is respectively equal to $q_1 - |v|$, $q_2 - |\alpha(v)|$ and $|v|$. In their interior, P_v is differentiable with constant gradient $(1, 0)$, $(0, 1)$ and $(0, 0)$ respectively. Explicitly, these sets are

- $\mathcal{A}(v) = \{(q_1, q_2) \in \mathbb{R}^2\,;\, q_1 \geq 2|v| \text{ and } q_2 \leq q_1 - |v| + |\alpha(v)|\}$,
- $\mathcal{B}(v) = \{(q_1, q_2) \in \mathbb{R}^2\,;\, q_2 \geq |v| + |\alpha(v)| \text{ and } q_2 \geq q_1 - |v| + |\alpha(v)|\}$,
- $\mathcal{C}(v) = \{(q_1, q_2) \in \mathbb{R}^2\,;\, q_1 \leq 2|v| \text{ and } q_2 \leq |v| + |\alpha(v)|\}$.

They are closed sectors of $\mathbb{R}^2$ with disjoint interiors and their union is $\mathbb{R}^2$. Figure 3 shows the three sets together with the value of P_v on each of them. It also shows by an arrow, the gradient of P_v in their interior (reduced to a point for $\mathcal{C}(v)$). Note in particular that

$$\mathcal{A}(\epsilon) = \{(q_1, q_2) \in \mathbb{R}^2\,;\, q_1 \geq 0 \text{ and } q_2 \leq q_1\}. \tag{3.10}$$

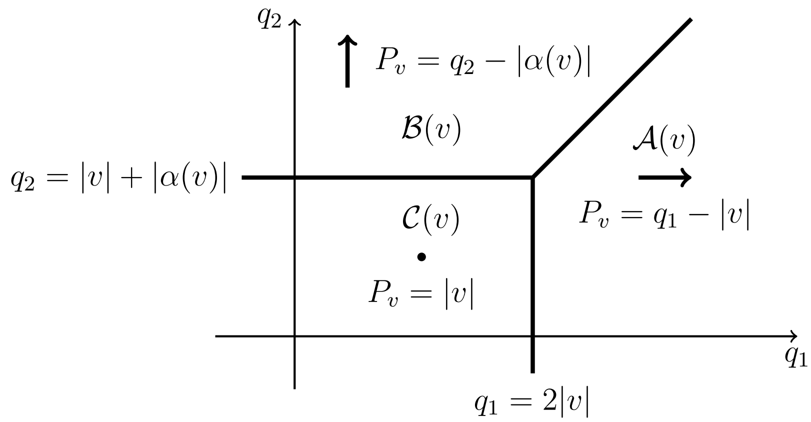

FIGURE 3. The function P_v attached to a prefix v of w_∞.

3.5. **The map P.** Using the function $\Phi\colon \mathbb{R}^3 \to \Delta_3$ from section 2, we construct a map $\mathbf{P} = (P_1, P_2, P_3)\colon \mathcal{A}(\epsilon) \to \Delta_3$ as follows. For each point $\mathbf{q} = (q_1, q_2)$ in $\mathcal{A}(\epsilon)$, we set

$$\mathbf{P}(\mathbf{q}) = \Phi(q_1 - |u|, q_2 + |u| - |w|, |w|) \tag{3.11}$$

where u is the largest prefix of w_∞ such that $\mathbf{q} \in \mathcal{A}(u)$, and w is the smallest prefix of w_∞ with $u < w$ such that $\mathbf{q} \in \mathcal{C}(w)$. Such prefixes exist because the condition $\mathbf{q} \in \mathcal{A}(u)$ requires $2|u| \leq q_1$, and if we choose $i \geq 1$ such that $q_2 \leq q_1 \leq 2F_i$ then $\mathbf{q} \in \mathcal{C}(w_i) =]\infty, 2F_i]^2$. By choice of u and w, we have $\mathbf{q} \in \mathcal{B}(v)$ for any word v with $u < v < w$. In Section 6, we will show that $|w| - |u| = F_\ell$ for some integer $\ell \geq 1$ and that, if $\ell > 1$, there exists a unique word v such that $u < v < w$ are consecutive elements of $\mathcal{V}_\ell$ with $|\alpha(v)| = F_\ell = |w| - |u|$ (see Proposition 6.7). In the latter case, we have

$$\mathbf{q} \in \mathcal{A}(u) \cap \mathcal{B}(v) \cap \mathcal{C}(w) \quad \text{and} \quad \mathbf{P}(\mathbf{q}) = \Phi(P_u(\mathbf{q}), P_v(\mathbf{q}), P_w(\mathbf{q})). \tag{3.12}$$

In Section 7, we view $\mathbf{P}$ as an example of what we call a *integral* 2*-parameter* 3*-system*. In particular, it is a continuous map whose sum of the components is

$$P_1(\mathbf{q}) + P_2(\mathbf{q}) + P_3(\mathbf{q}) = q_1 + q_2 \quad \text{for each } \mathbf{q} = (q_1, q_2) \in \mathcal{A}(\epsilon). \tag{3.13}$$

The following additional properties are established in sections 8 and 9 respectively.

Theorem 3.1. *For each integer $k \geq 4$, each $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2$ with*

$$0 \leq q_1 \leq 2F_{k-1} + 2F_{k-3} \quad \textit{and} \quad 0 \leq q_2 \leq \min\{q_1, 2F_{k-1}\}, \tag{3.14}$$

and each $j = 1, 2, 3$, we have $P_j(q_1 + 4F_{k-2}, q_2 + 2F_{k-2}) = P_j(q_1, q_2) + 2F_{k-2}$.

Theorem 3.2. *For each* $\mathbf{q} \in \mathcal{A}(\epsilon)$*, we have* $\|\mathbf{P}(\gamma\mathbf{q}) - \gamma\mathbf{P}(\mathbf{q})\| \le 40$*.*

3.6. **Main result.** For any matrix A with real coefficients, we denote by $\|A\|$ the maximum of the absolute values of its entries. This agrees with our convention for vectors in $\mathbb{R}^n$, for any positive integer n.

We say that a real number ξ is *extremal of* $\mathrm{GL}_2(\mathbb{Z})$*-type* if there exist an unbounded Fibonacci sequence of matrices $(W_i)_{i\ge1}$ in $\mathrm{GL}_2(\mathbb{Z})$, a matrix $M \in \mathrm{GL}_2(\mathbb{Q})$ with ${}^tM \ne \pm M$, and a constant $c \ge 1$ which, for each $i \ge 1$, satisfy the following properties:

(E1) the product $\mathbf{x}_i := W_i M_i^{-1}$ is a symmetric matrix, where $M_i = \begin{cases} M & \text{if } i \text{ is even,} \\ {}^tM & \text{if } i \text{ is odd;} \end{cases}$

(E2) $c^{-1}\|W_{i+1}\|\,\|W_i\| \le \|W_{i+2}\| \le c\|W_{i+1}\|\,\|W_i\|$;

(E3) $c^{-1}\|W_i\|^{-1} \le \|(\xi, -1)W_i\| \le c\|W_i\|^{-1}$.

According to [6, Theorem 2.2], these numbers ξ are simply those whose continued fraction expansion coincides, up to its first terms, with an infinite Fibonacci word $f_{u,v}$ on two non-commuting word u and v in $(\mathbb{N} \setminus \{0\})^*$. Thus, they are badly approximable and their set is stable under the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{R} \setminus \mathbb{Q}$ by fractional linear transformations. Note that we may always choose M with relatively prime integer coefficients. Then the products $\det(M)\mathbf{x}_i$ also have relatively prime integer coefficients.

For each integer $m \ge 1$, we denote by $\mathcal{E}_m$ the set of real numbers ξ such that the properties (E1)–(E3) hold for an unbounded Fibonacci sequence $(W_i)_{i\ge1}$ in $\mathrm{GL}_2(\mathbb{Z})$ and the choice of

$$M = \begin{pmatrix} m & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}). \tag{3.15}$$

Then the sequence $(\mathbf{x}_i)_{i\ge1}$ defined in (E1) is contained in $\mathrm{GL}_2(\mathbb{Z})$. In [5, Section 3], it is shown that $\mathcal{E}_m$ is non-empty for each m. We also denote by $\mathcal{E}_m^+$ the set of elements of $\mathcal{E}_m$ associated with a Fibonacci sequence $(W_i)_{i\ge1}$ in $\mathrm{SL}_2(\mathbb{Z})$. The set $\mathcal{E}_3^+$ is studied in [7] in connection with Markoff's theory in Diophantine approximation. In section 10, we show that it contains for example the number

$$\xi = [0, 1, 1, 2, 2, 1, 1, 2, 2, 2, 2, 1, 1, \dots] = [0, \mathbf{1}, f_{\mathbf{2},\mathbf{1}}] \tag{3.16}$$

where $\mathbf{1} = (1, 1)$, $\mathbf{2} = (2, 2)$, and $f_{\mathbf{2},\mathbf{1}}$ is the infinite Fibonacci word on $\mathbf{2}$ and $\mathbf{1}$ (as defined in section 3.2). However, by [7, Lemma 3.4], the set $\mathcal{E}_m^+$ is empty for $m \ne 3$. The main result of the present paper is the following.

Theorem 3.3. *Let* $\xi \in \mathcal{E}_m$ *for some integer* $m \ge 1$*, and let* $\boldsymbol{\xi} = (1, \xi, \xi^2)$*. Then there exist* $\rho > 0$ *and* $c > 0$ *such that*

$$\|\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) - \rho\mathbf{P}(\rho^{-1}\mathbf{q})\| \le c$$

for each $\mathbf{q} \in \mathcal{A}(\epsilon)$*.*

3.7. **A family of points.** Fix a choice of $\xi \in \mathcal{E}_m$ for an integer $m \ge 1$ and set $\boldsymbol{\xi} = (1, \xi, \xi^2)$. Choose also a corresponding unbounded Fibonacci sequence $(W_i)_{i\ge1}$ in $\mathrm{GL}_2(\mathbb{Z})$ satisfying the conditions (E1)–(E3) of section 3.6 for the matrix M given by (3.15). Let $(\mathbf{x}_i)_{i\ge1}$ be the sequence of symmetric matrices in $\mathrm{GL}_2(\mathbb{Z})$ defined in (E1). Following [4, 5], we identify each point $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Q}^3$ with the symmetric matrix

$$\mathbf{x} = \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix}.$$

We fix an alphabet $E = \{a, b\}$ with two letters and set

$$W_0 = W_1^{-1} W_2.$$

We denote by $\varphi\colon E^* \to \mathrm{GL}_2(\mathbb{Z})$ the morphism of monoids determined by the conditions $\varphi(b) = W_0$ and $\varphi(a) = W_1$. Then, in the notation of section 3.2, we observe that

$$\varphi(w_i) = W_i \quad \text{for each } i \geq 1.$$

For each non-empty word $v \in E^*$, we set

$$M(v) = \begin{cases} M & \text{if } v \text{ ends in } b, \\ {}^tM & \text{if } v \text{ ends in } a. \end{cases} \tag{3.17}$$

This yields $M(w_i) = M_i$ for each $i \geq 1$. We also define

$$\mathbf{x}(v) = \begin{pmatrix} x_0(v) & x_1(v) \\ x_1(v) & x_2(v) \end{pmatrix} \tag{3.18}$$

as the symmetric matrix with integer coefficients that has the same first column as the matrix $\varphi(v)M(v)^{-1}$ and satisfies $|x_1(v)\xi - x_2(v)| < 1/2$. The first condition determines $x_0(v)$ and $x_1(v)$, and the second then specifies uniquely $x_2(v)$ as $\xi \notin \mathbb{Q}$. We also view $\mathbf{x}(v)$ as the point $(x_0(v), x_1(v), x_2(v)) \in \mathbb{Z}^3$ as stated at the beginning of the section. Then, according to (2.3), its trajectory relative to $\boldsymbol{\xi}$ is the function $L_{\mathbf{x}(v)}\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$L_{\mathbf{x}(v)}(\mathbf{q}) = \max\{\log|x_0(v)|,\ q_1 + \log|x_0(v)\xi - x_1(v)|,\ q_2 + \log|x_0(v)\xi^2 - x_2(v)|\} \tag{3.19}$$

for each $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2$.

The tools of section 13 allows us to compute $\mathbf{x}(v)$ for each $v \in \mathcal{V}_\ell$ with ℓ large enough. In particular, we find that $\mathbf{x}(w_i) = \mathbf{x}_i$ for each large enough i. The following crucial result is proved in section 14, using the functions P_v of section 3.4.

Theorem 3.4. *With the above notation, there exist an integer $\ell_0 \geq 4$ and a constant $\rho > 0$ such that, for each integer $\ell \geq \ell_0$, the following properties hold.*

(i) *For any $v \in \mathcal{V}_\ell$ and any $\mathbf{q} \in \mathcal{A}(\epsilon)$, we have*

$$L_{\mathbf{x}(v)}(\mathbf{q}) = \rho P_v(\rho^{-1}\mathbf{q}) + \mathcal{O}_\xi(1),$$

where $\mathcal{O}_\xi(1)$ denotes a function of v and $\mathbf{q}$ whose absolute value is bounded above by a constant that depends only on ξ.

(ii) *For any triple of consecutive elements $u < v < w$ of $\mathcal{V}_\ell$, the points $\mathbf{x}(u)$, $\mathbf{x}(v)$, $\mathbf{x}(w)$ are linearly independent if and only if $v \notin \mathcal{V}_{\ell+1}$.*

3.8. **Proof of the main result.** Using the results of section 2 and taking for granted the statements of the preceding sections, Theorem 3.3 is proved as follows.

Choose $\rho > 0$ and ℓ_0 as in Theorem 3.4, and fix a point $\mathbf{q} = (q_1, q_2) \in \mathcal{A}(\epsilon)$. Then $\rho^{-1}\mathbf{q}$ also belongs to $\mathcal{A}(\epsilon)$, and so, according to section 3.5, we have

$$\rho\mathbf{P}(\rho^{-1}\mathbf{q}) = \Phi(\mathbf{p}) \quad \text{with} \quad \mathbf{p} = (q_1 - \rho|u|, q_2 + \rho|u| - \rho|w|, \rho|w|) \tag{3.20}$$

where u is the largest prefix of w_∞ such that $\rho^{-1}\mathbf{q} \in \mathcal{A}(u)$, and w is the smallest prefix of w_∞ with $u < w$ such that $\rho^{-1}\mathbf{q} \in \mathcal{C}(w)$. Moreover, $|w| - |u| = F_\ell$ for some positive integer ℓ. Furthermore, if $\ell > 1$, then there exists a unique word $v \in \,]u, w[$ such that $u < v < w$ are consecutive elements of $\mathcal{V}_\ell$ with

$$|\alpha(v)| = F_\ell = |w| - |u| \quad \text{and} \quad \rho^{-1}\mathbf{q} \in \mathcal{A}(u) \cap \mathcal{B}(v) \cap \mathcal{C}(w),$$

thus, by definition of P_u, P_v and P_w, we have

$$\mathbf{p} = \big(\rho P_u(\rho^{-1}\mathbf{q}), \rho P_v(\rho^{-1}\mathbf{q}), \rho P_w(\rho^{-1}\mathbf{q})\big).$$

Suppose first that $\ell \geq \ell_0$. Then, since $\ell_0 \geq 4$, the above formula for $\mathbf{p}$ holds and, by Theorem 3.4(i), we have

$$\mathbf{p} = \big(L_{\mathbf{x}(u)}(\mathbf{q}), L_{\mathbf{x}(v)}(\mathbf{q}), L_{\mathbf{x}(w)}(\mathbf{q})\big) + \mathcal{O}_\xi(1).$$

Since Φ is 1-Lipschitz, this yields

$$\rho\mathbf{P}(\rho^{-1}\mathbf{q}) = \Phi(\mathbf{p}) = \Phi\big(L_{\mathbf{x}(u)}(\mathbf{q}), L_{\mathbf{x}(v)}(\mathbf{q}), L_{\mathbf{x}(w)}(\mathbf{q})\big) + \mathcal{O}_\xi(1).$$

Moreover, as $v \notin \mathcal{V}_{\ell+1}$, the integer points $\mathbf{x}(u), \mathbf{x}(v), \mathbf{x}(w)$ are linearly independent by Theorem 3.4(ii), and so we have

$$\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) \leq \Phi\big(L_{\mathbf{x}(u)}(\mathbf{q}), L_{\mathbf{x}(v)}(\mathbf{q}), L_{\mathbf{x}(w)}(\mathbf{q})\big)$$

componentwise. Altogether, this means that

$$L_{\boldsymbol{\xi},j}(\mathbf{q}) \leq \rho P_j(\rho^{-1}\mathbf{q}) + \mathcal{O}_\xi(1) \quad (1 \leq j \leq 3).$$

We also observe from (3.20) that the sum of the coordinates of $\mathbf{p}$ is $q_1 + q_2$. Since the sum of the coordinates of $\Phi(\mathbf{p}) = \rho\mathbf{P}(\rho^{-1}\mathbf{q})$ is the same, we conclude from Lemma 2.2 that

$$\|\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) - \rho\mathbf{P}(\rho^{-1}\mathbf{q})\| = \mathcal{O}_\xi(1). \tag{3.21}$$

Suppose now that $\ell < \ell_0$. As $\rho^{-1}\mathbf{q} \in \mathcal{A}(u) \cap \mathcal{C}(w)$, we have

$$2|u| \leq \rho^{-1}q_1 \leq 2|w| \quad \text{and} \quad \rho^{-1}q_2 \leq \min\{\rho^{-1}q_1 - |u| + |\alpha(u)|, \, |w| + |\alpha(w)|\}.$$

Since $|w| - |u| = F_\ell$, we also have $\{u, w\} \not\subseteq \mathcal{V}_{\ell+3}$ according to the remark at the end of section 3.3 (or see Corollary 4.8). Thus,

$$|w| - |u| = F_\ell = \mathcal{O}_\xi(1) \quad \text{and} \quad \min\{|\alpha(u)|, |\alpha(w)|\} \leq F_{\ell+2} = \mathcal{O}_\xi(1).$$

So, the preceding estimates yield

$$q_2 \leq q_1/2 + \mathcal{O}_\xi(1) \quad \text{and} \quad \mathbf{p} = (q_1/2, q_2, q_1/2) + \mathcal{O}_\xi(1),$$

using the formula for $\mathbf{p}$ given in (3.20). Thus,

$$\rho\mathbf{P}(\rho^{-1}\mathbf{q}) = \Phi(q_2, q_1/2, q_1/2) + \mathcal{O}_\xi(1).$$

To estimate $\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q})$, we use the fact that ξ is badly approximable, as stated in section 3.6. Set $\mathbf{q}' = (q_1, q_2')$ where $q_2' = \min\{q_2, q_1/2\}$. By the above, we have $\|\mathbf{q} - \mathbf{q}'\| = |q_2 - q_2'| = \mathcal{O}_\xi(1)$. Then, Lemmas 2.1 and 2.3(ii) yield

$$\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) = \mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}') + \mathcal{O}_\xi(1) = \Phi(q_2', q_1/2, q_1/2) + \mathcal{O}_\xi(1) = \Phi(q_2, q_1/2, q_1/2) + \mathcal{O}_\xi(1).$$

Thus, (3.21) also holds in this case. This completes the proof of Theorem 3.3.

4. The maps $\bar{\iota}$ and $\bar{\alpha}$, and the sets $\overline{\mathcal{V}}_\ell$

In Section 3.3, we defined two maps ι and α from $[\epsilon, w_\infty[$ to itself. Using the natural bijection from $[\epsilon, w_\infty[$ to $\mathbb{N}$ given by the length of a word, these yield maps $\bar{\iota}$ and $\bar{\alpha}$ from $\mathbb{N}$ to $\mathbb{N}$. Similarly, the subsets $\mathcal{V}_\ell$ of $[\epsilon, w_\infty[$ correspond to subsets $\overline{\mathcal{V}}_\ell$ of $\mathbb{N}$. This section is devoted to the combinatorial properties of these simpler objects.

Here, the set $\mathcal{F}$, given by (3.5), is replaced by the set

$$\overline{\mathcal{F}} = \{F_i \,;\, i \ge -1\} = \{0, 1, 2, 3, 5, 8, 13, \dots\} \tag{4.1}$$

of all Fibonacci numbers, including 0. We define $\bar{\iota}\colon \mathbb{N} \to \mathbb{N}$ by

$$\bar{\iota}(x) = \begin{cases} x & \text{if } x \in \overline{\mathcal{F}}, \\ x - 2F_{i-2} & \text{if } F_i < x < F_{i+1} \text{ for an integer } i \ge 3. \end{cases}$$

We note that, for each $x \in \mathbb{N}$, we have $\bar{\iota}(x) \le x$ with equality if and only if $x \in \overline{\mathcal{F}}$. Thus, the sequence $(\bar{\iota}^k(x))_{k\ge1}$ of iterates of $\bar{\iota}$ evaluated at x is non-increasing and so, from some point on, it is constant, equal to some element of $\overline{\mathcal{F}}$. We thus obtain a map $\bar{\alpha}\colon \mathbb{N} \to \overline{\mathcal{F}}$ by sending x to this value:

$$\bar{\alpha}(x) = \lim_{k\to\infty} \bar{\iota}^k(x). \tag{4.2}$$

Clearly, we have $\bar{\alpha} \circ \bar{\iota} = \bar{\alpha}$. We also note that, for each $x \in \mathbb{N}$, we have $\bar{\alpha}(x) \le x$ with equality if and only if $x \in \overline{\mathcal{F}}$. The next table gives the values of the maps $\bar{\iota}$ and $\bar{\alpha}$ on $\mathbb{N} \cap [0, 13]$.

(4.3)

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\bar{\iota}(x)$	0	1	2	3	2	5	2	3	8	3	4	5	6	13
$\bar{\alpha}(x)$	0	1	2	3	2	5	2	3	8	3	2	5	2	13

For each integer $\ell \ge 1$, we also define

$$\overline{\mathcal{V}}_\ell = \{x \in \mathbb{N} \,;\, \bar{\alpha}(x) \ge F_\ell\}.$$

We say that elements $x_1 < x_2 < \cdots < x_k$ of $\overline{\mathcal{V}}_\ell$ are *consecutive* in $\overline{\mathcal{V}}_\ell$ if $x_1, x_2, \dots, x_k$ are all elements of $\overline{\mathcal{V}}_\ell \cap [x_1, x_k]$. By construction, we have

$$\bar{\iota}(|v|) = |\iota(v)|, \quad \bar{\alpha}(|v|) = |\alpha(v)| \quad \text{and} \quad \overline{\mathcal{V}}_\ell = \{|v| \,;\, v \in \mathcal{V}_\ell\} \tag{4.4}$$

for each $v \in [\epsilon, w_\infty[$ and each positive integer ℓ.

As the table (4.3) suggests, the map $\bar{\alpha}$ has the following property.

Lemma 4.1. *Let $x \in \mathbb{N}$ with $x \ge 2$. Then, $\bar{\alpha}(x) \ge 2$.*

Proof. It suffices to show that $\bar\imath(x) \ge 2$. This is clear if $x \in \overline{\mathcal{F}}$ because then $\bar\imath(x) = x \ge 2$. Otherwise, we have $F_i < x < F_{i+1}$ for some $i \ge 3$, and then $\bar\imath(x) = x - 2F_{i-2} > F_i - 2F_{i-2} = F_{i-3} \ge 1$, thus $\bar\imath(x) \ge 2$. □

Lemma 4.2. *We have* $\overline{\mathcal{V}}_1 = \mathbb{N} \setminus \{0\}$ *and* $\overline{\mathcal{V}}_2 = \mathbb{N} \setminus \{0, 1\}$.

Proof. By Lemma 4.1, we have $\mathbb{N} \setminus \{0, 1\} \subseteq \overline{\mathcal{V}}_2 \subseteq \overline{\mathcal{V}}_1 \subseteq \mathbb{N}$. The result follows since $0 \notin \overline{\mathcal{V}}_1$ while $1 \in \overline{\mathcal{V}}_1 \setminus \overline{\mathcal{V}}_2$. □

Lemma 4.3. *Let* $\ell \ge 2$ *be an integer. Then, the smallest element of* $\overline{\mathcal{V}}_\ell$ *is* F_ℓ*. Moreover, for each integer* $i \ge \ell$*, the map* $\bar\imath$ *restricts to an order preserving bijection from* $\overline{\mathcal{V}}_\ell \cap \,]F_i, F_{i+1}[$ *to* $\overline{\mathcal{V}}_\ell \cap \,]F_{i-3}, F_{i-1} + F_{i-3}[$.

Proof. For each $x \in \overline{\mathcal{V}}_\ell$, we have $x \ge \overline{\alpha}(x) \ge F_\ell$. Thus, F_ℓ is the smallest element of $\overline{\mathcal{V}}_\ell$. Let $i \ge \ell$ be an integer. By construction, the map $\bar\imath$ restricts to an order preserving bijection

$$\begin{array}{rcl} \mathbb{N} \cap \,]F_i, F_{i+1}[& \longrightarrow & \mathbb{N} \cap \,]F_{i-3}, F_{i-1} + F_{i-3}[\\ x & \longmapsto & x - 2F_{i-2} \end{array}$$

(with empty domain and codomain if $i = \ell = 2$). This yields the last assertion because $\bar\imath^{-1}(\overline{\mathcal{V}}_\ell) = \overline{\mathcal{V}}_\ell$. □

Lemma 4.4. *Let* $\ell \ge 2$ *be an integer. Then,* $\overline{\mathcal{V}}_\ell \cap [0, F_{\ell+3}]$ *consists of the 7 numbers*

$$F_\ell < F_{\ell+1} < F_{\ell+1} + F_{\ell-1} < F_{\ell+2} < F_{\ell+2} + F_{\ell-2} < F_{\ell+2} + F_\ell < F_{\ell+3}.$$

Proof. According to Lemma 4.3, we have

$$\overline{\mathcal{V}}_\ell \cap [0, F_\ell] = \{F_\ell\},$$

and $\bar\imath$ maps $\overline{\mathcal{V}}_\ell \cap \,]F_\ell, F_{\ell+1}[$ bijectively to $\overline{\mathcal{V}}_\ell \cap \,]F_{\ell-3}, F_{\ell-1} + F_{\ell-3}[\, = \emptyset$, thus

$$\overline{\mathcal{V}}_\ell \cap \,]F_\ell, F_{\ell+1}] = \{F_{\ell+1}\}.$$

Similarly, $\bar\imath$ maps $\overline{\mathcal{V}}_\ell \cap \,]F_{\ell+1}, F_{\ell+2}[$ bijectively to $\overline{\mathcal{V}}_\ell \cap \,]F_{\ell-2}, F_\ell + F_{\ell-2}[\, = \{F_\ell\}$, thus

$$\overline{\mathcal{V}}_\ell \cap \,]F_{\ell+1}, F_{\ell+2}] = \{2F_{\ell-1} + F_\ell, F_{\ell+2}\} = \{F_{\ell+1} + F_{\ell-1}, F_{\ell+2}\}.$$

Finally, $\bar\imath$ maps $\overline{\mathcal{V}}_\ell \cap \,]F_{\ell+2}, F_{\ell+3}[$ bijectively to $\overline{\mathcal{V}}_\ell \cap \,]F_{\ell-1}, F_{\ell+1} + F_{\ell-1}[\, = \{F_\ell, F_{\ell+1}\}$, thus

$$\overline{\mathcal{V}}_\ell \cap \,]F_{\ell+2}, F_{\ell+3}] = \{3F_\ell, 2F_\ell + F_{\ell+1}, F_{\ell+3}\} = \{F_{\ell+2} + F_{\ell-2}, F_{\ell+2} + F_\ell, F_{\ell+3}\}.$$ □

The next lemma presents a central property of the map $\overline{\alpha}$. It is the key to the proof of Theorem 3.1 in section 8. Below, we use it to derive several properties of the sets $\overline{\mathcal{V}}_\ell$.

Lemma 4.5. *Let* $k \ge 3$ *be an integer. We have*

$$\overline{\alpha}(x) \le \overline{\alpha}(x + 2F_{k-2}) \quad \textit{for each } x \in \mathbb{N} \cap [0, F_{k-1} + F_{k-3}],$$

with equality if $x \notin \overline{\mathcal{F}}$ *and* $x \ne F_{k-1} + F_{k-3}$.

Proof. We proceed by induction on k. If $k = 3$, the range for x is $\{0, 1, 2, 3\}$, contained in $\overline{\mathcal{F}}$, and, using the table (4.3), we find, as claimed, that $\overline{\alpha}(x) \le \overline{\alpha}(x + 2)$ for each x in that range. Now, suppose that $k \ge 4$. Let $x \in \mathbb{N} \cap [0, F_{k-1} + F_{k-3}]$, and set $y = x + 2F_{k-2}$. We note that

(i) if $x = F_{k-3}$, then $y = F_k$, so $\overline{\alpha}(y) = F_k > F_{k-3} = \overline{\alpha}(x)$;

(ii) if $x = F_{k-1} + F_{k-3}$, then $y = F_{k+1}$, so $\overline{\alpha}(y) = F_{k+1} > F_{k-2} = \overline{\alpha}(x)$;

(iii) if $x \in \,]F_{k-3}, F_{k-1} + F_{k-3}[$, then $y \in \,]F_k, F_{k+1}[$, so $\overline{\iota}(y) = x$, thus $\overline{\alpha}(y) = \overline{\alpha}(x)$.

So, we may assume that $x \in [0, F_{k-3}[$. Then, $y \in \,]F_{k-1}, F_k[$ and thus

$$\overline{\iota}(y) = y - 2F_{k-3} = x + 2F_{k-4} \quad \Longrightarrow \quad \overline{\alpha}(y) = \overline{\alpha}(x + 2F_{k-4}).$$

If $k = 4$, we are done because $x = 0 \in \mathcal{F}$ and $\overline{\alpha}(y) = 2 > \overline{\alpha}(x) = 0$. Hence, we may also assume that $k \geq 5$. Finally, as $x \in [0, F_{k-3} + F_{k-5}[$, we may further assume, by induction, that $\overline{\alpha}(x) \leq \overline{\alpha}(x + 2F_{k-4}) = \overline{\alpha}(y)$ with equality if $x \notin \mathcal{F}$, and we are done. □

Proposition 4.6. *Let $k, \ell \in \mathbb{N}$ with $k - 2 \geq \ell \geq 1$. Then, we have a bijection*

$$\begin{array}{rcl} \overline{\mathcal{V}}_\ell \cap [F_\ell, F_{k-1} + F_{k-3}] & \longrightarrow & \overline{\mathcal{V}}_\ell \cap [F_\ell + 2F_{k-2}, F_{k+1}] \\ x & \longmapsto & x + 2F_{k-2}. \end{array}$$

Proof. Set $A = [F_\ell, F_{k-1} + F_{k-3}]$ and $B = 2F_{k-2} + A = [F_\ell + 2F_{k-2}, F_{k+1}]$. By Lemma 4.5, translation by $2F_{k-2}$ maps $\overline{\mathcal{V}}_\ell \cap A$ injectively into $\overline{\mathcal{V}}_\ell \cap B$. To prove that it is surjective, choose $y \in \overline{\mathcal{V}}_\ell \cap B$ and let $x \in \mathbb{N} \cap A$ such that $y = x + 2F_{k-2}$. If $x \in \overline{\mathcal{F}}$, then $\overline{\alpha}(x) = x \geq F_\ell$. If $x = F_{k-1} + F_{k-3}$ and $x \notin \overline{\mathcal{F}}$, then $k \geq 4$ and $\overline{\alpha}(x) = F_{k-2} \geq F_\ell$. For any other value of x, Lemma 4.5 yields $\overline{\alpha}(x) = \overline{\alpha}(y) \geq F_\ell$. Thus, $x \in \overline{\mathcal{V}}_\ell \cap A$ in all cases. □

Proposition 4.7. *Let $\ell \geq 3$ be an integer and let $x_1 = F_\ell < x_2 < x_3 < \cdots$ be the elements of $\overline{\mathcal{V}}_\ell$ listed in increasing order. Then, the sequence*

$$(x_{i+1} - x_i)_{i \geq 1} = (F_{\ell-1}, F_{\ell-1}, F_{\ell-2}, F_{\ell-2}, F_{\ell-1}, F_{\ell-1}, \dots) \tag{4.5}$$

is the infinite Fibonacci word constructed on the words $(F_{\ell-1}, F_{\ell-1})$ and $(F_{\ell-2}, F_{\ell-2})$.

This is also true for $\ell = 2$ but not interesting since, by Lemma 4.2, we have $\overline{\mathcal{V}}_2 = \mathbb{N} \setminus \{0, 1\}$ and so (4.5) is the constant sequence $(1, 1, 1, \dots)$ when $\ell = 2$.

Proof of Proposition 4.7. For each integer $j \geq 1$, let $\lambda(j)$ denote the cardinality of the set $\overline{\mathcal{V}}_\ell \cap [F_\ell, F_\ell + 2F_{\ell+j-2}]$, and let

$$m_j = (x_{i+1} - x_i)_{1 \leq i < \lambda(j)}.$$

Using Lemma 4.4, we find that $x_1 = F_\ell$, $\lambda(1) = 3$, $\lambda(2) = 5$,

$$m_1 = (F_{\ell-1}, F_{\ell-1}) \quad \text{and} \quad m_2 = (F_{\ell-1}, F_{\ell-1}, F_{\ell-2}, F_{\ell-2}). \tag{4.6}$$

For $j \geq 1$, Proposition 4.6 applied with $k = \ell + j + 1$ yields a bijection

$$\begin{array}{rcl} \overline{\mathcal{V}}_\ell \cap [F_\ell, F_\ell + 2F_{\ell+j-2}] & \longrightarrow & \overline{\mathcal{V}}_\ell \cap [F_\ell + 2F_{\ell+j-1}, F_\ell + 2F_{\ell+j}] \\ x & \longmapsto & x + 2F_{\ell+j-1}. \end{array} \tag{4.7}$$

Since $F_\ell \in \overline{\mathcal{V}}_\ell$, this implies that $F_\ell + 2F_{\ell+j-1} \in \overline{\mathcal{V}}_\ell$ for each $j \geq 1$. This also extends to $j = 0$ since $F_\ell + 2F_{\ell-1} = F_{\ell+1} + F_{\ell-1} \in \overline{\mathcal{V}}_\ell$. Thus, for each $j \geq 1$, we have $F_\ell + 2F_{\ell+j-2} \in \overline{\mathcal{V}}_\ell$, so

$$x_{\lambda(j)} = F_\ell + 2F_{\ell+j-2},$$

and the bijection (4.7) amounts to

$$(x_i + 2F_{\ell+j-1})_{1 \leq i \leq \lambda(j)} = (x_i)_{\lambda(j+1) \leq i \leq \lambda(j+2)}.$$

In turn, this implies that

$$(x_{i+1} - x_i)_{1 \leq i < \lambda(j)} = (x_{i+1} - x_i)_{\lambda(j+1) \leq i < \lambda(j+2)},$$

which translates into the relation

$$m_{j+2} = m_{j+1}m_j \quad (j \geq 1).$$

The conclusion follows from this recurrence relation together with (4.6). □

Corollary 4.8. *Let $\ell \geq 3$ be an integer. For any pair of consecutive numbers $x < y$ in $\overline{\mathcal{V}}_\ell$, we have $y - x \in \{F_{\ell-2}, F_{\ell-1}\}$. There are infinitely many arithmetic progressions of length 3 with difference $F_{\ell-2}$ made of consecutive numbers in $\overline{\mathcal{V}}_\ell$, but no longer ones. There are infinitely many arithmetic progressions of length 5 with difference $F_{\ell-1}$ made of consecutive numbers in $\overline{\mathcal{V}}_\ell$, but no longer ones.*

Proof. The first assertion is a direct consequence of the proposition. The others follow from the proposition and the fact that the infinite Fibonacci word $f_{a,b} = abaababa\cdots$ on two letters a and b contains infinitely many letters b but no factor of the form bb, and that it contains infinitely factors aa but no factor aaa. Thus, the infinite Fibonacci word on $(F_{\ell-1}, F_{\ell-1})$ and $(F_{\ell-2}, F_{\ell-2})$ contains infinitely many subsequences of two consecutive terms equal to $F_{\ell-2}$ but no longer ones, and infinitely many subsequences of four consecutive terms equal to $F_{\ell-1}$ but no longer ones. □

Corollary 4.9. *Let $x < y$ be consecutive elements of $\overline{\mathcal{V}}_\ell$ for some integer $\ell \geq 2$. Then,*

$$\{x, y\} \not\subseteq \overline{\mathcal{V}}_{\ell+2}, \quad \{x, y\} \cap \overline{\mathcal{V}}_{\ell+1} \neq \emptyset \quad \textit{and} \quad |\overline{\alpha}(y) - \overline{\alpha}(x)| \geq y - x.$$

Proof. If $\ell \geq 3$, Corollary 4.8 yields $y - x \in \{F_{\ell-2}, F_{\ell-1}\}$. This also holds if $\ell = 2$ because then $y - x = 1$ by Lemma 4.2. Since $y - x \leq F_{\ell-1} < F_\ell$, it follows from Corollary 4.8 that $x < y$ are not consecutive elements of $\overline{\mathcal{V}}_{\ell+2}$, thus $\{x, y\} \not\subseteq \overline{\mathcal{V}}_{\ell+2}$.

Suppose that $\{x, y\} \cap \overline{\mathcal{V}}_{\ell+1} = \emptyset$. As the first two elements of $\overline{\mathcal{V}}_\ell$ are F_ℓ and $F_{\ell+1}$, we must have that $x > F_{\ell+1}$ and so there are consecutive $x' < y'$ in $\overline{\mathcal{V}}_{\ell+1}$ such that $x' < x < y < y'$. By the above, the differences $x - x'$, $y - x$ and $y' - y$ are all bounded below by $F_{\ell-2}$. Moreover, at least one of them is bounded below by $F_{\ell-1}$ because, otherwise, we would have $F_{\ell-1} > F_{\ell-2}$, thus $\ell \geq 3$, and $x' < x < y < y'$ would be 4 consecutive elements of $\overline{\mathcal{V}}_\ell$ in arithmetic progression with difference $F_{\ell-2}$, against Corollary 4.8. We conclude that $y' - x' \geq 2F_{\ell-2} + F_{\ell-1} > F_\ell$. This is impossible since the same corollary yields $y' - x' \leq F_\ell$. Thus, $\{x, y\} \cap \overline{\mathcal{V}}_{\ell+1} \neq \emptyset$.

Let m be the largest integer such that $\{x, y\} \subseteq \overline{\mathcal{V}}_m$. By the above, we have $m \in \{\ell, \ell+1\}$, and the preceeding reasoning yields $\{x, y\} \cap \overline{\mathcal{V}}_{m+1} \neq \emptyset$. Thus $\overline{\alpha}(x)$ and $\overline{\alpha}(y)$ are distinct Fibonacci numbers, the smallest of which is equal to F_ℓ or $F_{\ell+1}$. Consequently, we have $|\overline{\alpha}(y) - \overline{\alpha}(x)| \geq F_{\ell-1} \geq y - x$. □

The next result characterizes the elements of $\overline{\mathcal{V}}_{\ell+1}$ within $\overline{\mathcal{V}}_\ell$, for any integer $\ell \geq 3$.

Proposition 4.10. *Let $x < y < z$ be consecutive numbers in $\overline{\mathcal{V}}_\ell$ for some integer $\ell \geq 3$. Then, we have*

$$y \notin \overline{\mathcal{V}}_{\ell+1} \iff y - x \neq z - y. \tag{4.8}$$

When these equivalent conditions hold, we further have $z - x = \overline{\alpha}(y) = F_\ell$.

Note that (4.8) is false for $\ell = 2$. A counterexample is provided by any triple of consecutive integers $x < y < z$ with $x > 1$ and $\overline{\alpha}(y) = 2$. For then, $x < y < z$ are consecutive elements of $\overline{\mathcal{V}}_2$ with $y \notin \overline{\mathcal{V}}_3$ and $y - x = z - y = 1$.

Proof. Let $(x_i)_{i\geq 1}$ denote the sequence of elements of $\overline{\mathcal{V}}_\ell$ listed in increasing order, and let $(y_i)_{i\geq 1}$ denote the subsequence obtained by removing its first term x_1 and each x_i with $i \geq 2$ such that $x_i - x_{i-1} \neq x_{i+1} - x_i$. Since $x_1 = F_\ell \notin \overline{\mathcal{V}}_{\ell+1}$, proving the first part of the proposition amounts to showing that $(y_i)_{i\geq 1}$ lists the elements of $\overline{\mathcal{V}}_{\ell+1}$ in increasing order.

By Proposition 4.7, we have $x_2 - x_1 = x_3 - x_2 = F_{\ell-1}$, thus $y_1 = x_2 = F_{\ell+1}$ is indeed the smallest element of $\overline{\mathcal{V}}_{\ell+1}$. To go further, set

$$X = (x_{i+1} - x_i)_{i\geq 1}, \quad X' = (x_{i+1} - x_i)_{i\geq 2} \quad \text{and} \quad Y = (y_{i+1} - y_i)_{i\geq 1}.$$

By Proposition 4.7, we have $X = \lim_{i\to\infty} u_i$ where $(u_i)_{i\geq 1}$ is the Fibonacci sequence in $\{F_{\ell-2}, F_{\ell-1}\}^*$ starting with

$$u_1 = F_{\ell-1}F_{\ell-1} \quad \text{and} \quad u_2 = F_{\ell-1}F_{\ell-1}F_{\ell-2}F_{\ell-2}.$$

Since each u_i starts with $F_{\ell-1}$, we can write

$$u_i F_{\ell-1} = F_{\ell-1} v_i$$

for some $v_i \in \{F_{\ell-2}, F_{\ell-1}\}^*$. Then, $(v_i)_{i\geq 1}$ is the Fibonacci sequence in $\{F_{\ell-2}, F_{\ell-1}\}^*$ with

$$v_1 = F_{\ell-1}F_{\ell-1} \quad \text{and} \quad v_2 = F_{\ell-1}F_{\ell-2}F_{\ell-2}F_{\ell-1}.$$

We deduce that $X' = \lim_{i\to\infty} v_i = v_2v_1v_2v_2v_1\cdots$ is the infinite Fibonacci word on v_2 and v_1. Hence, the factors $F_{\ell-1}F_{\ell-2}$ and $F_{\ell-2}F_{\ell-1}$ do not overlap in X', and so Y is the infinite word on $\{F_{\ell-1}, F_\ell\}$ obtained from X' by replacing each of these factors by F_ℓ. Thus, Y is the infinite Fibonacci word on $F_\ell F_\ell$ and $F_{\ell-1}F_{\ell-1}$. As $y_1 = F_{\ell+1}$, we conclude from Proposition 4.7 that $(y_i)_{i\geq 1}$ lists the elements of $\overline{\mathcal{V}}_{\ell+1}$ in increasing order.

The second part of the proposition follows from this. Indeed, when both conditions hold in (4.8), we have $\overline{\alpha}(y) = F_\ell$ because $y \in \overline{\mathcal{V}}_\ell \setminus \overline{\mathcal{V}}_{\ell+1}$, and $z - x = F_\ell$ since $(y - x, z - y)$ is a permutation of $(F_{\ell-2}, F_{\ell-1})$. □

We conclude this section with two additional observations.

Proposition 4.11. *Let $\ell \geq 2$ be an integer and let $x < z$ be positive integers. Then, the following conditions are equivalent.*

(i) *$x < z$ are consecutive elements of $\overline{\mathcal{V}}_{\ell+1}$ but not consecutive elements of $\overline{\mathcal{V}}_\ell$;*

(ii) *there exists $y \in \overline{\mathcal{V}}_\ell \setminus \overline{\mathcal{V}}_{\ell+1}$ such that $x < y < z$ are consecutive elements of $\overline{\mathcal{V}}_\ell$;*

(iii) *$x < z$ are consecutive elements of $\overline{\mathcal{V}}_{\ell+1}$ and $z - x = F_\ell$.*

Proof. (i) ⇒ (ii): Suppose that (i) holds, and let y be the successor of x in $\overline{\mathcal{V}}_\ell$. Since $x < y < z$, this number y does not belong to $\overline{\mathcal{V}}_{\ell+1}$. So, by Corollary 4.9, its successor in $\overline{\mathcal{V}}_\ell$ belongs to $\overline{\mathcal{V}}_{\ell+1}$, hence it must be z. So (ii) holds.

(ii) ⇒ (iii): Under the condition (ii), Corollary 4.9 implies that $\{x, z\} \subseteq \overline{\mathcal{V}}_{\ell+1}$. Thus, $x < z$ are consecutive in $\overline{\mathcal{V}}_{\ell+1}$. Moreover, if $\ell \geq 3$, Proposition 4.10 yields $z - x = F_\ell$. This still holds when $\ell = 2$ for then $x < y < z$ are consecutive integers and so $z - x = 2 = F_\ell$.

(iii) $\Rightarrow$ (i): If (iii) holds, then $z - x = F_\ell \notin \{F_{\ell-2}, F_{\ell-1}\}$. So, $x < z$ cannot be consecutive in $\overline{\mathcal{V}}_\ell$ by Corollary 4.8 if $\ell \geq 3$, and by Lemma 4.2 if $\ell = 2$. □

Proposition 4.12. *Let $\ell \geq 2$ be an integer, and let $x < y < z$ be consecutive numbers in $\overline{\mathcal{V}}_\ell$, not all contained in $\overline{\mathcal{V}}_{\ell+1}$. Then, exactly one of the following situations holds:*

(i) $\overline{\alpha}(x) \geq F_{\ell+1}$, $\overline{\alpha}(y) = F_\ell$, $\overline{\alpha}(z) \geq F_{\ell+1}$ *and* $z - x = F_\ell$;

(ii) $\overline{\alpha}(x) = F_\ell$, $\overline{\alpha}(y) \geq F_{\ell+1}$, $\overline{\alpha}(z) = F_\ell$ *and* $y - x = z - y$;

(iii) $\overline{\alpha}(x) = F_\ell$, $\overline{\alpha}(y) = F_{\ell+1}$, $\overline{\alpha}(z) \geq F_{\ell+2}$ *and* $y - x = z - y = F_{\ell-1}$;

(iv) $\overline{\alpha}(x) \geq F_{\ell+2}$, $\overline{\alpha}(y) = F_{\ell+1}$, $\overline{\alpha}(z) = F_\ell$ *and* $y - x = z - y = F_{\ell-1}$.

Proof. We first note that the four conditions (i) to (iv) are mutually exclusive. So, we only have to show that they exhaust all possibilities. We also note that, if $\ell = 2$, then $x < y < z$ are consecutive integers, thus $y - x = z - y = 1 = F_1$ and $z - x = 2 = F_2$.

If $y \notin \overline{\mathcal{V}}_{\ell+1}$, then case (i) applies by Proposition 4.11.

Suppose from now on that $y \in \overline{\mathcal{V}}_{\ell+1}$. If $\ell \geq 3$, we have

$$y - x = z - y \in \{F_{\ell-2}, F_{\ell-1}\}$$

by Corollary 4.8 and Proposition 4.10. This is also true for $\ell = 2$ as noted above. Thus, if $x \notin \overline{\mathcal{V}}_{\ell+1}$ and $z \notin \overline{\mathcal{V}}_{\ell+1}$, then case (ii) applies.

Suppose that $x \notin \overline{\mathcal{V}}_{\ell+1}$ and $z \in \overline{\mathcal{V}}_{\ell+1}$. Then $y < z$ are consecutive both in $\overline{\mathcal{V}}_\ell$ and in $\overline{\mathcal{V}}_{\ell+1}$. So, Corollary 4.8 yields $z - y = F_{\ell-1}$. We also note that $x \neq F_\ell$ because otherwise, by Lemma 4.4, we would have $y = F_{\ell+1}$ and $z = F_{\ell+1} + F_{\ell-1}$, thus $\overline{\alpha}(z) = F_\ell$ against the hypothesis that $z \in \overline{\mathcal{V}}_{\ell+1}$. Hence, x admits a predecessor x' in $\overline{\mathcal{V}}_\ell$. Since $x \notin \overline{\mathcal{V}}_{\ell+1}$, Corollary 4.9 shows that $x' \in \overline{\mathcal{V}}_{\ell+1}$, thus $x' < y < z$ are consecutive elements of $\overline{\mathcal{V}}_{\ell+1}$ with $y - x' > y - x = z - y$. By Proposition 4.10, this implies that $y \notin \mathcal{V}_{\ell+2}$ and so $z \in \mathcal{V}_{\ell+2}$ by Corollary 4.9. Thus, case (iii) applies.

Finally, suppose that $x \in \overline{\mathcal{V}}_{\ell+1}$. Then the hypothesis implies that $z \notin \overline{\mathcal{V}}_{\ell+1}$, and we argue similarly as above. As $x < y$ are consecutive in $\overline{\mathcal{V}}_{\ell+1}$, we have $y - x = F_{\ell-1}$. Let z' be the successor of z in $\overline{\mathcal{V}}_\ell$. Since $z \notin \overline{\mathcal{V}}_{\ell+1}$, we have $z' \in \overline{\mathcal{V}}_{\ell+1}$, thus $x < y < z'$ are consecutive in $\overline{\mathcal{V}}_{\ell+1}$ with $y - x = z - y < z' - y$. This implies that $y \notin \overline{\mathcal{V}}_{\ell+2}$ and so $x \in \overline{\mathcal{V}}_{\ell+2}$. Thus, case (iv) applies. □

5. The maps ι and α, and the sets $\mathcal{V}_\ell$

Let the notation be as in section 3.3. In this section, we give an explicit description of the map $\iota\colon\ [\epsilon, w_\infty[\ \to [\epsilon, w_\infty[$. Then, we use it to refine some of the results of the previous section, and we prove the last assertion of section 3.3.

Before doing this, we note that, by Lemma 4.2, we have $\mathcal{V}_1 = [w_1, w_\infty[$ and $\mathcal{V}_2 = [w_2, w_\infty[$. We also note the following direct consequence of Lemma 4.4.

Lemma 5.1. *Let $\ell \geq 3$ be an integer. Then, $\mathcal{V}_\ell \cap [\epsilon, w_{\ell+3}]$ consists of the 7 words*

$$w_\ell < w_{\ell+1} < w_{\ell+1}w_{\ell-1} < w_{\ell+2} < w_{\ell+2}w_{\ell-2} < w_{\ell+2}w_\ell < w_{\ell+3}.$$

For each non-empty word v in E^*, we denote by v^* the word v deprived of its last letter, that is the prefix of v of length $|v|-1$. Induction based on the recurrence formula $w_{i+2} = w_{i+1}w_i$ $(i \geq 1)$ shows that

$$w_i = w_i^{**} f_i \quad \text{where} \quad f_i = \begin{cases} ab & \text{if } i \geq 2 \text{ is even,} \\ ba & \text{if } i \geq 2 \text{ is odd.} \end{cases} \tag{5.1}$$

For each $i \geq 2$, we define $\widetilde{w}_i$ to be the word obtained from w_i by permuting its last two letters. In view of the above, this means that

$$\widetilde{w}_i = w_i^{**} f_{i+1} \quad (i \geq 2). \tag{5.2}$$

It is a well known fact, attributed to J. Berstel, that w_i^{**} is a palindrome for each $i \geq 2$. This has the following useful consequence.

Lemma 5.2. *For each integer $i \geq 2$, the word w_i^{**} is a palindrome and we have*

$$w_{i+1} = w_{i-1}\widetilde{w}_i. \tag{5.3}$$

Proof. We proceed by induction on i. We first note that $w_2^{**} = \epsilon$ and $w_3^{**} = a$ are palindromes, that $w_3 = aba = w_1\widetilde{w}_2$, and that $w_4 = abaab = w_2\widetilde{w}_3$. Suppose that, for some $i \geq 3$, the words w_{i-1}^{**} and w_i^{**} are palindromes and that (5.3) holds. This yields $w_{i+1}^{**} = w_{i-1}^{**} f_{i-1} w_i^{**}$. Thus, the reverse of w_{i+1}^{**} is $w_i^{**} f_i w_{i-1}^{**} = w_i w_{i-1}^{**} = w_{i+1}^{**}$, showing that w_{i+1}^{**} is a palindrome. Moreover, since (5.3) yields $\widetilde{w}_{i+1} = w_{i-1}w_i$, we also find that $w_{i+2} = w_{i+1}w_i = w_i w_{i-1} w_i = w_i \widetilde{w}_{i+1}$, which completes the induction step. □

It follows from Lemma 5.2 that, for each $i \geq 2$, we have $w_{i+1}^{**} = w_{i-1}w_i^{**}$, hence w_i^{**} is the largest prefix v of w_∞ such that $w_{i-1}v$ is a prefix of w_∞.

Lemma 5.3. *Let $k \geq 4$ be an integer. Then,*

$$\begin{array}{rcl}]w_k, w_{k+1}[& \longrightarrow &]w_{k-3}, w_{k-1}w_{k-3}[\\ w & \longmapsto & \iota(w). \end{array} \tag{5.4}$$

is an order preserving bijection. For $w \in]w_k, w_{k+1}[$, we can compute $\iota(w)$ as follows.

(i) *If $w < w_k w_{k-2}^*$, then $w = w_k u$ for some $u \in \left]\epsilon, w_{k-2}^{**}\right]$ and $\iota(w) = w_{k-3}u$.*
(ii) *If $w = w_k w_{k-2}^*$, then $\iota(w) = w_{k-1}^*$.*
(iii) *If $w > w_k w_{k-2}^*$, then $w = w_k w_{k-2} u$ for some $u \in [\epsilon, w_{k-3}[$ and $\iota(w) = w_{k-1}u$.*

Proof. For each $w \in]w_k, w_{k+1}[$, we have $|\iota(w)| = \bar{\iota}(|w|) = |w| - 2F_{k-2}$. So, (5.4) is an order preserving bijection. To justify the explicit formulas, it suffices to note that, in each case, the value given for $\iota(w)$ is a prefix of w_∞ of length $|w| - 2F_{k-2}$. This is a direct computation for the length. That it is a prefix is immediate in cases (ii) and (iii). In case (i), it follows from the observation made right before the lemma. □

Lemma 5.4. *We have $\{\alpha(w_k^*), \alpha(w_{k+1}^*), \alpha(w_k w_{k-2}^*)\} \subseteq \{w_2, w_3\}$ for each integer $k \geq 3$.*

Proof. We proceed by induction. Table (3.7) gives $\alpha(w_3^*) = \alpha(w_4^*) = w_2$ and $\alpha(w_3 w_1^*) = w_3$. Suppose that $\alpha(w_{k-1}^*)$, $\alpha(w_k^*)$ and $\alpha(w_{k-1}w_{k-3}^*)$ lie in $\{w_2, w_3\}$ for some $k \geq 4$. Lemma 5.3 gives

$$\iota(w_{k+1}^*) = w_{k-1}w_{k-3}^* \quad \text{and} \quad \iota(w_k w_{k-2}^*) = w_{k-1}^*.$$

Thus, $\alpha(w_{k+1}^*) = \alpha(w_{k-1}w_{k-3}^*)$ and $\alpha(w_k w_{k-2}^*) = \alpha(w_{k-1}^*)$ also belong to $\{w_2, w_3\}$. □

Proposition 5.5. *Let $\ell \geq 4$ be an integer, and let $u < v$ be consecutive elements of $\mathcal{V}_\ell$. Then, we have $v = us$ for some suffix s in the set $\mathcal{S}_\ell := \{w_{\ell-2}, \widetilde{w}_{\ell-2}, w_{\ell-1}, \widetilde{w}_{\ell-1}\}$.*

Proof. By Lemma 5.1, the smallest element of $\mathcal{V}_\ell$ is w_ℓ and the next six elements of $\mathcal{V}_\ell$ are obtained by multiplying w_ℓ on the right successively by $w_{\ell-1}$, $w_{\ell-1}$, $w_{\ell-2}$, $w_{\ell-2}$, $\widetilde{w}_{\ell-1}$ and $w_{\ell-1}$. Thus, we may assume that $v > w_{\ell+3}$ and so $v \in]w_k, w_{k+1}]$ for some integer $k \geq \ell + 3$. Since $w_k \in \mathcal{V}_\ell$, we deduce that $\{u, v\} \subseteq [w_k, w_{k+1}]$. Since w_{k-3} and $w_{k-1}w_{k-3}$ belong to $\mathcal{V}_\ell$, the map

$$\begin{aligned} \pi\colon \mathcal{V}_\ell \cap [w_k, w_{k+1}] &\longrightarrow \mathcal{V}_\ell \cap [w_{k-3}, w_{k-1}w_{k-3}] \\ w &\longmapsto \pi(w) = \begin{cases} w_{k-3} & \text{if } w = w_k, \\ \iota(w) & \text{if } w_k < w < w_{k+1}, \\ w_{k-1}w_{k-3} & \text{if } w = w_{k+1}. \end{cases} \end{aligned}$$

is an order preserving bijection. In particular, $\pi(u) < \pi(v)$ are consecutive elements of $\mathcal{V}_\ell$ and so we may assume, by induction on $|v|$, that

$$\pi(v) = \pi(u)s \quad \text{for some } s \in \mathcal{S}_\ell. \tag{5.5}$$

We consider three cases.

(i) Suppose that $v < w_k w_{k-2}$. Since $\ell \geq 4$, Lemma 5.4 shows that $w_k w_{k-2}^* \notin \mathcal{V}_\ell$. So, we have $u = w_k u'$ and $v = w_k v'$ for some u', v' in $[\epsilon, w_{k-2}^{**}]$. Then Lemma 5.3 gives $\pi(u) = w_{k-3}u'$ and $\pi(v) = w_{k-3}v'$. By (5.5), this yields $v' = u's$, thus $v = us$.

(ii) Suppose that $v > w_k w_{k-2}$. Then, we have $u \geq w_k w_{k-2}$. So, $u = w_k w_{k-2} u'$ and $v = w_k w_{k-2} v'$ for some u', v' in $[\epsilon, w_{k-3}]$. Applying Lemma 5.3 gives $\pi(u) = w_{k-1}u'$ and $\pi(v) = w_{k-1}v'$. By (5.5), this implies that $v' = u's$, thus $v = us$.

(iii) Finally, suppose that $v = w_k w_{k-2}$. Since $w_k w_{k-2}^* \notin \mathcal{V}_\ell$, we have $u = w_k u'$ for some word $u' \in [\epsilon, w_{k-2}^{**}]$. Then, Lemma 5.3 gives $\pi(u) = w_{k-3}u'$ and $\pi(v) = w_{k-1}$. Thus, $\pi(v) = w_{k-3}\widetilde{w}_{k-2}$ by Lemma 5.2. By (5.5), we deduce that $\widetilde{w}_{k-2} = u's$, thus $w_{k-2} = u's'$ where s' is the reverse of s. We conclude that $v = w_k u's' = us'$ with $s' \in \mathcal{S}_\ell$. □

In Section 3.2, we defined θ to be the endomorphism of E^* determined by $\theta(a) = ab$ and $\theta(b) = a$. It has the following property.

Lemma 5.6. *For each integer $k \geq 2$, we have $\theta(w_k) = w_{k+1}$ and $\theta(\widetilde{w}_k) = \widetilde{w}_{k+1}$. Moreover, θ restricts to an order preserving map from $[\epsilon, w_\infty[$ to itself.*

Proof. The first formula holds for each $k \geq 1$, as stated in (3.3). This follows from the fact that $(\theta(w_i))_{i\geq 1}$ is a Fibonacci sequence in E^* with $\theta(w_1) = \theta(a) = ab = w_2$ and $\theta(w_2) = \theta(ab) = aba = w_3$. Since θ preserves the relation of prefix, we deduce that it maps $[\epsilon, w_k]$ to $[\epsilon, w_{k+1}]$ for each $k \geq 1$. Thus, it maps $[\epsilon, w_\infty[$ to itself.

To prove the second formula, we note that $\theta(ab) = aba$ and $\theta(ba) = aab$. This implies that, for each $k \geq 2$, we have $\theta(f_k) = af_{k+1}$, thus $w_{k+1} = \theta(w_k) = \theta(w_k^{**})af_{k+1}$, and so

$$\widetilde{w}_{k+1} = \theta(w_k^{**})af_k = \theta(w_k^{**}f_{k+1}) = \theta(\widetilde{w}_k). \qquad \square$$

In particular, the above lemma implies that, with the notation of Proposition 5.5, we have $\theta(\mathcal{S}_\ell) = \mathcal{S}_{\ell+1}$ for each integer $\ell \geq 4$. We can now prove the last assertion of Section 3.3.

Proposition 5.7. *Let $\ell \ge 4$ be an integer. Then, θ restricts to an order preserving bijection from $\mathcal{V}_\ell$ to $\mathcal{V}_{\ell+1}$.*

Proof. Let $(v_i)_{i\ge1}$ be the increasing sequence of elements of $\mathcal{V}_\ell$, and let $(v'_i)_{i\ge1}$ be that of $\mathcal{V}_{\ell+1}$. We need to show that $\theta(v_i) = v'_i$ for each $i \ge 1$. As v_i and v'_i are prefixes of w_∞, both $\theta(v_i)$ and v'_i are prefixes of w_∞ by Lemma 5.6. So, it suffices to show that $|\theta(v_i)| = |v'_i|$.

For each $i \ge 1$, write $v_{i+1} = v_i s_i$ and $v'_{i+1} = v'_i s'_i$ for words s_i and s'_i. By Proposition 4.7, the sequence $(|s_i|)_{i\ge1}$ is the infinite Fibonacci word on $(F_{\ell-1}, F_{\ell-1})$ and $(F_{\ell-2}, F_{\ell-2})$, while $(|s'_i|)_{i\ge1}$ is the infinite Fibonacci word on (F_ℓ, F_ℓ) and $(F_{\ell-1}, F_{\ell-1})$. Thus, for each $i \ge 1$, there is an integer $k \in \{\ell-2, \ell-1\}$ such that $|s_i| = F_k$ and $|s'_i| = F_{k+1}$. By Proposition 5.5, we have $s_i \in \{w_k, \widetilde{w}_k\}$ since $|s_i| = F_k$. Then, Lemma 5.6 gives $\theta(s_i) \in \{w_{k+1}, \widetilde{w}_{k+1}\}$, thus $|\theta(s_i)| = F_{k+1} = |s'_i|$.

Since $v_1 = w_\ell$ and $v'_1 = w_{\ell+1}$, we have $\theta(v_1) = v'_1$, thus $|\theta(v_i)| = |v'_i|$ for $i = 1$. We deduce by induction that $|\theta(v_i)| = |v'_i|$ for each $i \ge 1$ because, if this holds for some $i \ge 1$, then

$$|\theta(v_{i+1})| = |\theta(v_i)| + |\theta(s_i)| = |v'_i| + |s'_i| = |v'_{i+1}|. \qquad \square$$

Remark. The statement of the proposition fails for $\ell = 2$ and for $\ell = 3$. For example, for $v = w_6 w_4^* = w_6 w_3 a$, we have $\theta(v) = w_7 w_4 ab = w_7 w_5^*$. Using Lemma 5.3, we find that $\alpha(v) = \iota^2(v) = w_3$ and $\alpha(\theta(v)) = \iota^3(\theta(v)) = w_2$. So, $v \in \mathcal{V}_3$ while $\theta(v) \notin \mathcal{V}_3$. This shows that $\theta(\mathcal{V}_3) \not\subseteq \mathcal{V}_3$. A fortiori, we have $\theta(\mathcal{V}_2) \not\subseteq \mathcal{V}_3$ and $\theta(\mathcal{V}_3) \not\subseteq \mathcal{V}_4$.

Corollary 5.8. *For each $v \in \mathcal{V}_4$, we have $\alpha(\theta(v)) = \theta(\alpha(v))$.*

Proof. Let $v \in \mathcal{V}_4$. We have $\alpha(v) = w_\ell$ for some integer $\ell \ge 4$. Then $v \in \mathcal{V}_\ell \setminus \mathcal{V}_{\ell+1}$, and the proposition gives $\theta(v) \in \mathcal{V}_{\ell+1} \setminus \mathcal{V}_{\ell+2}$. So, $\alpha(\theta(v)) = w_{\ell+1} = \theta(\alpha(v))$. $\square$

6. A partition of the domain of $\mathbf{P}$

In this section, we study the map $\mathbf{P}$ of section 3.5 by decomposing its domain $\mathcal{A}(\epsilon)$ as a union of closed polygons with disjoint interiors, and by giving explicit formulas for $\mathbf{P}$ on each of these polygons. To this end, we introduce several definitions.

We define an *admissible line segment* to be a closed line segment in $\mathbb{R}^2$ with non-empty interior that is parallel to $(1,0)$, $(1,1)$ or $(0,1)$, with end points in $\mathbb{Z}^2$, i.e. a set of the form

$$\{(i+t, j)\,;\, t \in I\} \quad \text{or} \quad \{(i+t, j+t)\,;\, t \in I\} \quad \text{or} \quad \{(i, j+t)\,;\, t \in I\}$$

where $(i,j) \in \mathbb{Z}^2$ and where I is a closed subinterval of $\mathbb{R}$ with non-empty interior, possibly unbounded.

We define an *admissible polygon* to be a non-empty subset of $\mathbb{R}^2$ which is the closure of its interior, whose boundary (possibly empty) is a polygonal line made of admissible line segments intersecting at most in their end-points. These line segments are the *sides* of the polygon and their end-points are its *vertices*.

We say that two distinct admissible polygons are *compatible* if their intersection is either empty, or a common vertex of each, or a common side of each. We define a *polygonal partition* of an admissible polygon $\mathcal{A}$ to be a set S of admissible pairwise compatible polygons whose union is $\mathcal{A}$. This implies that each edge of $\mathcal{A}$ is an union of edges of polygons of S, however not necessarily the edge of a single polygon of S.

For example, the sector $\mathcal{A}(\epsilon)$ is an admissible polygon with a single vertex $(0,0)$, a vertical side $\{(0,t)\,;\,-\infty < t \le 0\}$, and a side $\{(t,t)\,;\,0 \le t < \infty\}$ of slope 1. The main result of this section is the following.

Theorem 6.1. *For each pair of consecutive words $u < v$ in $[\epsilon, w_\infty[$, the set*

$$\mathrm{Trap}(u,v) := \mathcal{A}(u) \cap \mathcal{C}(v) \tag{6.1}$$

is an unbounded admissible trapeze with 2 vertices. For each point $\mathbf{q} = (q_1, q_2)$ in it, we have

$$\mathbf{P}(\mathbf{q}) := \Phi(P_u(\mathbf{q}), q_2 - 1, P_v(\mathbf{q})) = \Phi(q_1 - |u|, q_2 - 1, |v|). \tag{6.2}$$

For each integer $\ell \ge 2$ and each triple of consecutive words $u < v < w$ in $\mathcal{V}_\ell$ with $v \notin \mathcal{V}_{\ell+1}$, the set

$$\mathrm{Cell}(u,v,w) := \mathcal{A}(u) \cap \mathcal{B}(v) \cap \mathcal{C}(w) \tag{6.3}$$

is a bounded admissible polygon with 4 or 5 vertices. For each $\mathbf{q} = (q_1, q_2)$ in it, we have

$$\mathbf{P}(\mathbf{q}) := \Phi(P_u(\mathbf{q}), P_v(\mathbf{q}), P_w(\mathbf{q})) = \Phi(q_1 - |u|, q_2 - |\alpha(v)|, |w|). \tag{6.4}$$

The polygons in (6.1) *and* (6.3) *are all distinct and form a partition of $\mathcal{A}(\epsilon)$.*

Note that the sets $\mathrm{Trap}(u,v)$ in (6.1) and $\mathrm{Cell}(u,v,w)$ in (6.3) are uniquely determined by v which, as a prefix of w_∞, is in turn determined by its length. So, it makes sense to denote them respectively as $\mathcal{T}_i$ and $\mathcal{R}_i$ where $i = |v|$. For $\mathcal{T}_i$, the range of i is $\mathbb{N} \setminus \{0\}$, while for $\mathcal{R}_i$, it is $\mathbb{N} \setminus \overline{\mathcal{F}}$ (for each $\ell \ge 1$, we cannot have $v = w_\ell$ because $w_\ell \in \mathcal{V}_\ell \setminus \mathcal{V}_{\ell+1}$ has no predecessor in $\mathcal{V}_\ell$). This shorter notation is used in Figure 4 below to illustrate the partition of $\mathcal{A}(\epsilon)$ given by the theorem in the range $0 \le q_1 \le 26$.

The proof of Theorem 6.1 requires several steps, leaving the formulas (6.2) and (6.4) for the end. In the process, we also prove, as Proposition 6.7, the assertion made at the beginning of section 3.5. We start by introducing some additional notation.

Definition 6.2. We denote by $\pi_1 \colon \mathbb{R}^2 \to \mathbb{R}$ the projection on the first coordinate:

$$\pi_1(\mathbf{q}) = q_1 \quad \text{for each } \mathbf{q} = (q_1, q_2) \in \mathbb{R}^2.$$

Definition 6.3. For each $v \in [\epsilon, w_\infty[$, we denote by

$$\mathbf{q}(v) = (q_1(v), q_2(v)) = (2|v|, |v| + |\alpha(v)|)$$

the common vertex of $\mathcal{A}(v)$, $\mathcal{B}(v)$ and $\mathcal{C}(v)$.

The 14 points $\mathbf{q}(v)$ with $v \in [\epsilon, w_6]$ are displayed as large dots in Figure 4.

Definition 6.4. Let $\ell \ge 1$ be an integer. For consecutive $u < v$ in $\{\epsilon\} \cup \mathcal{V}_\ell$, we define

$$\mathrm{Trap}(u,v) := \mathcal{A}(u) \cap \mathcal{C}(v). \tag{6.5}$$

We denote by T_ℓ the collection of these sets, and denote their union by

$$\mathrm{Layer}(\ell) = \mathrm{Trap}(\epsilon, w_\ell) \cup \mathrm{Trap}(w_\ell, w_{\ell+1}) \cup \cdots \tag{6.6}$$

When $\ell = 1$, we have $\{\epsilon\} \cup \mathcal{V}_\ell = [\epsilon, w_\infty[$, and so the sets $\mathrm{Trap}(u,v)$ defined above are the same as those in Theorem 6.1. The next lemma justifies this notation by showing that the latter are trapezes for any choice of ℓ. For $\ell = 1$, it proves the first assertion of Theorem 6.1.

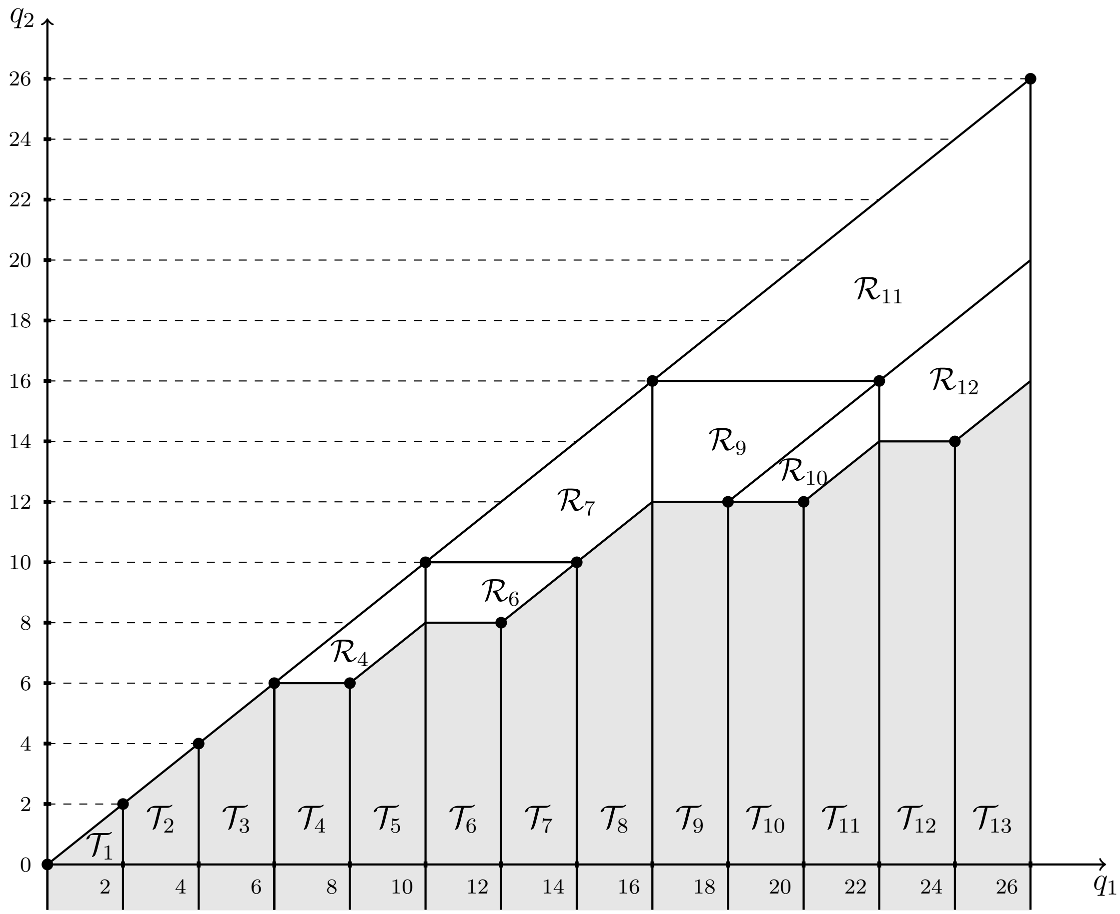

FIGURE 4. Partition of $\mathcal{A}(\epsilon)$ into admissible polygons for $q_1 \le 26$.

Lemma 6.5. *Let $\ell \ge 1$ be an integer, let $u < v$ be consecutive elements of $\{\epsilon\} \cup \mathcal{V}_\ell$, and let $\mathcal{T} = \mathrm{Trap}(u, v)$. Then, we have $\alpha(u) \ne \alpha(v)$ and $\mathcal{T}$ consists of the points $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2$ satisfying*

$$q_1(u) \le q_1 \le q_1(v) \quad \text{and} \quad q_2 \le \begin{cases} q_1 - q_1(u) + q_2(u) & \text{if } \alpha(u) < \alpha(v), \\ q_2(v) & \text{if } \alpha(u) > \alpha(v). \end{cases} \tag{6.7}$$

In both cases, $\mathcal{T}$ is a closed unbounded admissible trapeze with three sides and two vertices: two vertical sides, unbounded from below, each one ending in a vertex, and a line segment of slope 0 or 1 joining the two vertices. Moreover, $\pi_1(\mathcal{T}) = [q_1(u), q_1(v)]$.

We define the *left vertex* (resp. *right vertex*) of $\mathcal{T} = \mathrm{Trap}(u, v)$ to be the vertex of its left (resp. right) vertical side, and we define its *top side* to be the side of $\mathcal{T}$ joining these two vertices.

Proof of Lemma 6.5. It suffices to prove the first assertion. By definition, $\mathcal{T}$ consists of the points $(q_1, q_2) \in \mathbb{R}^2$ satisfying

$$q_1(u) \le q_1 \le q_1(v) \quad \text{and} \quad q_2 \le \min\{q_1 - q_1(u) + q_2(u), q_2(v)\}. \tag{6.8}$$

Let $q_1 \in [q_1(u), q_1(v)]$. We need to show that the upper bounds for q_2 are the same in (6.7) and (6.8). To this end, we note that, if $u > \epsilon$, the numbers $|u| < |v|$ are consecutive elements of $\overline{\mathcal{V}}_\ell$ and Corollary 4.9 gives

$$\big||\alpha(v)| - |\alpha(u)|\big| \geq |v| - |u|. \tag{6.9}$$

This also holds if $u = \epsilon$ for then $v = w_\ell$, so $\alpha(u) = u$ and $\alpha(v) = v$. Thus, we have $\alpha(u) \neq \alpha(v)$. If $\alpha(u) < \alpha(v)$, we deduce from (6.9) that

$$q_1 - q_1(u) + q_2(u) \leq 2|v| - |u| + |\alpha(u)| \leq |v| + |\alpha(v)| = q_2(v).$$

Otherwise, we have $\alpha(u) > \alpha(v)$, and so

$$q_1 - q_1(u) + q_2(u) \geq |u| + |\alpha(u)| \geq |v| + |\alpha(v)| = q_2(v). \qquad \square$$

The next lemma proves the third assertion of Theorem 6.1.

Lemma 6.6. *Let $\ell \geq 2$ be an integer, let $u < v < w$ be consecutive elements of $\mathcal{V}_\ell$ with $v \notin \mathcal{V}_{\ell+1}$, and let $\mathcal{R} = \mathrm{Cell}(u, v, w)$. Then, $u < w$ are consecutive elements of $\mathcal{V}_{\ell+1}$ with $|w| - |u| = F_\ell = |\alpha(v)|$, the point $\mathbf{q}(v)$ lies in the interior* $\mathrm{Trap}(u, w)$, *and we have*

$$\begin{aligned} \mathrm{Trap}(u, w) \cap \mathcal{A}(v) &= \mathrm{Trap}(v, w), \\ \mathrm{Trap}(u, w) \cap \mathcal{B}(v) &= \mathrm{Cell}(u, v, w), \\ \mathrm{Trap}(u, w) \cap \mathcal{C}(v) &= \mathrm{Trap}(u, v). \end{aligned} \tag{6.10}$$

Moreover, these three sets form a partition of $\mathrm{Trap}(u, w)$ *into admissible polygons. In particular, $\mathcal{R}$ is a bounded convex admissible polygon with 4 or 5 sides: the top sides of* $\mathrm{Trap}(u, v)$, $\mathrm{Trap}(v, w)$ *and* $\mathrm{Trap}(u, w)$, *the vertical line segment joining the left vertices of* $\mathrm{Trap}(u, v)$ *and* $\mathrm{Trap}(u, w)$ *when distinct, and the vertical line segment joining the right vertices of* $\mathrm{Trap}(v, w)$ *and* $\mathrm{Trap}(u, w)$ *when distinct. Moreover, $\pi_1(\mathcal{R}) = [q_1(u), q_1(w)]$.*

Thus, $\mathcal{R} = \mathrm{Cell}(u, v, w)$ has exactly three non-vertical sides. One of them is the top side of $\mathrm{Trap}(u, w)$. We call it the *top side* of $\mathcal{R}$. The other two are the top sides of $\mathrm{Trap}(u, v)$ and $\mathrm{Trap}(v, w)$. We call them the *bottom sides* of $\mathcal{R}$. The point $\mathbf{q}(v)$ is their common vertex. This is illustrated in Figure 5 when the top side of $\mathrm{Trap}(u, w)$ has slope 0. The configuration is similar when it has slope 1.

Proof of Lemma 6.6. By Proposition 4.11, $u < w$ are consecutive words in $\mathcal{V}_{\ell+1}$, so we may form the trapeze $\mathcal{T} = \mathrm{Trap}(u, w)$. We also have $|w| - |u| = F_\ell$. Since $\alpha(u) > \alpha(v)$ and $\alpha(w) > \alpha(v)$, Lemma 6.5 shows that the top side of $\mathrm{Trap}(u, v)$ is horizontal, that the top side of $\mathrm{Trap}(v, w)$ has slope 1, and that they share $\mathbf{q}(v)$ as a vertex.

If $\alpha(u) > \alpha(w)$, the top side of $\mathcal{T}$ is horizontal with right vertex $\mathbf{q}(w)$ above the right vertex of $\mathrm{Trap}(v, w)$, on the same vertical line. Thus, $\mathbf{q}(v)$ is an interior point of $\mathcal{T}$. This case is illustrated in Figure 5. Otherwise, we have $\alpha(u) < \alpha(w)$, so the top side of $\mathcal{T}$ has slope 1, with left vertex $\mathbf{q}(u)$ above the left vertex of $\mathrm{Trap}(u, v)$, on the same vertical line. Again, $\mathbf{q}(v)$ is an interior point of $\mathcal{T}$.

Since $\mathbf{q}(v) \in \mathcal{T} \subseteq \mathcal{A}(u)$, we have $\mathcal{A}(v) \subseteq \mathcal{A}(u)$ which yields the first formula in (6.10). Similarly, since $\mathbf{q}(v) \in \mathcal{T} \subseteq \mathcal{C}(w)$, we have $\mathcal{C}(v) \subseteq \mathcal{C}(w)$ which yields the third formula in (6.10). The middle formula follows directly from the definition of $\mathrm{Cell}(u, v, w)$.

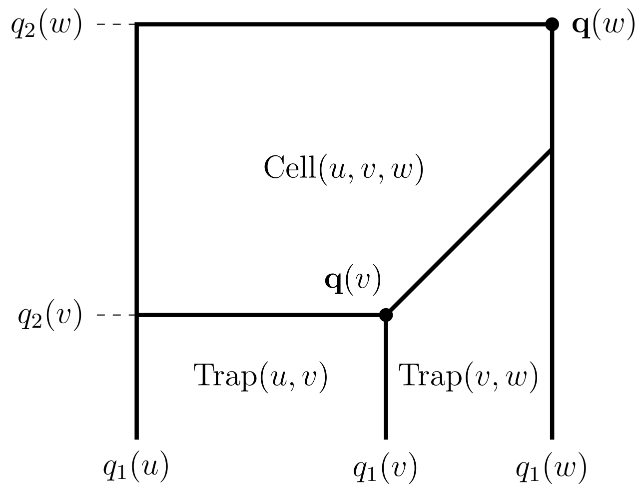

FIGURE 5. Partition of $\mathrm{Trap}(u,w)$ when $\alpha(w) < \alpha(u)$

Since $\mathcal{A}(v)$, $\mathcal{B}(v)$ and $\mathcal{C}(v)$ form a partition of $\mathbb{R}^2$ into admissible sectors with common vertex $\mathbf{q}(v)$ in the interior of $\mathcal{T}$, it follows that the three sets in (6.10) form a partition of $\mathcal{T}$ into admissible polygons, and the remaining assertions follow. □

We can now prove the assertion made at the beginning of section 3.5.

Proposition 6.7. *Let $\mathbf{q} \in \mathcal{A}(\epsilon)$, let u be the largest element of $[\epsilon, w_\infty[$ for which $\mathbf{q} \in \mathcal{A}(u)$, and let w be the smallest element of $]u, w_\infty[$ for which $\mathbf{q} \in \mathcal{C}(w)$. Then, exactly one of the following holds.*

(i) *The words $u < w$ are consecutive elements of $[\epsilon, w_\infty[$. We have $|w| - |u| = F_1 = 1$, $\mathbf{q} \in \mathrm{Trap}(u,w)$ and $\mathbf{q} \notin \mathcal{A}(w)$.*

(ii) *There is an integer $\ell \geq 2$ such that $u < w$ are consecutive elements of $\mathcal{V}_{\ell+1}$ with $|w| - |u| = F_\ell$, and there exists $v \in \mathcal{V}_\ell \setminus \mathcal{V}_{\ell+1}$ such that $u < v < w$ are consecutive in $\mathcal{V}_\ell$. The point $\mathbf{q}$ belongs to $\mathrm{Cell}(u,v,w)$ but not to its bottom sides, nor to $\mathcal{A}(w)$.*

Proof. As $(F_\ell)_{\ell \geq 1}$ is strictly increasing, the two assertions are mutually exclusive. So, it suffices to show that one of them applies.

Suppose first that $|u| \geq 2$. Then $\{u,w\} \subseteq \mathcal{V}_2 = [w_2, w_\infty[$, and so there is a largest integer $m \geq 2$ for which $\{u,w\} \subseteq \mathcal{V}_m$. We claim that u and w are consecutive elements of $\mathcal{V}_m$.

To prove this claim, suppose first that $u \notin \mathcal{V}_{m+1}$ and let v be the successor of u in $\mathcal{V}_m$. By Lemma 6.5, we have $\alpha(v) > \alpha(u) = w_m$ and the top side of $\mathrm{Trap}(u,v)$ has slope 1. Moreover, by the choice of u, we have $\mathbf{q} \notin \mathcal{A}(v)$, thus

$$\mathbf{q} \in \mathcal{A}(u) \setminus \mathcal{A}(v) = \mathrm{Trap}(u,v) \setminus \mathcal{A}(v) \subseteq \mathcal{C}(v). \tag{6.11}$$

Since $v \leq w$, this implies that $w = v$, so $u < w$ are consecutive in $\mathcal{V}_m$, as claimed. Suppose now that $u \in \mathcal{V}_{m+1}$ and let $v \geq u$ be the predecessor of w in $\mathcal{V}_m$. By the choice of m, we have $w \notin \mathcal{V}_{m+1}$. So, Lemma 6.5 yields $\alpha(v) > \alpha(w) = w_m$ and shows that the top side of $\mathrm{Trap}(v,w)$ is horizontal. If $v > u$, we also have $\mathbf{q} \notin \mathcal{C}(v)$ by the choice of w, and so

$$\mathbf{q} \in \mathcal{C}(w) \setminus \mathcal{C}(v) = \mathrm{Trap}(v,w) \setminus \mathcal{C}(v) \subseteq \mathcal{A}(v)$$

against the choice of u. Thus $v = u$, and the claim holds once again.

Let ℓ be the smallest integer with $1 \le \ell \le m-1$ such that u and w are consecutive elements of $\mathcal{V}_{\ell+1}$. If $\ell = 1$, then assertion (i) holds. If $\ell \ge 2$, then $u < w$ are not consecutive in $\mathcal{V}_\ell$. So, by Proposition 4.11, we have $|w| - |u| = F_\ell$ and there exists $v \in \mathcal{V}_\ell \setminus \mathcal{V}_{\ell+1}$ such that $u < v < w$ are consecutive in $\mathcal{V}_\ell$. Then (ii) holds because $\mathbf{q} \notin \mathcal{A}(v) \cup \mathcal{C}(v) \cup \mathcal{A}(w)$.

Finally, suppose that $|u| < 2$. If $u = \epsilon$, then (6.11) holds with $v = w_1$ since $\alpha(\epsilon) < \alpha(w_1)$, and so $w = w_1$. Otherwise, we have $u = w_1$ and (6.11) holds with $v = w_2$ since $\alpha(w_1) < \alpha(w_2)$, hence $w = w_2$. Thus, (i) applies. ☐

Definition 6.8. We denote by S_1 the set T_1 of all trapezes $\mathrm{Trap}(u, v)$ where $u < v$ are consecutive words in $[\epsilon, w_\infty[$. For each integer $\ell \ge 2$, we denote by S_ℓ the set of all polygons $\mathrm{Cell}(u, v, w)$ where $u < v < w$ are consecutive words in $\mathcal{V}_\ell$ with $v \notin \mathcal{V}_{\ell+1}$. We also set $S = \bigcup_{\ell \ge 1} S_\ell$.

Our next goal is to show that S is a partition of $\mathcal{A}(\epsilon)$, as asserted in Theorem 6.1. By Proposition 6.7, $\mathcal{A}(\epsilon)$ is the union of the polygons of S. So, it remains to prove that any pair of distinct polygons of S are compatible. The next result shows that the writing of an element of S, as in the above definition, is unique. It also shows that distinct polygons of S have distinct projections on the first coordinate axis.

Lemma 6.9. *Let $\mathcal{R} \in S$. Then, there is a unique integer $\ell \ge 1$ such that $\mathcal{R} \in S_\ell$.*

(i) *If $\ell = 1$, there is a unique pair of consecutive elements $u < w$ of $[\epsilon, w_\infty[$ such that $\mathcal{R} = \mathrm{Trap}(u, w)$.*

(ii) *If $\ell \ge 2$, there is a unique triple of consecutive elements $u < v < w$ of $\mathcal{V}_\ell$ with $v \notin \mathcal{V}_{\ell+1}$ such that $\mathcal{R} = \mathrm{Cell}(u, v, w)$.*

In both cases, we have

$$|w| - |u| = F_\ell \quad \textit{and} \quad \pi_1(\mathcal{R}) = [q_1(u), q_2(w)]. \tag{6.12}$$

Distinct polygons of S have distinct projections under π_1.

For example, the polygon $\mathcal{R}_9$ in Figure 4 has projection $\pi_1(\mathcal{R}_9) = [16, 22]$. Thus, it is $\mathrm{Cell}(u, v, w)$ where $|u| = 8$, $|v| = 9$ and $|w| = 11$. Since $|w| - |u| = 3 = F_3$, it belongs to S_3. Indeed, $8 < 9 < 11$ are consecutive elements of $\overline{\mathcal{V}}_3$ with $\alpha(9) = 3$, as table (4.3) shows.

Proof of Lemma 6.9. Suppose first that $\mathcal{R} \in S_1$. Then, $\mathcal{R} = \mathrm{Trap}(u, w)$ for consecutive elements $u < w$ of $[\epsilon, w_\infty[$. We have $|w| - |u| = 1 = F_1$ and Lemma 6.5 yields $\pi_1(\mathcal{R}) = [q_1(u), q_2(w)]$.

Suppose now that $\mathcal{R} \in S_\ell$ for some $\ell \ge 2$. Then, $\mathcal{R} = \mathrm{Cell}(u, v, w)$ for consecutive elements $u < v < w$ of $\mathcal{V}_\ell$ with $v \notin \mathcal{V}_{\ell+1}$. By Proposition 4.10, we have $|w| - |u| = F_\ell$. By Lemma 6.6, the top side of $\mathcal{R}$ is the top side of $\mathrm{Trap}(u, w)$. Hence, $\mathcal{R}$ has the same projection as $\mathrm{Trap}(u, w)$ under π_1, which is $[q_1(u), q_2(w)]$ by Lemma 6.5.

Thus, (6.12) holds in all cases. Hence, u and w are uniquely determined by $\pi_1(\mathcal{R})$, which in turn determine ℓ. If $\ell \ge 2$, then v is also determined as the successor of u in $\mathcal{V}_\ell$. ☐

Proposition 6.10. *Let $\ell \ge 1$ be an integer, and let $u < v < w$ be consecutive elements of $\{\epsilon\} \cup \mathcal{V}_\ell$. Then* $\mathrm{Trap}(u, v)$ *and* $\mathrm{Trap}(v, w)$ *are compatible trapezes: their intersection is both the right vertical side of* $\mathrm{Trap}(u, v)$ *and the left vertical side of* $\mathrm{Trap}(v, w)$.

Proof. By Lemma 6.5, the right vertex of $\mathrm{Trap}(u,v)$ is $(q_1(v), r)$ for some $r \in \mathbb{R}$ and the left vertex of $\mathrm{Trap}(v,w)$ is $(q_1(v), s)$ for some $s \in \mathbb{R}$. The lemma also allows us to compute r and s in terms of the numbers $x = |u|$, $y = |v|$, $z = |w|$, $\overline{\alpha}(x) = |\alpha(u)|$, $\overline{\alpha}(y) = |\alpha(v)|$ and $\overline{\alpha}(z) = |\alpha(w)|$. We simply need to show that $r = s$. Without loss of generality, we may assume that $u < v < w$ are not consecutive elements of $\{\epsilon\} \cup \mathcal{V}_{\ell+1}$.

If $|u| \geq 2$, we have $\ell \geq 2$, and $x < y < z$ are consecutive numbers in $\overline{\mathcal{V}}_\ell$, not all contained in $\overline{\mathcal{V}}_{\ell+1}$. So, Proposition 4.12 applies and yields four cases (i)–(iv) to consider. Figure 6 shows the trapezes $\mathrm{Trap}(u,v)$ and $\mathrm{Trap}(v,w)$ in each of these cases. Using Lemma 6.5, we find in all cases that $r = s$:

$$\begin{aligned}
\text{Case (i):}\quad & r = y + F_\ell = s;\\
\text{Case (ii):}\quad & r = (x + F_\ell) + 2(y - x) = z + F_\ell = s;\\
\text{Case (iii):}\quad & r = (x + F_\ell) + 2(y - x) = y + F_{\ell+1} = s;\\
\text{Case (iv):}\quad & r = y + F_{\ell+1} = z + F_\ell = s.
\end{aligned}$$

If $u = \epsilon$, then $v = w_\ell$, $w = w_{\ell+1}$, and we find $r = 2F_\ell = s$. Finally, if $u = w_1$, then $\ell = 1$, $v = w_2$, $w = w_3$, and $r = s = 4$. □

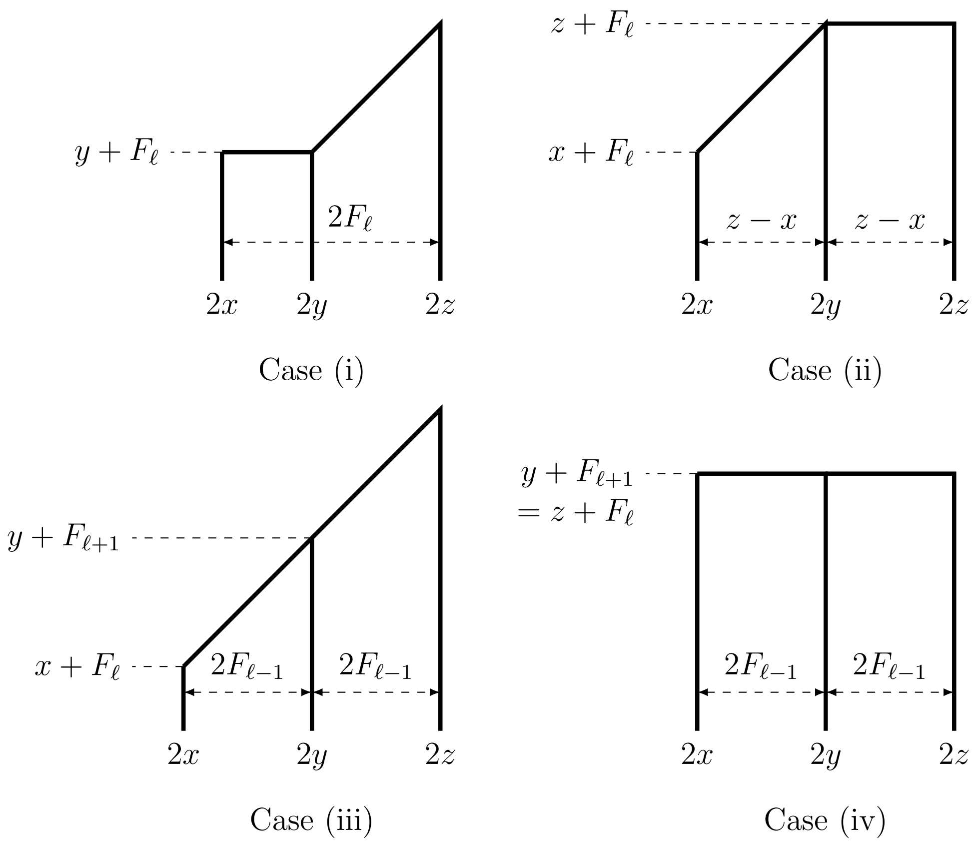

FIGURE 6. Four possible configurations for $\mathrm{Trap}(u,v)$ and $\mathrm{Trap}(v,w)$ for consecutive words $u < v < w$ in $\mathcal{V}_\ell$ with $\ell \geq 1$.

Corollary 6.11. *Let $\ell \geq 1$ be an integer. Any pair of distinct trapezes of T_ℓ are compatible. Any pair of distinct polygons of S_ℓ are compatible.*

Proof. The first assertion is clear since any pair of distinct trapezes of T_ℓ with non-empty intersection are as in the proposition.

For the second assertion, we may assume that $\ell \geq 2$ since $S_1 = T_1$. Let $\mathcal{R}$ and $\mathcal{R}'$ be distinct elements of S_ℓ. Write $\mathcal{R} = \mathrm{Cell}(u, v, w)$ and $\mathcal{R}' = \mathrm{Cell}(u', v', w')$ where $u < v < w$ and $u' < v' < w'$ are triples of consecutive elements of $\mathcal{V}_\ell$ with $v \notin \mathcal{V}_{\ell+1}$ and $v' \notin \mathcal{V}_{\ell+1}$. Without loss of generality, we may assume that $v < v'$ and so $w \leq u'$. Then $\mathcal{R} \cap \mathcal{R}'$ is empty if $w < u'$, and is contained in the vertical line L of abscissa $q_1(w) = q_1(u')$ if $w = u'$. Suppose that $w = u'$. By Lemma 6.6, $\mathcal{R} \cap L$ is the line segment joining the right vertex $\mathbf{r}$ of $\mathrm{Trap}(v, w)$ and the right vertex $\mathbf{s}$ of $\mathrm{Trap}(u, w)$, while $\mathcal{R}' \cap L$ is the line segment joining the left vertex $\mathbf{r}'$ of $\mathrm{Trap}(u', v')$ and the left vertex $\mathbf{s}'$ of $\mathrm{Trap}(u', w')$. These two line segments are the same because, by the proposition, we have $\mathbf{r} = \mathbf{r}'$ since $v < w = u' < v'$ are consecutive elements of $\mathcal{V}_\ell$, and $\mathbf{s} = \mathbf{s}'$ since $u < w = u' < w'$ are consecutive elements of $\mathcal{V}_{\ell+1}$. Thus, $\mathcal{R} \cap \mathcal{R}' = \mathcal{R} \cap L = \mathcal{R}' \cap L$ is a common side of $\mathcal{R}$ and $\mathcal{R}'$ if $\mathbf{r} \neq \mathbf{s}$, and a common vertex if $\mathbf{r} = \mathbf{s}$. In all cases, $\mathcal{R}$ and $\mathcal{R}'$ are compatible. □

Corollary 6.12. *Let $\ell \geq 1$ be an integer. Then,* Layer(ℓ) *is an admissible polygon and T_ℓ is a partition of it. The boundary of* Layer(ℓ) *is a polygonal line made of the vertical side of $\mathcal{A}(\epsilon)$ and the top sides of the trapezes in T_ℓ.*

This follows immediately from the preceding corollary and the definition of Layer(ℓ). For $\ell = 1$, this is illustrated in Figure 4 where Layer(1) is displayed in grey. Note that, if $u < v < w$ are consecutive words in $\{\epsilon\} \cup \mathcal{V}_\ell$ and if the top sides of $\mathrm{Trap}(u, v)$ and $\mathrm{Trap}(v, w)$ have the same slope, both of them are contained in the same side of Layer(ℓ). The next result provides an example of this.

Corollary 6.13. *Let $\ell \geq 1$ be an integer. Then,* $\mathrm{Trap}(\epsilon, w_\ell)$ *and* $\mathrm{Trap}(w_\ell, w_{\ell+1})$ *form a partition of* $\mathrm{Trap}(\epsilon, w_{\ell+1})$*, and so* $\mathrm{Trap}(\epsilon, w_{\ell+1}) \subseteq \mathrm{Layer}(\ell)$*. We also have* Layer(1) = Layer(2)*, and S_1 is a partition of* Layer(2)*.*

Proof. Since $\epsilon < w_\ell < w_{\ell+1}$ are consecutive elements of $\{\epsilon\} \cup \mathcal{V}_\ell$, the proposition shows that $\mathrm{Trap}(\epsilon, w_\ell)$ and $\mathrm{Trap}(w_\ell, w_{\ell+1})$ are compatible. Since $\alpha(\epsilon) < \alpha(w_\ell) < \alpha(w_{\ell+1})$, their top sides have slope 1, like the top side of $\mathrm{Trap}(\epsilon, w_{\ell+1})$. So, the former trapezes form a partition of the latter. This proves the first assertion. For $\ell = 1$, this gives $\mathrm{Trap}(\epsilon, w_2) \subseteq \mathrm{Layer}(1)$. As $\mathcal{V}_2 = [w_2, w_\infty[$, we conclude that Layer(2) = Layer(1). Since $S_1 = T_1$ is a partition of Layer(1), it is therefore a partition of Layer(2). □

Lemma 6.14. *Let $\ell \geq 2$ be an integer and let $u < v < w$ be consecutive elements of $\mathcal{V}_\ell$ with $v \notin \mathcal{V}_{\ell+1}$. Then* $\mathrm{Cell}(u, v, w) \cap \mathrm{Layer}(\ell)$ *is the union of the bottom sides of* $\mathrm{Cell}(u, v, w)$*.*

Proof. This follows immediately from Lemma 6.6 since the portion of Layer(ℓ) between the vertical line segments of abscissa $q_1(u)$ and $q_1(w)$ is $\mathrm{Trap}(u, v) \cup \mathrm{Trap}(v, w)$. □

Lemma 6.15. *Let $u < w$ be consecutive elements of $\mathcal{V}_{\ell+1}$ for some integer $\ell \geq 1$, and let $\mathcal{E}$ be the top side of $\mathcal{T} = \mathrm{Trap}(u, w)$. Then $\mathcal{E}$ is a side of a unique polygon $\mathcal{R} \in S_1 \cup \cdots \cup S_\ell$, and it is the top side of $\mathcal{R}$.*

Proof. If $\mathcal{E}$ is a side of some $\mathcal{R} \in S_1 \cup \cdots \cup S_\ell$ then it is its top side because $\mathcal{R}$ is contained in Layer$(\ell + 1)$ and $\mathcal{E}$ lies on the boundary of Layer$(\ell + 1)$. Thus, $\pi_1(\mathcal{R}) = \pi_1(\mathcal{T})$ and so, by Lemma 6.9, there is at most one such $\mathcal{R}$.

To show the existence of $\mathcal{R}$, we may assume without loss of generality that ℓ is the smallest positive integer such that $u < w$ are consecutive in $\mathcal{V}_{\ell+1}$. If $\ell = 1$, we may take $\mathcal{R} = \mathcal{T}$. If $\ell \geq 2$, then $u < w$ are not consecutive in $\mathcal{V}_\ell$ and so, by Proposition 4.11, there exists $v \in \mathcal{V}_\ell \setminus \mathcal{V}_{\ell+1}$ such that $u < v < w$ are consecutive in $\mathcal{V}_\ell$. Then Lemma 6.6 shows that the top side of $\mathcal{R} = \mathrm{Cell}(u, v, w)$ is $\mathcal{E}$. □

The next result proves the last assertion of Theorem 6.1.

Proposition 6.16. *The set $S_1 \cup \cdots \cup S_{\ell-1}$ is a partition of* $\mathrm{Layer}(\ell)$ *for each integer $\ell \geq 2$. Moreover, $S = \bigcup_{\ell=1}^{\infty} S_\ell$ is a partition of $\mathcal{A}(\epsilon)$.*

Proof. Since the sets $\mathrm{Layer}(\ell)$ with $\ell \geq 2$ form an increasing sequence whose union is $\mathcal{A}(\epsilon)$, it suffices to show the first assertion. We do this by induction on ℓ.

For $\ell = 2$, the statement follows from Corollary 6.13.

Now, suppose that $S_1 \cup \cdots \cup S_{\ell-1}$ is a partition of $\mathrm{Layer}(\ell)$ for some $\ell \geq 2$. By Lemma 6.6, each trapeze in $T_{\ell+1}$ either belongs to T_ℓ or decomposes as the union of two trapezes of T_ℓ and a polygon of S_ℓ. Thus $\mathrm{Layer}(\ell + 1)$ is the union of $\mathrm{Layer}(\ell)$ and of the polygons of S_ℓ. Hence, it is the union of the polygons in $S_1 \cup \cdots \cup S_\ell$.

To complete the induction step, it remains to show that any pair of polygons $\mathcal{R} \neq \mathcal{R}'$ in $S_1 \cup \cdots \cup S_\ell$ are compatible. By hypothesis, this is true if they both belong to $S_1 \cup \cdots \cup S_{\ell-1}$. By Corollary 6.11, this is also true if they both belong to S_ℓ. So, we may assume that $\mathcal{R} \in S_\ell$ and that $\mathcal{R}' \in S_1 \cup \cdots \cup S_{\ell-1}$. We may further assume that $\mathcal{R}$ and $\mathcal{R}'$ intersect.

Since $\mathcal{R}' \subseteq \mathrm{Layer}(\ell)$, the set $\mathcal{R} \cap \mathcal{R}'$ is contained in $\mathcal{R} \cap \mathrm{Layer}(\ell)$ which, by Lemma 6.14, is the union of the two bottom sides of $\mathcal{R}$. As these have distinct slopes and as $\mathcal{R} \cap \mathcal{R}'$ is convex, that intersection is contained in a single bottom side $\mathcal{E}$ of $\mathcal{R}$. By Lemma 6.15, $\mathcal{E}$ is a side of a unique $\mathcal{R}'' \in S_1 \cup \cdots \cup S_{\ell-1}$. If $\mathcal{R}' = \mathcal{R}''$, then $\mathcal{R} \cap \mathcal{R}' = \mathcal{E}$ is a common side of $\mathcal{R}$ and $\mathcal{R}'$, and we are done. Otherwise, $\mathcal{E}$ is not a side of $\mathcal{R}'$. As $\mathcal{R}'$ and $\mathcal{R}''$ are compatible polygons by our induction hypothesis, it follows that $\mathcal{R} \cap \mathcal{R}' = \mathcal{E} \cap \mathcal{R}'$ is a common vertex of $\mathcal{R}'$ and $\mathcal{E}$, thus also a vertex of $\mathcal{R}$, and again we are done. □

We also have a similar result for trapezes.

Proposition 6.17. *Let $\mathcal{T} \in \bigcup_{\ell=1}^{\infty} T_\ell$. Then, $\{\mathcal{R} \in S\,;\, \mathcal{R} \subseteq \mathcal{T}\}$ is a partition of $\mathcal{T}$.*

Proof. By Proposition 6.16, the polygons in S are pairwise compatible. Thus, it suffices to show that $\mathcal{T}$ is a union of polygons of S. This is automatic if $\mathcal{T} \in T_1$ because $T_1 = S_1 \subseteq S$. Suppose that $\mathcal{T} \in T_{\ell+1}$ for some $\ell \geq 1$. Then, we have $\mathcal{T} \in T_\ell$, or $\mathcal{T} = \mathrm{Trap}(\epsilon, w_{\ell+1})$ is the union of two trapezes in T_ℓ by Corollary 6.12, or we have $\ell \geq 2$ and $\mathcal{T}$ is the union of an element of S_ℓ and two elements of T_ℓ by Lemma 6.6. As we may assume, by induction, that each element of T_ℓ is a union of polygons of S, the same is true for $\mathcal{T}$. □

By Lemmas 6.5 and 6.6, the following result completes the proof of Theorem 6.1.

Proposition 6.18. *Let $\mathcal{R} \in S$, let $u, w \in [\epsilon, w_\infty[$ such that $\pi_1(\mathcal{R}) = [q_1(u), q_1(w)]$, and let $\mathbf{q} = (q_1, q_2) \in \mathcal{R}$. Then, we have*

$$\mathbf{P}(\mathbf{q}) = \Phi(q_1 - |u|, q_2 + |u| - |w|, |w|). \tag{6.13}$$

Proof. We proceed by induction on the integer $\ell \geq 1$ for which $\mathcal{R} \in S_\ell$.

Suppose first that $\ell = 1$. Then, $u < w$ are consecutive in $[\epsilon, w_\infty[$, and $\mathcal{R} = \mathrm{Trap}(u, w)$. If $\mathbf{q} \notin \mathcal{A}(w)$, then u is the largest element of $[\epsilon, w_\infty[$ for which $\mathbf{q} \in \mathcal{A}(u)$ and w is the smallest element of $]u, w_\infty[$ for which $\mathbf{q} \in \mathcal{C}(w)$. Hence, formula (3.11) for $\mathbf{P}(\mathbf{q})$ applies and gives (6.13). If $\mathbf{q} \in \mathcal{A}(w)$, then $\mathbf{q}$ belongs to $\mathrm{Trap}(w, w') \setminus \mathcal{A}(w')$ where w' is the successor of w in $[\epsilon, w_\infty[$, and the preceding yields

$$\mathbf{P}(\mathbf{q}) = \Phi(q_1 - |w|, q_2 + |w| - |w'|, |w'|).$$

This yields (6.13) because $q_1 = 2|w|$ and $|w'| - |w| = 1 = |w| - |u|$, thus

$$(q_1 - |w|, q_2 + |w| - |w'|, |w'|) = (|w|, q_2 + |u| - |w|, q_1 - |u|).$$

Suppose now that $\ell \geq 2$. Then, by Lemma 6.9, we have $|w| - |v| = F_\ell$ and $\mathcal{R} = \mathrm{Cell}(u, v, w)$ for some $v \in \mathcal{V}_\ell \setminus \mathcal{V}_{\ell+1}$ such that $u < v < w$ are consecutive elements of $\mathcal{V}_\ell$. Let $\mathcal{E}_0$ denote the top side of $\mathrm{Trap}(u, v)$, and $\mathcal{E}_1$, the top side of $\mathrm{Trap}(v, w)$. By Lemma 6.6, $\mathcal{E}_0$ and $\mathcal{E}_1$ are the bottom sides of $\mathcal{R}$. Moreover, $u < w$ are consecutive elements of $\mathcal{V}_{\ell+1}$, and $\mathbf{q} \in \mathrm{Trap}(u, w)$. We consider several possibilities.

1. Suppose that $\mathbf{q} \notin \mathcal{E}_0 \cup \mathcal{E}_1$. Then we have $\mathbf{q} \notin \mathrm{Layer}(\ell)$ by Lemma 6.14. Let u' be the largest element of $[\epsilon, w_\infty[$ such that $\mathbf{q} \in \mathcal{A}(u')$ and let w' be the smallest element of $]u', w_\infty[$ such that $\mathbf{q} \in \mathcal{C}(w')$. Since $\mathbf{q} \in \mathrm{Trap}(u, w)$, we have $u \leq u' < w'$. By Proposition 6.7, $u' < w'$ are consecutive elements of $\mathcal{V}_{k+1}$ for some integer $k \geq 1$ (because $u' \geq u \geq w_2$), and so $\mathbf{q} \in \mathrm{Trap}(u', w') \subseteq \mathrm{Layer}(k+1)$. Thus, we must have $k \geq \ell$ and therefore $\{u', w'\} \subseteq \mathcal{V}_{\ell+1}$. We divide this case into two sub-cases.

a) If $\mathbf{q} \notin \mathcal{A}(w)$, then we have $u \leq u' < w' \leq w$. As $u < w$ are consecutive elements of $\mathcal{V}_{\ell+1}$, we deduce that $u' = u$ and $w' = w$, and then (6.13) holds by definition of $\mathbf{P}$.

b) If instead $\mathbf{q} \in \mathcal{A}(w)$, then $u' = w$ and, by Proposition 6.10, $\mathbf{q}$ belongs to the left vertical side of $\mathrm{Trap}(w, w'')$ where w'' is the successor of w in $\mathcal{V}_{\ell+1}$. Hence, we must have $w' = w''$. So, $w < w'$ are consecutive in $\mathcal{V}_{\ell+1}$, but not consecutive in $\mathcal{V}_\ell$ because $\mathbf{q} \notin \mathrm{Layer}(\ell)$, thus $|w'| - |w| = F_\ell = |w| - |u|$ by Proposition 4.11. By definition of $\mathbf{P}$, we thus have

$$\mathbf{P}(\mathbf{q}) = \Phi(q_1 - |w|, q_2 + |w| - |w'|, |w'|) = \Phi(|w|, q_2 + |u| - |w|, q_1 - |u|)$$

where the second equality uses $q_1 = q_1(w) = 2|w|$. Thus, (6.13) also holds in this case.

2. Suppose that $\mathbf{q} \in \mathcal{E}_0$. By Lemma 6.15, $\mathcal{E}_0$ is the top side of a unique $\mathcal{R}' \in S_1 \cup \cdots \cup S_{\ell-1}$. Moreover, we have $\pi_1(\mathcal{R}') = \pi_1(\mathcal{E}_0) = [q_1(u), q_1(v)]$. So, by induction, we may assume that

$$\mathbf{P}(\mathbf{q}) = \Phi(q_1 - |u|, q_2 + |u| - |v|, |v|).$$

This yields (6.13) because $q_2 = q_2(v) = |v| + F_\ell = |v| + |w| - |u|$, and so

$$(q_1 - |u|, q_2 + |u| - |v|, |v|) = (q_1 - |u|, |w|, q_2 + |u| - |w|).$$

3. Finally, suppose that $\mathbf{q} \in \mathcal{E}_1$. Then, similarly as in the previous case, $\mathcal{E}_1$ is the top side of a unique $\mathcal{R}'' \in S_1 \cup \cdots \cup S_{\ell-1}$ and we have $\pi_1(\mathcal{R}'') = \pi_1(\mathcal{E}_1) = [q_1(v), q_1(w)]$. So, by induction, we may assume that

$$\mathbf{P}(\mathbf{q}) = \Phi(q_1 - |v|, q_2 + |v| - |w|, |w|).$$

Then (6.13) follows because $q_1 - q_2 = q_1(v) - q_2(v) = |v| - F_\ell = |v| + |u| - |w|$, and so

$$(q_1 - |v|, q_2 + |v| - |w|, |w|) = (q_2 + |u| - |w|, q_1 - |u|, |w|).$$

□

7. Additional properties of the map P

In this section, we study in more detail the map $\mathbf{P}$ and we look more closely at its first component P_1. We also introduce a notion of integral 2-parameter 3-system which applies to $\mathbf{P}$ and extends that of integral 3-system from [5].

Proposition 7.1. *Let $\mathbf{P} = (P_1, P_2, P_3)\colon \mathcal{A}(\epsilon) \to \mathbb{R}^3$ be as in section 6, and let $\mathcal{R} \in S$. Then, there is a unique triple $(a, b, c) \in \mathbb{Z}^3$ with $c = a + b$ such that*

$$\mathbf{P}(\mathbf{q}) = \Phi(q_1 - a, q_2 - b, c) \quad \text{for each } \mathbf{q} = (q_1, q_2) \in \mathcal{R}. \tag{7.1}$$

If $\mathcal{R} \neq \mathrm{Trap}(\epsilon, w_1)$, there is also a unique point $\mathbf{r}$ of $\mathcal{R}$ such that $P_1(\mathbf{r}) = P_2(\mathbf{r}) = P_3(\mathbf{r})$. If moreover $\mathcal{R} \neq \mathrm{Trap}(w_1, w_2)$, then $\mathbf{r}$ is an interior point of $\mathcal{R}$, and the sets

$$\mathcal{R}_{i,j,k} = \{\mathbf{q} = (q_1, q_2) \in \mathcal{R}\,;\, P_i(\mathbf{q}) = q_1 - a, P_j(\mathbf{q}) = q_2 - b, P_k(\mathbf{q}) = c\}$$

attached to the six permutations (i, j, k) of $(1, 2, 3)$ form a partition of $\mathcal{R}$ into admissible convex polygons with $\mathbf{r}$ as a common vertex. If $\mathcal{R}$ is $\mathrm{Trap}(\epsilon, w_1)$ or $\mathrm{Trap}(w_1, w_2)$, then the $\mathcal{R}_{i,j,k}$ with non-empty interior provide a partition of $\mathcal{R}$ into admissible convex polygons.

Proof. We will simply treat the case where $\mathcal{R} \in S_\ell$ for an integer $\ell \geq 2$. The reasoning is similar and simpler for the trapezes of S_1. Thus, we assume that $\mathcal{R} = \mathrm{Cell}(u, v, w)$ for consecutive elements $u < v < w$ of $\mathcal{V}_\ell$ with $v \notin \mathcal{V}_{\ell+1}$. Then, Theorem 6.1 yields (7.1) with $(a, b, c) = (|u|, |\alpha(v)|, |w|)$. By Lemma 6.6, we have $|w| - |u| = F_\ell = |\alpha(v)|$, thus $c = a + b$. Since $\mathcal{R}$ has non-empty interior, the condition (7.1) uniquely determines (a, b, c). As shown in Figure 7 when the top side of $\mathcal{R}$ is horizontal, the lines of equation $q_1 - a = c$, $q_2 - b = c$ and $q_1 - a = q_2 - b$ cut respectively in their middle the top sides of $\mathrm{Trap}(u, w)$, $\mathrm{Trap}(v, w)$ and $\mathrm{Trap}(u, v)$. Their intersection point $\mathbf{r} = (a + c, b + c)$ is therefore an interior point of $\mathcal{R}$ and the only point of $\mathcal{R}$ where P_1, P_2 and P_3 coincide. These lines induce a partition of $\mathcal{R}$ into six convex polygons on which the differences $(q_1 - a) - c$, $(q_2 - b) - c$ and $(q_1 - a) - (q_2 - b)$ are everywhere ≥ 0 or ≤ 0. By definition of $\mathbf{P}$, this is equivalent to $(q_1 - a, q_2 - b, c) = (P_i(\mathbf{q}), P_j(\mathbf{q}), P_k(\mathbf{q}))$ for a fixed permutation (i, j, k) of $(1, 2, 3)$. We get a different permutation for each polygon, as indicated in Figure 7. □

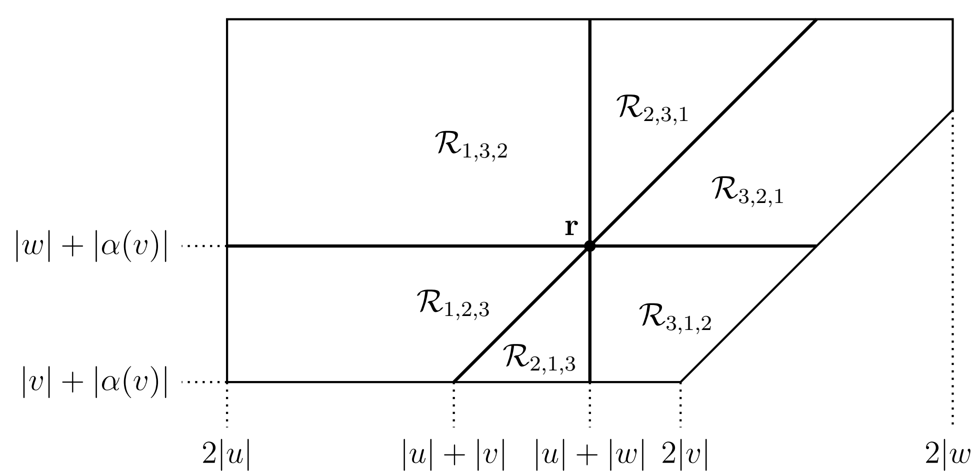

Figure 7. Partition of $\mathcal{R}$ into six polygons on which $\mathbf{P}$ is affine

In the notation of the proposition, we have

$$P_1(\mathbf{q}) = \begin{cases} q_1 - a & \text{if } \mathbf{q} \in \mathcal{R}_{1,2,3} \cup \mathcal{R}_{1,3,2}, \\ q_2 - b & \text{if } \mathbf{q} \in \mathcal{R}_{2,1,3} \cup \mathcal{R}_{3,1,2}, \\ c & \text{if } \mathbf{q} \in \mathcal{R}_{2,3,1} \cup \mathcal{R}_{3,2,1}. \end{cases}$$

Thus, we obtain a 3-dimensional picture of the graph of P_1 over a trapeze $\mathrm{Trap}(\epsilon, w_\ell)$ by partitioning this trapeze into polygons $\mathcal{R} \in S$ and then by colouring the corresponding $\mathcal{R}_{i,j,k}$ in light grey when $i = 1$, in medium grey when $j = 1$ and in dark grey when $k = 1$. The result is shown in Figure 8 for $\ell = 7$. On the connected regions in light grey, $P_1(\mathbf{q}) - q_1$ is constant; on those in medium grey, $P_1(\mathbf{q}) - q_2$ is constant; on those in dark grey, $P_1(\mathbf{q})$ is constant. This picture shows some symmetries. One notes for example that the colouring is the same, up to translation, in the region surrounded by solid lines as in the region surrounded by dashed lines. This is an illustration of Theorem 3.1 with $k = 6$.

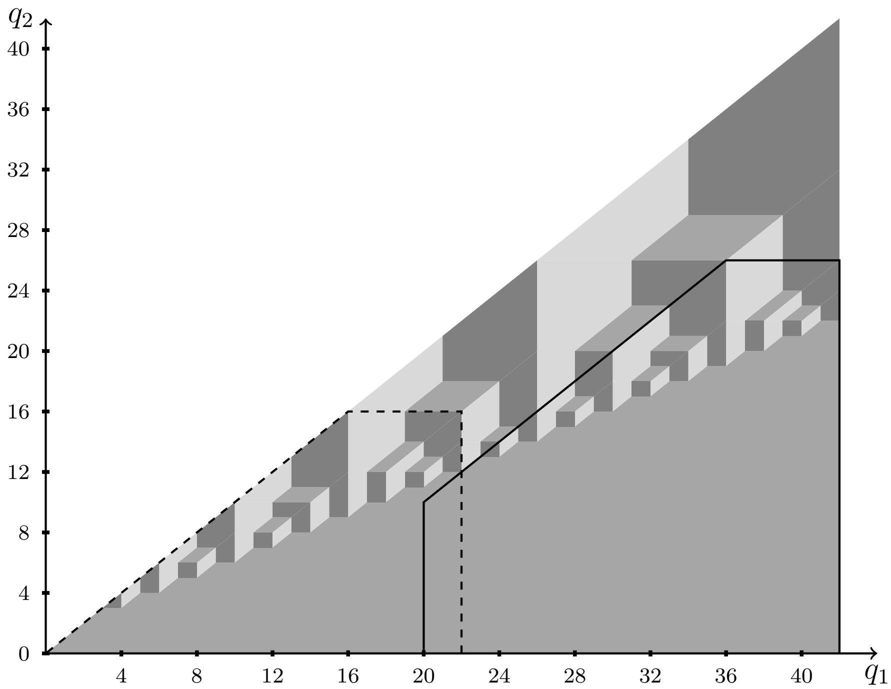

FIGURE 8. The graph of $P_1(q_1, q_2)$ for $0 \le q_2 \le q_1 \le 2F_7 = 42$

The smallest admissible polygons are the admissible triangles with horizontal and vertical sides of length 1, namely the triangles with set of vertices

$$V = \{(m, n), (m+1, n), (m+1, n+1)\} \quad \text{or} \quad V' = \{(m, n), (m, n+1), (m+1, n+1)\}$$

for some $(m, n) \in \mathbb{Z}^2$. We call them the *basic triangles*. We say that a basic triangle $\mathcal{T}$ is respectively of *lower* or *upper* type if its set of vertices is of the form V or V' respectively, that is if $\mathcal{T}$ lies respectively below or above its hypotenuse.

The basic triangles are pairwise compatible, each basic triangle of a type sharing common sides with 3 basic triangles of the other type (see Figure 9). They form a partition of $\mathbb{R}^2$ as defined in section 6. In general, the set of basic triangles contained in any given admissible polygon $\mathcal{A}$ is a partition of $\mathcal{A}$.

Lemma 7.2. *Let $\mathcal{T}$ be a basic triangle and let $(a,b,c)\in\mathbb{Z}^3$. Then the map from $\mathcal{T}$ to $\mathbb{R}^3$ sending each $\mathbf{q}=(q_1,q_2)\in\mathcal{T}$ to $\Phi(q_1-a,q_2-b,c)$ is affine.*

Proof. For any interior point $\mathbf{q}=(q_1,q_2)$ of $\mathcal{T}$, none of the numbers q_1, q_2 and q_1-q_2 is an integer, so the coordinates of (q_1-a,q_2-b,c) are all distinct, and their order is independent of $\mathbf{q}$. Hence, there is a linear map $\sigma\colon\mathbb{R}^3\to\mathbb{R}^3$ that permutes the coordinates in $\mathbb{R}^3$ such that $\Phi(q_1-a,q_2-b,c)=\sigma(q_1-a,q_2-b,c)$ for each interior point of $\mathcal{T}$, and thus, by continuity, for each point of $\mathcal{T}$. □

By Proposition 7.1, the next two results apply to the map $\mathbf{P}\colon\mathcal{A}(\epsilon)\to\mathbb{R}^3$ of section 6.

Lemma 7.3. *Let $\mathcal{A}$ be an admissible polygon, let T denote the set of basic triangles contained in $\mathcal{A}$, and let $\mathbf{P}=(P_1,P_2,P_3)\colon\mathcal{A}\to\mathbb{R}^3$ be a function. Suppose that, for each $\mathcal{T}\in T$, there exists $(a,b,c)\in\mathbb{Z}^3$ with $c=a+b$ such that*

$$\mathbf{P}(q_1,q_2)=\Phi(q_1-a,q_2-b,c)\quad\textit{for any }(q_1,q_2)\in\mathcal{T}.\tag{7.2}$$

Then $\mathbf{P}$ is continuous and satisfies

$$P_1(\mathbf{q})+P_2(\mathbf{q})+P_3(\mathbf{q})=q_1+q_2\quad\textit{for any }\mathbf{q}=(q_1,q_2)\in\mathcal{A}.\tag{7.3}$$

If $\mathcal{A}$ is convex then $\mathbf{P}$ is 1-Lipschitz.

Proof. By (7.2), the map $\mathbf{P}$ is continuous on each $\mathcal{T}\in T$ (for the relative topology of $\mathcal{T}$). Thus, it is continuous on $\mathcal{A}$ because, for each $\mathbf{q}\in\mathcal{A}$, there are at most six triangles $\mathcal{T}\in T$ with $\mathbf{q}\in\mathcal{T}$, and their union is a neighbourhood of $\mathbf{q}$. Formula (7.2) also implies (7.3) for each $\mathbf{q}\in\mathcal{T}$ with $\mathcal{T}\in T$, thus for each $\mathbf{q}\in\mathcal{A}$. Finally, suppose that $\mathcal{A}$ is convex and let $\mathbf{q},\mathbf{q}'\in\mathcal{A}$. Write $\mathbf{q}'-\mathbf{q}=(u_1,u_2)$. Then, for each $i=1,2,3$, the function $f_i\colon[0,1]\to\mathcal{A}$ given by $f_i(t)=P_i(\mathbf{q}+t(\mathbf{q}'-\mathbf{q}))$ is continuous and piecewise linear with slopes 0, u_1 or u_2, thus

$$|P_i(\mathbf{q}')-P_i(\mathbf{q})|=|f_i(1)-f_i(0)|\le\max\{|u_1|,|u_2|\}=\|\mathbf{q}'-\mathbf{q}\|,$$

and so $\|\mathbf{P}(\mathbf{q}')-\mathbf{P}(\mathbf{q})\|\le\|\mathbf{q}'-\mathbf{q}\|$. □

Lemma 7.4. *Let the notation and hypotheses be as in Lemma 7.3. Suppose that $\mathcal{T}\in T$ is a lower basic triangle, that $\mathcal{T}'\in T$ is an upper basic triangle, and that they share a common side $\mathcal{E}$. Let $(a,b,c)\in\mathbb{Z}^3$ with $c=a+b$ such that (7.2) holds, and let $(a',b',c')\in\mathbb{Z}^3$ with $c'=a'+b'$ such that*

$$\mathbf{P}(q_1,q_2)=\Phi(q_1-a',q_2-b',c')\quad\textit{for each }(q_1,q_2)\in\mathcal{T}'.$$

Suppose further that $(a',b',c')\neq(a,b,c)$. Then,

(i) *$(a',b',c')=(m-c,b,m-a)$ if $\mathcal{E}$ is contained in the vertical line $q_1=m$;*
(ii) *$(a',b',c')=(a,m-c,m-b)$ if $\mathcal{E}$ is contained in the horizontal line $q_2=m$;*
(ii) *$(a',b',c')=(b+m,a-m,c)$ if $\mathcal{E}$ is contained in the line $q_1-q_2=m$.*

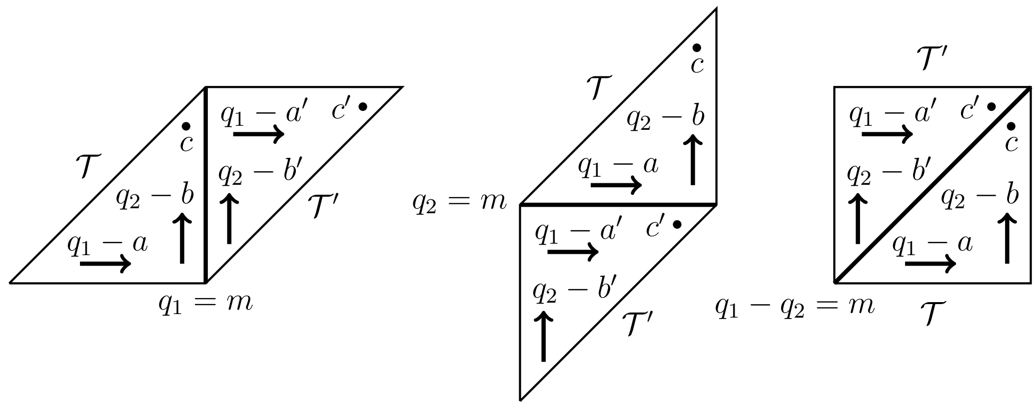

FIGURE 9. Basic triangles meeting in a common side

The three cases are illustrated in Figure 9. In each, the formula given for (a', b', c') follows from the hypotheses that $(a', b', c') \neq (a, b, c)$ and that $\Phi(q_1 - a, q_2 - b, c) = \Phi(q_1 - a', q_2 - b', c')$ for all $(q_1, q_2) \in \mathcal{E}$.

We propose the following notion.

Definition 7.5. An integral 2-parameter 3-system is a function $\mathbf{P}: \mathcal{A} \to \mathbb{R}^3$ as in Lemma 7.3, such that, under the hypotheses of Lemma 7.4, we have $c' > c$ in case (i), $b' < b$ in case (ii), and $a' < a$ in case (iii).

It is easily seen that, if $\mathbf{P}: \mathbb{R}^2 \to \mathbb{R}^3$ is an integral 2-parameter 3-system on $\mathbb{R}^2$, then the maps $q \mapsto P(q, 0)$ and $q \mapsto P(0, q)$ are both integral 3-systems in the sense of [5]. We conclude with the following result.

Proposition 7.6. *The map* $\mathbf{P}: \mathcal{A}(\epsilon) \to \mathbb{R}^3$ *of section 6 is an integral* 2*-parameter* 3*-system.*

Proof. Let the notation and hypotheses be as in Lemma 7.4, and let $\mathcal{R}, \mathcal{R}' \in S$ such that $\mathcal{T} \subseteq \mathcal{R}$ and $\mathcal{T}' \subseteq \mathcal{R}'$. Since $(a', b', c') \neq (a, b, c)$, Proposition 7.1 gives $\mathcal{R} \neq \mathcal{R}'$. So the common side $\mathcal{E}$ of $\mathcal{T}$ and $\mathcal{T}'$ is contained in a common side of $\mathcal{R}$ and $\mathcal{R}'$. If $\mathcal{R} = \mathrm{Trap}(\epsilon, w_1)$, we have $\mathcal{R}' = \mathrm{Trap}(w_1, w_2)$, $(a, b, c) = (0, 1, 1)$, $(a', b', c') = (1, 1, 2)$, and case (i) applies with $m = 2$. If $\mathcal{R} = \mathrm{Trap}(w_1, w_2)$, we have $\mathcal{R}' = \mathrm{Trap}(w_2, w_3)$, $(a, b, c) = (1, 1, 2)$, $(a', b', c') = (2, 1, 3)$, and case (i) applies with $m = 4$. In both instances, we note that $c' > c$, as needed. If $\mathcal{R}$ is not one of these trapezes, then, by Proposition 7.1, the functions $q_1 - a$, $q_2 - b$ and c coincide in an interior point of $\mathcal{R}$. So, in case (i), we have $q_1 - a = c$ for some $q_1 < m$, thus $c' = m - a > c$. In case (ii), we have $q_2 - b = c$ for some $q_2 > m$, thus $b' = m - c < b$. In case (iii), we have $q_1 - a = q_2 - b$ for some $(q_1, q_2) \in \mathcal{R}$ with $q_1 - q_2 > m$, thus $a - b > m$ and so $a' = b + m < a$. □

It would be interesting to know if, for any matrix $A \in \mathrm{GL}_3(\mathbb{R})$, there exists an integral 2-parameter 3-system $\mathbf{P}: \mathbb{R}^2 \to \mathbb{R}^3$ such that $\mathbf{L}_A(0, q_1, q_2) - \mathbf{P}(q_1, q_2)$ is a bounded function of $(q_1, q_2) \in \mathbb{R}^2$, and conversely if, for any integral 2-parameter 3-system $\mathbf{P}: \mathbb{R}^2 \to \mathbb{R}^3$, there exists $A \in \mathrm{GL}_3(\mathbb{R})$ with the same property.

8. Proof of Theorem 3.1

Fix an integer $k \geq 4$. Since $q_2(w_{k-1}) = q_2(w_{k-1}w_{k-3}) = 2F_{k-1}$, the conditions (3.14) on a point $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2$ amount to

$$\mathbf{q} \in \mathrm{Trap}(\epsilon, w_{k-2}) \cup \mathrm{Trap}(w_{k-2}, w_{k-1}) \cup \mathrm{Trap}(w_{k-1}, w_{k-1}w_{k-3}). \tag{8.1}$$

Fix such a point $\mathbf{q}$. We need to show that

$$\mathbf{P}(\mathbf{p}+\mathbf{q}) = \mathbf{r} + \mathbf{P}(\mathbf{q}) \quad \text{where} \quad \mathbf{p} = (4F_{k-2}, 2F_{k-2}) \text{ and } \mathbf{r} = (2F_{k-2}, 2F_{k-2}, 2F_{k-2}).$$

To this end, choose a trapeze $\mathcal{T}$ in the right hand side of (8.1) for which $\mathbf{q} \in \mathcal{T}$. By Proposition 6.17, there exists $\mathcal{R} \in S$ such that $\mathbf{q} \in \mathcal{R} \subseteq \mathcal{T}$. So, $\pi_1(\mathcal{R}) = [q_1(u), q_1(w)]$ for some $u < w$ in $[\epsilon, w_{k-1}w_{k-3}]$. Moreover, since $\mathcal{R} \subseteq \mathcal{T} \subseteq \mathrm{Layer}(k-2)$, Proposition 6.16 implies that $\mathcal{R} \in S_\ell$ for some $\ell \in \{1, \dots, k-3\}$. Let u' and w' be the prefixes of w_∞ with

$$|u'| = |u| + 2F_{k-2} \quad \text{and} \quad |w'| = |w| + 2F_{k-2}. \tag{8.2}$$

We claim that there exists $\mathcal{R}' \in S_\ell$ such that

$$\pi_1(\mathcal{R}') = [q_1(u'), q_1(w')] \quad \text{and} \quad \mathbf{p} + \mathcal{R} \subseteq \mathcal{R}'. \tag{8.3}$$

If we admit this, then $\mathbf{p} + \mathbf{q} \in \mathcal{R}'$ and, using (8.2) and Proposition 6.18, we find

$$\begin{aligned} \mathbf{P}(\mathbf{p}+\mathbf{q}) &= \Phi(q_1 + 4F_{k-2} - |u'|, q_2 + 2F_{k-2} + |u'| - |w'|, |w'|) \\ &= \Phi(q_1 - |u| + 2F_{k-2}, q_2 + |u| - |w| + 2F_{k-2}, |w| + 2F_{k-2}) = \mathbf{r} + \mathbf{P}(\mathbf{q}), \end{aligned}$$

as needed.

To prove the claim, we first note that, by Lemma 4.5, we have $|\alpha(u)| \leq |\alpha(u')|$ and $|\alpha(w)| \leq |\alpha(w')|$, thus $\mathbf{p} + \mathbf{q}(u) \in \mathcal{A}(u')$ and $\mathbf{p} + \mathbf{q}(w) \in \mathcal{C}(w')$, so

$$\mathbf{p} + \mathcal{A}(u) \subseteq \mathcal{A}(u') \quad \text{and} \quad \mathbf{p} + \mathcal{C}(w) \subseteq \mathcal{C}(w'). \tag{8.4}$$

If $\ell = 1$, the words $u < w$ are consecutive prefixes of w_∞, and we have $\mathcal{R} = \mathrm{Trap}(u, w)$. Then, by (8.2), the words $u' < w'$ are also consecutive prefixes of w_∞ and we may form $\mathcal{R}' = \mathrm{Trap}(u', w') \in S_1$. It satisfies (8.3) because by (8.4), we have

$$\mathbf{p} + \mathcal{R} = (\mathbf{p} + \mathcal{A}(u)) \cap (\mathbf{p} + \mathcal{C}(w)) \subseteq \mathcal{A}(u') \cap \mathcal{C}(w') = \mathcal{R}'.$$

Otherwise, we have $\ell \geq 2$ and, by Lemma 6.9, there exists $v \in \mathcal{V}_\ell \setminus \mathcal{V}_{\ell+1}$ such that $u < v < w$ are consecutive elements of $\mathcal{V}_\ell$, and $\mathcal{R} = \mathrm{Cell}(u, v, w)$. We note that $v \notin \mathcal{F}$ because the only element of $\mathcal{F}$ in $\mathcal{V}_\ell \setminus \mathcal{V}_{\ell+1}$ is w_ℓ and $v > u \geq w_\ell$. Since we also have $v < w \leq w_{k-1}w_{k-3}$, we deduce from Lemma 4.5 that the prefix v' of w_∞ with $|v'| = |v| + 2F_{k-2}$ satisfies $|\alpha(v)| = |\alpha(v')|$, thus $v' \in \mathcal{V}_\ell \setminus \mathcal{V}_{\ell+1}$ and $\mathbf{p} + \mathbf{q}(v) = \mathbf{q}(v')$, so

$$\mathbf{p} + \mathcal{B}(v) = \mathcal{B}(v'). \tag{8.5}$$

By Proposition 4.6, the words $u' < v' < w'$ are consecutive elements of $\mathcal{V}_\ell$, so we may form $\mathcal{R}' = \mathrm{Cell}(u', v', w') \in S_\ell$. On the basis of (8.4) and (8.5), we conclude as above that (8.3) holds.

9. Proof of Theorem 3.2

We begin by establishing three lemmas, the first two of which concern the morphism of monoids $\theta\colon E^* \to E^*$ defined in section 3.2.

Lemma 9.1. *For each $v \in [\epsilon, w_\infty[$, we have $\big||\theta(v)| - \gamma|v|\big| \le 2$.*

Proof. More precisely, we will show, by induction on $|v|$, that

$$\big||\theta(v)| - \gamma|v|\big| + \max\{1, |v|\}^{-1} \le 2 \tag{9.1}$$

for each $v \in [\epsilon, w_\infty[$. If $|v| = 0$, then $\epsilon = v = \theta(v)$, so $|\theta(v)| = 0$ and (9.1) holds. If $|v| = F_k$ for some positive integer k, then $v = w_k$ and $\theta(v) = w_{k+1}$, so $|\theta(v)| = F_{k+1}$. Since F_{k+1}/F_k is a convergent of γ in reduced form, we have

$$|F_{k+1} - \gamma F_k| \le 1/F_k.$$

Thus, the left hand side of (9.1) is at most $2/F_k \le 2$, and we are done.

We may therefore assume that $F_k < |v| < F_{k+1}$ for some $k \ge 3$, thus $v = w_k u$ with $u \in]\epsilon, w_{k-1}[$. Then, we have $\theta(v) = w_{k+1}\theta(u)$, and so

$$\big||\theta(v)| - \gamma|v|\big| \le |F_{k+1} - \gamma F_k| + \big||\theta(u)| - \gamma|u|\big|.$$

By induction, we may assume that $\big||\theta(u)| - \gamma|u|\big| \le 2 - 1/|u|$. So, we obtain

$$\big||\theta(v)| - \gamma|v|\big| + 1/|v| \le 1/F_k + 2 - 1/|u| + 1/|v| = 1/F_k + 2 - F_k/(|u|\,|v|)$$

because $|v| - |u| = F_k$. Since $|u|\,|v| < F_{k-1}F_{k+1} = F_k^2 \pm 1$, we also have $|u|\,|v| \le F_k^2$ and the desired estimate (9.1) follows. □

Lemma 9.2. *Let $v \in \mathcal{V}_4$ and let $v' = \theta(v)$. Then, $\|\mathbf{q}(v') - \gamma\mathbf{q}(v)\| \le 4$.*

Proof. Since $v \in \mathcal{V}_4$, Corollary 5.8 gives $\alpha(v') = \theta(\alpha(v))$. Thus, by Lemma 9.1,

$$\|\mathbf{q}(v') - \gamma\mathbf{q}(v)\| \le 2\max\big\{\big||v'| - \gamma|v|\big|, \big||\alpha(v')| - \gamma|\alpha(v)|\big|\big\} \le 4.$$ □

Lemma 9.3. *Let $\mathbf{q} = (q_1, q_2) \in \mathcal{A}(\epsilon)$ with $q_2 \le q_1/2$. Then, $\|\mathbf{P}(\mathbf{q}) - (q_2, q_1/2, q_1/2)\| \le 1$.*

Proof. Choose consecutive words $u < v$ in $[\epsilon, w_\infty[$ with $q_1(u) \le q_1 \le q_1(v)$. Then, $\mathbf{q}$ belongs to $\mathrm{Trap}(u, v)$ because

$$\min\{q_2(u) + q_1 - q_1(u), q_2(v)\} \ge \min\{q_1 - |u|, |v|\} \ge q_1/2 \ge q_2.$$

Thus, by Theorem 6.1, we have $\mathbf{P}(\mathbf{q}) = \Phi(\mathbf{r})$ where $\mathbf{r} = (q_1 - |u|, q_2 - 1, |v|)$. As Φ is 1-Lipschitz, this yields

$$\|\mathbf{P}(\mathbf{q}) - (q_2, q_1/2, q_1/2)\| = \|\Phi(\mathbf{r}) - \Phi(q_1/2, q_2, q_1/2)\| \le \|\mathbf{r} - (q_1/2, q_2, q_1/2)\| \le 1.$$ □

Proof of Theorem 3.2. Let $\mathbf{q} = (q_1, q_2) \in \mathcal{A}(\epsilon)$. We need to show that

$$\|\mathbf{P}(\gamma\mathbf{q}) - \gamma\mathbf{P}(\mathbf{q})\| \le 40.$$

If $q_2 \le q_1/2$, Lemma 9.3 applies to $\mathbf{q}$ and to $\gamma\mathbf{q}$. Thus, setting $\mathbf{r} = (q_2, q_1/2, q_1/2)$, we find

$$\|\mathbf{P}(\gamma\mathbf{q}) - \gamma\mathbf{P}(\mathbf{q})\| \le \|\mathbf{P}(\gamma\mathbf{q}) - \gamma\mathbf{r}\| + \gamma\|\mathbf{r} - \mathbf{P}(\mathbf{q})\| \le 1 + \gamma.$$

Suppose now that $q_2 > q_1/2$ and $\mathbf{q} \in \mathrm{Layer}(4)$. The above special case gives

$$\|\mathbf{P}(\gamma\mathbf{q}') - \gamma\mathbf{P}(\mathbf{q}')\| \le 1 + \gamma \quad \text{where} \quad \mathbf{q}' = (q_1, q_1/2). \tag{9.2}$$

Since $\mathbf{q} \in \mathrm{Layer}(4)$, we have $\mathbf{q} \in \mathrm{Trap}(u, v)$ for consecutive $u < v$ in $\{\epsilon\} \cup \mathcal{V}_4$, and then

$$2|u| \le q_1 \le 2|v| \quad \text{and} \quad q_2 \le \min\{q_1 - |u| + |\alpha(u)|, |v| + |\alpha(v)|\}.$$

If $u \neq \epsilon$, Corollary 4.8 gives $|v| - |u| \le F_3 = 3$, and Corollary 4.9 gives $\{u, v\} \not\subseteq \mathcal{V}_6$, thus $\min\{|\alpha(u)|, |\alpha(v)|\} \le F_5 = 8$, and so

$$q_2 - q_1/2 \le \max\{q_1/2 - |u|, |v| - q_1/2\} + 8 \le |v| - |u| + 8 \le 11.$$

If $u = \epsilon$, we have $v = w_4$, thus $q_2 - q_1/2 \le q_1/2 \le |v| = 5$. Hence, in all cases, we have $0 \le q_2 - q_1/2 \le 11$. As $\mathbf{P}$ is 1-Lipschitz by Proposition 7.1 and Lemma 7.3, we deduce that

$$\|\mathbf{P}(\mathbf{q}) - \mathbf{P}(\mathbf{q}')\| \le 11 \quad \text{and} \quad \|\mathbf{P}(\gamma\mathbf{q}) - \mathbf{P}(\gamma\mathbf{q}')\| \le 11\gamma.$$

Using (9.2), we conclude that $\|\mathbf{P}(\gamma\mathbf{q}) - \gamma\mathbf{P}(\mathbf{q})\| \le 1 + 23\gamma \le 40$, as claimed.

Finally, suppose that $\mathbf{q} \notin \mathrm{Layer}(4)$. By Proposition 6.10, there exist an integer $\ell \ge 4$ and a polygon $\mathcal{R} \in S_\ell$ such that $\mathbf{q} \in \mathcal{R}$. Let $u < v < w$ be consecutive elements of $\mathcal{V}_\ell$ with $v \notin \mathcal{V}_{\ell+1}$ such that $\mathcal{R} = \mathrm{Cell}(u, v, w)$, and set

$$u' = \theta(u), \quad v' = \theta(v) \quad \text{and} \quad w' = \theta(w).$$

By Proposition 5.7, the words $u' < v' < w'$ are consecutive elements of $\mathcal{V}_{\ell+1}$ with $v' \notin \mathcal{V}_{\ell+2}$. Thus, we may form $\mathcal{R}' = \mathrm{Cell}(u', v', w') \in S_{\ell+1}$. Using Lemma 9.2, we find

$$\begin{aligned}
\gamma\mathcal{A}(u) &= \gamma\mathbf{q}(u) - \mathbf{q}(u') + \mathcal{A}(u') \subseteq [-4, 4]^2 + \mathcal{A}(u'),\\
\gamma\mathcal{B}(v) &= \gamma\mathbf{q}(v) - \mathbf{q}(v') + \mathcal{B}(v') \subseteq [-4, 4]^2 + \mathcal{B}(v'),\\
\gamma\mathcal{C}(w) &= \gamma\mathbf{q}(w) - \mathbf{q}(w') + \mathcal{C}(w') \subseteq [-4, 4]^2 + \mathcal{C}(w').
\end{aligned}$$

Taking term by term intersections, we deduce that

$$\gamma\mathcal{R} \subseteq [-8, 8]^2 + \mathcal{R}'$$

because the sides of $\mathcal{A}(u')$, $\mathcal{B}(v')$ and $\mathcal{C}(w')$ are horizontal, vertical, or have slope 1. Thus, there exists $\mathbf{q}' = (q_1', q_2') \in \mathcal{R}'$ such that

$$\|\gamma\mathbf{q} - \mathbf{q}'\| \le 8. \tag{9.3}$$

As $\mathbf{P}$ is 1-Lipschitz, this implies that

$$\|\mathbf{P}(\gamma\mathbf{q}) - \mathbf{P}(\mathbf{q}')\| \le 8. \tag{9.4}$$

Since $\mathbf{q} \in \mathcal{R}$ and $\mathbf{q}' \in \mathcal{R}'$, Theorem 6.1 gives $\mathbf{P}(\mathbf{q}) = \Phi(\mathbf{r})$ and $\mathbf{P}(\mathbf{q}') = \Phi(\mathbf{r}')$ where

$$\mathbf{r} = (q_1 - |u|, q_2 + |u| - |w|, |w|) \quad \text{and} \quad \mathbf{r}' = (q_1' - |u'|, q_2' + |u'| - |w'|, |w'|).$$

As Φ is 1-Lipschitz and commute with scalar multiplication, we deduce that

$$\|\mathbf{P}(\mathbf{q}') - \gamma\mathbf{P}(\mathbf{q})\| \le \|\mathbf{r}' - \gamma\mathbf{r}\| \le 12,$$

where the second estimate uses (9.3) and Lemma 9.1. Combining this with (9.4), we conclude that $\|\mathbf{P}(\gamma\mathbf{q}) - \gamma\mathbf{P}(\mathbf{q})\| \le 20$.

10. A specific example

Using the terminology of section 3.6, we show below that the real number ξ given by (3.16) belongs to $\mathcal{E}_3^+$, by computing explicitly an associated Fibonacci sequence in $\mathrm{SL}_2(\mathbb{Z})$. Although this is not needed for the rest of the paper, it provides a concrete example of a number ξ to which our results apply, independently of [6]. We start by recalling some general facts from continued fraction theory.

Let $\sigma\colon (\mathbb{N}\setminus\{0\})^* \to \mathrm{GL}_2(\mathbb{Z})$ be the morphism of monoids such that

$$\sigma(a) = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for each } a \in \mathbb{N}\setminus\{0\}.$$

By [6, Corollary 4.2], its image is $\mathcal{S}\cup\{I\}$ where I is the identity of $\mathrm{GL}_2(\mathbb{Z})$ and

$$\mathcal{S} = \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})\,;\, a \ge \max\{b,c\} \text{ and } \min\{b,c\} \ge d \ge 0 \Big\}.$$

Moreover, its restriction $\sigma\colon (\mathbb{N}\setminus\{0\})^* \to \mathcal{S}\cup\{I\}$ is an isomorphism of monoids. It follows that, for each word $v \in (\mathbb{N}\setminus\{0\})^*$, the matrix $\sigma(v)$ is symmetric if and only if v is a palindrome.

Let $(a_i)_{i\ge1}$ be a sequence of positive integers and let $\xi \in\,]0,1[$ be the real number with continued fraction expansion

$$\xi = [0, a_1, a_2, a_3 \dots] = 1/(a_1 + 1/(a_2 + \cdots)).$$

For each integer $k \ge 1$, we have

$$\sigma(a_1\cdots a_k) = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} q_k & q_{k-1} \\ p_k & p_{k-1} \end{pmatrix}$$

where $p_k/q_k = [0, a_1, \dots, a_k]$ is the k-th convergent of ξ in reduced form (and $p_0/q_0 = 0$), thus

$$\|(\xi,-1)\sigma(a_1\cdots a_k)\| = |q_{k-1}\xi - p_{k-1}| \asymp q_k^{-1} = \|\sigma(a_1\cdots a_k)\|^{-1} \tag{10.1}$$

with absolute implied constants (see [10, Chapter I]).

Proposition 10.1. *Let $\xi = [0, \mathbf{1}, f_{\mathbf{2},\mathbf{1}}]$ where $\mathbf{1} = (1,1)$ and $\mathbf{2} = (2,2)$. Then ξ belongs to $\mathcal{E}_3^+$ with associated Fibonacci sequence $(W_i)_{i\ge0}$ in $\mathrm{SL}_2(\mathbb{Z})$ starting with*

$$W_0 = \sigma(\mathbf{1}) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad W_1 = \sigma(\mathbf{1})\sigma(\mathbf{2})\sigma(\mathbf{1})^{-1} = \begin{pmatrix} 7 & -2 \\ 4 & -1 \end{pmatrix}. \tag{10.2}$$

Moreover, let $E = \{a,b\}$ be an alphabet on two letters, let $w_\infty = f_{a,b}$, and let $\varphi\colon E^* \to \mathrm{SL}_2(\mathbb{Z})$ be the morphism of monoids such that $\varphi(a) = W_1$ and $\varphi(b) = W_0$. Then, for each $v \in [\epsilon, w_\infty[$, we have

$$\|(\xi,-1)\varphi(v)\| \asymp \|\varphi(v)\|^{-1}. \tag{10.3}$$

In (10.3) and in the proof below, all implied constants are explicitly computable numbers.

Proof. We first note that, for each $v \in E^*$, we have

$$\varphi(v) = \sigma(\mathbf{1})\sigma(\tau(v))\sigma(\mathbf{1})^{-1} \tag{10.4}$$

where $\tau\colon E^* \to (\mathbb{N}\setminus\{0\})^*$ is the morphism of monoids such that $\tau(a) = \mathbf{2}$ and $\tau(b) = \mathbf{1}$. Indeed, both sides of this equality define morphisms from E^* to $\mathrm{SL}_2(\mathbb{Z})$ which, in view of (10.2) agree for $v = a$ and $v = b$. We also note that $W_i = \varphi(w_i)$ for each $i \geq 0$, where $(w_i)_{i\geq 0}$ is the Fibonacci sequence in E^* with $w_0 = b$ and $w_1 = a$. Thus, (10.4) yields

$$W_i = \sigma(\mathbf{1})\sigma(\tau(w_i))\sigma(\mathbf{1})^{-1} \quad (i \geq 0). \tag{10.5}$$

For $v \in [\epsilon, w_\infty[$, the word $(\mathbf{1}, \tau(v))$ is a prefix of $(\mathbf{1}, f_{\mathbf{2},\mathbf{1}})$ and so, using (10.4), the general estimate (10.1) yields

$$\|(\xi,-1)\varphi(v)\| \asymp \|(\xi,-1)\sigma(\mathbf{1},\tau(v))\| \asymp \|\sigma(\mathbf{1},\tau(v))\|^{-1} \asymp \|\varphi(v)\|^{-1},$$

which proves (10.3). For the choice of $v = w_i$, this gives $\|(\xi,-1)W_i\| \asymp \|W_i\|^{-1}$ for each $i \geq 0$. Thus, the sequence $(W_i)_{i\geq 0}$ fulfils condition (E3) from section 3.6.

By (10.5), we have $\|W_i\| \asymp \|\sigma(\tau(w_i))\|$ for each $i \geq 0$. Since $\|A\|\,\|B\| \leq \|AB\| \leq 2\|A\|\,\|B\|$ for any $A, B \in \mathcal{S}$ and since $\sigma(\tau(w_i)) \in \mathcal{S}$ for each $i \geq 1$, we deduce that

$$\begin{aligned}\|W_{i+2}\| \asymp \|\sigma(\tau(w_{i+2}))\| &= \|\sigma(\tau(w_{i+1}))\sigma(\tau(w_i))\| \\ &\asymp \|\sigma(\tau(w_{i+1}))\|\|\sigma(\tau(w_i))\| \asymp \|W_{i+1}\|\|W_i\|\end{aligned}$$

for each $i \geq 0$. Thus, $(W_i)_{i\geq 0}$ is unbounded and fulfils condition (E2).

Finally, we note that

$$\sigma(\mathbf{2}) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = \sigma(\mathbf{1})M = {}^tM\sigma(\mathbf{1}) \quad \text{where} \quad M = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}. \tag{10.6}$$

For each $i \geq 0$, let $\mathbf{x}_i = W_i M_i^{-1}$ where M_i is defined as in condition (E1) for the above choice of M. For $i \geq 2$, we claim that

$$\mathbf{x}_i = \sigma(\mathbf{1}, \tau(w_i^{**}), \mathbf{1}). \tag{10.7}$$

Indeed, if i is even, w_i ends in ab. Then, using (10.5) and (10.6), we find that

$$\mathbf{x}_i = \sigma(\mathbf{1})\sigma(\tau(w_i^{**}))\sigma(\mathbf{2})M^{-1} = \sigma(\mathbf{1})\sigma(\tau(w_i^{**}))\sigma(\mathbf{1}).$$

Otherwise, i is odd, w_i ends in ba, and we find similarly that

$$\mathbf{x}_i = \sigma(\mathbf{1})\sigma(\tau(w_i^{**}))\sigma(\mathbf{1})\sigma(\mathbf{2})\sigma(\mathbf{1})^{-1}({}^tM)^{-1} = \sigma(\mathbf{1})\sigma(\tau(w_i^{**}))\sigma(\mathbf{1}).$$

Moreover, w_i^{**} is a palindrome in E^* by Lemma 5.2. Since $\tau(a) = \mathbf{2}$ and $\tau(b) = \mathbf{1}$ are palindromes in $(\mathbb{N}\setminus\{0\})^*$, we deduce that $(\mathbf{1}, \tau(w_i^{**}), \mathbf{1})$ is a palindrome in $(\mathbb{N}\setminus\{0\})^*$. So, by (10.7), the matrix $\mathbf{x}_i$ is symmetric for $i \geq 2$. A direct computation shows that $\mathbf{x}_0$ and $\mathbf{x}_1$ are also symmetric. Thus, (E1) is fulfilled as well and so $\xi \in \mathcal{E}_3^+$. ☐

A Markoff triple is a solution in positive integers $\mathbf{m} = (m, m_1, m_2)$ of the Markoff equation $m^2 + m_1^2 + m_2^2 = 3mm_1m_2$, up to permutation. Theorem 3.6 of [7] provides an explicit bijection $\mathbf{m} \mapsto \xi_{\mathbf{m}}$ from the set of all Markoff triples $\mathbf{m}$ with $\mathbf{m} \neq (1,1,1)$ to the set $\mathcal{E}_3^+ \cup \,]1/2, 1[$. It can be shown that the number ξ of Proposition 10.1 is $\xi_{\mathbf{m}}$ for $\mathbf{m} = (2,1,1)$. Thus, by [7, Corollary 5.10], its Lagrange constant is $1/3$, the largest possible value for an irrational non-quadratic real number.

11. Preliminary estimates

It follows from [6, Theorem 2.1] that any Fibonacci sequence $(W_i)_{i\ge1}$ in $\mathrm{GL}_2(\mathbb{Z})$ associated to an extremal number ξ of $\mathrm{GL}_2(\mathbb{Z})$-type satisfies $\|W_{i+1}\| \asymp \|W_i\|^\gamma$. The goal of this section is to prove the following result which provides a sharper estimate as well as additional properties of this sequence and of the corresponding sequence of symmetric matrices $(\mathbf{x}_i)_{i\ge1}$.

Proposition 11.1. *Let ξ be an extremal real number of $\mathrm{GL}_2(\mathbb{Z})$-type. Fix an unbounded Fibonacci sequence $(W_i)_{i\ge1}$ in $\mathrm{GL}_2(\mathbb{Z})$, a matrix $M\in\mathrm{GL}_2(\mathbb{Q})$ and a sequence of symmetric matrices $(\mathbf{x}_i)_{i\ge1}$ in $\mathrm{GL}_2(\mathbb{Q})$ satisfying conditions (E1)–(E3) from section 3.6. Set*

$$t_i = \mathrm{trace}(W_i) \quad \text{and} \quad d_i = \det(W_i) \tag{11.1}$$

for each integer $i\ge1$. Set also

$$J=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad \Xi=\begin{pmatrix}1&\xi\\ \xi&\xi^2\end{pmatrix} \quad \text{and} \quad \theta_0=\mathrm{trace}(\Xi M). \tag{11.2}$$

Then we have $\theta_0\neq0$ and, for each $i\ge1$,

(i) $\mathbf{x}_{i+2}=W_{i+1}\mathbf{x}_i=W_i\mathbf{x}_{i+1}$,

(ii) $\mathbf{x}_{i+3}=t_{i+1}\mathbf{x}_{i+2}-d_{i+1}\mathbf{x}_i$,

(iii) $t_{i+3}=t_{i+1}t_{i+2}-d_{i+1}t_i$,

(iv) $\det(\mathbf{x}_i,\mathbf{x}_{i+1},\mathbf{x}_{i+2})=\pm\det(M)^{-2}\mathrm{trace}(MJ)\neq0$.

Moreover, there exists $\rho>0$ and $i_0\ge1$ such that, for each $i\ge i_0$, we have $t_i\neq0$,

$$\log|t_i|=\rho F_i+\mathcal{O}(\gamma^{-i}) \quad \text{and} \quad W_i=\theta_0^{-1}t_i\Xi M_i+\mathcal{O}(|t_i|^{-1}). \tag{11.3}$$

The main novelty is the first estimate in (11.3). In formula (iv), we identify each

$$\mathbf{x}_k=\begin{pmatrix}x_{k,0}&x_{k,1}\\x_{k,1}&x_{k,2}\end{pmatrix}\in\mathrm{GL}_2(\mathbb{Q})$$

with the triple $\mathbf{x}_k=(x_{k,0},x_{k,1},x_{k,2})\in\mathbb{Q}^3$. Then, $\det(\mathbf{x}_i,\mathbf{x}_{i+1},\mathbf{x}_{i+2})$ represents the determinant of the 3×3 matrix whose rows are the triples $\mathbf{x}_i$, $\mathbf{x}_{i+1}$ and $\mathbf{x}_{i+2}$.

Proof of Proposition 11.1. For each integer $i\ge1$, we find

$$\mathbf{x}_{i+2}=W_{i+2}M_i^{-1}=W_{i+1}\mathbf{x}_i=\mathbf{x}_{i+1}M_{i+1}\mathbf{x}_i.$$

Taking the transpose, we deduce that $\mathbf{x}_{i+2}=\mathbf{x}_iM_i\mathbf{x}_{i+1}=W_i\mathbf{x}_{i+1}$. This proves (i) and yields

$$\mathbf{x}_{i+3}=W_{i+1}\mathbf{x}_{i+2}=W_{i+1}^2\mathbf{x}_i \quad (i\ge1).$$

Moreover, the Cayley-Hamilton theorem gives $W_{i+1}^2=t_{i+1}W_{i+1}-d_{i+1}I$, where I is the identity of $\mathrm{GL}_2(\mathbb{Z})$. Substituting this into the formula for $\mathbf{x}_{i+3}$, we obtain (ii).

We also note that, for each $k\ge1$, the matrices $\mathbf{x}_kM$, and $\mathbf{x}_k\,{}^tM={}^t(M\mathbf{x}_k)$ have the same trace t_k. Thus multiplying both sides of (ii) on the right by M and taking traces yields (iii).

By Formula (2.1) in [4], we have

$$\det(\mathbf{x}_i,\mathbf{x}_{i+1},\mathbf{x}_{i+2})=\mathrm{trace}(J\mathbf{x}_iJ\mathbf{x}_{i+2}J\mathbf{x}_{i+1}).$$

Using $\mathbf{x}_{i+2} = \mathbf{x}_i M_i \mathbf{x}_{i+1}$ and $\mathbf{x}_k J \mathbf{x}_k = \det(\mathbf{x}_k) J = \pm \det(M)^{-1} J$ for each $k \geq 1$, we deduce that

$$\det(\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}) = \mathrm{trace}(J\mathbf{x}_i J \mathbf{x}_i M_i \mathbf{x}_{i+1} J \mathbf{x}_{i+1}) = \pm \det(M)^{-2} \mathrm{trace}(M_i J),$$

which implies (iv) upon noting that $\mathrm{trace}({}^t\!MJ) = -\mathrm{trace}(MJ) \neq 0$ since ${}^t\!M \neq M$.

Let $c \geq 1$ be a constant for which (E2) and (E3) hold. We have

$$c^{-1}\|W_{i+2}\| \geq (c^{-1}\|W_{i+1}\|)(c^{-1}\|W_i\|) \quad (i \geq 1). \tag{11.4}$$

Since $(W_i)_{i\geq 1}$ is unbounded, there exists an index $k \geq 2$ such that $\|W_k\| \geq ec^2$. As we have $\|W_{k-1}\| \geq 1$, applying (11.4) with $i = k-1$ and $i = k$ yields $c^{-1}\|W_{k+2}\| \geq c^{-1}\|W_{k+1}\| \geq e$. Then, by induction on i, we obtain

$$c^{-1}\|W_{k+i+1}\| \geq c^{-1}\|W_{k+i}\| \geq \exp(F_{i-1}) \quad (i \geq 1). \tag{11.5}$$

Since $\|(\xi, -1)W_i\| \leq c\|W_i\|^{-1}$, we have $\|(\xi, -1)\mathbf{x}_i\| \ll \|W_i\|^{-1}$ and so,

$$\mathbf{x}_i = x_{i,0}\Xi + \mathcal{O}(\|W_i\|^{-1}). \tag{11.6}$$

As the entries of $\mathbf{x}_i$ are rational numbers with a common denominator $d \geq 1$ independent of i, this implies that $x_{i,0} \neq 0$ for each large enough i and that

$$t_i = \mathrm{trace}(\mathbf{x}_i M) = x_{i,0}\theta_0 + \mathcal{O}(\|W_i\|^{-1}). \tag{11.7}$$

If $\theta_0 = 0$, this implies that $\mathrm{trace}(\mathbf{x}_i M) = 0$ for each large enough i. However, it follows from (iv) that, for each $i \geq 1$, the matrices $\mathbf{x}_i$, $\mathbf{x}_{i+1}$ and $\mathbf{x}_{i+2}$ span the vector space of 2×2 symmetric matrices $\mathbf{x}$ with coefficients in $\mathbb{Q}$. Thus, $\mathrm{trace}(\mathbf{x}M) = 0$ for all those $\mathbf{x}$ and so ${}^t\!M = -M$, against the hypothesis. We conclude that $\theta_0 \neq 0$.

Since $\theta_0 \neq 0$, we deduce from (11.5), (11.6) and (11.7) that there exists an integer $i_0 > k$ such that, for each $i \geq i_0$, both t_i and $x_{i,0}$ are non-zero with

$$\|W_i\| \asymp \|\mathbf{x}_i\| \asymp |x_{i,0}| \asymp |t_i|, \tag{11.8}$$

and the second estimate in (11.3) follows.

To prove the first estimate in (11.3), we note that, for $i \geq i_0$, we have

$$|t_{i+2}| \gg |t_{i+1}| \gg |t_i| \gg \exp(F_{i-1-k}) \gg \exp(i)$$

by (11.5) and (11.8). Then, the recurrence relation (iii) yields

$$\left| \frac{t_{i+3}}{t_{i+1}t_{i+2}} - 1 \right| = \left| \frac{t_i}{t_{i+1}t_{i+2}} \right| \ll \frac{1}{|t_{i+2}|} \ll \exp(-i),$$

from which we deduce that

$$\left| \log \left| \frac{t_{i+3}}{t_{i+1}t_{i+2}} \right| \right| \ll \exp(-i) \leq \gamma^{-2i}. \tag{11.9}$$

For each $i \geq i_0$, we set

$$\rho_i = \gamma^{-i} \log|t_i| \quad \text{and} \quad \delta_i = \rho_{i+2} - \rho_{i+1}.$$

With this notation, we find that

$$\gamma^{-i-3} \log \left| \frac{t_{i+3}}{t_{i+1}t_{i+2}} \right| = \rho_{i+3} - \gamma^{-1}\rho_{i+2} - \gamma^{-2}\rho_{i+1} = \delta_{i+1} + \gamma^{-2}\delta_i.$$

Thus, for $i \geq i_0$, (11.9) translates into $\left|\delta_{i+1} + \gamma^{-2}\delta_i\right| \leq c_1\gamma^{-3i-3}$ for a constant $c_1 > 0$ which is independent of i, and so

$$\gamma^{2(i+1)}|\delta_{i+1}| \leq \gamma^{2i}|\delta_i| + c_1\gamma^{-i-1} \quad (i \geq i_0).$$

Thus, there is a constant $c_2 > 0$ such that $\gamma^{2i}|\delta_i| \leq c_2$ for each $i \geq i_0$. Hence, $(\rho_i)_{i \geq i_0}$ is a Cauchy sequence in $\mathbb{R}$ with

$$|\rho_i - \rho_{i+1}| = |\delta_{i-1}| \leq c_2\gamma^{-2i+2} \quad (i \geq i_0 + 1).$$

So, it converges to a real number ρ_∞ with $|\rho_i - \rho_\infty| \ll \gamma^{-2i}$ for $i \geq i_0 + 1$, and then

$$\log|t_i| = \rho_\infty\gamma^i + \mathcal{O}(\gamma^{-i}) \quad (i \geq i_0 + 1).$$

By Binet's formula (3.1), we also have $\gamma^i = \sqrt{5}\gamma^{-1}F_i + \mathcal{O}(\gamma^{-i})$ for $i \geq 0$. Substituting this into the previous estimate yields the first part of (11.3) with $\rho = \sqrt{5}\gamma^{-1}\rho_\infty$. □

12. The first two coordinates of the approximation points

In this section, we provide estimates for $|x_0(v)|$ and $|x_0(v)\xi - x_1(v)|$ for the points $\mathbf{x}(v)$ with $v \in]\epsilon, w_\infty[$ attached to a number $\xi \in \mathcal{E}_m$ with $m \geq 1$, as defined in section 3.7. We first establish a general result which applies to any extremal number of $\mathrm{GL}_2(\mathbb{Z})$-type.

Proposition 12.1. *Let the notation be as in Proposition 11.1, let $E = \{a, b\}$ be an alphabet of two letters, and let $\varphi\colon E^* \to \mathrm{GL}_2(\mathbb{Z})$ be the morphism of monoids such that $\varphi(a) = W_1$ and $\varphi(b) = W_0 := W_1^{-1}W_2$. For each $v \in]\epsilon, w_\infty[$, we have*

(i) $\|\varphi(v)\| \asymp \exp(\rho|v|)$,

(ii) $\|(\xi, -1)\varphi(v)\| \asymp \|\varphi(v)\|^{-1}$.

Proof. Part (ii) follows from [6, Theorem 2.3] since $|\det\varphi(v)| = 1$ for each v. It can also be deduced, with additional work, from the proof of [6, Proposition 4.3]. An independent argument is provided by Proposition 10.1 for the number ξ given by (3.16).

To prove part (i), we proceed as in the proof of [6, Lemma 5.2]. Using (11.3) in Proposition 11.1, we first note that, for each $i \geq 1$, we have

$$\exp(-\rho|w_i|)\varphi(w_i) = \exp(-\rho F_i)W_i = A_i + R_i \tag{12.1}$$

where $\|R_i\| \leq c_1\gamma^{-i}$ for a constant $c_1 \geq 1$, and where $A_i = \pm\theta_0^{-1}\Xi M_i$ belongs to

$$\mathcal{A} = \left\{ \pm I, \pm\theta_0^{-1}\Xi M, \pm\theta_0^{-1}\Xi\, {}^tM \right\}.$$

Since $\Xi M \Xi = \Xi\, {}^tM \Xi = \theta_0 \Xi$, the set $\mathcal{A}$ is stable under multiplication. Choose $c_2 \geq 1$ such that $c_2^{-1} \leq \|A\| \leq c_2$ for each $A \in \mathcal{A}$, and choose an integer ℓ such that $\gamma^{\ell-1} \geq 16c_1c_2^3$.

Any $v \in]w_\ell, w_\infty[$ can be written as a product

$$v = w_{i_1} \cdots w_{i_s} u$$

for a decreasing sequence of integers $i_1 > \cdots > i_s$ with $i_s > \ell$, and some $u \in [\epsilon, w_\ell]$. Set $w = w_{i_1} \cdots w_{i_s}$. Then, using (12.1), we obtain

$$\exp(-\rho|w|)\varphi(w) = (A_{i_1} + R_{i_1}) \cdots (A_{i_s} + R_{i_s}) = A + R$$

where $A = A_{i_1} \cdots A_{i_s} \in \mathcal{A}$ and where R is a sum, indexed by the non-empty subsequences $(j_1, \dots, j_t)$ of $(i_1, \dots, i_s)$, of products of the form $B_1 R_{j_1} \cdots B_t R_{j_t} B_{t+1}$ with $B_1, \dots, B_{t+1} \in \mathcal{A}$. As the norm of such a product is at most $c_3^t \| R_{j_1} \| \cdots \| R_{j_t} \|$ with $c_3 = 4c_2^2$, we find

$$\| R \| \le \prod_{k=1}^{s} (1 + c_3 \| R_{i_k} \|) - 1 \le \exp\Big(c_3 \sum_{i=\ell+1}^{\infty} \| R_i \| \Big) - 1 \le \exp\big(c_1 c_3 \gamma^{-\ell+1} \big) - 1.$$

Since $c_1 c_3 \gamma^{-\ell+1} \le (4c_2)^{-1} \le 1/2$, this gives $\| R \| \le (2c_2)^{-1} \le \| A \|/2$, thus

$$\| \exp(-\rho|v|)\varphi(v) \| \asymp \| \exp(-\rho|w|)\varphi(w) \| = \| A + R \| \asymp \| A \| \asymp 1,$$

and so $\| \varphi(v) \| \asymp \exp(\rho|v|)$. This last estimate also holds if $v \in]\epsilon, w_\ell]$. □

Corollary 12.2. *Let $\xi \in \mathcal{E}_m$ for some integer $m \ge 1$. There exists $\rho > 0$ such that, for each $v \in]\epsilon, w_\infty[$, the point $\mathbf{x}(v)$ defined in section 3.7 satisfies*

$$\max\{1, |x_0(v)|\} \asymp \exp(\rho|v|) \quad \textit{and} \quad |x_0(v)\xi - x_1(v)| \asymp \exp(-\rho|v|).$$

Proof. Let $v \in]\epsilon, w_\infty[$. Since $\mathbf{x}(v)$ has the same first column as $\varphi(v)M(v)^{-1} \in \mathrm{GL}_2(\mathbb{Z})$, we have $(x_0(v), x_1(v)) \ne (0, 0)$ and the proposition yields

$$\max\{1, |x_0(v)|\} \le \| \varphi(v)M(v)^{-1} \| \asymp \exp(\rho|v|),$$
$$|x_0(v)\xi - x_1(v)| \le \| (\xi, -1)\varphi(v)M(v)^{-1} \| \asymp \exp(-\rho|v|).$$

As ξ is badly approximable, we also have $\max\{1, |x_0(v)|\}|x_0(v)\xi - x_1(v)| \gg 1$, and the conclusion follows. □

We take this opportunity to fill a small gap in the proof of [6, Proposition 4.3], itself a crucial step towards [6, Theorem 2.2]. The argument there involves an unspecified real number c_i whose absolute value is tacitly assumed to be bounded away from 0 when i is large enough. To show that this is indeed the case, one notes that, in the notation of the proof, we have $c_i = c^2 y_{i,2}$ where $c \ne 0$ is independent of i and defined by the condition $(1/\xi, 1)^t U_k^{-1} = c(r, 1)$. Since $y_{i,2}$ is a non-zero integer for each large enough i, we conclude that $|c_i| \ge c^2$ for all those i.

13. Recurrence relations

Let the notation and hypotheses be as in section 3.7. In particular, ξ is a fixed number in $\mathcal{E}_m$ for some integer $m \ge 1$, and both Proposition 11.1 and Corollary 12.2 apply. In this section, we complement Corollary 12.2 by estimating $\| (\xi, -1)\mathbf{x}(v) \|$ from above for each $v \in \mathcal{V}_\ell$ with $\ell \ge 4$ large enough. We also prove Theorem 3.4 (ii).

We say that two symmetric matrices A and B in $\mathrm{Mat}_{2\times 2}(\mathbb{Z})$ are *equivalent*, and we write $A \equiv B$, if they have the same first column. With this notation, we recall that, for each non-empty word $v \in E^*$, the matrix $\mathbf{x}(v)$ is the unique symmetric matrix in $\mathrm{Mat}_{2\times 2}(\mathbb{Z})$ which satisfies

$$\mathbf{x}(v) \equiv \varphi(v)M(v)^{-1} \quad \text{and} \quad |x_1(v)\xi - x_2(v)| < 1/2. \tag{13.1}$$

To estimate $\| (\xi, -1)\mathbf{x}(v) \|$, we will first construct recursively, for each $v \in \mathcal{V}_4$, a symmetric matrix $\mathbf{x}^{\mathrm{alg}}(v)$ in $\mathrm{Mat}_{2\times 2}(\mathbb{Z})$ with $\mathbf{x}^{\mathrm{alg}}(v) \equiv \mathbf{x}(v)$. Then, we will estimate $\| (\xi, -1)\mathbf{x}^{\mathrm{alg}}(v) \|$ from above and show that, for each $v \in \mathcal{V}_\ell$ with $\ell \ge 4$ large enough, this norm is less than

$1/2$, thus $\mathbf{x}^{\mathrm{alg}}(v) = \mathbf{x}(v)$. As a consequence, for each triple of consecutive elements $u < v < w$ of $\mathcal{V}_\ell$ with ℓ large enough, we will derive an explicit relation of linear dependence between $\mathbf{x}(u)$, $\mathbf{x}(v)$ and $\mathbf{x}(w)$ when $v \in \mathcal{V}_{\ell+1}$, while we will show that the determinant of these three points is ± 2 when $v \notin \mathcal{V}_{\ell+1}$, thereby proving Theorem 3.4 (ii).

We start with three simple lemmas.

Lemma 13.1. *We have* $\mathbf{x}(vw_i) = \mathbf{x}(v\widetilde{w}_i)$ *for each* $v \in E^*$ *and each* $i \geq 2$.

Proof. Let $u \in E^*$, and let $U = \varphi(u)$. By definition, we have

$$\mathbf{x}(uab) \equiv UW_1W_0M^{-1} = UW_2M_2^{-1} = U\mathbf{x}_2.$$

Proposition 11.1 (i) also gives $W_1\mathbf{x}_2 = W_2\mathbf{x}_1$, thus $\mathbf{x}_2 = W_0\mathbf{x}_1$ and so we find that

$$\mathbf{x}(uba) \equiv UW_0W_1\,{}^tM^{-1} = UW_0\mathbf{x}_1 = U\mathbf{x}_2.$$

This shows that $\mathbf{x}(uab) \equiv \mathbf{x}(uba)$ and so $\mathbf{x}(uab) = \mathbf{x}(uba)$. Applying this to $u = vw_i^{**}$ for an integer $i \geq 2$ and a word $v \in E^*$, this yields $\mathbf{x}(vw_i) = \mathbf{x}(v\widetilde{w}_i)$. □

Lemma 13.2. *For each integer* $i \geq 2$*, the matrices* $\varphi(\widetilde{w}_i)$ *and* $\varphi(w_i)$ *have the same characteristic polynomial.*

Proof. For $i \geq 3$, Lemma 5.2 gives $w_i = w_{i-2}\widetilde{w}_{i-1}$ which implies that $\widetilde{w}_i = w_{i-2}w_{i-1}$. Since we also have $w_i = w_{i-1}w_{i-2}$, we deduce that

$$\varphi(\widetilde{w}_i) = W_{i-2}W_{i-1} = W_{i-2}\varphi(w_i)W_{i-2}^{-1}.$$

The last formulas remain true for $i = 2$. Thus, $\varphi(\widetilde{w}_i)$ and $\varphi(w_i)$ are conjugate matrices, and so their characteristic polynomials are the same. □

The third lemma below relies on the specific form of the matrix M, given by (3.15).

Lemma 13.3. *For any non-empty words* $u, v \in E^*$ *we have*

$$\varphi(u)M(v)^{-1} \equiv \begin{cases} \mathbf{x}(u) & \text{if } M(u) = M(v), \\ -\mathbf{x}(u) & \text{otherwise.} \end{cases}$$

Proof. This follows from the definition if $M(u) = M(v)$. Otherwise, we have $M(v) = {}^tM(u)$. Since

$$M^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & m \end{pmatrix} \equiv \begin{pmatrix} 0 & -1 \\ 1 & -m \end{pmatrix} = -{}^tM^{-1},$$

we deduce that $M(v)^{-1} \equiv -M(u)^{-1}$, thus $\varphi(u)M(v)^{-1} \equiv -\varphi(u)M(u)^{-1} \equiv -\mathbf{x}(u)$. □

The next result is the key to our analysis.

Proposition 13.4. *Let* $\ell \geq 4$ *be an integer and let* $u < v < w$ *be consecutive words in* $\mathcal{V}_\ell$ *with* $v \in \mathcal{V}_{\ell+1}$. *Then, we have* $|w| - |v| = |v| - |u| = F_i$ *for some* $i \in \{\ell - 2, \ell - 1\}$, *and*

$$\mathbf{x}(w) \equiv \begin{cases} t_i\mathbf{x}(v) - d_i\mathbf{x}(u) & \text{if } M(u) = M(v), \\ t_i\mathbf{x}(v) + d_i\mathbf{x}(u) & \text{otherwise,} \end{cases} \tag{13.2}$$

where $t_i = \mathrm{trace}(W_i)$ *and* $d_i = \det(W_i)$.

Proof. The fact that $|w| - |v| = |v| - |u| = F_i$ for some $i \in \{\ell-2, \ell-1\}$ follows from Corollary 4.8 and Proposition 4.10. According to Proposition 5.5, this implies that $v = us$ and $w = vs'$ for some $s, s' \in \{w_i, \widetilde{w}_i\}$. By Lemma 13.1, we have $\mathbf{x}(vs) = \mathbf{x}(vs')$, thus

$$\mathbf{x}(w) = \mathbf{x}(vs) = \mathbf{x}(us^2).$$

As $M(us^2) = M(us) = M(v)$, we deduce that

$$\mathbf{x}(w) \equiv \varphi(us^2)M(v)^{-1} = \varphi(u)\varphi(s)^2 M(v)^{-1}.$$

Since $s \in \{w_i, \widetilde{w}_i\}$ and $i \geq 2$, Lemma 13.2 shows that $\varphi(s)$ has the same characteristic polynomial as $\varphi(w_i) = W_i$, and so the Cayley-Hamilton theorem gives

$$\varphi(s)^2 = t_i\varphi(s) - d_i I$$

where I denotes the 2×2 identity matrix. Altogether, this yields

$$\mathbf{x}(w) \equiv \varphi(u)\big(t_i\varphi(s) - d_i I\big)M(v)^{-1} = t_i\varphi(v)M(v)^{-1} - d_i\varphi(u)M(v)^{-1},$$

and (13.2) follows using Lemma 13.3. □

We will prove below that the congruence (13.2) is in fact an equality when ℓ is large enough. To show this, we first construct "algebraic" points $\mathbf{x}^{\mathrm{alg}}(w)$ for each $w \in \mathcal{V}_4$.

Corollary 13.5. *The following recurrence process constructs, for each $w \in \mathcal{V}_4$, a symmetric matrix $\mathbf{x}^{\mathrm{alg}}(w)$ in $\mathrm{Mat}_{2\times 2}(\mathbb{Z})$ with $\mathbf{x}^{\mathrm{alg}}(w) \equiv \mathbf{x}(w)$.*

(i) *If $w \in \mathcal{F}$, then $w = w_i$ for some $i \geq 4$, and we set $\mathbf{x}^{\mathrm{alg}}(w) = \mathbf{x}_i$.*

(ii) *If $w \notin \mathcal{F}$, then $\alpha(w) = w_\ell$ for some integer $\ell \geq 4$, and w is at least the third element of $\mathcal{V}_\ell$. Thus, we can find $u, v \in \mathcal{V}_\ell$ such that $u < v < w$ are consecutive elements of $\mathcal{V}_\ell$. Since $w \notin \mathcal{V}_{\ell+1}$, Corollary 4.9 implies that $v \in \mathcal{V}_{\ell+1}$ and so, by Proposition 13.4, we have $|w| - |v| = |v| - |u| = F_i$ for some $i \in \{\ell-2, \ell-1\}$. Then, we define*

$$\mathbf{x}^{\mathrm{alg}}(w) = \begin{cases} t_i\mathbf{x}^{\mathrm{alg}}(v) - d_i\mathbf{x}^{\mathrm{alg}}(u) & \text{if } M(u) = M(v), \\ t_i\mathbf{x}^{\mathrm{alg}}(v) + d_i\mathbf{x}^{\mathrm{alg}}(u) & \text{otherwise.} \end{cases} \tag{13.3}$$

Proof. In case (i), this is because $\mathbf{x}_i = W_iM_i^{-1} = \varphi(w_i)M(w_i)^{-1} \in \mathrm{GL}_2(\mathbb{Z})$ is symmetric and so $\mathbf{x}_i \equiv \mathbf{x}(w_i)$, for each $i \geq 1$. In case (ii), we may assume, by induction on the length, that $\mathbf{x}^{\mathrm{alg}}(u) \equiv \mathbf{x}(u)$ and $\mathbf{x}^{\mathrm{alg}}(v) \equiv \mathbf{x}(v)$ are symmetric 2×2 integral matrices. As $\mathbf{x}^{\mathrm{alg}}(w)$ is an integral linear combination of these, it is also a symmetric 2×2 integral matrix, and we have $\mathbf{x}^{\mathrm{alg}}(w) \equiv \mathbf{x}(w)$ by Proposition 13.4. □

Thus, by construction, (13.3) holds for each triple of consecutive elements $u < v < w$ of $\mathcal{V}_\ell$ with $\ell \geq 4$ and $w \notin \mathcal{V}_{\ell+1}$. This contrasts with the congruence (13.2) which holds for the larger set of triples of consecutive elements $u < v < w$ of $\mathcal{V}_\ell$ with $\ell \geq 4$ and $v \in \mathcal{V}_{\ell+1}$.

By definition, we have $\mathbf{x}^{\mathrm{alg}}(w_\ell) = \mathbf{x}_\ell$ for each $\ell \geq 4$. Below, we compute $\mathbf{x}^{\mathrm{alg}}(w)$ for the next simplest families of prefixes w of w_∞.

Lemma 13.6. *For each integer $\ell \geq 4$, we have*

$$\mathbf{x}^{\mathrm{alg}}(w_{\ell+1}w_{\ell-1}) = \mathbf{y}_\ell := t_{\ell-1}\mathbf{x}_{\ell+1} + d_{\ell-1}\mathbf{x}_\ell,$$
$$\mathbf{x}^{\mathrm{alg}}(w_{\ell+2}w_{\ell-2}) = \mathbf{z}_\ell := t_{\ell-2}\mathbf{x}_{\ell+2} + d_{\ell-2}\mathbf{y}_\ell.$$

Proof. Fix a choice of $\ell \geq 4$. By Lemma 5.1, the words $w_\ell < w_{\ell+1} < w_{\ell+1}w_{\ell-1}$ are the first three elements of $\mathcal{V}_\ell$. Since $w_{\ell+1}w_{\ell-1} \notin \mathcal{V}_{\ell+1}$, they satisfy the hypotheses of Corollary 13.5 (ii) with $i = \ell - 1$. As $M(w_\ell) = M_\ell \neq M_{\ell+1} = M(w_{\ell+1})$, we deduce that

$$\mathbf{x}^{\mathrm{alg}}(w_{\ell+1}w_{\ell-1}) = t_{\ell-1}\mathbf{x}^{\mathrm{alg}}(w_{\ell+1}) + d_{\ell-1}\mathbf{x}^{\mathrm{alg}}(w_\ell) = t_{\ell-1}\mathbf{x}_{\ell+1} + d_{\ell-1}\mathbf{x}_\ell$$

which is denoted $\mathbf{y}_\ell$. Lemma 5.1 also shows that $w_{\ell+1}w_{\ell-1} < w_{\ell+2} < w_{\ell+2}w_{\ell-2}$ are consecutive elements of $\mathcal{V}_\ell$. Since $w_{\ell+2}w_{\ell-2} \notin \mathcal{V}_{\ell+1}$, they satisfy the hypothesis of Corollary 13.5 (ii) with $i = \ell - 2$. As $M(w_{\ell+1}w_{\ell-1}) = M_{\ell-1} \neq M_{\ell+2} = M(w_{\ell+2})$, we deduce that

$$\mathbf{x}^{\mathrm{alg}}(w_{\ell+2}w_{\ell-2}) = t_{\ell-2}\mathbf{x}^{\mathrm{alg}}(w_{\ell+2}) + d_{\ell-2}\mathbf{x}^{\mathrm{alg}}(w_{\ell+1}w_{\ell-1}) = t_{\ell-2}\mathbf{x}_{\ell+2} + d_{\ell-2}\mathbf{y}_\ell$$

which is denoted $\mathbf{z}_\ell$. □

We can now proceed to our main estimate.

Proposition 13.7. *Let ρ be as in Proposition 11.1. Then, there is a constant $c > 0$ such that, for any $v \in \mathcal{V}_4$, we have*

$$\delta(v) := \|\,(\xi, -1)\mathbf{x}^{\mathrm{alg}}(v)\,\| \exp(\rho|\alpha(v)|) \leq c.$$

Proof. For any pair of integers k, ℓ with $4 \leq \ell \leq k$, we set

$$d_k(\ell) = \max\{\delta(v)\,;\, v \in \mathcal{V}_\ell \text{ and } v \leq w_k\}.$$

We need to show that the non-decreasing sequence $\big(d_k(4)\big)_{k\geq 4}$ is bounded from above. We proceed in two steps.

Step 1. By estimates (11.3) in Proposition 11.1, there exist $c_1, c_2 > 0$ such that

$$|t_\ell| \leq (1 + c_1\gamma^{-\ell})\exp(\rho F_\ell),$$
$$\delta(w_\ell) = \|\,(\xi, -1)\mathbf{x}_\ell\,\| \exp(\rho F_\ell) \leq c_2,$$

for each integer $\ell \geq 1$, because $\mathbf{x}^{\mathrm{alg}}(w_\ell) = W_\ell M_\ell^{-1} = \mathbf{x}_\ell$ and $(\xi, -1)\Xi = 0$. Then, for any $\ell \geq 4$, Lemma 13.6 yields

$$\begin{aligned}\delta(w_{\ell+1}w_{\ell-1}) &= \|\,(\xi, -1)\mathbf{y}_\ell\,\| \exp(\rho F_\ell)\\ &\leq (1 + c_1\gamma^{-\ell+1})\delta(w_{\ell+1}) + \delta(w_\ell) \leq (2 + c_1)c_2,\\ \delta(w_{\ell+2}w_{\ell-2}) &= \|\,(\xi, -1)\mathbf{z}_\ell\,\| \exp(\rho F_\ell)\\ &\leq (1 + c_1\gamma^{-\ell+2})\delta(w_{\ell+2}) + \delta(w_{\ell+1}w_{\ell-1}) \leq (3 + 2c_1)c_2.\end{aligned}$$

By Lemma 5.1, this implies that, for any $\ell \geq 4$, we have

$$\delta(v) \leq c_3 \quad \text{for each } v \in \mathcal{V}_\ell \cap [\epsilon, w_{\ell+3}] \tag{13.4}$$

where $c_3 = (3 + 2c_1)c_2$, and so $d_{\ell+3}(\ell) \leq c_3$.

Step 2. We claim that there is a constant $c_4 > 0$ such that

$$\delta(v) \leq (1 + c_4\gamma^{-\ell})d_k(\ell + 1) \tag{13.5}$$

for each $v \in \mathcal{V}_\ell \cap\,]w_{\ell+3}, w_k]$ with $4 \leq \ell \leq k - 4$.

If we take this for granted, then, in view of (13.4), we obtain

$$\max\{c_3, d_k(\ell)\} \leq (1 + c_4\gamma^{-\ell})\max\{c_3, d_k(\ell + 1)\}$$

for each $k \geq 8$ and each $\ell = 4, \dots, k-4$. As $d_k(k-3) \leq c_3$, this gives, as needed

$$d_k(4) \leq c := c_3 \prod_{i=4}^{\infty} (1 + c_4\gamma^{-\ell}) < \infty.$$

To prove the claim, we may assume that $v \notin \mathcal{V}_{\ell+1}$ because otherwise $\delta(v) \leq d_k(\ell+1)$ and (13.5) is automatic. Then, we have $v \in \mathcal{V}_\ell \setminus \mathcal{V}_{\ell+1}$ and, since $v > w_{\ell+3}$, there is a maximal sequence of consecutive elements of $\mathcal{V}_\ell$

$$v_1 < u_1 < \cdots < v_s < u_s$$

of even cardinality $2s$ with $s \geq 2$, such that

$$v = v_s \quad \text{and} \quad \{v_2, \dots, v_s\} \subseteq \mathcal{V}_\ell \setminus \mathcal{V}_{\ell+1}.$$

As $w_{\ell+2}w_\ell < w_{\ell+3}$ are consecutive elements of $\mathcal{V}_\ell$ contained in $\mathcal{V}_{\ell+1}$, we must have $v_1 \geq w_{\ell+2}w_\ell$ and so $v_1 \in \mathcal{V}_{\ell+1}$ by maximality of the sequence. Moreover, for each $i = 2, \dots, s$, the words $u_{i-1} < v_i < u_i$ are consecutive in $\mathcal{V}_\ell$ with $v_i \notin \mathcal{V}_{\ell+1}$. By Proposition 4.11, this implies that $u_{i-1} < u_i$ are consecutive elements of $\mathcal{V}_{\ell+1}$ with $|u_i| - |u_{i-1}| = F_\ell$. Thus, $|u_1| < |u_2| < \cdots < |u_s|$ is an arithmetic progression made of consecutive elements of $\overline{\mathcal{V}}_{\ell+1}$. By Corollary 4.8, this implies that $s \leq 5$. If $s \geq 3$, then, by Proposition 4.10, we further have $u_i \in \mathcal{V}_{\ell+2}$ for $i = 2, \dots, s-1$.

For each $i = 1, \dots, s-1$, the words $v_i < u_i < v_{i+1}$ are consecutive elements of $\mathcal{V}_\ell$ with $v_{i+1} \notin \mathcal{V}_{\ell+1}$. So, Corollary 13.5 (ii) yields

$$\mathbf{x}^{\mathrm{alg}}(v_{i+1}) = t_k \mathbf{x}^{\mathrm{alg}}(u_i) \pm \mathbf{x}^{\mathrm{alg}}(v_i)$$

for some $k = k_i \in \{\ell-2, \ell-1\}$. For $i = 1$, we have $\{u_1, v_1\} \subseteq \mathcal{V}_{\ell+1}$. Thus,

$$\begin{aligned} \delta(v_2) &\leq (|t_{\ell-1}|\delta(u_1) + \delta(v_1)) \exp(\rho F_\ell - \rho F_{\ell+1}) \\ &\leq (1 + c_1\gamma^{-\ell+1})\delta(u_1) + \exp(-\rho F_{\ell-1})\delta(v_1) \\ &\leq \big(1 + c_1\gamma^{-\ell+1} + \exp(-\rho F_{\ell-1})\big)\, d_k(\ell+1). \end{aligned} \tag{13.6}$$

When $2 \leq i \leq s-1$, we have $\{v_i, v_{i+1}\} \subseteq \mathcal{V}_\ell \setminus \mathcal{V}_{\ell+1}$ and $u_i \in \mathcal{V}_{\ell+2}$, thus

$$\delta(v_{i+1}) \leq |t_k|\delta(u_i) \exp(\rho F_\ell - \rho F_{\ell+2}) + \delta(v_i).$$

Since $|t_k| \leq (1+c_1)\exp(\rho F_{\ell-1})$ and $\delta(u_i) \leq d_k(\ell+1)$, this yields

$$\delta(v_{i+1}) \leq \delta(v_i) + (1+c_1)\exp(-\rho F_\ell) d_k(\ell+1) \quad (2 \leq i \leq s-1). \tag{13.7}$$

As $v = v_s$ with $s \leq 5$, we conclude from (13.6) and (13.7), that (13.5) holds for a constant c_4 depending only on c_1 and ρ. □

Corollary 13.8. *There is an integer $\ell_1 \geq 4$ with the following properties.*

(i) *For any integer $\ell \geq \ell_1$ and any $v \in \mathcal{V}_\ell$, we have*

$$\mathbf{x}(v) = \mathbf{x}^{\mathrm{alg}}(v), \quad \|(\xi, -1)\mathbf{x}(v)\| \leq c\exp(-\rho|\alpha(v)|) \quad \text{and} \quad x_0(v) \neq 0. \tag{13.8}$$

(ii) *For any integer $\ell \geq \ell_1$ and any triple of consecutive words $u < v < w$ in $\mathcal{V}_\ell$ with $v \in \mathcal{V}_{\ell+1}$, we have $|w| - |v| = |v| - |u| = F_i$ for some $i \in \{\ell-2, \ell-1\}$, and*

$$\mathbf{x}(w) = \begin{cases} t_i\mathbf{x}(v) - d_i\mathbf{x}(u) & \text{if } M(u) = M(v), \\ t_i\mathbf{x}(v) + d_i\mathbf{x}(u) & \text{otherwise.} \end{cases} \tag{13.9}$$

Thus, the congruence (13.2) is an equality when $\ell \geq \ell_1$, as claimed earlier.

Proof of Corollary 13.8. By estimates (11.3) of Proposition 11.1, there is an integer $\ell_1 \geq 4$ such that $|t_i| \leq 2\exp(\rho F_i)$ for each $i \geq \ell_1$. We choose this ℓ_1 so that $3c\exp(-\rho F_{\ell_1}) < 1/2$, where $c \geq 1$ is the constant of the preceding proposition.

(i) For $v \in \mathcal{V}_\ell$ with $\ell \geq \ell_1$, we have $|\alpha(v)| \geq F_\ell \geq F_{\ell_1}$ and the preceding proposition yields

$$\|(\xi, -1)\mathbf{x}^{\mathrm{alg}}(v)\| \leq c\exp(-\rho|\alpha(v)|) < 1/2.$$

As $\mathbf{x}^{\mathrm{alg}}(v) \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ is a symmetric integral matrix, (13.8) follows.

(ii) Let $\ell \geq \ell_1$ and let $u < v < w$ be consecutive elements of $\mathcal{V}_\ell$ with $v \in \mathcal{V}_{\ell+1}$. By Proposition 13.4, we have $|w| - |v| = |v| - |u| = F_i$ for some $i \in \{\ell-2, \ell-1\}$ and $\mathbf{x}(w) \equiv \mathbf{y}$ where $\mathbf{y}$ denotes the right hand side of (13.9). By part (i) proved above, (13.8) applies to both u and v, thus

$$\begin{aligned}\|(\xi, -1)\mathbf{y}\| &\leq c|t_i|\exp(-\rho|\alpha(v)|) + c\exp(-\rho|\alpha(u)|)\\ &\leq 2c\exp(\rho F_{\ell-1} - \rho F_{\ell+1}) + c\exp(-\rho F_\ell) = 3c\exp(-\rho F_\ell) < 1/2.\end{aligned}$$

Since $\mathbf{y}$ is symmetric with integer coefficients, we conclude that $\mathbf{x}(w) = \mathbf{y}$. ☐

We conclude with the following complement.

Proposition 13.9. *Let ℓ_1 be as in Corollary 13.8, and let $u < v < w$ be consecutive words in $\mathcal{V}_\ell$ with $v \notin \mathcal{V}_{\ell+1}$, for an integer $\ell \geq \ell_1$. Then, we have*

$$\det(\mathbf{x}(u), \mathbf{x}(v), \mathbf{x}(w)) = \pm 2.$$

Proof. We proceed by induction on $|v|$. Since $v \notin \mathcal{V}_{\ell+1}$, Lemma 5.1 implies that $v \geq w_{\ell+1}w_{\ell-1}$.

To start, suppose that $v = w_{\ell+1}w_{\ell-1}$. Then we have $u = w_{\ell+1}$ and $w = w_{\ell+2}$. As $\ell \geq \ell_1$, Corollary 13.8 (i) applies to u, v and w. Thus, we find

$$\det(\mathbf{x}(u), \mathbf{x}(v), \mathbf{x}(w)) = \det(\mathbf{x}_{\ell+1}, \mathbf{y}_\ell, \mathbf{x}_{\ell+2}) = \pm\det(\mathbf{x}_\ell, \mathbf{x}_{\ell+1}, \mathbf{x}_{\ell+2}) = \pm 2,$$

using Lemma 13.6 and then Proposition 11.1(iv).

Assume from now on that $v > w_{\ell+1}w_{\ell-1}$. By Lemma 5.1, we have $v \geq w_{\ell+2}w_{\ell-2}$ and we can extend $u < v < w$ to a sequence of consecutive words

$$u' < v' < u < v < w$$

in $\mathcal{V}_\ell$. Since $v \notin \mathcal{V}_{\ell+1}$, Proposition 4.11 shows that $u < w$ are consecutive elements of $\mathcal{V}_{\ell+1}$ with $|w| - |u| = F_\ell$. Moreover, since $v' < u < v$ are consecutive elements of $\mathcal{V}_\ell$ with $u \in \mathcal{V}_{\ell+1}$, Corollary 13.8 (ii) gives $\mathbf{x}(v) = t_i\mathbf{x}(u) \pm \mathbf{x}(v')$ for some $i \in \{\ell-2, \ell-1\}$, and therefore

$$\det(\mathbf{x}(u), \mathbf{x}(v), \mathbf{x}(w)) = \pm\det(\mathbf{x}(v'), \mathbf{x}(u), \mathbf{x}(w)). \tag{13.10}$$

Suppose first that $v' \notin \mathcal{V}_{\ell+1}$. Then, arguing as above, we find that $u' < u$ are consecutive elements of $\mathcal{V}_{\ell+1}$ with $|u| - |u'| = F_\ell$. Thus, $u' < u < w$ are consecutive elements of $\mathcal{V}_{\ell+1}$ with $|w| - |u| = |u| - |u'| = F_\ell$. By Proposition 4.10, this implies that $u \in \mathcal{V}_{\ell+2}$, and so Corollary 13.8 gives $\mathbf{x}(w) = t_\ell\mathbf{x}(u) \pm \mathbf{x}(u')$. Substituting this into the right hand side of (13.10), we deduce that

$$\det(\mathbf{x}(u), \mathbf{x}(v), \mathbf{x}(w)) = \pm\det(\mathbf{x}(u'), \mathbf{x}(v'), \mathbf{x}(u)).$$

As $u' < v' < u$ are consecutive elements of $\mathcal{V}_\ell$ with $v' \notin \mathcal{V}_{\ell+1}$ and $|v'| < |v|$, we may assume by induction that the determinant in the right hand side of this equality is ± 2, and we are done.

Finally, suppose that $v' \in \mathcal{V}_{\ell+1}$. Then $v' < u < w$ are consecutive elements of $\mathcal{V}_{\ell+1}$. Since $v' < u$ are also consecutive in $\mathcal{V}_\ell$, Corollary 4.8 gives $|u| - |v'| = F_{\ell-1}$, thus $|u| - |v'| < F_\ell = |w| - |u|$, and so $u \notin \mathcal{V}_{\ell+2}$ by Proposition 4.10. As $|u| < |v|$, we may assume by induction that the determinant in the right hand side of (13.10) is ± 2, and we are done once again. □

Combining this proposition with Corollary 13.8 (ii) yields the following qualitative statement, and thus proves Theorem 3.4 (ii).

Corollary 13.10. *For consecutive words $u < v < w$ in $\mathcal{V}_\ell$ with $\ell \ge \ell_1$, the points $\mathbf{x}(u)$, $\mathbf{x}(v)$ and $\mathbf{x}(w)$ are linearly independent if and only if $v \notin \mathcal{V}_{\ell+1}$.*

14. Proof of Theorem 3.4

Let the notation be as in section 3.7, let ρ be as in Proposition 11.1 for the given $\xi \in \mathcal{E}_m$, and let ℓ_1 be as in Corollary 13.8. In view of Corollary 13.10, it remains to prove that there exists an integer ℓ with $\ell \ge \ell_1$ such that, for each $v \in \mathcal{V}_\ell$ and each $\mathbf{q} \in \mathcal{A}(\epsilon)$, we have $L_{\mathbf{x}(v)}(\mathbf{q}) = \rho P_v(\rho^{-1}\mathbf{q}) + \mathcal{O}_\xi(1)$, where $\mathcal{O}_\xi(1)$ stands for a function of v and $\mathbf{q}$ whose absolute value is bounded above by a constant that depends only on ξ.

For each $v \in]\epsilon, w_\infty[$, we set

$$\Delta(v) = (\Delta_0(v), \Delta_1(v), \Delta_2(v)) = (x_0(v), x_0(v)\xi - x_1(v), x_0(v)\xi^2 - x_2(v)),$$

so that, for any $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2$, we have

$$L_{\mathbf{x}(v)}(\mathbf{q}) = \max\{\log|\Delta_0(v)|, q_1 + \log|\Delta_1(v)|, q_2 + \log|\Delta_2(v)|\}.$$

By Corollaries 12.2 and 13.8 (i), for each $v \in \mathcal{V}_\ell$ with $\ell \ge \ell_1$, we have

$$\begin{aligned} &\log|\Delta_0(v)| = \rho|v| + \mathcal{O}_\xi(1),\\ &\log|\Delta_1(v)| = -\rho|v| + \mathcal{O}_\xi(1), \qquad (14.1)\\ &\log|\Delta_2(v)| \le \log\|(\xi,-1)\mathbf{x}(v)\| + \mathcal{O}_\xi(1) \le -\rho|\alpha(v)| + \mathcal{O}_\xi(1). \end{aligned}$$

We claim that we may further choose ℓ so that

$$\log|\Delta_2(v)| \ge -\rho|\alpha(v)| + \mathcal{O}_\xi(1) \quad \text{for all } v \in \mathcal{V}_\ell \setminus \mathcal{F}. \qquad (14.2)$$

If we take this for granted, then, for $v \in \mathcal{V}_\ell \setminus \mathcal{F}$ and $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2$, we obtain

$$L_{\mathbf{x}(v)}(\mathbf{q}) = \max\{\rho|v|, q_1 - \rho|v|, q_2 - \rho|\alpha(v)|\} + \mathcal{O}_\xi(1) = \rho P_v(\rho^{-1}\mathbf{q}) + \mathcal{O}_\xi(1).$$

This still holds for $v \in \mathcal{F}$ and $\mathbf{q} \in \mathcal{A}(\epsilon)$, as we then have $\alpha(v) = v$ and $q_2 \le q_1$, thus,

$$q_2 + \log|\Delta_2(v)| \le q_1 - \rho|v| + \mathcal{O}_\xi(1) = q_1 + \log|\Delta_1(v)| + \mathcal{O}_\xi(1).$$

To prove (14.2), choose $v \in \mathcal{V}_{\ell_1} \setminus \mathcal{F}$ and let $\ell \ge \ell_1$ such that $\alpha(v) = w_\ell$. Then, $v \ne w_\ell$, and so v is the middle term of a triple of consecutive elements $u < v < w$ of $\mathcal{V}_\ell$. As $v \notin \mathcal{V}_{\ell+1}$, Proposition 13.9 gives

$$2 = |\det(\mathbf{x}(u), \mathbf{x}(v), \mathbf{x}(w))| = |\det(\Delta(u), \Delta(v), \Delta(w))|.$$

The determinant on the right is a sum of six products $\pm\Delta_i(u)\Delta_j(v)\Delta_k(w)$ where (i,j,k) runs through the permutations of $(0,1,2)$. Since the estimates (14.1) also apply to u and w in place of v, we find that, for $(i,j,k) \neq (1,2,0)$, these products tend to 0 as ℓ go to infinity, uniformly in v. For example, we have

$$\log|\Delta_2(u)\Delta_1(v)\Delta_0(w)| \le \rho(-|\alpha(u)| - |v| + |w|) + \mathcal{O}_\xi(1) \le -\rho F_\ell + \mathcal{O}_\xi(1)$$

because $|v| \ge |u| + F_{\ell-2}$ by Corollary 4.8, $|\alpha(u)| \ge F_{\ell+1}$ by Corollary 4.9, and finally $|w| - |u| = |\alpha(v)| = F_\ell$ by Proposition 4.10. Thus, if ℓ is large enough, we obtain

$$\begin{aligned} 0 \le \log|\Delta_1(u)\Delta_2(v)\Delta_0(w)| &= \rho(|w| - |u|) + \log|\Delta_2(v)| + \mathcal{O}_\xi(1) \\ &= \rho|\alpha(v)| + \log|\Delta_2(v)| + \mathcal{O}_\xi(1) \end{aligned}$$

which gives (14.2).

15. Weighted exponents of approximation

For each $\boldsymbol{\xi} = (1, \xi_1, \xi_2) \in \mathbb{R}^3$ and each $\sigma \in [0, \infty[$, we define $\lambda_\sigma(\boldsymbol{\xi})$ (resp. $\widehat{\lambda}_\sigma(\boldsymbol{\xi})$) as the supremum of all real numbers $\lambda > 0$ such that the inequalities

$$|x_0| \le Q, \quad |x_0\xi_1 - x_1| \le Q^{-\lambda+1} \quad \text{and} \quad |x_0\xi_2 - x_2| \le Q^{-\sigma\lambda+1} \tag{15.1}$$

admit a non-zero solution $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3$ for arbitrarily large values of $Q \ge 1$ (resp. for all sufficiently large values of $Q \ge 1$). These are essentially the usual weighted exponents of approximation to $\boldsymbol{\xi}$, as in [2] for example, except for the additive constant 1 in the exponents of Q. This allows us to define equivalently $\lambda_\sigma(\boldsymbol{\xi})$ as the supremum of all $\lambda > 0$ such that

$$|\xi_1 - x_1/x_0| \le x_0^{-\lambda} \quad \text{and} \quad |\xi_2 - x_2/x_0| \le x_0^{-\sigma\lambda}$$

for infinitely many $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3$ with $x_0 > 0$. This modification also makes the exponents easier to handle via the following result.

Lemma 15.1. *For $\boldsymbol{\xi}$ and σ as above we have*

$$\lambda_\sigma(\boldsymbol{\xi})^{-1} = \liminf_{q\to\infty} \frac{L_{\boldsymbol{\xi},1}(q,\sigma q)}{q} \quad \text{and} \quad \widehat{\lambda}_\sigma(\boldsymbol{\xi})^{-1} = \limsup_{q\to\infty} \frac{L_{\boldsymbol{\xi},1}(q,\sigma q)}{q}.$$

Proof. For $Q = e^q$ with $q \ge 0$, the condition that (15.1) admits a non-zero solution in $\mathbb{Z}^3$ is equivalent to asking that $L_{\boldsymbol{\xi},1}(\lambda q, \sigma\lambda q) \le q$. Thus, $\lambda_\sigma(\boldsymbol{\xi})$ (resp. $\widehat{\lambda}_\sigma(\boldsymbol{\xi})$) is also the supremum of all $\lambda > 0$ such that $L_{\boldsymbol{\xi},1}(q,\sigma q)/q < 1/\lambda$ for arbitrarily large values of $q > 0$ (resp. for all sufficiently large $q > 0$), and the formulas follow. □

We conclude this paper with the following computation.

Theorem 15.2. *Let $\boldsymbol{\xi} = (1, \xi, \xi^2)$ where $\xi \in \mathcal{E}_m$ for some positive integer m. Then, we have $\lambda_\sigma(\boldsymbol{\xi}) = 2$ for each $\sigma \in [0,1]$, and*

$$\widehat{\lambda}_\sigma(\boldsymbol{\xi}) = \begin{cases} \gamma & \text{if } 1 - \gamma^{-4} \le \sigma \le 1, \\ (1+\gamma^{-2})/\sigma & \text{if } 5/(2\gamma^2+1) \le \sigma \le 1 - \gamma^{-4}, \\ (2\gamma^2+1)/(\gamma^2+1) & \text{if } 1 - \gamma^{-3} \le \sigma \le 5/(2\gamma^2+1). \end{cases} \tag{15.2}$$

Proof. Let $\sigma \in [0,1]$. Since ξ is badly approximable, (15.1) has no non-zero solution $\mathbf{x} \in \mathbb{Z}^3$ for $\xi_1 = \xi$ and $\lambda > 2$ when Q is large enough. On the hand, since $\xi \in \mathcal{E}_m$, conditions (E1)–(E3) of section 3.6 apply and yield $\|(\xi,-1)\mathbf{x}_i\| \asymp \|\mathbf{x}_i\|^{-1}$ for each $i \geq 1$. Thus, for the current point $\boldsymbol{\xi}$ and for any given λ with $0 < \lambda < 2$, the point $\mathbf{x} = \mathbf{x}_i$ satisfies (15.1) with $Q = \|\mathbf{x}_i\|$ for each large enough i. This shows that $\lambda_\sigma(\boldsymbol{\xi}) = 2$.

Choosing ρ as in Theorem 3.3, we find by Lemma 15.1

$$\widehat{\lambda}_\sigma(\boldsymbol{\xi})^{-1} = \limsup_{q\to\infty} \frac{\rho P_1(\rho^{-1}q, \sigma\rho^{-1}q)}{q} = \limsup_{q\to\infty} \frac{P_1(q,\sigma q)}{q},$$

thus

$$\widehat{\lambda}_\sigma(\boldsymbol{\xi})^{-1} = \limsup_{k\to\infty} \bar{\varphi}_k(\sigma) \quad \text{where} \quad \bar{\varphi}_k(\sigma) = \max\{P_1(q,\sigma q)/q\,;\, 2F_k \leq q \leq 2F_{k+1}\}.$$

Let $k \geq 5$ be an arbitrarily large integer. To estimate $\bar{\varphi}_k(\sigma)$, we set

$$\mathcal{R} = \mathrm{Cell}(w_k, w_k w_{k-4}, w_k w_{k-2}) \in S_{k-2} \quad \text{and} \quad \mathcal{R}' = \mathrm{Cell}(w_k, w_k w_{k-2}, w_{k+1}) \in S_{k-1}.$$

We denote by $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ the subsets of $\mathcal{R}$ made of the points $\mathbf{q} = (q_1, q_2)$ where $P_1(\mathbf{q})$ is given respectively by $q_1 - F_k$, $q_2 - F_{k-2}$, and $F_k + F_{k-2}$. As explained right after Proposition 7.1, these are admissible polygons with a common vertex $\mathbf{r}$, and they form a partition of $\mathcal{R}$, as illustrated in Figure 10. Similarly, we denote respectively by $\mathcal{R}'_1$, $\mathcal{R}'_2$ and

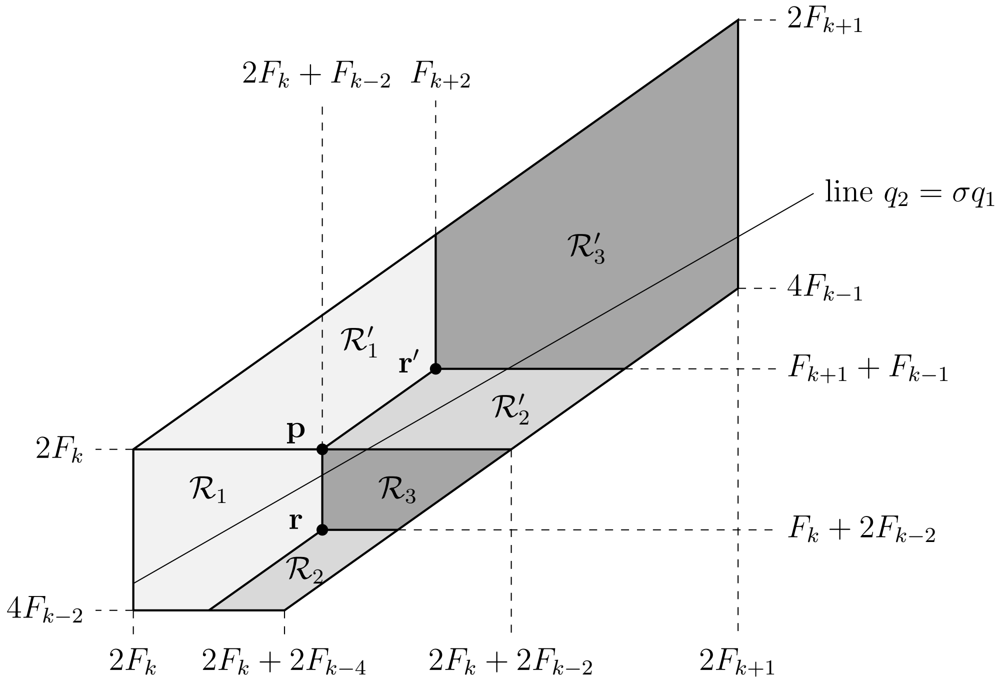

FIGURE 10. P_1 on $\mathrm{Cell}(w_k, w_k w_{k-4}, w_k w_{k-2}) \cup \mathrm{Cell}(w_k, w_k w_{k-2}, w_{k+1})$

$\mathcal{R}'_3$ the subsets of $\mathcal{R}'$ where $P_1(\mathbf{q})$ is given respectively by $q_1 - F_k$, $q_2 - F_{k-1}$ and F_{k+1}, and we denote by $\mathbf{r}'$ the common vertex of these polygons. We also define $\mathbf{p}$ as the mid-point of the top side of $\mathcal{R}$. Upon setting $\mu(\mathbf{q}) = q_2/q_1$ for each $\mathbf{q} = (q_1, q_2) \in \mathcal{A}(\epsilon)$, we find

$$\mu(\mathbf{r}') = \frac{F_{k+1} + F_{k-1}}{F_{k+2}} > \mu(\mathbf{p}) = \frac{2F_k}{2F_k + F_{k-2}} > \mu := \max\left\{\frac{2F_{k-2}}{F_k}, \frac{2F_{k-1}}{F_{k+1}}\right\} > \mu(\mathbf{r})$$

for each large enough k. When $\sigma \geq \mu$, the line $q_2 = \sigma q_1$ meets the left vertical side of $\mathcal{R}$ and the right vertical side of $\mathcal{R}'$. In between, it remains in $\mathcal{R} \cup \mathcal{R}'$. As this line crosses the regions $\mathcal{R}_1$, $\mathcal{R}'_1$ or $\mathcal{R}'_2$, the ratio $P_1(q, \sigma q)/q$ increases. As it crosses $\mathcal{R}_3$ or $\mathcal{R}'_3$, the same ratio decreases. So, this ratio is maximal at the point where the line meets the left vertical side of $\mathcal{R}_3$ or the left vertical side of $\mathcal{R}'_3$ or the horizontal bottom side of $\mathcal{R}'_3$. As P_1 is constant equal to $F_k + F_{k-2}$ on $\mathcal{R}_3$ and constant equal to F_{k+1} on $\mathcal{R}'_3$, we deduce that

$$\bar{\varphi}_k(\sigma) = \begin{cases} \dfrac{F_{k+1}}{F_{k+2}} & \text{if } \mu(\mathbf{r}') \leq \sigma \leq 1, \\ \dfrac{F_{k+1}\sigma}{F_{k+1} + F_{k-1}} & \text{if } \mu(\mathbf{p}) \leq \sigma \leq \mu(\mathbf{r}'), \\ \max\left\{ \dfrac{F_k + F_{k-2}}{2F_k + F_{k-2}}, \dfrac{F_{k+1}\sigma}{F_{k+1} + F_{k-1}} \right\} & \text{if } \mu \leq \sigma \leq \mu(\mathbf{p}). \end{cases}$$

Letting k go to infinity, we deduce that

$$\widehat{\lambda}_\sigma(\boldsymbol{\xi})^{-1} = \begin{cases} \dfrac{1}{\gamma} & \text{if } \dfrac{\gamma^2+1}{\gamma^3} \leq \sigma \leq 1, \\ \dfrac{\gamma^2\sigma}{\gamma^2+1} & \text{if } \dfrac{2\gamma^2}{2\gamma^2+1} \leq \sigma \leq \dfrac{\gamma^2+1}{\gamma^3}, \\ \max\left\{ \dfrac{\gamma^2+1}{2\gamma^2+1}, \dfrac{\gamma^2\sigma}{\gamma^2+1} \right\} & \text{if } \dfrac{2}{\gamma^2} \leq \sigma \leq \dfrac{2\gamma^2}{2\gamma^2+1}, \end{cases}$$

which rewrites as (15.2). □

With additional work, the same method enables one to calculate $\widehat{\lambda}_\sigma(\boldsymbol{\xi})$ for any given $\sigma \in]1/2, 1]$. For $\sigma \in]0, 1/2]$, Lemma 2.3 yields trivially $\widehat{\lambda}_\sigma(\boldsymbol{\xi}) = \sigma^{-1}$ (take $\mathbf{x} = (0, 0, 1)$ in (15.1)).

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References

[1] H. Davenport and W. M. Schmidt, Approximation to real numbers by algebraic integers. *Acta Arith.* **15** (1969), 393–416.

[2] O. N. German, Multiparametric geometry of numbers and its application to splitting transference theorems. *Monatsh. Math.* **197** (2022), 579–606.

[3] P. M. Gruber and C. G. Lekkerkerker, *Geometry of numbers*, North-Holland, 1987.

[4] D. Roy, Approximation to real numbers by cubic algebraic integers I. *Proc. London Math. Soc.* **88** (2004), 42–62.

[5] D. Roy, Approximation to real numbers by cubic algebraic integers II. *Ann. of Math.* **158** (2003), 1081–1087.

[6] D. Roy, On the continued fraction expansion of a class of numbers. in: *Diophantine approximation, Festschrift for Wolfgang Schmidt*, Developments in Math. **16**, Eds: H. P. Schlickewei, K. Schmidt and R. Tichy, Springer-Verlag, 2008, 347–361.

[7] D. Roy, Markoff-Lagrange spectrum and extremal numbers. *Acta Math.* **206** (2011), 325–362.

[8] D. Roy, Construction of points realizing the regular systems of Wolfgang Schmidt and Leonard Summerer, *J. Théor. Nombres Bordeaux* **27** (2015), 591–603.

[9] D. Roy, On Schmidt and Summerer parametric geometry of numbers, *Ann. of Math.* **182** (2015), 739–786.

[10] W. M. Schmidt, *Diophantine approximation*, Lecture Note in Math., vol. 785, Springer-Verlag, 1980.

[11] W. M. Schmidt, On parametric geometry of numbers. *Acta Arith.* **195** (2020), 383–414.

[12] W. M. Schmidt and L. Summerer, Parametric geometry of numbers and applications. *Acta Arith.* **140** (2009), 67–91.

[13] W. M. Schmidt and L. Summerer, Diophantine approximation and parametric geometry of numbers, *Monatsh. Math.* **169** (2013), 51–104.

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