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On the ring of approximation triples attached to a class of extremal real numbers ☆

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ABSTRACT

We attach a ring of sequences to each number from a certain class of extremal real numbers, and we study the properties of this ring both from an analytic point of view by exhibiting elements with specific behaviors, and also from an algebraic point of view by identifying it with the quotient of a polynomial ring over $\mathbb{Q}$. The link between these points of view relies on combinatorial results of independent interest. We apply this theory to estimate the dimension of a certain space of sequences satisfying prescribed growth constraints.

1. INTRODUCTION

Let $\gamma = (1+\sqrt{5})/2$ denote the golden ratio. In [3], Davenport and Schmidt proved that, for each real number ξ which is neither rational nor quadratic irrational, there exists a constant $c > 0$ with the property that, for arbitrarily large real numbers X, the system of inequalities

$$(1) \qquad |x_0| \leqslant X, \quad |x_0\xi - x_1| \leqslant cX^{-1/\gamma}, \qquad \left|x_0\xi^2 - x_2\right| \leqslant cX^{-1/\gamma}$$

has no non-zero solution $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3$. Because of this, we say that a real number ξ is *extremal* if it is neither rational nor quadratic irrational and if there exists a constant $c' > 0$ such that the system (1) with c replaced by c' has a non-zero solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$ for each $X \geqslant 1$. The existence of such numbers

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is established in [7,8], showing in particular that the exponent $1/\gamma$ in the result of Davenport and Schmidt is best possible. Among these numbers are all real numbers whose continued fraction expansion is the infinite Fibonacci word constructed on an alphabet consisting of two different positive integers [7] (or a generalized such word constructed on two non-commuting words in positive integers [11]). This connection with symbolic dynamics is extended by Laurent and Bugeaud in [2], and stressed even further in recent work of Fischler [4,5].

Schmidt's subspace theorem implies that any extremal real number is transcendental (see [12, Chapter VI, Theorem 1B]). Using a quantitative version of the subspace theorem, Adamczewski and Bugeaud even produced a measure of transcendence for these numbers, showing that, in terms of Mahler's classification, they are either S- or T-numbers [1, Theorem 4.6]. The purpose of the present paper is to provide tools which may eventually lead to sharper measures of approximation to extremal real numbers either by all algebraic numbers or by more restricted types of algebraic numbers (like in [9]).

As shown in [8], any extremal real number comes with rigid sequences of integer triples (x_0, x_1, x_2) satisfying a stronger approximation property than that required by (1). In the next section, we show that, for each extremal real number in some large family, this naturally gives rise to a finitely generated ring of sequences over $\mathbb{Q}$. We study this ring in Sections 2 and 3, both from an analytic point of view by exhibiting elements with specific behaviors, and also from an algebraic point of view by showing that it is isomorphic to the quotient of a polynomial ring in six variables over $\mathbb{Q}$ by an ideal I with three explicitly given generators. The link between these points of view relies on two combinatorial results of independent interest that are stated at the beginning of Sections 2 and 3, and proved in Section 4. In Section 5, we apply this theory to estimate the dimension of a certain space of sequences with restricted growth. Following a suggestion of Daniel Daigle, we conclude in Section 6 with a complementary result showing that the ideal I mentioned above is a prime ideal of rank 3 and thus, that the ring of sequences attached to the extremal real numbers under study is an integral domain of transcendence degree 3.

2. THE RING OF APPROXIMATION TRIPLES

2.1. A combinatorial result

We denote by $\mathbb{N}$ the set of non-negative integers, and by $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ the set of positive integers. We also denote by $f : \mathbb{Z} \to \mathbb{Z}$ the function satisfying

$$f(0) = f(1) = 1 \quad \text{and} \quad f(i+2) = f(i+1) + f(i) \quad \text{for each } i \in \mathbb{Z}, \tag{2}$$

so that $(f(i))_{i\in\mathbb{N}}$ is simply the Fibonacci sequence. This function admits the symmetry

$$f(-i-2) = (-1)^i f(i) \quad \text{for each } i \in \mathbb{Z}. \tag{3}$$

In order not to interrupt the flow of the discussion later, we start by stating the following crucial combinatorial result whose proof is postponed to Section 4 and whose relevance will be made clear shortly.

Theorem 2.1. *Let $d \in \mathbb{N}$. For each $s \in \mathbb{N}$, denote by $\chi_d(s)$ the number of points $(m, n) \in \mathbb{Z}^2$ for which the conditions*

$$(4) \qquad (m,n) = \sum_{k=1}^{s} \big(f(-i_k), f(-i_k - 1)\big) \quad \textit{and} \quad \sum_{k=1}^{s} f(i_k) \leqslant d$$

admit a solution integers $0 \leqslant i_1 \leqslant \cdots \leqslant i_s$, and s is maximal with this property. Then,

$$(5) \qquad \chi_d(s) = \begin{cases} 2s+1 & \textit{if } 0 \leqslant s < d, \\ d+1 & \textit{if } s = d, \\ 0 & \textit{if } s > d. \end{cases}$$

Here, and throughout the rest of this paper, we agree that an empty sum is zero, so that for $(m, n) = (0, 0)$, the conditions (4) are satisfied with $s = 0$.

For our purposes, we need to recast this result in the following context. Consider the subring $\mathbb{Z}[\gamma]$ of $\mathbb{R}$ generated by γ. Since $\gamma = 1 + 1/\gamma$ and $(1/\gamma)^2 = 1 - (1/\gamma)$, we note that $\mathbb{Z}[\gamma] = \mathbb{Z}[1/\gamma] = \mathbb{Z} \oplus \mathbb{Z} \cdot (1/\gamma)$ is a free $\mathbb{Z}$-module with basis $\{1, 1/\gamma\}$. The formulas

$$(6) \qquad \gamma^{-i} = f(-i) + f(-i-1)/\gamma = (-1)^i \big(f(i-2) - f(i-1)/\gamma\big),$$

which follow from a quick recurrence argument, show that, for any $\alpha = m + n/\gamma \in \mathbb{Z}[\gamma]$, the conditions (4) are equivalent to

$$(7) \qquad \alpha = \gamma^{-i_1} + \cdots + \gamma^{-i_s} \quad \text{and} \quad f(i_1) + \cdots + f(i_s) \leqslant d.$$

For each $d \in \mathbb{N}$, let E_d denote the set of points $\alpha \in \mathbb{Z}[\gamma]$ for which these conditions admit a solution in integers $0 \leqslant i_1 \leqslant \cdots \leqslant i_s$ for some $s \in \mathbb{N}$ and, for each $\alpha \in E_d$, let $s_d(\alpha)$ denote the largest value of s for which such a solution exists. Then, Theorem 2.1 can be restated by saying that, for each pair of integers $d, s \in \mathbb{N}$, the number of elements α of E_d with $s_d(\alpha) = s$ is the integer $\chi_d(s)$ given by (5). It is in this form that Theorem 2.1 will be proved in Section 4, together with a more compact description of the sets E_d.

2.2. A class of extremal numbers

We first recall that a real number ξ is extremal if and only if there exists an unbounded sequence of points $\mathbf{x}_k = (x_{k,0}, x_{k,1}, x_{k,2})$ in $\mathbb{Z}^3$, indexed by integers $k \geqslant 1$ in $\mathbb{N}^*$, and a constant $c_1 \geqslant 1$ such that, for each $k \geqslant 1$, the first coordinate $x_{k,0}$ of $\mathbf{x}_k$ is non-zero and we have

(E1) $c_1^{-1}|x_{k,0}|^{\gamma} \leqslant |x_{k+1,0}| \leqslant c_1|x_{k,0}|^{\gamma}$,

(E2) $\max\{|x_{k,0}\xi - x_{k,1}|, |x_{k,0}\xi^2 - x_{k,2}|\} \leqslant c_1|x_{k,0}|^{-1}$,
(E3) $1 \leqslant |x_{k,0}x_{k,2} - x_{k,1}^2| \leqslant c_1$,
(E4) $1 \leqslant |\det(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{x}_{k+2})| \leqslant c_1$.

This follows from Theorem 5.1 of [8] upon noting that the condition (E2) forces the maximum norm of $\mathbf{x}_k$ to behave like $|x_{k,0}|$. As in [8], it is convenient to identify each triple $\mathbf{x} = (x_0, x_1, x_2)$ of elements of a commutative ring with the symmetric matrix

$$\mathbf{x} = \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix}.$$

Then, the condition (E3) reads simply as $1 \leqslant |\det(\mathbf{x}_k)| \leqslant c_1$. We also recall that, for a given extremal real number ξ, the corresponding sequence $(\mathbf{x}_k)_{k\geqslant 1}$ is unique up to its first terms and up to multiplication of its terms by non-zero rational numbers with bounded numerators and denominators (see Proposition 4.1 of [10]). Moreover, for such ξ and such a sequence $(\mathbf{x}_k)_{k\geqslant 1}$ viewed as symmetric matrices, Corollary 4.3 of [10] ensures the existence of a non-symmetric and non-skew-symmetric 2×2 matrix

$$M = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \tag{8}$$

with integer coefficients such that, for each sufficiently large $k \geqslant 1$, the matrix $\mathbf{x}_{k+2}$ is a rational multiple of $\mathbf{x}_{k+1}M_{k+1}\mathbf{x}_k$ where

$$M_{k+1} = \begin{cases} M & \text{if } k \text{ is odd,} \\ {}^tM & \text{if } k \text{ is even,} \end{cases} \tag{9}$$

and where ${}^tM$ denotes the transpose of M. The present paper deals with a special class of extremal numbers.

Hypothesis 2.2. *In the sequel, we fix an extremal real number ξ, a corresponding sequence $(\mathbf{x}_k)_{k\geqslant 1}$ and a corresponding matrix M satisfying the additional property that, for each $k \geqslant 1$, we have*

$$\det(\mathbf{x}_k) = 1 \quad \textit{and} \quad \mathbf{x}_{k+2} = \mathbf{x}_{k+1}M_{k+1}\mathbf{x}_k. \tag{10}$$

The first condition $\det(\mathbf{x}_k) = 1$ is restrictive as there exist extremal real numbers with no corresponding sequence $(\mathbf{x}_k)_{k\geqslant 1}$ in $\mathrm{SL}_2(\mathbb{Z})$, but it is not empty as it is fulfilled by any real number whose continued fraction expansion is given by an infinite Fibonacci word constructed on two non-commuting words in positive integers, provided that both words have even length (see [11]; compare also with Definition 2.4 of [9] where the weaker requirement $\det(\mathbf{x}_k) = \pm 1$ is imposed). The second condition however is no real additional restriction. It is achieved by omitting the first terms of the sequence $(\mathbf{x}_k)_{k\geqslant 1}$ if necessary, by choosing M so that $\mathbf{x}_3 = \mathbf{x}_2 M \mathbf{x}_1$ and then by multiplying recursively each $\mathbf{x}_k$ with $k \geqslant 4$ by ± 1 so that the second equality in (10) holds for each $k \geqslant 1$. In particular, we have $\det(M) = 1$. Since ${}^tM \neq \pm M$, this in turn implies that we have $a_{1,1} \neq 0$ or $a_{2,2} \neq 0$.

2.3. A subring of the ring of sequences

Let S denote the ring of sequences of real numbers $(a_k)_{k\geqslant 1}$ indexed by the set $\mathbb{N}^*$ of positive integers, with component-wise addition and multiplication, and let $\mathfrak{S}$ denote the quotient of S by the ideal S_0 of sequences with finitely many non-zero terms. Two sequences $(a_k)_{k\geqslant 1}$ and $(b_k)_{k\geqslant 1}$ in S thus represent the same element of $\mathfrak{S}$ if and only if $a_k = b_k$ for each sufficiently large integer k. We view $\mathbb{R}$ as a subring of $\mathfrak{S}$ by identifying each $x \in \mathbb{R}$ with the image of the constant sequence $(x)_{k\geqslant 1}$ modulo S_0. This gives $\mathfrak{S}$ the structure of an $\mathbb{R}$-algebra. By restriction of scalars, we may also view $\mathfrak{S}$ as a $\mathbb{Q}$-algebra. In particular, given elements $\mathfrak{A}_1, \dots, \mathfrak{A}_\ell$ of $\mathfrak{S}$ we can form the sub-$\mathbb{Q}$-algebra $\mathbb{Q}[\mathfrak{A}_1, \dots, \mathfrak{A}_\ell]$ that they generate.

For each $i \in \mathbb{Z}$ and each $j = 0, 1, 2$, we define an element $\mathfrak{X}_j^{(i)}$ of $\mathfrak{S}$ by

$$\mathfrak{X}_j^{(i)} = \text{ class of } (x_{2k+i,j})_{k\geqslant 1} \quad \text{in } \mathfrak{S}, \tag{11}$$

where for definiteness we agree that $x_{2k+i,j} = 0$ when $2k + i \leqslant 0$, although the resulting element of $\mathfrak{S}$ is independent of this choice. Clearly, $\mathfrak{X}_j^{(i+2)}$ differs simply from $\mathfrak{X}_j^{(i)}$ by a shift, but nevertheless they are quite different from the algebraic point of view. For each $i \in \mathbb{Z}$, we also define a triple

$$\mathfrak{X}^{(i)} = \big(\mathfrak{X}_0^{(i)}, \mathfrak{X}_1^{(i)}, \mathfrak{X}_2^{(i)}\big) \in \mathfrak{S}^3. \tag{12}$$

Identifying these triples with 2×2 symmetric matrices according to our general convention, and using (9) to extend the definition of M_{k+1} to all integers k, Hypothesis 2.2 gives

$$\det\big(\mathfrak{X}^{(i)}\big) = 1 \quad \text{and} \quad \mathfrak{X}^{(i+2)} = \mathfrak{X}^{(i+1)} M_{i+1} \mathfrak{X}^{(i)} \tag{13}$$

for each $i \in \mathbb{Z}$. Our goal in this paper is to study the sub-$\mathbb{Q}$-algebra $\mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]$ of $\mathfrak{S}$. For this purpose, we form the polynomial ring $\mathbb{Q}[\mathbf{X}, \mathbf{X}^*]$ in six indeterminates

$$\mathbf{X} = (X_0, X_1, X_2) \quad \text{and} \quad \mathbf{X}^* = \big(X_0^*, X_1^*, X_2^*\big).$$

We first note that $\mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]$ contains the coordinates of $\mathfrak{X}^{(i)}$ for each $i \in \mathbb{Z}$.

Lemma 2.3. *For each $i \in \mathbb{Z}$, the coordinates of $\mathfrak{X}^{(-i)}$ can be written as values at the point $(\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)})$ of polynomials of $\mathbb{Q}[\mathbf{X}, \mathbf{X}^*]$ that are separately homogeneous of degree $|f(i-2)|$ in $\mathbf{X}$ and homogeneous of degree $|f(i-1)|$ in $\mathbf{X}^*$.*

Proof. Since each product $\mathfrak{X}^{(i)} M_i$ has determinant 1, its inverse is its adjoint matrix. Thus, upon replacing i by $-i-2$ in the second formula of (13), we find

$$\mathfrak{X}^{(-i-2)} = \operatorname{Adj}\big(\mathfrak{X}^{(-i-1)} M_{-i-1}\big) \mathfrak{X}^{(-i)},$$

showing that, for each $i \in \mathbb{Z}$, the coordinates of $\mathfrak{X}^{(-i-2)}$ are bilinear forms in $(\mathfrak{X}^{(-i-1)}, \mathfrak{X}^{(-i)})$. The formula (13) also shows that, for each $i \in \mathbb{Z}$, the coordinates of $\mathfrak{X}^{(i+2)}$ are bilinear in $(\mathfrak{X}^{(i+1)}, \mathfrak{X}^{(i)})$. By recurrence, this implies that, for each $i \geqslant 0$, the coordinates of $\mathfrak{X}^{(-i)}$ (resp. $\mathfrak{X}^{(i)}$) are values at the point $(\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)})$ of bi-homogeneous polynomials of bi-degree $(f(i-2), f(i-1))$ (resp. of bi-degree $(f(i), f(i-1))$). The conclusion follows since, for each integer $i \geqslant 0$, the formula (3) gives $f(i) = |f(-i-2)|$ and $f(i-1) = |f(-i-1)|$. $\square$

The preceding result implicitly uses the natural surjective ring homomorphism

$$\begin{aligned} \pi : \mathbb{Q}[\mathbf{X}, \mathbf{X}^*] &\to \mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}] \\ P(\mathbf{X}, \mathbf{X}^*) &\mapsto P(\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}). \end{aligned} \tag{14}$$

Our first goal is to describe the kernel of this map.

Lemma 2.4. *The kernel of π contains the ideal I of $\mathbb{Q}[\mathbf{X}, \mathbf{X}^*]$ generated by the polynomials*

$$\begin{aligned} &\det(\mathbf{X}) - 1 = X_0X_2 - X_1^2 - 1, \\ &\det(\mathbf{X}^*) - 1 = X_0^*X_2^* - (X_1^*)^2 - 1, \\ &\Phi(\mathbf{X}, \mathbf{X}^*) = a_{1,1} \begin{vmatrix} X_0^* & X_1^* \\ X_0 & X_1 \end{vmatrix} + a_{1,2} \begin{vmatrix} X_1^* & X_2^* \\ X_0 & X_1 \end{vmatrix} \\ &\qquad\qquad + a_{2,1} \begin{vmatrix} X_0^* & X_1^* \\ X_1 & X_2 \end{vmatrix} + a_{2,2} \begin{vmatrix} X_1^* & X_2^* \\ X_1 & X_2 \end{vmatrix}. \end{aligned} \tag{15}$$

Proof. The first equality in (13) tells us that $\det(\mathfrak{X}^{(i)}) = 1$ for each $i \in \mathbb{Z}$. Applying this with $i = 0$ and $i = -1$, we deduce that $\det(\mathbf{X}) - 1$ and $\det(\mathbf{X}^*) - 1$ belong to the kernel of π. On the other hand, the second equality in (13) gives $\mathfrak{X}^{(1)} = \mathfrak{X}^{(0)} M \mathfrak{X}^{(-1)}$ and so

$$\begin{pmatrix} \mathfrak{X}_0^{(1)} & \mathfrak{X}_1^{(1)} \\ \mathfrak{X}_1^{(1)} & \mathfrak{X}_2^{(1)} \end{pmatrix} = \begin{pmatrix} \mathfrak{X}_0^{(0)} & \mathfrak{X}_1^{(0)} \\ \mathfrak{X}_1^{(0)} & \mathfrak{X}_2^{(0)} \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} \mathfrak{X}_0^{(-1)} & \mathfrak{X}_1^{(-1)} \\ \mathfrak{X}_1^{(-1)} & \mathfrak{X}_2^{(-1)} \end{pmatrix}.$$

In particular, the matrix product on the right-hand side gives rise to a symmetric matrix. This fact translates into $\Phi(\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}) = 0$, and so $\Phi \in \ker(\pi)$. $\square$

We will see below that I is precisely the kernel of π. In Section 6 we will provide an alternative proof of this, suggested by Daniel Daigle, showing moreover that I is a prime ideal of $\mathbb{Q}[\mathbf{X}, \mathbf{X}^*]$ of rank 3, and therefore that $\mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]$ is an integral domain of transcendence degree 3 over $\mathbb{Q}$.

2.4. Asymptotic behaviors

The units of $\mathfrak{S}$ are the elements of $\mathfrak{S}$ which are represented by sequences $(a_k)_{k \geqslant 1}$ with $a_k \neq 0$ for each sufficiently large k. We define an equivalence relation $\sim$ on the group $\mathfrak{S}^*$ of units of $\mathfrak{S}$ by writing $\mathfrak{A} \sim \mathfrak{B}$ when $\mathfrak{A}$ and $\mathfrak{B}$ are represented respectively by sequences $(a_k)_{k\geqslant 1}$ and $(b_k)_{k\geqslant 1}$ with $\lim_{k\to\infty} a_k/b_k = 1$. Then, for each $i \in \mathbb{Z}$, the condition (E2) implies that

$$\mathfrak{X}_1^{(i)} \sim \xi\mathfrak{X}_0^{(i)} \quad \text{and} \quad \mathfrak{X}_2^{(i)} \sim \xi^2\mathfrak{X}_0^{(i)}. \tag{16}$$

For this reason, we regard the points $\mathfrak{X}^{(i)}$ as generic (projective) approximations to the triple $(1,\xi,\xi^2)$ and, in view of Lemma 2.3, we say that $\mathbb{Q}[\mathfrak{X}^{(0)},\mathfrak{X}^{(-1)}]$ is the ring of approximation triples to $(1,\xi,\xi^2)$. Since the second formula in (13) gives

$$\mathfrak{X}_0^{(i+2)} = \begin{pmatrix}\mathfrak{X}_0^{(i+1)} & \mathfrak{X}_1^{(i+1)}\end{pmatrix} M_{i+1} \begin{pmatrix}\mathfrak{X}_0^{(i)} \\ \mathfrak{X}_1^{(i)}\end{pmatrix},$$

we also find that

$$\mathfrak{X}_0^{(i+2)} \sim \theta\mathfrak{X}_0^{(i+1)}\mathfrak{X}_0^{(i)}, \tag{17}$$

where

$$\theta = \begin{pmatrix}1 & \xi\end{pmatrix} M_{i+1} \begin{pmatrix}1 \\ \xi\end{pmatrix} = a_{1,1} + (a_{1,2}+a_{2,1})\xi + a_{2,2}\xi^2$$

is independent of i and non-zero because ξ is transcendental over $\mathbb{Q}$ and M is not skew-symmetric.

For each $d \in \mathbb{N}$, we denote by $\mathbb{Q}[\mathbf{X},\mathbf{X}^*]_{\leqslant d}$ the subspace of $\mathbb{Q}[\mathbf{X},\mathbf{X}^*]$ consisting of all polynomials of total degree at most d, and by $\mathbb{Q}[\mathfrak{X}^{(0)},\mathfrak{X}^{(-1)}]_{\leqslant d}$ its image under the evaluation map π. The next lemma provides a variety of elements of the latter set, with explicit behavior.

Lemma 2.5. *Let $d \in \mathbb{N}$. For each $\alpha = m + n/\gamma \in E_d$ and each integer j with $0 \leqslant j \leqslant 2s_d(\alpha)$, there exists an element $\mathfrak{M}_{\alpha,j}$ of $\mathbb{Q}[\mathfrak{X}^{(0)},\mathfrak{X}^{(-1)}]_{\leqslant d}$ with*

$$\mathfrak{M}_{\alpha,j} \sim \theta^{m+n-s_d(\alpha)}\xi^j\big(\mathfrak{X}_0^{(0)}\big)^m\big(\mathfrak{X}_0^{(-1)}\big)^n.$$

Proof. By recurrence, we deduce from (17) that, for each $i \in \mathbb{Z}$, we have

$$\mathfrak{X}_0^{(i)} \sim \theta^{f(i+1)-1}\big(\mathfrak{X}_0^{(0)}\big)^{f(i)}\big(\mathfrak{X}_0^{(-1)}\big)^{f(i-1)}.$$

Now, let $\alpha = m + n/\gamma \in E_d$, let $s = s_d(\alpha)$, and let j be an integer with $0 \leqslant j \leqslant 2s$. By definition, there exist integers $0 \leqslant i_1 \leqslant \cdots \leqslant i_s$ satisfying (7). Choose also integers $j_1,\ldots,j_s \in \{0,1,2\}$ such that $j_1+\cdots+j_s = j$. Then, we find

$$\prod_{k=1}^{s}\mathfrak{X}_{j_k}^{(-i_k)} \sim \xi^j\prod_{k=1}^{s}\theta^{f(-i_k+1)-1}\big(\mathfrak{X}_0^{(0)}\big)^{f(-i_k)}\big(\mathfrak{X}_0^{(-1)}\big)^{f(-i_k-1)}.$$

Since (7) and (4) are equivalent, the product on the right is simply $\theta^{m+n-s}\xi^j \times (\mathfrak{X}_0^{(0)})^m(\mathfrak{X}_0^{(-1)})^n$. The conclusion then follows by observing that, according to

Lemma 2.3, the product on the left is the value at $(\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)})$ of some bi-homogeneous polynomial of $\mathbb{Q}[\mathbf{X}, \mathbf{X}^*]$ with bi-degree $\sum_{k=1}^{s}(f(i_k-2), f(i_k-1))$, and thus with total degree $\sum_{k=1}^{s} f(i_k) \leqslant d$. $\square$

In fact, we claim that the elements $\mathfrak{M}_{\alpha,j}$ constructed in the preceding lemma form a basis of $\mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]_{\leqslant d}$. To prove this, we will first show that they are linearly independent over $\mathbb{Q}$, and count them using Theorem 2.1. This will provide a lower bound for the dimension of $\mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]_{\leqslant d}$. Next, using the fact that the ideal I is contained in the kernel of the evaluation map π, we will find that the same number is also an upper bound for this dimension. This will prove our claim and will bring other consequences as well. We now proceed to the first step of this programme.

2.5. Growth estimates

Let $\mathfrak{S}_+$ denote the subgroup of $\mathfrak{S}^*$ whose elements are represented by sequences with positive terms. Given $\mathfrak{A}, \mathfrak{B} \in \mathfrak{S}_+$, we write $\mathfrak{A} \ll \mathfrak{B}$ or $\mathfrak{B} \gg \mathfrak{A}$ if there exists a constant $c > 0$ such that corresponding sequences $(a_k)_{k\geqslant 1}$ and $(b_k)_{k\geqslant 1}$ satisfy $a_k \leqslant c b_k$ for each sufficiently large index k. We write $\mathfrak{A} \asymp \mathfrak{B}$ if we both have $\mathfrak{A} \ll \mathfrak{B}$ and $\mathfrak{A} \gg \mathfrak{B}$. The latter is an equivalence relation on $\mathfrak{S}_+$, and the condition (E1) in Section 2.2 can be expressed in the form

$$\left|\mathfrak{X}_0^{(i+1)}\right| \asymp \left|\mathfrak{X}_0^{(i)}\right|^{\gamma}$$

for each $i \in \mathbb{Z}$, where the absolute value and exponentiation are taken "component-wise". In particular, for any $d \in \mathbb{N}$, $\alpha = m + n/\gamma \in E_d$ and $j \in \{0, 1, \ldots, 2s_d(\alpha)\}$, the element $\mathfrak{M}_{\alpha,j}$ of $\mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]_{\leqslant d}$ provided by Lemma 2.5 satisfies

$$|\mathfrak{M}_{\alpha,j}| \asymp \left|\mathfrak{X}_0^{(0)}\right|^{m} \left|\mathfrak{X}_0^{(-1)}\right|^{n} \asymp \left|\mathfrak{X}_0^{(0)}\right|^{m+n/\gamma} = \left|\mathfrak{X}_0^{(0)}\right|^{\alpha}.$$

Since each E_d is a finite set of positive real numbers, this leads to the following conclusion.

Lemma 2.6. *Let $d \in \mathbb{N}$, let $r_{\alpha,j}$ $(\alpha \in E_d,\ 0 \leqslant j \leqslant 2s_d(\alpha))$ be rational numbers not all zero, let $\alpha' = m + n/\gamma$ be the largest element of E_d for which at least one of the numbers $r_{\alpha',j}$ $(0 \leqslant j \leqslant 2s_d(\alpha'))$ is non-zero, and put $s' = s_d(\alpha')$. Then, with the notation of Lemma* 2.5*, the linear combination $\mathfrak{A} = \sum_{\alpha\in E_d} \sum_{j=0}^{2s_d(\alpha)} r_{\alpha,j} \mathfrak{M}_{\alpha,j}$ satisfies*

$$\mathfrak{A} \sim \left(\sum_{j=0}^{2s'} r_{\alpha',j} \xi^{j}\right) \theta^{m+n-s'} \left(\mathfrak{X}_0^{(0)}\right)^{m} \left(\mathfrak{X}_0^{(-1)}\right)^{n} \quad \textit{and} \quad |\mathfrak{A}| \asymp \left|\mathfrak{X}_0^{(0)}\right|^{\alpha'}.$$

We are now ready to complete the first step of the programme outlined at the end of the Section 2.4.

Lemma 2.7. *Let $d \in \mathbb{N}$. The elements $\mathfrak{M}_{\alpha,j}$ $(\alpha \in E_d,\ j = 0, \ldots, 2s_d(\alpha))$ constructed in Lemma* 2.5 *form a $\mathbb{Q}$-linearly independent subset of $\mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]_{\leqslant d}$ with cardinality* $(4d^3 + 6d^2 + 8d + 3)/3$.

Proof. Lemma 2.6 shows that the elements $\mathfrak{M}_{\alpha,j}$ are linearly independent over $\mathbb{Q}$. By Theorem 2.1, their number is

$$\sum_{\alpha\in E_d}\bigl(2s_d(\alpha)+1\bigr)=\sum_{s=0}^{d}\chi_d(s)(2s+1)=(d+1)(2d+1)+\sum_{s=0}^{d-1}(2s+1)^2. \qquad \square$$

2.6. Computation of an Hilbert function

We introduce a new variable U, make the ring $\mathbb{Q}[\mathbf{X},\mathbf{X}^*,U]$ into a graded ring for the total degree, and denote by I_1 the homogeneous ideal of this ring generated by

$$\det(\mathbf{X})-U^2,\qquad \det\bigl(\mathbf{X}^*\bigr)-U^2,\qquad \Phi\bigl(\mathbf{X},\mathbf{X}^*\bigr). \tag{18}$$

Then, for each $d\in\mathbb{N}$, Lemma 2.4 ensures that we have a surjective $\mathbb{Q}$-linear map

$$\begin{aligned}\bigl(\mathbb{Q}\bigl[\mathbf{X},\mathbf{X}^*,U\bigr]/I_1\bigr)_d&\to\mathbb{Q}\bigl[\mathfrak{X}^{(0)},\mathfrak{X}^{(-1)}\bigr]_{\leqslant d},\\ P\bigl(\mathbf{X},\mathbf{X}^*,U\bigr)+I_1&\mapsto P\bigl(\mathfrak{X}^{(0)},\mathfrak{X}^{(-1)},1\bigr).\end{aligned} \tag{19}$$

In particular, this gives

$$\dim_{\mathbb{Q}}\mathbb{Q}\bigl[\mathfrak{X}^{(0)},\mathfrak{X}^{(-1)}\bigr]_{\leqslant d}\leqslant H(I_1;d):=\dim_{\mathbb{Q}}\bigl(\mathbb{Q}\bigl[\mathbf{X},\mathbf{X}^*,U\bigr]/I_1\bigr)_d. \tag{20}$$

Thus, in order to complete the programme outlined at the end of the Section 2.4, it remains to compute the Hilbert function $H(I_1;d)$ of I_1. We achieve this by showing first that the generators (18) of I_1 form a regular sequence in $\mathbb{Q}[\mathbf{X},\mathbf{X}^*,U]$.

Recall that a *regular sequence* in a ring R is a finite sequence of elements $a_1,\dots,a_n$ of R such that, for $i=1,\dots,n$, the multiplication by a_i in the quotient $R/(a_1,\dots,a_{i-1})$ is injective (with the convention that $(a_1,\dots,a_{i-1})=(0)$ for $i=0$). If R is a polynomial ring in m variables over a field, then, for any integer n with $1\leqslant n\leqslant m$, a sequence of n homogeneous polynomials $a_1,\dots,a_n$ of R is regular if and only if the ideal $(a_1,\dots,a_n)$ that it generates has rank (or codimension) equal to n. In that case, any permutation of $a_1,\dots,a_n$ is a regular sequence (see [6, Corollary 5.2.17]).

Lemma 2.8. *The polynomials* $\det(\mathbf{X})$, $\det(\mathbf{X}^*)$ *and* $\Phi(\mathbf{X},\mathbf{X}^*)$ *form a regular sequence in* $\mathbb{Q}[\mathbf{X},\mathbf{X}^*]$.

Proof. Put $R:=\mathbb{Q}[\mathbf{X},\mathbf{X}^*]$. Since $\det(\mathbf{X})$ and $\det(\mathbf{X}^*)$ are relatively prime, they form a regular sequence in R. Moreover, the ideal that they generate is the kernel of the endomorphism of R which maps X_i to $X_0^{2-i}X_1^i$ and X_i^* to $(X_0^*)^{2-i}(X_1^*)^i$ for $i=0,1,2$. The conclusion follows by observing that the image of $\Phi(\mathbf{X},\mathbf{X}^*)$ under this map is

$$\bigl(X_1X_0^*-X_0X_1^*\bigr)\bigl(a_{1,1}X_0X_0^*+a_{1,2}X_0X_1^*+a_{2,1}X_1X_0^*+a_{2,2}X_1X_1^*\bigr),$$

which is a non-zero polynomial. $\square$

We can now turn to the ideal I_1.

Lemma 2.9. *The generators* (18) *of* I_1 *form a regular sequence in* $\mathbb{Q}[\mathbf{X}, \mathbf{X}^*, U]$. *For each* $d \in \mathbb{N}$, *we have*

$$H(I_1; d) = \big(4d^3 + 6d^2 + 8d + 3\big)/3.$$

Proof. Since the natural isomorphism $\mathbb{Q}[\mathbf{X}, \mathbf{X}^*, U]/(U) \to \mathbb{Q}[\mathbf{X}, \mathbf{X}^*]$ induced by the specialization $U \mapsto 0$ maps the sequence of polynomials (18) to the regular sequence of R studied in Lemma 2.8, we deduce that U, $\det(\mathbf{X}) - U^2$, $\det(\mathbf{X}^*) - U^2$ and $\Phi(\mathbf{X}, \mathbf{X}^*)$ form a regular sequence in $\mathbb{Q}[\mathbf{X}, \mathbf{X}^*, U]$. Since these are homogeneous polynomials, it follows that the last three of them, which generate I_1, form a regular sequence. Since the latter are homogeneous of degree 2 and since $\mathbb{Q}[\mathbf{X}, \mathbf{X}^*, U]$ is a polynomial ring in 7 variables, the Hilbert series of the ideal I_1 is given by

$$\sum_{d=0}^{\infty} H(I_1; d)T^d = \frac{(1-T^2)^3}{(1-T)^7} = \frac{(1+T)^3}{(1-T)^4} = (1+T)^3 \sum_{d=0}^{\infty} \binom{d+3}{3} T^d$$

(see for example [6, Corollary 5.2.17]), and a short computation completes the proof. □

2.7. Conclusion

Combining the above result with (20) and Lemma 2.7, we obtain finally the following theorem.

Theorem 2.10. *Let* $d \in \mathbb{N}$. *Then, the map* (19) *is an isomorphism of vector spaces over* $\mathbb{Q}$, *and the elements* $\mathfrak{M}_{\alpha, j}$ ($\alpha \in E_d$, $j = 0, \ldots, 2s_d(\alpha)$) *constructed in Lemma* 2.5 *form a basis of* $\mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]_{\leqslant d}$. *The dimension of the latter vector space is* $(4d^3 + 6d^2 + 8d + 3)/3$.

Applying first Lemma 2.6 and then the growth estimates of Section 2.5, we deduce from this the following two consequences.

Corollary 2.11. *Let* $d \in \mathbb{N}$. *For any non-zero element* $\mathfrak{A}$ *of* $\mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]_{\leqslant d}$, *there exists a point* $\alpha = m + n/\gamma \in E_d$ *and a polynomial* $A \in \mathbb{Q}[T]$ *of degree at most* $2s_d(\alpha)$ *such that*

$$\mathfrak{A} \sim \theta^{m+n-s_d(\alpha)} A(\xi) \big(\mathfrak{X}_0^{(0)}\big)^m \big(\mathfrak{X}_0^{(-1)}\big)^n.$$

Corollary 2.12. *For any non-zero element* $\mathfrak{A}$ *of* $\mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]$, *there exists a point* $\alpha \in \mathbb{Z}[\gamma]$ *such that*

$$\mathfrak{A} \asymp \big|\mathfrak{X}_0^{(0)}\big|^{\alpha}.$$

The map $\mathfrak{A} \mapsto \alpha$ *is a rank two valuation on the ring* $\mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]$.

Finally, we note that, for each $d \in \mathbb{N}$, the linear map (19) factors through the map from $(\mathbb{Q}[\mathbf{X}, \mathbf{X}^*]_{\leqslant d} + I)/I$ to $\mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]_{\leqslant d}$ induced by π. Since the former is an isomorphism, the latter is also an isomorphism, and so we have the following corollary.

Corollary 2.13. *The ideal I defined in Lemma* 2.4 *is the kernel of the evaluation map π from $\mathbb{Q}[\mathbf{X}, \mathbf{X}^*]$ to $\mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]$.*

3. ANALOGOUS RESULTS IN BI-DEGREE

The following result is analogous to Theorem 2.1.

Theorem 3.1. *Let $\mathbf{d} = (d_1, d_2) \in \mathbb{N}^2$. For each $s \in \mathbb{N}$, let $\chi_{\mathbf{d}}(s) = \chi_{d_1,d_2}(s)$ denote the number of points $(m, n) \in \mathbb{Z}^2$ for which the conditions*

$$(21) \qquad \begin{aligned} &(m, n) = \sum_{k=1}^{s} \big(f(-i_k), f(-i_k - 1)\big), \\ &\sum_{k=1}^{s} f(i_k - 2) \leqslant d_1 \quad \text{and} \quad \sum_{k=1}^{s} f(i_k - 1) \leqslant d_2 \end{aligned}$$

admit a solution integers $0 \leqslant i_1 \leqslant \cdots \leqslant i_s$, and s is maximal with this property. Then,

$$(22) \qquad \chi_{d_1,d_2}(s) = \begin{cases} 2\min\{d_1, d_2, s, d_1 + d_2 - s\} + 1 & \text{if } 0 \leqslant s \leqslant d_1 + d_2, \\ 0 & \text{if } s > d_1 + d_2. \end{cases}$$

Note that $\chi_{d_1,d_2}(s)$ possesses several symmetries. For each $(d_1, d_2) \in \mathbb{N}^2$ and each $s = 0, \ldots, d_1 + d_2$, it satisfies

$$\chi_{d_1,d_2}(s) = \chi_{d_2,d_1}(s) \quad \text{and} \quad \chi_{d_1,d_2}(s) = \chi_{d_1,d_2}(d_1 + d_2 - s).$$

As in Section 2, we note that, for a point $\alpha = m + n/\gamma \in \mathbb{Z}[\gamma]$, the conditions (21) are equivalent to

$$(23) \qquad \alpha = \sum_{k=1}^{s} \gamma^{-i_k}, \qquad \sum_{k=1}^{s} f(i_k - 2) \leqslant d_1 \quad \text{and} \quad \sum_{k=1}^{s} f(i_k - 1) \leqslant d_2.$$

For each $\mathbf{d} \in \mathbb{N}^2$, we denote by $E_{\mathbf{d}}$ the set of $\alpha \in \mathbb{Z}[\gamma]$ for which these conditions admit a solution in integers $0 \leqslant i_1 \leqslant \cdots \leqslant i_s$ for some $s \in \mathbb{N}$ and, for each $\alpha \in E_{\mathbf{d}}$, we denote by $s_{\mathbf{d}}(\alpha)$ the largest such s. Then, Theorem 3.1 tells us that, for given $\mathbf{d} \in \mathbb{N}^2$ and $s \in \mathbb{N}$, the number of elements α of $E_{\mathbf{d}}$ with $s_{\mathbf{d}}(\alpha) = s$ is $\chi_{\mathbf{d}}(s)$ given by (22). This will be proved in Section 4.

For each $\mathbf{d} = (d_1, d_2) \in \mathbb{N}^2$, we also denote by $\mathbb{Q}[\mathbf{X}, \mathbf{X}^*]_{\leqslant \mathbf{d}}$ the set of polynomials of $\mathbb{Q}[\mathbf{X}, \mathbf{X}^*]$ with degree at most d_1 in $\mathbf{X}$ and degree at most d_2 in $\mathbf{X}^*$. We also write $\mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]_{\leqslant \mathbf{d}}$ for the image of that set under the evaluation map π defined by (14). We can now state and prove the following bi-degree analog of Theorem 2.10.

Theorem 3.2. *Let* $\mathbf{d} = (d_1, d_2) \in \mathbb{N}^2$. *For each* $\alpha = m + n/\gamma \in E_{\mathbf{d}}$ *and each* $j = 0, \dots, 2s_{\mathbf{d}}(\alpha)$, *there exists an element* $\mathfrak{M}'_{\alpha,j}$ *of* $\mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]_{\leqslant \mathbf{d}}$ *with*

$$\mathfrak{M}'_{\alpha,j} \sim \theta^{m+n-s_{\mathbf{d}}(\alpha)} \xi^j \big(\mathfrak{X}_0^{(0)}\big)^m \big(\mathfrak{X}_0^{(-1)}\big)^n.$$

Any such choice of elements, one for each pair (α, j), *provides a basis of* $\mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]_{\leqslant \mathbf{d}}$. *This vector space has dimension*

$$(d_1 + d_2 + 1)(2d_1d_2 + d_1 + d_2 + 1).$$

Proof. The existence of the elements $\mathfrak{M}'_{\alpha,j}$ is established exactly as in the proof of Lemma 2.5, upon replacing everywhere the symbol d by $\mathbf{d}$, using (23) and (21) instead of (7) and (4). Fix such a choice of elements. The fact that they are linearly independent over $\mathbb{Q}$ is proved as in Section 2.5, upon observing that the statement of Lemma 2.6 still holds when d is replaced by $\mathbf{d}$ and $\mathfrak{M}_{\alpha,j}$ by $\mathfrak{M}'_{\alpha,j}$. According to Theorem 3.1, they form a set of cardinality $\sum_{s=0}^{d_1+d_2} \chi_{\mathbf{d}}(s)(2s+1)$. Since $\chi_{\mathbf{d}}(s) = \chi_{\mathbf{d}}(d_1 + d_2 - s)$ for $s = 0, \dots, d_1 + d_2$, this cardinality is also given by

$$\begin{aligned}
&\frac{1}{2} \sum_{s=0}^{d_1+d_2} \Big(\chi_{\mathbf{d}}(s)(2s+1) + \chi_{\mathbf{d}}(s)\big(2(d_1 + d_2 - s) + 1\big)\Big) \\
&\quad = (d_1 + d_2 + 1) \sum_{s=0}^{d_1+d_2} \chi_{\mathbf{d}}(s).
\end{aligned}$$

A short computation based on the formula (22) shows that the right-most sum is equal to $2d_1d_2 + d_1 + d_2 + 1$ (an alternative approach is to note that this sum is the cardinality of $E_{\mathbf{d}}$ and to use Corollary 4.5). Thus the elements $\mathfrak{M}'_{\alpha,j}$ span a subspace of dimension $(d_1 + d_2 + 1)(2d_1d_2 + d_1 + d_2 + 1)$. To complete the proof, it remains only to show that the dimension of $\mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]_{\leqslant \mathbf{d}}$ is no more than this. To that end, we proceed as in Section 2.6. We introduce two new indeterminates V and V^* and, for each $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$, we denote by $\mathbb{Q}[\mathbf{X}, V, \mathbf{X}^*, V^*]_{\mathbf{n}}$ the subspace of $\mathbb{Q}[\mathbf{X}, V, \mathbf{X}^*, V^*]$ whose elements are homogeneous in $(\mathbf{X}, V)$ of degree n_1 and homogeneous in $(\mathbf{X}^*, V^*)$ of degree n_2. This makes the polynomial ring $R_2 := \mathbb{Q}[\mathbf{X}, V, \mathbf{X}^*, V^*]$ into a $\mathbb{N}^2$-graded ring. Let I_2 denote the bi-homogeneous ideal of R_2 generated by

$$\det(\mathbf{X}) - V^2, \qquad \det\big(\mathbf{X}^*\big) - \big(V^*\big)^2, \qquad \Phi\big(\mathbf{X}, \mathbf{X}^*\big). \tag{24}$$

Lemma 2.4 ensures that we have a surjective $\mathbb{Q}$-linear map in each bi-degree $\mathbf{n}$

$$\begin{aligned}
&\big(\mathbb{Q}\big[\mathbf{X}, V, \mathbf{X}^*, V^*\big]/I_2\big)_{\mathbf{n}} \to \mathbb{Q}\big[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}\big]_{\leqslant \mathbf{n}}, \\
&\ P\big(\mathbf{X}, V, \mathbf{X}^*, V^*\big) + I_2 \mapsto P\big(\mathfrak{X}^{(0)}, 1, \mathfrak{X}^{(-1)}, 1\big).
\end{aligned} \tag{25}$$

In particular, this gives $\dim_{\mathbb{Q}} \mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]_{\leqslant \mathbf{d}} \leqslant H(I_2; \mathbf{d})$ where $H(I_2; \mathbf{n})$ stands for the Hilbert function of I_2 at $\mathbf{n}$, namely the dimension of the domain of the linear map (25). As in the proof of Lemma 2.9, we deduce from Lemma 2.8 that the generators (24) of I_2 form a regular sequence in R_2. Since these generators are bi-homogeneous of bi-degree $(2,0)$, $(0,2)$ and $(1,1)$, and since the grading of R_2 involves two sets of 4 variables, we deduce that the Hilbert series of I_2 is

$$\begin{aligned}
&\sum_{n_1,n_2\in\mathbb{N}} H(I_2;n_1,n_2)T_1^{n_1}T_2^{n_2}\\
&\quad=\frac{(1-T_1^2)(1-T_2^2)(1-T_1T_2)}{(1-T_1)^4(1-T_2)^4}\\
&\quad=(1-T_1T_2)\sum_{n_1,n_2\in\mathbb{N}}(n_1+1)^2(n_2+1)^2T_1^{n_1}T_2^{n_2}
\end{aligned}$$

(see [6, Proposition 5.8.13 and Theorem 5.8.15]). This completes the proof as it implies that

$$\begin{aligned}
H(I_2;\mathbf{d})&=(d_1+1)^2(d_2+1)^2-d_1^2d_2^2\\
&=(d_1+d_2+1)(2d_1d_2+d_1+d_2+1). \qquad\square
\end{aligned}$$

Note that this result implies that the statement of Corollary 2.11 still holds in bi-degree, with d replaced by $\mathbf{d}$.

4. COMBINATORIAL STUDY

This section is devoted to the proof of Theorems 2.1 and 3.1. As mentioned in Section 2.1, we work within the ring $\mathbb{Z}[\gamma]$. We define

$$E=\{\alpha\in\mathbb{Z}[\gamma];\alpha\geqslant 0\}\quad\text{and}\quad E^*=E\setminus\{0\},$$

and note that, since $\gamma>0$, the sets E_d and $E_{\mathbf{d}}$ defined respectively in Sections 2.1 and 3 are subsets of E. Our first goal is to provide a more explicit description of these.

4.1. A partition

We first establish a partition of E^*.

Proposition 4.1. *The sets*

$$E^{(+)}=\left\{m+n\gamma^{-1};m,n\geqslant 1\right\}$$

and

$$E^{(i)}=\left\{m\gamma^{-i}+n\gamma^{-i-2};m\geqslant 1,n\geqslant 0\right\}\quad \textit{for } i\geqslant 0$$

form a partition $E^*=E^{(+)}\coprod\left(\coprod_{i=0}^{\infty}E^{(i)}\right)$ *of* E^*.

Proof. Consider the bijection $\varphi:\mathbb{Z}[\gamma]\to\mathbb{Z}^2$ which maps a point $m+n/\gamma$ to its pair of coordinates (m,n) relative to the basis $\{1,1/\gamma\}$ of $\mathbb{Z}[\gamma]$. It identifies E^* with the set of non-zero points (m,n) of $\mathbb{Z}^2$ whose argument in polar coordinates satisfies

$$-\arctan(\gamma)<\arg(m,n)<\pi-\arctan(\gamma).$$

Using the formulas (6), a quick recurrence argument shows that, for each index $i \geqslant 0$, the determinant of the points $\varphi(\gamma^{-i})$ and $\varphi(\gamma^{-i-2})$ is

$$\begin{vmatrix} f(i-2) & -f(i-1) \\ f(i) & -f(i+1) \end{vmatrix} = (-1)^{i+1}.$$

This means that $\{\varphi(\gamma^{-i}), \varphi(\gamma^{-i-2})\}$ forms a basis of $\mathbb{Z}^2$ for each $i \geqslant 0$. Since the points $\varphi(\gamma^{-2i})$ have positive first coordinate, it also means that $\arg\varphi(\gamma^{-2i})$ is a strictly decreasing function of $i \geqslant 0$ starting from $\arg\varphi(\gamma^0) = 0$. Finally, since the points $\varphi(\gamma^{-2i-1})$ have positive second coordinate, it tells us that $\arg\varphi(\gamma^{-2i-1})$ is a strictly increasing function of $i \geqslant 0$ starting from $\arg\varphi(\gamma^{-1}) = \pi/2$. In other words, we have

$$\begin{aligned} \cdots < \arg\varphi\big(\gamma^{-4}\big) < \arg\varphi\big(\gamma^{-2}\big) < \arg\varphi\big(\gamma^{0}\big) \\ < \arg\varphi\big(\gamma^{-1}\big) < \arg\varphi\big(\gamma^{-3}\big) < \arg\varphi\big(\gamma^{-5}\big) < \cdots. \end{aligned}$$

We conclude from this that a point α of E^* belongs to $E^{(+)}$ if and only if $\arg\varphi(\gamma^0) < \arg\varphi(\alpha) < \arg\varphi(\gamma^{-1})$, and that it belongs to $E^{(i)}$ for some $i \geqslant 0$ if and only if $\arg\varphi(\alpha)$ lies between $\arg\varphi(\gamma^{-i})$ and $\arg\varphi(\gamma^{-i-2})$, with the first end point included and the second excluded. In particular the sets $E^{(+)}$ and $E^{(i)}$ with $i \geqslant 0$ are all disjoint. They cover E^* because the fact that $\lim_{j\to\infty} f(j)/f(j-1) = \gamma$ implies that $\arg\varphi(\gamma^{-2i})$ and $\arg\varphi(\gamma^{-2i-1})$ tend respectively to $-\arctan(\gamma)$ and $\pi - \arctan(\gamma)$ as $i \to \infty$. $\square$

With our convention that an empty sum is zero, this implies the corollary:

Corollary 4.2. *Any $\alpha \in E$ can be written in the form*

$$\alpha = \gamma^{-i_1} + \cdots + \gamma^{-i_s} \tag{26}$$

for a choice of integers $s \geqslant 0$ and $0 \leqslant i_1 \leqslant \cdots \leqslant i_s$.

We say that a finite non-decreasing sequence of non-negative integers $\mathbf{i} = (i_1, \ldots, i_s)$ is a *representation* of a point α of E if it satisfies the condition (26). In particular, the only representation of the point 0 is the empty sequence.

4.2. Degree and bi-degree

For any finite non-decreasing sequence of non-negative integers $\mathbf{i} = (i_1, \ldots, i_s)$, we define

$$d(\mathbf{i}) = \sum_{k=1}^{s} f(i_k), \qquad d_1(\mathbf{i}) = \sum_{k=1}^{s} f(i_k - 2), \qquad d_2(\mathbf{i}) = \sum_{k=1}^{s} f(i_k - 1),$$
$$\mathbf{d}(\mathbf{i}) = \big(d_1(\mathbf{i}), d_2(\mathbf{i})\big) \quad \text{and} \quad s(\mathbf{i}) = s.$$

We say that $d(\mathbf{i})$, $\mathbf{d}(\mathbf{i})$ and $s(\mathbf{i})$ are respectively the *degree*, *bi-degree* and *size* of the point $\mathbf{i}$, while $d_1(\mathbf{i})$ and $d_2(\mathbf{i})$ are respectively the *first* and *second partial degrees* of $\mathbf{i}$. For the empty sequence, all these integers are zero. We also put a partial order on $\mathbb{N}^2$ by writing $(m,n) \leqslant (m',n')$ if $m \leqslant m'$ and $n \leqslant n'$. We can now state and prove the following proposition.

Proposition 4.3. *Let $\alpha = m + n/\gamma \in E^*$ and let $\mathbf{i} = (i_1, \dots, i_s)$ be a representation of α. Then we have $d(\mathbf{i}) \geqslant |m| + |n|$ and $\mathbf{d}(\mathbf{i}) \geqslant (|m|, |n|)$. Both inequalities are equalities if $i_s \leqslant 1$ or if $i_1, \dots, i_s$ share the same parity. Otherwise, they become strict inequalities. Moreover, we have $d_2(\mathbf{i}) > |n|$ if $i_1, \dots, i_s$ contains a pair of positive integers not of the same parity.*

Proof. Since $\alpha = \gamma^{-i_1} + \cdots + \gamma^{-i_s}$, the formulas (6) imply that

$$m = \sum_{k=1}^{s} (-1)^{i_k} f(i_k - 2) \quad \text{and} \quad n = \sum_{k=1}^{s} (-1)^{i_k+1} f(i_k - 1).$$

From this we deduce that

$$|m| \leqslant \sum_{k=1}^{s} f(i_k - 2) = d_1(\mathbf{i}) \quad \text{and} \quad |n| \leqslant \sum_{k=1}^{s} f(i_k - 1) = d_2(\mathbf{i}),$$

and the conclusion follows because $f(-2) = 1$, $f(-1) = 0$ and $f(i) \geqslant 1$ for each $i \in \mathbb{N}$. □

Since, by Proposition 4.1, each $\alpha \in E^*$ admits a representation $\mathbf{i} = (i_1, \dots, i_s)$ with $i_s \leqslant 1$ or with $i_1, \dots, i_s$ of the same parity, we deduce that:

Corollary 4.4. *Each $\alpha = m + n/\gamma \in E$ admits a representation with largest degree $d(\alpha) := |m| + |n|$ and largest bi-degree $\mathbf{d}(\alpha) := (|m|, |n|)$.*

We say that the quantities $d(\alpha)$ and $\mathbf{d}(\alpha)$ defined in the above corollary are respectively the *degree* and *bi-degree* of α.

Let $d \in \mathbb{N}$ and $\mathbf{d} = (d_1, d_2) \in \mathbb{N}^2$. In Section 2.1 (resp. Section 3), we defined E_d (resp. $E_{\mathbf{d}}$) as the set of points which admit a representation of degree $\leqslant d$ (resp. of bi-degree $\leqslant \mathbf{d}$). According to the corollary, it can also be described as the set of elements of E with degree $\leqslant d$ (resp. with bi-degree $\leqslant \mathbf{d}$):

$$E_d = \{m + n/\gamma \in E;\ |m| + |n| \leqslant d\}$$

(resp. $E_{\mathbf{d}} = \{m + n/\gamma \in E;\ |m| \leqslant d_1,\ |n| \leqslant d_2\}$).

We can now easily compute the cardinality of these sets.

Corollary 4.5. *Let $d \in \mathbb{N}$ and $\mathbf{d} = (d_1, d_2) \in \mathbb{N}^2$. Then, we have $|E_d| = d^2 + d + 1$ and $|E_{\mathbf{d}}| = 2d_1d_2 + d_1 + d_2 + 1$.*

Proof. Denote by $\mathcal{L}$ the set of all non-zero points (m,n) in $\mathbb{Z}^2$ satisfying $|m| + |n| \leqslant d$ (resp. $|m| \leqslant d_1$ and $|n| \leqslant d_2$). Define also $\mathcal{L}^+$ to be the set of points (m,n) in $\mathcal{L}$

for which $m + n/\gamma > 0$. Then, $E_d \setminus \{0\}$ (resp. $E_{\mathbf{d}} \setminus \{0\}$) is in bijection with $\mathcal{L}^+$. As the sets $\mathcal{L}^+$ and $-\mathcal{L}^+$ form a partition of $\mathcal{L}$ in two subsets of the same cardinality, it follows that the cardinality of E_d (resp. of $E_{\mathbf{d}}$) is $1 + |\mathcal{L}|/2$, and the conclusion follows upon noting that $|\mathcal{L}|$ is $2d(d+1)$ (resp. $(2d_1+1)(2d_2+1)-1$). $\square$

4.3. Representations by quads

Let $\alpha \in E$. For each $d \in \mathbb{N}$ such that $\alpha \in E_d$, we define the *size* $s_d(\alpha)$ *of* α *relative to* d to be the largest size of a representation of α of degree $\leqslant d$ (see Section 2.1). Similarly, for each $\mathbf{d} \in \mathbb{N}^2$ such that $\alpha \in E_{\mathbf{d}}$, we define the *size* $s_{\mathbf{d}}(\alpha)$ *of* α *relative to* $\mathbf{d}$ to be the largest size of a representation of α of bi-degree $\leqslant \mathbf{d}$ (see Section 3). The next proposition shows that, in order to compute the various degrees and sizes of α, it suffices to consider only representations of the form

$$\alpha = a\gamma^{-i} + b\gamma^{-i-1} + c\gamma^{-i-2} \tag{27}$$

with $i, a, b, c \in \mathbb{N}$, and $a \geqslant 1$ if $\alpha \neq 0$.

Proposition 4.6. *Let $d \in \mathbb{N}^*$ and let $\mathbf{d} \in \mathbb{N}^2 \setminus \{(0,0)\}$. Each $\alpha \in E_d \setminus \{0\}$ admits a representation $\mathbf{i} = (i_1, \dots, i_s)$ with degree $d(\mathbf{i}) \leqslant d$ and size $s = s_d(\alpha)$ for which $i_s \leqslant i_1 + 2$. Similarly, each $\alpha \in E_{\mathbf{d}} \setminus \{0\}$ admits a representation $\mathbf{i} = (i_1, \dots, i_s)$ with bi-degree $\mathbf{d}(\mathbf{i}) \leqslant \mathbf{d}$ and size $s = s_{\mathbf{d}}(\alpha)$ for which $i_s \leqslant i_1 + 2$.*

Proof. Let $\alpha \in E_d \setminus \{0\}$. Put $s = s_d(\alpha)$, and choose a representation $\mathbf{i} = (i_1, \dots, i_s)$ of α of size s with minimal degree. We claim that $\mathbf{i}$ has all the required properties. First it satisfies $d(\mathbf{i}) \leqslant d$ by definition of $s_d(\alpha)$. It remains to show that $i_s \leqslant i_1 + 2$.

To show this, we first observe that, for any pair of integers (p, k) with $k \geqslant 1$, we have

$$\gamma^{-p} + \gamma^{-p-2k-1} = \left(\sum_{i=0}^{k-1} \gamma^{-p-2i-1}\right) + \gamma^{-p-2k+1},$$

$$\gamma^{-p} + \gamma^{-p-2k-2} = \gamma^{-p-2} + \left(\sum_{i=1}^{k} \gamma^{-p-2i}\right) + \gamma^{-p-2k}.$$

Assuming that $i_s \geqslant i_1 + 3$, these formulas show that the point $\beta = \gamma^{-i_1} + \gamma^{-i_s}$ admits a representation $\mathbf{j} = (j_1, \dots, j_t)$ with coordinates of the same parity as i_s, size $t = 2$ if $i_s = i_1 + 3$, and size $t \geqslant 3$ if $i_s > i_1 + 3$. In this case, Proposition 4.3 gives $d(\mathbf{j}) = d(\beta)$ and also $d(\beta) \leqslant d(i_1, i_s)$ with the strict inequality if $i_s = i_1 + 3$. Then, upon reorganizing terms in the decomposition

$$\alpha = \left(\gamma^{-j_1} + \cdots + \gamma^{-j_t}\right) + \left(\gamma^{-i_2} + \cdots + \gamma^{-i_{s-1}}\right),$$

we get a representation $\mathbf{i}'$ of α with degree $d(\mathbf{i}') = d(\mathbf{i}) + d(\beta) - d(i_1, i_s)$ and size $s(\mathbf{i}') = s + t - 2$. If $i_s = i_1 + 3$, we have $d(\mathbf{i}') < d(\mathbf{i})$ and $s(\mathbf{i}') = s$ in contradiction with the choice of $\mathbf{i}$. If $i_s > i_1 + 3$, we find that $d(\mathbf{i}') \leqslant d(\mathbf{i}) \leqslant d$ and $s(\mathbf{i}') > s = s_d(\alpha)$ in contradiction with the definition of $s_d(\alpha)$. Thus, we must have $i_s \leqslant i_1 + 2$.

This proves the first assertion of the proposition. The proof of the second assertion is the same provided that one replaces everywhere the word "degree" by "bi-degree", and the symbol d by $\mathbf{d}$. $\square$

We define a *quad* q to be an expression of the form $q = (i; a, b, c)$ with $i, a, b, c \in \mathbb{N}$ and $a \geqslant 1$. We say that a quad q as above represents a point $\alpha \in E$ if it satisfies (27). Identifying it with the sequence formed by a occurrences of i followed by b occurrences of $i+1$ and c occurrences of $i+2$, the various notions of degree and size translate to

$$\begin{aligned} &d_1(q) = af(i-2) + bf(i-1) + cf(i), \\ &d_2(q) = af(i-1) + bf(i) + cf(i+1), \\ &d(q) = d_1(q) + d_2(q), \\ &\mathbf{d}(q) = \big(d_1(q),\ d_2(q)\big) \quad \text{and} \quad s(q) = a + b + c. \end{aligned} \tag{28}$$

In this context, Proposition 4.6 shows that for any $\alpha \in E^*$ and any integer $d \geqslant d(\alpha)$ (resp. any integer pair $\mathbf{d} \geqslant \mathbf{d}(\alpha)$), the integer $s_d(\alpha)$ (resp. $s_{\mathbf{d}}(\alpha)$) is the largest size of a quad of degree $\leqslant d$ (resp. of bi-degree $\leqslant \mathbf{d}$) which represents α.

4.4. Sequences of quads

For each $\alpha \in E^*$, we denote by Q_α the set of quads which represent α, and, for each $\mathbf{d} \in \mathbb{N}^2$, we denote by $Q_{\mathbf{d}}$ the set of quads of bi-degree $\mathbf{d}$. Although we use the same letter for both kinds of sets, the nature of the subscript should in practice remove any ambiguity. As we will see, these families have similar properties. We start with those of the first kind.

Proposition 4.7. *Let $\alpha \in E^*$. The set Q_α of all quads representing α is an infinite set whose elements have distinct size. If we order its elements by increasing size, then their sizes form an increasing sequence of consecutive integers while their degrees, bi-degrees, and second partial degrees form strictly increasing sequences in $\mathbb{N}$, $\mathbb{N}^2$ and $\mathbb{N}$ respectively. The element of Q_α of smallest size is the quad of degree $d(\alpha)$ and bi-degree $\mathbf{d}(\alpha)$ associated to the representation of α given by Proposition* 4.1.

Proof. The relation $\gamma^{-i} = \gamma^{-i-1} + \gamma^{-i-2}$ shows that, for each $q = (i; a, b, c) \in Q_\alpha$, the quad

$$\theta(q) = \begin{cases} (i; a-1, b+1, c+1) & \text{if } a \geqslant 2, \\ (i+1; b+1, c+1, 0) & \text{if } a = 1 \end{cases} \tag{29}$$

also represents α. This defines an injective map θ from Q_α to itself, which increases the size of a quad by 1. Since the size of any quad is finite and non-negative, this implies that any $q \in Q_\alpha$ can be written in a unique way in the form $q = \theta^j(q_0)$ where $j \in \mathbb{N}$ and where q_0 is an element of Q_α which does not belong to the image of θ. The latter condition on q_0 means that it is of the form $q_0 = (0; a, b, 0)$ with

$a, b \geqslant 1$ or $q_0 = (i; a, 0, c)$ with $a \geqslant 1$. According to Proposition 4.1, there exists exactly one representation of α of that form and, by Proposition 4.3, it has degree $d(\alpha)$ and bi-degree $\mathbf{d}(\alpha)$. Thus q_0 is the element of Q_α of smallest size, and we can organize Q_α in a sequence $(\theta^j(q_0))_{j\geqslant 0}$ where the size increases by steps of 1. Along this sequence, the degree, bi-degree and second partial degree are strictly increasing, since for any $q = (i; a, b, c) \in Q_\alpha$, the formula (29) implies that we have $\mathbf{d}(\theta(q)) = \mathbf{d}(q) + (2f(i-1), 2f(i))$. $\square$

The proof of the above proposition provides an explicit recursive way of constructing the elements of Q_α by order of increasing size: given any $q \in Q_\alpha$ the next element is $\theta(q)$. We will not use this explicit formula in the sequel, except in the proof of the second corollary below.

Corollary 4.8. *Let $d \in \mathbb{N}^*$, $\mathbf{d} \in \mathbb{N}^2 \setminus \{(0,0)\}$, and $\alpha \in E^*$. If $\alpha \in E_d$, then $s_d(\alpha)$ is the size of the quad of largest degree $\leqslant d$ which represents α. If $\alpha \in E_{\mathbf{d}}$, then $s_{\mathbf{d}}(\alpha)$ is the size of the quad of largest bi-degree $\leqslant \mathbf{d}$ which represents α.*

Proof. Suppose that $\alpha \in E_d$. Then, by Proposition 4.6, the size $s_d(\alpha)$ of α relative to d is the largest size achieved by a quad of degree $\leqslant d$ in Q_α. By Proposition 4.7, this is also the size of the quad of largest degree $\leqslant d$ in Q_α. The proof of the assertion in bi-degree $\mathbf{d}$ is similar. $\square$

Corollary 4.9. *Let $d \in \mathbb{N}^*$ and $\alpha \in E_d$. There exists one and only one representative $(i; a, b, c)$ of α which satisfies*

$$d - 2f(i+1) < af(i) + bf(i+1) + cf(i+2) \leqslant d. \tag{30}$$

For this choice of quad, one has $s_d(\alpha) = a + b + c$.

Proof. By Corollary 4.8 and the remark following Proposition 4.7, the integer $s_d(\alpha)$ is the size of the unique quad q in Q_α satisfying $d(q) \leqslant d < d(\theta(q))$. Upon writing $q = (i; a, b, c)$ and using the formula (29) for $\theta(q)$, the latter inequality translates into (30), and the conclusion follows. $\square$

Proposition 4.10. *Let $\mathbf{d} = (d_1, d_2) \in \mathbb{N}^2 \setminus \{(0,0)\}$. The set $Q_{\mathbf{d}}$ of all quads of bi-degree $\mathbf{d}$ is a finite non-empty set whose elements have distinct size. If we order its elements by decreasing size, then their sizes form a decreasing sequence of consecutive integers while the points of $E_{\mathbf{d}}$ that they represent form a strictly decreasing sequence of positive real numbers. The element of $Q_{\mathbf{d}}$ of largest size has size $d_1 + d_2$ and represents the point $d_1 + d_2/\gamma$, while the element of $Q_{\mathbf{d}}$ of smallest size represents the point $|d_1 - d_2/\gamma|$, both points being of bi-degree $\mathbf{d}$. The elements of $Q_{\mathbf{d}}$ of intermediate sizes represent points of bi-degree $\leqslant (d_1, d_2 - 1)$.*

Proof. The set $Q_{\mathbf{d}}$ is not empty as it contains the quad

$$q_0 = \begin{cases} (0; d_1, d_2, 0) & \text{if } d_1 > 0, \\ (1; d_2, 0, 0) & \text{if } d_1 = 0. \end{cases}$$

Since $f(-2) = 1$, $f(-1) = 0$ and $f(j) \geqslant 1$ for each $j \geqslant 0$, the formula (28) for the bi-degree shows that q_0 is the only element of $Q_{\mathbf{d}}$ if $d_1 = 0$ or if $d_2 = 0$. As the proposition is easily verified in that case, we may assume that d_1 and d_2 are positive.

Define $Q_{\mathbf{d}}^+$ to be the set of quads $q = (i; a, b, c)$ of $Q_{\mathbf{d}}$ with $b \geqslant 1$. Then, under our present assumptions, $Q_{\mathbf{d}}^+$ is not empty as it contains the point q_0. Moreover, the recurrence relation for the function f combined with (28) shows that one defines a map $\psi : Q_{\mathbf{d}}^+ \to Q_{\mathbf{d}}$ by sending a quad $q = (i; a, b, c) \in Q_{\mathbf{d}}^+$ to

$$\psi(q) = \begin{cases} (i; a-1, b-1, c+1) & \text{if } a \geqslant 2, \\ (i+1; b-1, c+1, 0) & \text{if } a = 1 \text{ and } b \geqslant 2, \\ (i+2; c+1, 0, 0) & \text{if } a = b = 1. \end{cases} \tag{31}$$

This map is injective and decreases the size of a quad by 1. Thus any $q \in Q_{\mathbf{d}}$ can be written in a unique way in the form $q = \psi^j(q_0')$ where $j \in \mathbb{N}$ and where q_0' is an element of $Q_{\mathbf{d}}$ which does not belong to the image of ψ. This means that q_0' is of the form $(0; a, b, 0)$ with $a, b \geqslant 1$ or $(1; a, 0, 0)$ with $a \geqslant 1$, and thus that $q_0' = q_0$ since its bi-degree is $\mathbf{d}$.

The above discussion shows that we can organize $Q_{\mathbf{d}}$ in a sequence $(\psi^j(q_0))_{j=0}^t$ where the size decreases by steps of 1, starting from the element q_0 of $Q_{\mathbf{d}}$ of largest size $d_1 + d_2$, and ending with the element $q_t := \psi^t(q_0)$ of smallest size. Since the quad q_t does not belong to $Q_{\mathbf{d}}^+$, it has the form $(i; a, 0, c)$ for some $i \geqslant 0$. In the notation of Proposition 4.1, it thus represents a point of $E^{(i)} \subset E \setminus E^{(+)}$ which, by Proposition 4.3, has bi-degree exactly $\mathbf{d}$. Since $d_1 + d_2/\gamma \in E^{(+)}$ and $|d_1 - d_2/\gamma| \in E \setminus E^{(+)}$ are the only points of E of bi-degree $\mathbf{d}$, we conclude that q_0 and q_t are respectively the representations of $d_1 + d_2/\gamma$ and $|d_1 - d_2/\gamma|$ coming from Proposition 4.1. All intermediate quads $\psi^j(q_0)$ with $j = 1, \dots, t-1$ belong to $Q_{\mathbf{d}}^+ \setminus \{q_0\}$. They have the form $(i; a, b, c)$ with $i = 0$ and $a, b, c \geqslant 1$, or with $i \geqslant 1$ and $a, b \geqslant 1$. Therefore, by Proposition 4.3, they represent points of bi-degree $\leqslant (d_1, d_2 - 1)$. Finally, for any given $q = (i; a, b, c) \in Q_{\mathbf{d}}^+$, the quad $\psi(q)$ given by (31) represents the point $(a-1)\gamma^{-i} + (b-1)\gamma^{-i-1} + (c+1)\gamma^{-i-2}$ of E^* which, as a real number, is smaller than the point $a\gamma^{-i} + b\gamma^{-i-1} + c\gamma^{-i-2}$ represented by q. Thus the points of E^* represented by quads in $Q_{\mathbf{d}}$ decrease (in absolute value) with the size of these quads. □

4.5. Proof of Theorem 3.1

We prove it in the following form.

Theorem 4.11. *Let $\mathbf{d} = (d_1, d_2) \in \mathbb{N}^2$. For each integer $s \geqslant 0$, define*

$$E_{\mathbf{d}}(s) = \{\alpha \in E_{\mathbf{d}}; s_{\mathbf{d}}(\alpha) = s\}.$$

Then, for $s > d_1 + d_2$, this set is empty while for $0 \leqslant s \leqslant d_1 + d_2$ its cardinality is

$$|E_{\mathbf{d}}(s)| = 2\min\{d_1, d_2, s, d_1 + d_2 - s\} + 1. \tag{32}$$

Proof. We fix a choice of $d_1 \geqslant 0$ and prove the theorem by recurrence on $d_2 \geqslant 0$. For $d_2 = 0$, we have $E_{d_1,0} = \{0, 1, \ldots, d_1\}$ and $s_{d_1,0}(i) = i$ for $i = 0, 1, \ldots, d_1$. Therefore $E_{d_1,0}(s)$ has cardinality 1 for $0 \leqslant s \leqslant d_1$ and is empty for $s > d_1$, as asserted by the theorem. Suppose now that $d_2 > 0$ and that the statement of the theorem holds in bi-degree $(d_1, d_2 - 1)$.

Fix an integer $s \geqslant 0$. In order to establish the formula in bi-degree (d_1, d_2), we first compare the sets $E_{d_1,d_2}(s)$ and $E_{d_1,d_2-1}(s)$.

(1) According to Corollary 4.8, the points of $E_{d_1,d_2}(s)$ which do not belong to $E_{d_1,d_2-1}(s)$ are the elements α of E for which the quad of Q_α of largest bi-degree $\leqslant (d_1, d_2)$ has size s but does not have bi-degree $\leqslant (d_1, d_2 - 1)$. They are therefore the points of E which are represented by an element of Q_{i,d_2} of size s for some integer i with $0 \leqslant i \leqslant d_1$. Since by Proposition 4.7 the quads representing the same point have distinct second partial degrees, and since by Proposition 4.10 each Q_{i,d_2} contains at most one element of size s, we conclude that the cardinality of the set $E_{d_1,d_2}(s) \setminus E_{d_1,d_2-1}(s)$ is the number of indices i with $0 \leqslant i \leqslant d_1$ such that Q_{i,d_2} contains an element of size s.

(2) According again to Corollary 4.8, the points of $E_{d_1,d_2-1}(s)$ which do not belong to $E_{d_1,d_2}(s)$ are the points α of E_{d_1,d_2-1} for which Q_α contains both a quad of size s and bi-degree $\leqslant (d_1, d_2 - 1)$, and a quad of size $s + 1$ and bi-degree (i, d_2) for some i with $0 \leqslant i \leqslant d_1$. The first condition however is redundant because if $\alpha \in E_{d_1,d_2-1}$ is represented by a quad of size $s + 1$ and bi-degree (i, d_2) with $0 \leqslant i \leqslant d_1$, then as the second partial degree increases with the size in Q_α while the first partial degree does not decrease (by Proposition 4.7), the quad of Q_α with largest bi-degree $\leqslant (d_1, d_2 - 1)$ must have size s. Thus, the set $E_{d_1,d_2-1}(s) \setminus E_{d_1,d_2}(s)$ consists of the points of E_{d_1,d_2-1} which are represented by an element of Q_{i,d_2} of size $s + 1$ for some i with $0 \leqslant i \leqslant d_1$. Moreover, according to Proposition 4.10, for a given $i \in \mathbb{N}$ all elements of Q_{i,d_2} represent points of bi-degree $\leqslant (i, d_2 - 1)$ except for the ones of smallest or largest size. Therefore, the cardinality of $E_{d_1,d_2-1}(s) \setminus E_{d_1,d_2}(s)$ is the number of indices i with $0 \leqslant i \leqslant d_1$ such that Q_{i,d_2} contains an element of size s and an element of size $s + 2$.

Combining the conclusions of (1) and (2), we obtain that the cardinality of $E_{d_1,d_2}(s)$ is equal to that of $E_{d_1,d_2-1}(s)$ plus the number of indices i with $0 \leqslant i \leqslant d_1$ such that Q_{i,d_2} contains an element of size s but no element of size $s + 2$. Since, by Proposition 4.10, the largest size of an element of Q_{i,d_2} is $i + d_2$, the latter condition on i amounts to either $i + d_2 = s$ or both $i + d_2 = s + 1$ and $i \neq 0$ (so that Q_{i,d_2} contains at least two elements and thus contains an element of size s). This provides the recurrence relation

$$|E_{d_1,d_2}(s)| = |E_{d_1,d_2-1}(s)| + \begin{cases} 0 & \text{if } s < d_2 \text{ or } s > d_1 + d_2, \\ 1 & \text{if } s = d_1 + d_2, \\ 2 & \text{if } d_2 \leqslant s < d_1 + d_2. \end{cases}$$

Combining this with the induction hypothesis for $|E_{d_1,d_2-1}(s)|$, we get $|E_{d_1,d_2}(s)| = 0$ if $s > d_1 + d_2$ and $|E_{d_1,d_2}(s)| = 1$ if $s = d_1 + d_2$. If $s < d_1 + d_2$, it also provides the required value (32) for $|E_{d_1,d_2}(s)|$ because the difference

$$\min\{d_1, d_2, s, d_1 + d_2 - s\} - \min\{d_1, d_2 - 1, s, d_1 + d_2 - 1 - s\}$$

is 0 if $\min\{d_1, s\} < \min\{d_2, d_1 + d_2 - s\}$ and 1 otherwise. Since $\min\{d_2, d_1 + d_2 - s\} = d_2 - s + \min\{d_1, s\}$, this difference is therefore 0 if $s < d_2$ and is 1 if $d_2 \leqslant s < d_1 + d_2$. $\square$

4.6. Proof of Theorem 2.1

Similarly, we prove Theorem 2.1 in the following form.

Theorem 4.12. *Let $d \in \mathbb{N}$. For each integer $s \geqslant 0$, define*

$$E_d(s) = \{\alpha \in E_d; s_d(\alpha) = s\}.$$

Then, for $s > d$, this set is empty while for $0 \leqslant s \leqslant d$ its cardinality is

$$|E_d(s)| = \begin{cases} 2s + 1 & \text{if } 0 \leqslant s < d, \\ d + 1 & \text{if } s = d. \end{cases}$$

Proof. We proceed by recurrence on d. For $d = 0$, we have $E_0 = \{0\}$ and since $s_0(0) = 0$, the theorem is verified in that case. Suppose now that $d > 0$ and that the conclusion of the theorem holds in smaller degree.

Fix an integer $s \geqslant 0$. Arguing in a similar way as in the proof of Theorem 4.11, we find that:

(1) $E_d(s) \setminus E_{d-1}(s)$ consists of the points of E which are represented by a quad of $Q_{i,d-i}$ of size s for some integer i with $0 \leqslant i \leqslant d$; its cardinality is the number of indices i with $0 \leqslant i \leqslant d$ such that $Q_{i,d-i}$ contains an element of size s;
(2) $E_{d-1}(s) \setminus E_d(s)$ consists of the points of E_{d-1} which are represented by an element of $Q_{i,d-i}$ of size $s + 1$ for some i with $0 \leqslant i \leqslant d$; its cardinality is the number of indices i with $0 \leqslant i \leqslant d$ such that $Q_{i,d-i}$ contains an element of size s and an element of size $s + 2$.

Thus, the cardinality of $E_d(s)$ is equal to that of $E_{d-1}(s)$ plus the number of indices i with $0 \leqslant i \leqslant d$ such that $Q_{i,d-i}$ contains an element of size s but no element of size $s + 2$. Since the largest size of an element of $Q_{i,d-i}$ is d, the latter condition amounts to either $d = s$ or both $d = s + 1$ and $i \neq 0$. This gives the recurrence relation

$$|E_d(s)| = |E_{d-1}(s)| + \begin{cases} 0 & \text{if } s < d - 1 \text{ or } s > d, \\ d & \text{if } s = d - 1, \\ d + 1 & \text{if } s = d, \end{cases}$$

and from there the conclusion follows. $\square$

The following result illustrates how the theory developed in Sections 2 and 4 can be used to derive dimension estimates of the type that one requires in the construction of auxiliary polynomials.

Theorem 5.1. *Let $d \in \mathbb{N}^*$ and $\delta \in \mathbb{R}$ with $0 < \delta \leqslant \gamma d$. Define $V_d(\delta)$ to be the set of sequences $\mathfrak{A} \in \mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]_{\leqslant d}$ satisfying $|\mathfrak{A}| \ll |\mathfrak{X}_0^{(0)}|^{\delta}$. Then, $V_d(\delta)$ is a subspace of $\mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]_{\leqslant d}$ and its dimension satisfies*

$$c_1(d\delta)^{3/2} \leqslant \dim_{\mathbb{Q}} V_d(\delta) \leqslant 1 + c_2(d\delta)^{3/2} \tag{33}$$

for appropriate positive constants c_1 and c_2 depending only on ξ.

Proof. It is clear that $V_d(\delta)$ is a vector space over $\mathbb{Q}$. Combining Lemma 2.6 with Theorem 2.10 and then using Corollary 4.9 we find that his dimension is

$$\dim_{\mathbb{Q}} V_d(\delta) = \sum_{\{\alpha \in E_d;\, \alpha \leqslant \delta\}} \big(2s_d(\alpha) + 1\big) = 1 + \sum_{S} \big(2(a+b+c) + 1\big),$$

where the rightmost sum runs over the set S of all quads $(i; a, b, c)$ satisfying the system of inequalities

$$d - 2f(i+1) < af(i) + bf(i+1) + cf(i+2) \leqslant d, \tag{34}$$

$$a\gamma^{-i} + b\gamma^{-i-1} + c\gamma^{-i-2} \leqslant \delta. \tag{35}$$

For each $i \in \mathbb{N}$, let S_i denote the set of triples $(a, b, c) \in \mathbb{N}^3$ satisfying both $a \geqslant 1$ and the first condition (34). In the computations below, we freely use the fact that $\gamma^{i-1} \leqslant f(i) \leqslant \gamma^i$ for each $i \in \mathbb{N}$ (as one easily shows by recurrence on i).

Let $i \in \mathbb{N}$. We first note that S_i is empty if $f(i) > d$ (since we require $a \geqslant 1$). Assume that $f(i) \leqslant d$. Then each $(a, b, c) \in S_i$ satisfies $d/(2f(i+2)) \leqslant a+b+c \leqslant d/f(i)$ and so

$$\begin{aligned}
&\frac{d}{\gamma^{i+2}} \leqslant \frac{d}{f(i+2)} \leqslant 2(a+b+c)+1 \leqslant \frac{2d}{f(i)} + 1 \leqslant \frac{3d}{f(i)} \leqslant \frac{3d}{\gamma^{i-1}},\\
&\frac{d}{2\gamma^{2i+4}} \leqslant \frac{d}{2f(i+2)}\gamma^{-i-2} \leqslant a\gamma^{-i} + b\gamma^{-i-1} + c\gamma^{-i-2}\\
&\qquad\qquad\qquad\qquad \leqslant \frac{d}{f(i)}\gamma^{-i} \leqslant \frac{d}{\gamma^{2i-1}}.
\end{aligned}$$

The second chain of inequalities implies that if some $(a, b, c) \in S_i$ satisfies (35), then we must have $d \leqslant 2\gamma^{2i+4}\delta$. On the other hand, it also implies that any $(a, b, c) \in S_i$ satisfies (35) as soon as $d \leqslant \gamma^{2i-1}\delta$. Therefore, we obtain

$$1 + \sum_{i \in J} |S_i| \frac{d}{\gamma^{i+2}} \leqslant \dim_{\mathbb{Q}} V_d(\delta) \leqslant 1 + \sum_{i \in I} |S_i| \frac{3d}{\gamma^{i-1}},$$

where I denotes the set of integers $i \geqslant 0$ such that $f(i) \leqslant d$ and $d \leqslant 2\gamma^{2i+4}\delta$, and J the set of integers $i \geqslant 0$ such that $f(i) \leqslant d$ and $d \leqslant \gamma^{2i-1}\delta$.

Again, let $i \in \mathbb{N}$. For each pair of integers $a \geqslant 1$ and $c \geqslant 0$ with $af(i)+cf(i+2) \leqslant d$, there are exactly two choices of integer $b \geqslant 0$ such that (a,b,c) satisfies (34), or equivalently such that $(a,b,c) \in S_i$. This means that the cardinality $|S_i|$ of S_i is twice the number of points $(a,c) \in \mathbb{N}^* \times \mathbb{N}$ satisfying $af(i)+cf(i+2) \leqslant d$. Since $f(i) \leqslant d$ for each i in I or J, this number is non-zero and a short computation provides absolute constants $c_3 > 0$ and $c_4 > 0$ such that $|S_i| \leqslant c_3 d^2\gamma^{-2i}$ for each $i \in I$ and $|S_i| \geqslant c_4 d^2 \gamma^{-2i}$ for each $i \in J$.

If I is not empty, it contains a smallest element i_0 and so the above considerations give

$$\dim_{\mathbb{Q}} V_d(\delta) \leqslant 1 + 3c_3 d^3 \sum_{i=i_0}^{\infty} \gamma^{-3i+1} \leqslant 1 + 4\gamma c_3 \big(d\gamma^{-i_0}\big)^3.$$

Since $d \leqslant 2\gamma^{2i_0+4}\delta$, we find that $d\gamma^{-i_0} \leqslant (2\gamma^4 d\delta)^{1/2}$ and so we obtain the estimate $\dim_{\mathbb{Q}} V_d(\delta) \leqslant 1 + c_2(d\delta)^{3/2}$ with $c_2 = 12\gamma^7 c_3$. If I is empty, $V_d(\delta)$ has dimension 1 and this inequality still holds.

Let j_0 denote the integer for which $\gamma^{2j_0-3}\delta < d \leqslant \gamma^{2j_0-1}\delta$. Since $\delta \leqslant \gamma d$, we have $j_0 \geqslant 0$, and thus j_0 belongs to J if and only if $f(j_0) \leqslant d$. In that case, using $\gamma^{2j_0-3}\delta < d$, we find that $d\gamma^{-j_0} \geqslant \gamma^{-3/2}(d\delta)^{1/2}$ and so

$$\dim_{\mathbb{Q}} V_d(\delta) \geqslant |Q_{j_0}| \frac{d}{\gamma^{j_0+2}} \geqslant \gamma^{-2} c_4 \big(d\gamma^{-j_0}\big)^3 \geqslant \gamma^{-7} c_4 (d\delta)^{3/2}.$$

If $j_0 \notin J$, then we have $d < f(j_0) \leqslant \gamma^{j_0}$, thus $d\delta \leqslant \gamma^{j_0}\delta \leqslant \gamma^{-j_0+3} d \leqslant \gamma^3$ and therefore

$$\dim_{\mathbb{Q}} V_d(\delta) \geqslant 1 \geqslant \gamma^{-9/2}(d\delta)^{3/2},$$

showing that the lower bound in (33) holds with $c_1 = \min\{\gamma^{-7}c_4, \gamma^{-9/2}\}$. □

6. A COMPLEMENTARY RESULT

Following a suggestion of Daniel Daigle, we show the following theorem.

Theorem 6.1. *The ideal I defined in Lemma* 2.4 *is a prime ideal of rank* 3 *of the ring* $R = \mathbb{Q}[\mathbf{X}, \mathbf{X}^*]$.

This provides a proof of Corollary 2.13 which is independent of the combinatorial arguments of Section 4. Indeed, it follows easily from the considerations of Section 2.4–2.5 that $\mathfrak{X}_0^{(0)}$, $\mathfrak{X}_0^{(-1)}$ and $\mathfrak{X}_1^{(-1)}$ are elements of $\mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]$ which are algebraically independent over $\mathbb{Q}$. On the other hand, the evaluation map $\pi : R \to \mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]$ defined by (14) induces a surjective ring homomorphism $\bar{\pi} : R/I \to \mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]$ and, if we take for granted Theorem 6.1, the quotient R/I is an integral domain of transcendence degree 3 over $\mathbb{Q}$. Therefore, as Daniel Daigle remarked, this means that $\bar{\pi}$ is an isomorphism. This not only proves Corollary 2.13 but also the following statement:

Corollary 6.2. *The ring* $\mathbb{Q}[\mathfrak{X}^{(0)}, \mathfrak{X}^{(-1)}]$ *is an integral domain of transcendence degree* 3 *over* $\mathbb{Q}$.

In order to prove Theorem 6.1, we first note that we may assume, without loss of generality, that the coefficient $a_{2,2}$ of the matrix M is non-zero. Indeed, as we saw at the end of Section 2.2, at least one of the coefficients $a_{1,1}$ or $a_{2,2}$ of M is non-zero. If $a_{2,2} = 0$, then $a_{1,1} \neq 0$ and we replace I by its image under the ring automorphism of R which sends X_i to X_{2-i} and X_i^* to X_{2-i}^* for $i = 0, 1, 2$. This automorphism fixes the first two generators $\det(\mathbf{X}) - 1$ and $\det(\mathbf{X}^*) - 1$ of I and maps $\Phi(\mathbf{X}, \mathbf{X}^*)$ to a polynomial of the same form with the coefficient $a_{2,2}$ replaced by $-a_{1,1} \neq 0$.

Now, let V, V^* and W be indeterminates over R. We put a $\mathbb{N}^3$-grading on the ring $R_3 := R[V, V^*, W]$ by requesting that each variable is multi-homogeneous with multi-degree:

$$\begin{aligned} &\deg(X_i) = (1, 0, i), &&\deg\big(X_i^*\big) = (0, 1, i) &&\text{for } i = 0, 1, 2, \\ &\deg(V) = (1, 0, 1), &&\deg\big(V^*\big) = (0, 1, 1) &&\text{and} \quad \deg(W) = (0, 0, 1). \end{aligned}$$

A polynomial P in R_3 is thus multi-homogeneous of multi-degree (d_1, d_2, d_3) if and only if, in the usual sense, it is homogeneous of degree d_1 in $(\mathbf{X}, V)$, homogeneous of degree d_2 in $(\mathbf{X}^*, V^*)$, and if its image under the specialization $X_i \mapsto W^i X_i$, $X_i^* \mapsto W^i X_i^*$ $(i = 0, 1, 2)$, $V \mapsto WV$ and $V^* \mapsto WV^*$ belongs to $W^{d_3} R[V, V^*]$. In this case, d_3 is called the *weight* of P.

Let I_3 denote the ideal of R_3 generated by

$$\begin{aligned} F &= \det(\mathbf{X}) - V^2, \\ F^* &= \det\big(\mathbf{X}^*\big) - \big(V^*\big)^2, \\ G &= a_{1,1} \begin{vmatrix} X_0^* & X_1^* \\ X_0 & X_1 \end{vmatrix} W^2 + \left(a_{1,2} \begin{vmatrix} X_1^* & X_2^* \\ X_0 & X_1 \end{vmatrix} + a_{2,1} \begin{vmatrix} X_0^* & X_1^* \\ X_1 & X_2 \end{vmatrix} \right) W \\ &\quad + a_{2,2} \begin{vmatrix} X_1^* & X_2^* \\ X_1 & X_2 \end{vmatrix}. \end{aligned}$$

Since F, F^* and G are respectively multi-homogeneous of multi-degree $(2, 0, 2)$, $(0, 2, 2)$ and $(1, 1, 3)$, the ideal I_3 is multi-homogeneous. By construction, it is mapped to I under the R-linear ring homomorphism from R_3 to R sending V, V^* and W to 1. Moreover, any element of I is the image of a multi-homogeneous element of I_3 under that map. Therefore, in order to prove Theorem 6.1, it suffices to show that I_3 is a prime ideal of R_3. In preparation to this, we first establish the following lemma where, for any $f \in R_3$, we define $(I_3 : f) = \{a \in R_3; af \in I_3\}$.

Lemma 6.3. *The polynomials* F, F^* *and* G *form a regular sequence in* R_3 *and we have* $(I_3 : X_0) = (I_3 : X_0^*) = I_3$.

Proof. We claim more precisely that the polynomials X_0, X_0^*, W, F, F^*, G form a regular sequence in R_3. If we take this for granted, then F, F^*, G and X_0 form a regular sequence in R_3 and so $(I_3 : X_0) = I_3$. Similarly, F, F^*, G and X_1 form a regular sequence in R_3 and so $(I_3 : X_1) = I_3$.

Since X_0, X_0^* and W are variables from R_3, our claim is equivalent to the assertion that

$$\bar{F} = -X_1^2 - V^2, \qquad \bar{F}^* = -\left(X_1^*\right)^2 - \left(V^*\right)^2 \quad \text{and} \quad \bar{G} = a_{2,2}\begin{vmatrix} X_1^* & X_2^* \\ X_1 & X_2 \end{vmatrix}$$

form a regular sequence in the ring $\bar{R} = \mathbb{Q}[X_1, X_2, X_1^*, X_2^*, V, V^*]$, where for a polynomial P in R_3 we denote by $\bar{P}$ its image under the ring homomorphism from R_3 to $\bar{R}$ which maps X_0, X_0^* and W to 0, and all other variables to themselves. This assertion in turn follows from the fact that $\bar{F}$ is monic in V and independent of V^*, that $\bar{F}^*$ is monic in V^* and independent of V, and that $\bar{G}$ is non-zero and independent of both V and V^*. $\square$

We now complete the proof of Theorem 6.1 by showing the following lemma.

Lemma 6.4. *The ideal I_3 is prime of rank* 3.

Proof. Let S denote the multiplicative subset of R_3 generated by X_0 and X_0^*. By Lemma 6.3, we have $(I_3 : f) = I_3$ for each $f \in S$. So, it is equivalent to prove that $S^{-1}I_3$ is a prime ideal in the localized ring $S^{-1}R_3$. Since

$$X_0^{-1}F = X_2 - X_0^{-1}\left(X_1^2 + V^2\right)$$

and

$$\left(X_0^*\right)^{-1}F^* = X_2^* - \left(X_0^*\right)^{-1}\left(\left(X_1^*\right)^2 + \left(V^*\right)^2\right),$$

this amounts simply to showing that G is mapped to a prime element of $S^{-1}R_3$ under the ring endomorphism of $S^{-1}R_3$ which sends X_2 to $X_0^{-1}(X_1^2 + V^2)$, X_2^* to $(X_0^*)^{-1}((X_1^*)^2 + (V^*)^2)$, and all other variables to themselves. The image of G takes the form $(X_0X_0^*)^{-1}H$ where

$$\begin{aligned} H = a_{1,1}\begin{vmatrix} X_0^* & X_1^* \\ X_0 & X_1 \end{vmatrix} X_0X_0^*W^2 + a_{1,2}\begin{vmatrix} X_0^*X_1^* & (X_1^*)^2 + (V^*)^2 \\ X_0 & X_1 \end{vmatrix} X_0W \\ + a_{2,1}\begin{vmatrix} X_0^* & X_1^* \\ X_0X_1 & X_1^2 + V^2 \end{vmatrix} X_0^*W + a_{2,2}\begin{vmatrix} X_0^*X_1^* & (X_1^*)^2 + (V^*)^2 \\ X_0X_1 & X_1^2 + V^2 \end{vmatrix}. \end{aligned}$$

Since R_3 is a unique factorization domain, we are reduced to showing that H is an irreducible element of R_3. Moreover, since H is multi-homogeneous of multi-degree $(2, 2, 3)$, it suffices to prove that H has no non-constant multi-homogeneous divisor of multi-degree $< (2, 2, 3)$. Let H_0 denote the constant coefficient of H, viewed as a polynomial in W. Since $a_{2,2} \neq 0$, it is non-zero. It takes the form $H_0 = a(V^*)^2 - b$

where a and b are relatively prime elements of $R[V]$ such that ab is not a square in $R[V]$ (ab is divisible by X_0 but not by X_0^2). Therefore H_0 is irreducible. If A is a multi-homogeneous divisor of H of multi-degree $< (2,2,3)$, then the constant coefficient A_0 of A (as a polynomial in W) is a divisor of H_0. Since H_0 is irreducible and multi-homogeneous of the same multi-degree $(2,2,3)$ as H, it follows that A_0 is a constant and therefore that A itself is a constant.

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