ACTA ARITHMETICA
Online First version

Diophantine approximation with constraints

by

Jérémy Champagne (Waterloo, ON) and Damien Roy (Ottawa, ON)

1. Introduction. Given an integer $n \geq 2$ and an arbitrary point $\mathbf{u} \in \mathbb{R}^n$ with linearly independent coordinates over $\mathbb{Q}$, we know that there are infinitely many integer points $\mathbf{x} \in \mathbb{Z}^n$ for which

$$|\mathbf{x} \cdot \mathbf{u}| \leq c_1 \|\mathbf{x}\|^{-(n-1)} \tag{1.1}$$

for a constant $c_1 > 0$ that depends only on $\mathbf{u}$. Here the dot represents the usual scalar product in $\mathbb{R}^n$, and the norm is the associated Euclidean norm $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$. This is a result of Dirichlet [13, Chapter II, Corollary 1D], and it is best possible in the sense that there are points $\mathbf{u} \in \mathbb{R}^n$ with linearly independent coordinates over $\mathbb{Q}$ which satisfy $|\mathbf{x} \cdot \mathbf{u}| \geq c_2 \|\mathbf{x}\|^{-(n-1)}$ for any non-zero $\mathbf{x} \in \mathbb{Z}^n$, with another constant $c_2 > 0$. The latter occurs for example when the coordinates of $\mathbf{u}$ form a basis of a real number field of degree n [13, Chapter II, Theorem 4A].

Of course, any point $\mathbf{x} \in \mathbb{Z}^n$ of large norm that satisfies (1.1) makes a small angle with the maximal subspace $\mathbf{u}^\perp$ of $\mathbb{R}^n$ orthogonal to $\mathbf{u}$. In the present paper, we study how the right hand side of (1.1) has to be modified when $\mathbf{x}$ is required to make a small angle with a fixed proper non-zero subspace V of $\mathbb{R}^n$. This line of research was initiated in 1976 by W. M. Schmidt [12] and followed by several authors [16, 1, 6, 7, 2]. Our first main result is the following statement, where $\mathrm{dist}(\mathbf{x}, V)$ denotes the sine of the angle between a non-zero point $\mathbf{x}$ and a non-zero subspace V in $\mathbb{R}^n$.

Theorem 1.1. *Let m, n be integers with $m \geq 1$ and $n \geq m+2$, and let V be a subspace of $\mathbb{R}^n$ of dimension $m+1$. Set*

$$\rho = \rho_m = \frac{m + \sqrt{m^2 + 4m}}{2}. \tag{1.2}$$

2020 *Mathematics Subject Classification*: Primary 11J13; Secondary 11H50, 11J25.
Key words and phrases: Diophantine approximation, sign constraints, angular constraints, parametric geometry of numbers, geometry of numbers.
Received 31 October 2022.
Published online *.

(1) *For each point* $\mathbf{u}$ *of* $\mathbb{R}^n$ *whose coordinates are linearly independent over* $\mathbb{Q}$ *and each pair of numbers* $\delta, \epsilon > 0$, *there exists a non-zero point* $\mathbf{x}$ *of* $\mathbb{Z}^n$ *with*

$$\operatorname{dist}(\mathbf{x}, V) \le \delta \quad \text{and} \quad |\mathbf{x} \cdot \mathbf{u}| \le \epsilon \|\mathbf{x}\|^{-\rho}. \tag{1.3}$$

(2) *Conversely, let* $\psi\colon [1,\infty) \to (0,\infty)$ *be any unbounded non-decreasing function and let* $\delta = 1/\max\{4n, 24(n-m)\}$. *There exists a point* $\mathbf{u}$ *of* $\mathbb{R}^n$ *with linearly independent coordinates over* $\mathbb{Q}$ *such that at most finitely many non-zero points* $\mathbf{x}$ *of* $\mathbb{Z}^n$ *satisfy*

$$\operatorname{dist}(\mathbf{x}, V) \le \delta \quad \text{and} \quad |\mathbf{x} \cdot \mathbf{u}| \le \psi(\|\mathbf{x}\|)^{-1} \|\mathbf{x}\|^{-\rho}. \tag{1.4}$$

For fixed $\mathbf{u}$ and δ as in part (1) of the theorem, we obtain infinitely many non-zero points $\mathbf{x} \in \mathbb{Z}^n$ with $\operatorname{dist}(\mathbf{x}, V) \le \delta$ and $|\mathbf{x} \cdot \mathbf{u}| \le \|\mathbf{x}\|^{-\rho}$ by letting ϵ tend to zero. On the other hand, if we choose $\psi(t) = t^\eta$ for some $\eta > 0$, then part (2) provides $\mathbf{u}$ and δ for which only finitely many non-zero points $\mathbf{x} \in \mathbb{Z}^n$ have $\operatorname{dist}(\mathbf{x}, V) \le \delta$ and $|\mathbf{x} \cdot \mathbf{u}| \le \|\mathbf{x}\|^{-\rho-\eta}$. Thus the exponent ρ in (1.3) is best possible. Since

$$\rho - m = \frac{\rho}{\rho+1} \in (0,1), \tag{1.5}$$

we have $m < \rho < m+1$, and so ρ is strictly smaller than Dirichlet's exponent $n-1$ in (1.1).

Corollary 1.2. *The statement of Theorem* 1.1 *remains true if the condition* $\operatorname{dist}(\mathbf{x}, V) \le \delta$ *in* (1.3) *and in* (1.4) *is replaced by* $\operatorname{dist}(\mathbf{x}, W) \le \delta$ *where* $W = V \cap \mathbf{u}^\perp$.

In fact, as we will see, this provides an equivalent form of the theorem. Note that, when V is defined over $\mathbb{Q}$, we have $V \not\subseteq \mathbf{u}^\perp$ and so $\dim(W) = m$, in the notation of the above corollary.

Theorem 1.1 extends results of several authors. In the case where $m = 1$ and

$$V = \{(x_1, \dots, x_n) \in \mathbb{R}^n \,;\, x_2 = \cdots = x_n\}, \tag{1.6}$$

we will see that it admits the following consequence, where

$$\gamma = \rho_1 = (1 + \sqrt{5})/2 \simeq 1.618$$

denotes the golden ratio.

Corollary 1.3. *Let* $n \ge 3$ *be an integer. For each point* $\mathbf{u}$ *of* $\mathbb{R}^n$ *with linearly independent coordinates over* $\mathbb{Q}$ *and for each* $\epsilon > 0$, *there exists a point* $\mathbf{x} = (x_1, \dots, x_n)$ *of* $\mathbb{Z}^n$ *with*

$$x_2, \dots, x_n > 0 \quad \text{and} \quad |\mathbf{x} \cdot \mathbf{u}| \le \epsilon \|\mathbf{x}\|^{-\gamma}. \tag{1.7}$$

On the other hand, for each unbounded non-decreasing function $\psi\colon [1,\infty) \to (0,\infty)$, *there exists a point* $\mathbf{u}$ *of* $\mathbb{R}^n$ *with linearly independent coordinates over* $\mathbb{Q}$ *for which at most finitely many points* $\mathbf{x} = (x_1, \dots, x_n)$ *of* $\mathbb{Z}^n$ *satisfy*

$$x_2, \dots, x_n > 0 \quad \text{and} \quad |\mathbf{x} \cdot \mathbf{u}| \le \psi(\|\mathbf{x}\|)^{-1} \|\mathbf{x}\|^{-\gamma}. \tag{1.8}$$

For $n = 3$, the first part of the corollary is the original result of Schmidt [12, Theorem 1] from 1976. Later, in [14, Section 5], Schmidt conjectured that, in that case, one could replace γ in (1.7) by any number smaller than 2. This was disproved in 2012 by Moshchevitin who showed, by an ingenious construction in [6], that it cannot be replaced by a number larger than the largest real root of $x^4 - 2x^2 - 4x + 1$, which is approximately 1.947. Motivated by this, the second author proved in [7, Corollary] that $\rho = \gamma$ is best possible in the wider context of Theorem 1.1, part (2), for $m = 1$ and $n = 3$. Earlier, in [12, Remark (F)], Schmidt had observed that, for $n \geq 3$, the exponent γ cannot be replaced by a number larger than 2.

In the general case, we will see that the choice of

$$V = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,;\, x_{m+2} = \cdots = x_n = 0\} \tag{1.9}$$

yields the following statement.

Corollary 1.4. *Let m, n be integers with $1 \leq m \leq n - 2$ and let $\rho = \rho_m$ as in Theorem 1.1. For each point $\mathbf{u}$ of $\mathbb{R}^n$ with linearly independent coordinates over $\mathbb{Q}$ and each choice of $\delta, \epsilon > 0$, there exists a non-zero point $\mathbf{x} = (x_1, \ldots, x_n)$ of $\mathbb{Z}^n$ which satisfies the conditions*

$$\begin{aligned} &\max\{|x_{m+2}|, \ldots, |x_n|\} \leq \delta \max\{|x_2|, \ldots, |x_n|\}, \\ &|\mathbf{x} \cdot \mathbf{u}| \leq \epsilon \|\mathbf{x}\|^{-\rho}. \end{aligned} \tag{1.10}$$

On the other hand, for each unbounded non-decreasing function $\psi\colon [1, \infty) \to (0, \infty)$, there exists a point $\mathbf{u}$ of $\mathbb{R}^n$ with linearly independent coordinates over $\mathbb{Q}$ for which at most finitely many points $\mathbf{x} = (x_1, \ldots, x_n)$ of $\mathbb{Z}^n$ satisfy the conditions

$$\begin{aligned} &\max\{|x_1|, \ldots, |x_{m+1}|\} = \max\{|x_1|, \ldots, |x_n|\}, \\ &|\mathbf{x} \cdot \mathbf{u}| \leq \psi(\|\mathbf{x}\|)^{-1} \|\mathbf{x}\|^{-\rho}. \end{aligned} \tag{1.11}$$

For $n = m + 2$ and any $m \geq 1$, the first part of the corollary was established by Thurnheer in 1990 [16, Theorem 1(b)], along with other interesting results. For $n = 3$ and $m = 1$, this result is equivalent to that of Schmidt mentioned above. At the level of exponents, we deduce the following statement.

Corollary 1.5. *Let m, n be integers with $1 \leq m \leq n - 2$. For each point $\mathbf{u}$ of $\mathbb{R}^n$ with $\mathbb{Q}$-linearly independent coordinates, let $\rho(\mathbf{u})$ denote the supremum of all ρ for which the inequality $|\mathbf{x} \cdot \mathbf{u}| \leq \|\mathbf{x}\|^{-\rho}$ has infinitely many solutions $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ with*

$$\max\{|x_{m+2}|, \ldots, |x_n|\} < \max\{|x_2|, \ldots, |x_{m+1}|\}. \tag{1.12}$$

Then $\rho(\mathbf{u}) \geq \rho_m$ for all those $\mathbf{u}$, and $\rho(\mathbf{u}) = \rho_m$ for uncountably many unit vectors $\mathbf{u}$ among them.

In [1, Theorem 1], Bugeaud and Kristensen showed that $\rho(\mathbf{u}) = n-1$ for almost all $\mathbf{u}$ with respect to the Lebesgue measure, and that each number $\rho \geq m+1$ is equal to $\rho(\mathbf{u})$ for uncountably many $\mathbf{u}$ with first coordinate equal to 1 ($\rho(\mathbf{u})$ is a projective invariant). They further posed two problems about the existence of points $\mathbf{u}$ with $\mathbb{Q}$-linearly independent coordinates satisfying $\rho(\mathbf{u}) < m+1$. Since $\rho_m < m+1$, the above corollary shows that ρ_m is the smallest such value, answering positively their first problem and negatively the other.

The 2021 MSc thesis of the first author establishes Theorem 1.1 for $m = 1$ and any $n \geq 3$. Part (1) is proved as [2, Theorem 4.2.4], and part (2) as [2, Theorem 4.3.1]. For the first part, it is reasonable that the exponent $\rho = \gamma$ which works for $n = 3$ also works for any $n \geq 3$ as one expects more freedom in the choice of the integer points $\mathbf{x}$. Indeed this is how part (1) is proved there, by reduction to the case $n = 3$ due to Schmidt. However, it is surprising that this exponent remains best possible, independently of n. The construction that shows its optimality and proves part (2) of the theorem generalizes the construction elaborated in [7], and exhibits additional features (see [2, Lemma 4.3.10]).

In Section 4, we prove part (1) of Theorem 1.1 for the general case $n \geq m+2$ by reducing to Thurnheer's result in the special case where $n = m+2$. We also provide proofs of the four corollaries assuming that the full theorem holds. In Section 7, we propose a simplification of the proof of Thurnheer from [16] along the lines of [2, Chapter 5]. It involves a reasoning which is reminiscent of the theory of continued fractions and remarkably similar to that of Davenport and Schmidt in [4] although our goal is different. Moreover, it allows us to treat at once the cases $m = 1$ and $m \geq 2$ which were analyzed separately by Thurnheer in [16]. Preliminaries on the projective distance are gathered in Section 3.

To prove part (2) of the theorem in the general case, we introduce a new construction in parametric geometry of numbers which involves angular constraints. As this requires additional discussion, we describe it in the next section.

2. Parametric geometry of numbers with angular constraints. Fix an integer $n \geq 2$. Parametric geometry of numbers, as developed in [15] and [8], studies in logarithmic scale the successive minima of the parametric families of compact symmetric convex subsets of $\mathbb{R}^n$,

$$\mathcal{C}_{\mathbf{u}}(Q) = \{\mathbf{x} \in \mathbb{R}^n \,;\, \|\mathbf{x}\| \leq 1 \text{ and } |\mathbf{x} \cdot \mathbf{u}| \leq Q^{-1}\|\mathbf{u}\|\} \quad (Q \geq 1), \tag{2.1}$$

attached to non-zero points $\mathbf{u} \in \mathbb{R}^n$ (in the setting of [8]). Thus, its object of study consists of the maps

$$\mathbf{L}_{\mathbf{u}} \colon [0, \infty) \to \mathbb{R}^n, \quad q \mapsto (L_{\mathbf{u},1}(q), \dots, L_{\mathbf{u},n}(q)),$$

where, for each $j = 1, \dots, n$ and $q \geq 0$, the number $L_{\mathbf{u},j}(q)$ represents the logarithm of the jth minimum of $\mathcal{C}_{\mathbf{u}}(e^q)$, namely the smallest real number L for which $e^L\mathcal{C}_{\mathbf{u}}(e^q)$ contains at least j linearly independent points of $\mathbb{Z}^n$. The main result of the theory asserts that, modulo bounded functions, the set of these maps coincides with a simpler set of functions called n-systems (or $(n,0)$-systems) whose definition is purely combinatorial (see [8, Section 2.5]). We can even use smaller sets of functions, the rigid n-systems of a given mesh $c > 0$.

To recall their definition from [8, Section 1], let

$$\varDelta_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \, ; \, x_1 \leq \dots \leq x_n\}$$

denote the set of n-tuples of real numbers in non-decreasing order and let $\varPhi_n \colon \mathbb{R}^n \to \varDelta_n$ denote the continuous map which sends an n-tuple to its permutation in $\varDelta_n$. The following definitions are adapted from [8, Definitions 1.1 and 1.2].

Definition 2.1. Let $c > 0$, and let $s \in \{\infty, 1, 2, \dots\}$. A *canvas with mesh c and cardinality s* in $\mathbb{R}^n$ is a triple consisting of a sequence $(\mathbf{a}^{(i)})_{0 \leq i < s}$ of points in $\varDelta_n$ together with sequences $(k_i)_{0 \leq i < s}$ and $(\ell_i)_{0 \leq i < s}$ of integers such that, for each index i with $0 \leq i < s$,

(C1) the coordinates $(a_1^{(i)}, \dots, a_n^{(i)})$ of $\mathbf{a}^{(i)}$ form a strictly increasing sequence of positive integer multiples of c,
(C2) we have $1 \leq k_0 < \ell_0 = n$ and $1 \leq k_i < \ell_i \leq n$ if $i \geq 1$,
(C3) if $i + 1 < s$, we further have $k_i \leq \ell_{i+1}$, $a_{\ell_{i+1}}^{(i)} + c \leq a_{\ell_{i+1}}^{(i+1)}$ and

$$(a_1^{(i+1)}, \dots, \widehat{a_{\ell_{i+1}}^{(i+1)}}, \dots, a_n^{(i+1)}) = (a_1^{(i)}, \dots, \widehat{a_{k_i}^{(i)}}, \dots, a_n^{(i)}) \tag{2.2}$$

where the hat on a coordinate means that it is omitted.

The only difference with [8, Definition 1.1] is the strict inequality $k_0 < \ell_0$ in condition (C2), which is better suited to our current purposes.

Condition (C3) means that, when $i + 1 < s$, the point $\mathbf{a}^{(i+1)}$ is obtained from the preceding point $\mathbf{a}^{(i)}$ by replacing one of its coordinates by a larger multiple of c, different from its other coordinates, and then by reordering the new n-tuple.

Definition 2.2. To each canvas of mesh $c > 0$ as in Definition 2.1, we associate the function $\mathbf{P} \colon [q_0, \infty) \to \varDelta_n$ given by

$$\mathbf{P}(q) = \varPhi_n(a_1^{(i)}, \dots, \widehat{a_{k_i}^{(i)}}, \dots, a_n^{(i)}, a_{k_i}^{(i)} + q - q_i) \quad (0 \leq i < s, \, q_i \leq q < q_{i+1}),$$

where $q_i = a_1^{(i)} + \dots + a_n^{(i)}$ $(0 \leq i < s)$ and $q_s = \infty$ if $s < \infty$. We say that such a function is a *rigid n-system with mesh c* and that $(q_i)_{0 \leq i < s}$ is its sequence of *switch numbers*.

Since $a^{(i)}_{k_i} + q_{i+1} - q_i = a^{(i+1)}_{\ell_{i+1}}$ when $i+1 < s$, such a map $\mathbf{P}$ is continuous. Moreover, upon writing $\mathbf{P}(q) = (P_1(q), \dots, P_n(q))$ for each $q \ge q_0$, we see that

(S1) $P_1, \dots, P_n$ are continuous and piecewise linear on $[q_0, \infty)$ with slopes 0 and 1;
(S2) $0 \le P_1(q) \le \cdots \le P_n(q)$ and $P_1(q) + \cdots + P_n(q) = q$ for each $q \ge q_0$;
(S3) if, for some $j \in \{1, \dots, n-1\}$, the sum $P_1 + \cdots + P_j$ changes slope from 1 to 0 at a point $q > q_0$, then $P_j(q) = P_{j+1}(q)$.

Thus $\mathbf{P}$ is an $(n, 0)$-system in the sense of [8, Section 2.5]. The switch numbers q_i with $i > 0$ may also be characterized as the points $q > q_0$ at which one of the sums $P_1 + \cdots + P_j$ changes slope from 0 to 1. Then the smallest such index j is k_i. From a graphical point of view, it also follows from the definition that, whenever $0 \le i < s$, the union of the graphs of $P_1, \dots, P_n$ over $[q_i, q_{i+1})$ consists of $n-1$ horizontal line segments and one line segment of slope 1. When $i \ge 1$, the condition $k_i < \ell_i$ in (C2) means that the line segment of slope 1 over $[q_{i-1}, q_i]$ ends at $(q_i, a^{(i)}_{\ell_i})$, above the starting point $(q_i, a^{(i)}_{k_i})$ of the line segment of slope 1 over $[q_i, q_{i+1})$.

Since the convex bodies $\mathcal{C}_{\mathbf{u}}(q)$ depend only on the class of $\mathbf{u}$ in projective space, we may restrict to unit vectors $\mathbf{u}$. The main result [8, Theorem 1.3] then reads as follows.

Theorem 2.3. *Let $c > 0$. For each unit vector $\mathbf{u}$ of $\mathbb{R}^n$, there exists a rigid system $\mathbf{P}\colon [q_0, \infty) \to \varDelta_n$ with mesh c such that $\mathbf{L}_{\mathbf{u}} - \mathbf{P}$ is bounded on $[q_0, \infty)$. Conversely, for each rigid system $\mathbf{P}\colon [q_0, \infty) \to \varDelta_n$ with mesh c, there exists a unit vector $\mathbf{u}$ in $\mathbb{R}^n$ such that $\mathbf{L}_{\mathbf{u}} - \mathbf{P}$ is bounded on $[q_0, \infty)$.*

Here, we propose a new and simpler construction of the unit vector $\mathbf{u}$ which involves angular constraints and yields a simple self-contained proof of the second part of Theorem 1.1. It assumes that the n-system $\mathbf{P}$ satisfies some additional properties, but, even with this restriction, we will see that it suffices for some important applications of the theory. To state our result, we need some additional notation.

Definition 2.4. Let $\mathbf{u} \in \mathbb{R}^n$ be a unit vector. The *trajectory* of a non-zero integer point $\mathbf{x} \in \mathbb{Z}^n$ (relative to the family $\mathcal{C}_{\mathbf{u}}(Q)$) is the map $L_{\mathbf{u}}(\mathbf{x}, \cdot)$ from $[0, \infty)$ to $\mathbb{R}$ given by

$$L_{\mathbf{u}}(\mathbf{x}, q) = \max\{\log \|\mathbf{x}\|, q + \log |\mathbf{x} \cdot \mathbf{u}|\} \quad (q \ge 0). \tag{2.3}$$

The *trajectory* of a linearly independent n-tuple $\underline{\mathbf{x}} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$ of points of $\mathbb{Z}^n$ is the map $\mathbf{L}_{\mathbf{u}}(\underline{\mathbf{x}}, \cdot)\colon [0, \infty) \to \varDelta_n$ given, for each $q \ge 0$, by

$$\mathbf{L}_{\mathbf{u}}(\underline{\mathbf{x}}, q) = (L_{\mathbf{u},1}(\underline{\mathbf{x}}, q), \dots, L_{\mathbf{u},n}(\underline{\mathbf{x}}, q)) := \varPhi_n(L_{\mathbf{u}}(\mathbf{x}_1, q), \dots, L_{\mathbf{u}}(\mathbf{x}_n, q)),$$

so that the coordinates $L_{\mathbf{u},j}(\underline{\mathbf{x}}, q)$ of $\mathbf{L}_{\mathbf{u}}(\underline{\mathbf{x}}, q)$ are the numbers $L_{\mathbf{u}}(\mathbf{x}_j, q)$ written in non-decreasing order.

Fix a number $q \geq 0$. It follows from the formula (2.1) for $\mathcal{C}_{\mathbf{u}}(Q)$ that, for each non-zero $\mathbf{x} \in \mathbb{Z}^n$, the number $L_{\mathbf{u}}(\mathbf{x}, q)$ given by (2.3) is the smallest L for which $\mathbf{x} \in e^L \mathcal{C}_{\mathbf{u}}(e^q)$. Thus, we have $L_{\mathbf{u},1}(q) \leq L_{\mathbf{u}}(\mathbf{x}, q)$. More generally, we deduce that

$$L_{\mathbf{u},j}(q) \leq L_{\mathbf{u},j}(\underline{\mathbf{x}}, q) \quad \text{for } j = 1, \dots, n, \tag{2.4}$$

for each linearly independent n-tuple $\underline{\mathbf{x}}$ of points of $\mathbb{Z}^n$. Since $\mathcal{C}_{\mathbf{u}}(e^q)$ is a compact subset of $\mathbb{R}^n$, we can even find such an $\underline{\mathbf{x}}$ that realizes the n minima of $\mathcal{C}_{\mathbf{u}}(e^q)$ in the sense that $\mathbf{L}_{\mathbf{u}}(q) = \mathbf{L}_{\mathbf{u}}(\underline{\mathbf{x}}, q)$. Thus, in the componentwise ordering on $\mathbb{R}^n$ defined by

$$(t_1, \dots, t_n) \leq (t'_1, \dots, t'_n) \iff t_1 \leq t'_1, \, \dots, \, t_n \leq t'_n,$$

the n-tuple $\mathbf{L}_{\mathbf{u}}(q)$ is the minimum of $\mathbf{L}_{\mathbf{u}}(\underline{\mathbf{x}}, q)$ over all possible choices of $\underline{\mathbf{x}}$.

DEFINITION 2.5. Let $\mathbf{P}\colon [q_0, \infty) \to \Delta_n$ be a rigid n-system with mesh c as in Definition 2.2. Consider its associated canvas as in Definition 2.1, and let $(q_i)_{0 \leq i < s}$ denote its sequence of switch numbers. We say that a sequence $(\underline{\mathbf{x}}^{(i)})_{0 \leq i < s}$ of bases of $\mathbb{Z}^n$ written $\underline{\mathbf{x}}^{(i)} = (\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)})$ is *coherent with* $\mathbf{P}$ if, for each $i \geq 0$ with $i + 1 < s$, we have

(1) $(\mathbf{x}_1^{(i+1)}, \dots, \widehat{\mathbf{x}_{\ell_{i+1}}^{(i+1)}}, \dots, \mathbf{x}_n^{(i+1)}) = (\mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)})$,

(2) $\mathbf{x}_{\ell_{i+1}}^{(i+1)} \in \mathbf{x}_{k_i}^{(i)} + \langle \mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_{\ell_{i+1}}^{(i)} \rangle_{\mathbb{Z}}$.

Note that, if $\underline{\mathbf{x}}^{(i)}$ is a basis of $\mathbb{Z}^n$ for some i with $1 \leq i+1 < s$, then any choice of vectors $\mathbf{x}_1^{(i+1)}, \dots, \mathbf{x}_n^{(i+1)}$ satisfying conditions (1) and (2) also form a basis of $\mathbb{Z}^n$. So, provided that $\underline{\mathbf{x}}^{(0)}$ is a basis of $\mathbb{Z}^n$, all subsequent n-tuples satisfying these conditions also form bases of $\mathbb{Z}^n$.

As for the terminology, we say that $(\underline{\mathbf{x}}^{(i)})_{0 \leq i < s}$ is coherent with $\mathbf{P}$ because condition (1) in Definition 2.5 has the same form as condition (2.2) in (C3). This proves very useful for induction purposes. Similarly, we have the following notion of a coherent system of directions for $\mathbf{P}$.

DEFINITION 2.6. Let the notation be as in Definition 2.5, and let $\underline{\mathbf{v}} = (\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$ be a linearly independent $(n-1)$-tuple of unit vectors of $\mathbb{R}^n$. Set $\underline{\mathbf{v}}^{(0)} = (\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{v}_{k_0})$ and, for each integer $i \geq 0$ with $i + 1 < s$, define recursively $\underline{\mathbf{v}}^{(i+1)} = (\mathbf{v}_1^{(i+1)}, \dots, \mathbf{v}_n^{(i+1)})$ by the conditions

(1) $(\mathbf{v}_1^{(i+1)}, \dots, \widehat{\mathbf{v}_{\ell_{i+1}}^{(i+1)}}, \dots, \mathbf{v}_n^{(i+1)}) = (\mathbf{v}_1^{(i)}, \dots, \widehat{\mathbf{v}_{k_i}^{(i)}}, \dots, \mathbf{v}_n^{(i)})$,

(2) $\mathbf{v}_{\ell_{i+1}}^{(i+1)} = \mathbf{v}_{k_{i+1}}^{(i+1)}$.

We say that $(\underline{\mathbf{v}}^{(i)})_{0\le i<s}$ is the *coherent sequence of directions for* $\mathbf{P}$ *attached to* $\underline{\mathbf{v}}$.

Note that the definition of $\underline{\mathbf{v}}^{(0)}$ uses the strict inequality $k_0 < \ell_0 = n$ in condition (C2). Moreover, relations (1) and (2) uniquely determine $\underline{\mathbf{v}}^{(i+1)}$ in terms of $\underline{\mathbf{v}}^{(i)}$ for $i \ge 0$, because $k_{i+1} < \ell_{i+1}$ by condition (C2). Thus, for each integer $i \ge 0$ with $i < s$, we have $\mathbf{v}^{(i)}_{\ell_i} = \mathbf{v}^{(i)}_{k_i}$ and the $(n-1)$-tuples $(\mathbf{v}^{(i)}_1, \dots, \widehat{\mathbf{v}^{(i)}_{k_i}}, \dots, \mathbf{v}^{(i)}_n)$ and $(\mathbf{v}^{(i)}_1, \dots, \widehat{\mathbf{v}^{(i)}_{\ell_i}}, \dots, \mathbf{v}^{(i)}_n)$ are permutations of $\underline{\mathbf{v}} = (\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$.

In the statement of our second main result below, we use the standard Euclidean norm on the exterior powers of $\mathbb{R}^n$ (recalled in Section 3) to define a constant θ, and we denote by $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ the canonical basis of $\mathbb{R}^n$.

Theorem 2.7. *Let* $\underline{\mathbf{v}} = (\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$ *be a linearly independent* $(n-1)$*-tuple of unit vectors of* $\mathbb{R}^n$*, and let* $\delta \in \mathbb{R}$ *with*

$$0 < \delta \le \frac{\theta}{4n} \quad \textit{where} \quad \theta = \|\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{n-1}\|.$$

There is a constant $\kappa > 0$ *depending only on* $\underline{\mathbf{v}}$ *and* δ *with the following property. Let* $\mathbf{P} = (P_1, \dots, P_n)\colon [q_0, \infty) \to \Delta_n$ *be a rigid* n*-system of mesh* $c > 0$ *with associated canvas* $((\mathbf{a}^{(i)})_{0\le i<s}, (k_i)_{0\le i<s}, (\ell_i)_{0\le i<s})$ *and sequence of switch numbers* $(q_i)_{0\le i<s}$ *as in Definitions* 2.1 *and* 2.2. *Suppose that we have* $a^{(0)}_1 = P_1(q_0) \ge \kappa$ *and that at least one of the following conditions holds:*

$$c \ge \log(8/\delta) \quad \textit{and} \quad \ell_i = n \textit{ for each integer } i \textit{ with } 0 \le i < s; \tag{2.5}$$

$$c \ge \log 2 \quad \textit{and} \quad \sum_{i=1}^{s} \exp(q_{i-1} - q_i) < \delta/4. \tag{2.6}$$

Then, for the coherent sequence of directions $(\underline{\mathbf{v}}^{(i)})_{i\ge0}$ *attached to* $\underline{\mathbf{v}}$*, there is a unit vector* $\mathbf{u}$ *of* $\mathbb{R}^n$ *and a coherent sequence of bases* $(\underline{\mathbf{x}}^{(i)})_{i\ge0}$ *of* $\mathbb{Z}^n$ *such that, for each index* i *with* $0 \le i < s$*, each* $j = 1, \dots, n$*, and each* $q \in [q_i, q_{i+1})$*, we have*

$$\mathrm{dist}(\mathbf{x}^{(i)}_j, \mathbf{v}^{(i)}_j) \le \delta, \tag{2.7}$$

$$|\log \|\mathbf{x}^{(i)}_j\| - a^{(i)}_j| \le \log 2, \tag{2.8}$$

$$\big|\log |\mathbf{x}^{(i)}_{k_i} \cdot \mathbf{u}| - a^{(i)}_{k_i} + q_i\big| \le c_2 \quad \textit{if } q_{i+1} > q_i + \log 2 + c_2, \tag{2.9}$$

$$L_{\mathbf{u},j}(q) \le L_{\mathbf{u},j}(\underline{\mathbf{x}}^{(i)}, q) \le P_j(q) + c_1 \le L_{\mathbf{u},j}(q) + c_2, \tag{2.10}$$

where $c_1 = \log(32/\theta^2)$ *and* $c_2 = nc_1 + \log(n!)$. *When* $\underline{\mathbf{v}} = (\mathbf{e}_1, \dots, \mathbf{e}_{n-1})$*, we have* $\theta = 1$ *and one can take* $\kappa = \log(6/\delta)$.

For $0 \le i < s$ and $q \in [q_i, q_{i+1})$, inequalities (2.10) imply that, in some order which depends on q, the basis vectors $\mathbf{x}^{(i)}_1, \dots, \mathbf{x}^{(i)}_n$ realize the successive minima of $\mathcal{C}_{\mathbf{u}}(e^q)$ up to the factor $\exp(c_2)$, while, by (2.7), each of them stands at distance of at most δ from a vector among $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$. Since (2.10) holds independently of the choice of i and q, it also implies that

$$\|\mathbf{P}(q) - \mathbf{L}_{\mathbf{u}}(q)\|_\infty \le c_2 \quad \text{for each } q \ge q_0. \tag{2.11}$$

In particular, if P_1 is unbounded, then $L_{\mathbf{u},1}$ is also unbounded and thus the coordinates of $\mathbf{u}$ must be linearly independent over $\mathbb{Q}$.

As we will see in Section 5, the construction of the point $\mathbf{u}$ is particularly simple for a rigid n-system that meets condition (2.6). When $s = \infty$, this requires that the series $\sum_{i=1}^{\infty} \exp(q_{i-1} - q_i)$ is bounded and thus that the difference $q_i - q_{i-1}$ tends to infinity with i. Although this is restrictive, it holds for the important class of *self-similar rigid n-systems*, namely the rigid n-systems $\mathbf{P}\colon [q_0, \infty) \to \mathit{\Delta}_n$ which, for some $\rho > 1$, satisfy

$$\mathbf{P}(\rho q) = \rho \mathbf{P}(q) \quad \text{for each } q \ge q_0.$$

If such $\mathbf{P}$ has mesh c, then, for each $h \ge 0$, the restriction of $\mathbf{P}$ to the interval $[\rho^h q_0, \infty)$ is self-similar and rigid with mesh $\rho^h c$. Then, for h large enough, this restriction fulfills conditions (2.6).

In [10], it is shown that the spectrum of the standard exponents of Diophantine approximation can be computed in terms of n-systems only, and that the self-similar rigid n-systems yield a dense subset of that spectrum. The simplicity of the construction of points attached to such n-systems could possibly help in estimating the Hausdorff dimension of sets of points whose exponents lie in a given region of the spectrum.

The construction of the point $\mathbf{u}$ is more delicate for a rigid n-system that satisfies condition (2.5). This is also done in Section 5. Although this condition restricts the shape of the n-system, it is well adapted to our purpose. In Section 6, we apply this result to specific rigid n-systems that satisfy condition (2.5) to produce the points $\mathbf{u}$ that are needed in the second part of Theorem 1.1.

3. Projective distance. Fix an integer $n \ge 2$. We view $\mathbb{R}^n$ as a Euclidean space for the usual scalar product, denoted by a dot, so that the canonical basis $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ of $\mathbb{R}^n$ is orthonormal. More generally, for each integer k with $1 \le k \le n$, we endow $\bigwedge^k \mathbb{R}^n$ with the unique structure of Euclidean space for which the products $\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}$ with $1 \le i_1 < \cdots < i_k \le n$ form an orthonormal basis of that space, and we denote by $\|\boldsymbol{\alpha}\|$ the Euclidean norm of a vector $\boldsymbol{\alpha}$ in that space. Then the well-known Hadamard inequality tells us that

$$\|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k\| \le \|\mathbf{x}_1\| \cdots \|\mathbf{x}_k\|$$

for any choice of vectors $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^n$.

We define the *projective distance* between two non-zero points $\mathbf{x}, \mathbf{y}$ of $\mathbb{R}^n$ by

$$\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \frac{\|\mathbf{x} \wedge \mathbf{y}\|}{\|\mathbf{x}\| \, \|\mathbf{y}\|}.$$

It depends only on the classes of $\mathbf{x}$ and $\mathbf{y}$ in the projective space over $\mathbb{R}^n$ and represents the sine of the acute angle between the lines spanned by these vectors. In particular, it is a symmetric function of $\mathbf{x}$ and $\mathbf{y}$. Moreover, it is well-known that it satisfies the triangle inequality

$$\operatorname{dist}(\mathbf{x}, \mathbf{z}) \le \operatorname{dist}(\mathbf{x}, \mathbf{y}) + \operatorname{dist}(\mathbf{y}, \mathbf{z})$$

for any non-zero points $\mathbf{x}, \mathbf{y}, \mathbf{z}$ of $\mathbb{R}^n$ [11, Section 8, equation (3)]. The following property will also be useful.

LEMMA 3.1. *Let $\mathbf{u}, \mathbf{u}'$ be unit vectors of $\mathbb{R}^n$ with $\mathbf{u} \cdot \mathbf{u}' \ge 0$. Then*

$$\operatorname{dist}(\mathbf{u}, \mathbf{u}') \le \|\mathbf{u} - \mathbf{u}'\| \le 2 \operatorname{dist}(\mathbf{u}, \mathbf{u}').$$

Moreover, any point $\mathbf{x} \in \mathbf{u}^\perp$ satisfies

$$|\mathbf{x} \cdot \mathbf{u}'| \le 2\|\mathbf{x}\| \operatorname{dist}(\mathbf{u}, \mathbf{u}'). \tag{3.1}$$

Proof. The first estimate is well-known and amounts to the inequalities $\sin(\theta) \le 2\sin(\theta/2) \le 2\sin(\theta)$ where $\theta \in [0, \pi/2]$ is the angle between $\mathbf{u}$ and $\mathbf{u}'$. Then (3.1) follows since, for $\mathbf{x} \in \mathbf{u}^\perp$, we find $|\mathbf{x} \cdot \mathbf{u}'| = |\mathbf{x} \cdot (\mathbf{u}' - \mathbf{u})| \le \|\mathbf{x}\| \, \|\mathbf{u} - \mathbf{u}'\|$ by Hadamard's inequality. ∎

We define the projective distance from a non-zero point $\mathbf{x}$ of $\mathbb{R}^n$ to a non-zero subspace V of $\mathbb{R}^n$ as the infimum of the distances between $\mathbf{x}$ and non-zero points $\mathbf{y}$ of V. As explained in [8, Section 4], this infimum, denoted $\operatorname{dist}(\mathbf{x}, V)$, is in fact a minimum, and [8, Lemma 4.2] provides the following formula where $\operatorname{proj}_{V^\perp}$ denotes the orthogonal projection on the orthogonal complement $V^\perp$ of V in $\mathbb{R}^n$.

LEMMA 3.2. *For any non-zero point $\mathbf{x}$ of $\mathbb{R}^n$ and any non-zero subspace V of $\mathbb{R}^n$, we have*

$$\operatorname{dist}(\mathbf{x}, V) = \frac{\|\operatorname{proj}_{V^\perp}(\mathbf{x})\|}{\|\mathbf{x}\|} = \frac{\|\mathbf{x} \wedge \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_m\|}{\|\mathbf{x}\| \, \|\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_m\|},$$

where $(\mathbf{v}_1, \dots, \mathbf{v}_m)$ is any basis of V over $\mathbb{R}$.

Finally, for any non-zero subspaces V_1 and V_2 of $\mathbb{R}^n$, we define the distance $\operatorname{dist}(V_1, V_2)$ from V_1 to V_2 as the supremum of the numbers $\operatorname{dist}(\mathbf{x}, V_2)$ with $\mathbf{x} \in V_1 \setminus \{0\}$. As explained in [8, Section 4], this supremum is in fact a maximum. It is not symmetric in V_1 and V_2, for example when $V_1 \subsetneq V_2$. However, as shown in [8, Lemma 4.3], it has the following properties.

Lemma 3.3. *For any non-zero point* $\mathbf{x}$ *of* $\mathbb{R}^n$ *and any non-zero subspaces* V, V_1, V_2 *of* $\mathbb{R}^n$, *we have*

$$\operatorname{dist}(\mathbf{x}, V_2) \le \operatorname{dist}(\mathbf{x}, V_1) + \operatorname{dist}(V_1, V_2),$$
$$\operatorname{dist}(V, V_2) \le \operatorname{dist}(V, V_1) + \operatorname{dist}(V_1, V_2).$$

In this paper, we mainly need estimates for subspaces of $\mathbb{R}^n$ of codimension 1. For these, [8, Lemma 4.4] gives the following alternative formula which implies that, among themselves, the distance is symmetric.

Lemma 3.4. *For each* $j = 1, 2$, *let* U_j *be a subspace of* $\mathbb{R}^n$ *of codimension* 1 *and let* $\mathbf{u}_j$ *be a unit vector of* $U_j^\perp$. *Then* $\operatorname{dist}(U_1, U_2) = \operatorname{dist}(\mathbf{u}_1, \mathbf{u}_2)$.

The next result provides an explicit formula for this distance in a situation that we will encounter later. It follows directly from [8, Lemma 4.7] as the height of $\mathbb{R}^n$ is 1.

Lemma 3.5. *Let* $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ *be a basis of* $\mathbb{Z}^n$, *and let* k, ℓ *be integers with* $1 \le k < \ell \le n$. *For the subspaces*

$$U_1 = \langle \mathbf{x}_1, \dots, \widehat{\mathbf{x}_\ell}, \dots, \mathbf{x}_n \rangle_{\mathbb{R}} \quad \text{and} \quad U_2 = \langle \mathbf{x}_1, \dots, \widehat{\mathbf{x}_k}, \dots, \mathbf{x}_n \rangle_{\mathbb{R}},$$

we have

$$\operatorname{dist}(U_1, U_2) = \frac{\|\mathbf{x}_1 \wedge \cdots \wedge \widehat{\mathbf{x}_k} \wedge \cdots \wedge \widehat{\mathbf{x}_\ell} \wedge \cdots \wedge \mathbf{x}_n\|}{\|\mathbf{x}_1 \wedge \cdots \wedge \widehat{\mathbf{x}_k} \wedge \cdots \wedge \mathbf{x}_n\| \, \|\mathbf{x}_1 \wedge \cdots \wedge \widehat{\mathbf{x}_\ell} \wedge \cdots \wedge \mathbf{x}_n\|}.$$

For each m-tuple $\underline{\mathbf{v}} = (\mathbf{v}_1, \dots, \mathbf{v}_m)$ of non-zero vectors of $\mathbb{R}^n$ with $1 \le m \le n$, we define

$$\Theta(\underline{\mathbf{v}}) = \frac{\|\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_m\|}{\|\mathbf{v}_1\| \cdots \|\mathbf{v}_m\|} \in [0, 1]. \tag{3.2}$$

This normalized volume depends only on the classes of $\mathbf{v}_1, \dots, \mathbf{v}_m$ in the projective space over $\mathbb{R}^n$. We have $\Theta(\underline{\mathbf{v}}) = 0$ if and only if $\underline{\mathbf{v}}$ is linearly dependent, and $\Theta(\underline{\mathbf{v}}) = 1$ if and only if it is orthogonal. The following result provides a measure of continuity of this map.

Lemma 3.6. *Let* $\underline{\mathbf{v}} = (\mathbf{v}_1, \dots, \mathbf{v}_m)$ *be a linearly independent* m-*tuple of vectors of* $\mathbb{R}^n$ *for some* $m \in \{1, \dots, n\}$, *let* $\delta \in \mathbb{R}$ *with* $0 \le \delta < \Theta(\underline{\mathbf{v}})/(2m)$, *and let* $\underline{\mathbf{x}} = (\mathbf{x}_1, \dots, \mathbf{x}_m)$ *be an* m-*tuple of non-zero vectors of* $\mathbb{R}^n$ *with* $\operatorname{dist}(\mathbf{x}_j, \mathbf{v}_j) \le \delta$ *for each* $j = 1, \dots, m$. *Then* $\underline{\mathbf{x}}$ *is linearly independent over* $\mathbb{R}$ *and*

$$|\Theta(\underline{\mathbf{x}}) - \Theta(\underline{\mathbf{v}})| \le 2m\delta.$$

Upon setting $W = \langle \mathbf{x}_1, \dots, \mathbf{x}_m \rangle_{\mathbb{R}}$ *and* $V = \langle \mathbf{v}_1, \dots, \mathbf{v}_m \rangle_{\mathbb{R}}$, *we also have*

$$\operatorname{dist}(W, V) \le 2m\delta/\Theta(\underline{\mathbf{x}}). \tag{3.3}$$

Proof. We may assume that, for $j = 1, \dots, m$, we have $\|\mathbf{v}_j\| = \|\mathbf{x}_j\| = 1$ and $\mathbf{x}_j \cdot \mathbf{v}_j \ge 0$ so that Lemma 3.1 gives $\|\mathbf{x}_j - \mathbf{v}_j\| \le 2\delta$. Set

$$\boldsymbol{\alpha}_j = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_j \wedge \mathbf{x}_{j+1} \wedge \cdots \wedge \mathbf{x}_m \quad \text{for } j = 0, \dots, m.$$

For each $j = 0, \dots, m-1$, we find that

$$\|\boldsymbol{\alpha}_j - \boldsymbol{\alpha}_{j+1}\| = \|\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_j \wedge (\mathbf{x}_{j+1} - \mathbf{v}_{j+1}) \wedge \mathbf{x}_{j+2} \wedge \cdots \wedge \mathbf{x}_m\| \le 2\delta,$$

thus $|\Theta(\underline{\mathbf{x}}) - \Theta(\underline{\mathbf{v}})| = \big|\|\boldsymbol{\alpha}_0\| - \|\boldsymbol{\alpha}_m\|\big| \le \|\boldsymbol{\alpha}_0 - \boldsymbol{\alpha}_m\| \le 2m\delta$. As $2m\delta < \Theta(\underline{\mathbf{v}})$, this implies that $\Theta(\underline{\mathbf{x}}) > 0$, and so $\underline{\mathbf{x}}$ is linearly independent over $\mathbb{R}$.

To prove the last assertion of the lemma, choose any non-zero m-tuple $(a_1, \dots, a_m)$ in $\mathbb{R}^m$, and form the points

$$\mathbf{x} = a_1\mathbf{x}_1 + \cdots + a_m\mathbf{x}_m \in W \quad \text{and} \quad \mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m \in V.$$

For each $j = 1, \dots, m$, we find, using Hadamard's inequality, that

$$\|\mathbf{x}\| \ge \|\mathbf{x} \wedge \mathbf{x}_1 \wedge \cdots \wedge \widehat{\mathbf{x}_j} \wedge \cdots \wedge \mathbf{x}_m\| = |a_j|\Theta(\underline{\mathbf{x}}).$$

This implies that

$$\|\mathbf{x} - \mathbf{v}\| \le 2\delta(|a_1| + \cdots + |a_m|) \le 2m\delta\|\mathbf{x}\|/\Theta(\underline{\mathbf{x}}),$$

and so

$$\operatorname{dist}(\mathbf{x}, V) = \frac{\|\operatorname{proj}_{V^\perp}(\mathbf{x})\|}{\|\mathbf{x}\|} \le \frac{\|\mathbf{x} - \mathbf{v}\|}{\|\mathbf{x}\|} \le 2m\delta/\Theta(\underline{\mathbf{x}}).$$

As $\mathbf{x}$ can be any non-zero point of W, this gives (3.3). ∎

The lemma below will be needed in Section 5 to prove the second part of Theorem 1.1.

LEMMA 3.7. *Let r be an integer with $1 \le r \le n$, let $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ be an orthonormal basis of $\mathbb{R}^n$ and let*

$$V = \langle \mathbf{v}_1 + \cdots + \mathbf{v}_r, \mathbf{v}_{r+1}, \dots, \mathbf{v}_n \rangle_{\mathbb{R}}.$$

Suppose that non-zero vectors $\mathbf{x}, \mathbf{x}_1, \dots, \mathbf{x}_r \in \mathbb{R}^n$ satisfy

$$\operatorname{dist}(\mathbf{x}, V) \le \delta \quad \textit{and} \quad \max_{1 \le j \le r} \operatorname{dist}(\mathbf{x}_j, \mathbf{v}_j) \le \delta$$

for some $\delta \in \mathbb{R}$ with $0 \le \delta \le 1/(24r)$, and that

$$\mathbf{x} = a_1\mathbf{x}_1 + \cdots + a_r\mathbf{x}_r$$

for some $a_1, \dots, a_r \in \mathbb{R}$. Then, for each $j = 1, \dots, r$, we have

$$\frac{\|\mathbf{x}\|}{2\sqrt{r}} \le \|a_j\mathbf{x}_j\| \le \frac{2\|\mathbf{x}\|}{\sqrt{r}}.$$

Proof. Upon dividing $\mathbf{x}, \mathbf{x}_1, \dots, \mathbf{x}_r$ by their norms, we may assume that these vectors have norm 1. Then, upon multiplying each of them appropriately by ± 1, we may further assume, by Lemma 3.1, that

$$\|\mathbf{x} - \mathbf{v}\| \le 2\delta \quad \text{and} \quad \max_{1 \le j \le r} \|\mathbf{x}_j - \mathbf{v}_j\| \le 2\delta$$

for a unit vector $\mathbf{v}$ of V which is closest to $\mathbf{x}$, of the form

$$\mathbf{v} = b(\mathbf{v}_1 + \cdots + \mathbf{v}_r) + \mathbf{w}$$

with $b \geq 0$ and $\mathbf{w} \in W$, where $W = \langle \mathbf{v}_{r+1}, \dots, \mathbf{v}_n \rangle_{\mathbb{R}}$. Upon setting

$$\mathbf{y} = a_1\mathbf{v}_1 + \cdots + a_r\mathbf{v}_r \in W^{\perp} \quad \text{and} \quad A = 1 + |a_1| + \cdots + |a_r|,$$

we find

$$\begin{aligned} \|\mathbf{v} - \mathbf{y}\| &= \|(\mathbf{v} - \mathbf{x}) + a_1(\mathbf{x}_1 - \mathbf{v}_1) + \cdots + a_r(\mathbf{x}_r - \mathbf{v}_r)\| \leq 2A\delta, \\ |a_j - b| &= |(\mathbf{y} - \mathbf{v}) \cdot \mathbf{v}_j| \leq \|\mathbf{v} - \mathbf{y}\| \quad \text{for } j = 1, \dots, r, \\ |b\sqrt{r} - 1| &= \big|\|b(\mathbf{v}_1 + \cdots + \mathbf{v}_r)\| - \|\mathbf{v}\|\big| \leq \|\mathbf{w}\|. \end{aligned}$$

Since $\mathbf{w} = \mathrm{proj}_W(\mathbf{v}) = \mathrm{proj}_W(\mathbf{v} - \mathbf{y})$, we also have $\|\mathbf{w}\| \leq \|\mathbf{v} - \mathbf{y}\|$ and so the preceding three estimates give

$$\max_{1 \leq j \leq r} |a_j - 1/\sqrt{r}| \leq (1 + 1/\sqrt{r})\|\mathbf{v} - \mathbf{y}\| \leq 4A\delta. \tag{3.4}$$

In turn, by definition of A, this implies that

$$A \leq 1 + r(1/\sqrt{r} + 4A\delta) \leq 2\sqrt{r} + A/6$$

since $4r\delta \leq 1/6$. So, we obtain the upper bound $A \leq 3\sqrt{r}$, which substituted in (3.4) gives

$$\max_{1 \leq j \leq r} |a_j - 1/\sqrt{r}| \leq (12r\delta)/\sqrt{r} \leq 1/(2\sqrt{r}),$$

and so $1/(2\sqrt{r}) \leq |a_j| \leq 2/\sqrt{r}$ for $j = 1, \dots, r$. ∎

4. Proof of the first part of Theorem 1.1 and deduction of the corollaries. In this section, we prove the first part of our first main Theorem 1.1 and, assuming that its second part holds as well, we deduce the corollaries stated in the introduction. To this end, we work with a fixed integer $n \geq 2$, and we identify $\mathrm{GL}_n(\mathbb{Z})$ with the subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of the linear operators T on $\mathbb{R}^n$ with $T(\mathbb{Z}^n) = \mathbb{Z}^n$. The first lemma below is our main tool.

LEMMA 4.1. *Let $\mathbf{v}_1, \dots, \mathbf{v}_m$ be linearly independent vectors of $\mathbb{R}^n$ for some integer m with $0 \leq m < n$, and let $\delta > 0$. Then there exists $T \in \mathrm{GL}_n(\mathbb{Z})$ such that* $\mathrm{dist}(T(\mathbf{e}_j), \mathbf{v}_j) \leq \delta$ *for each index j with $1 \leq j \leq m$.*

Note that the restriction $m < n$ is needed in general. For $m = 1$, the lemma is a consequence of a much stronger result of Erdős [5].

Proof of Lemma 4.1. We are going to prove, by induction on m, that there is a basis $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ of $\mathbb{Z}^n$ such that $\mathrm{dist}(\mathbf{x}_j, \mathbf{v}_j) \leq \delta$ for each j with $1 \leq j \leq m$. The result then follows by taking for T the element of $\mathrm{GL}_n(\mathbb{Z})$ which sends $\mathbf{e}_j$ to $\mathbf{x}_j$ for each $j = 1, \dots, n$.

For $m = 0$, we can take any basis $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ of $\mathbb{Z}^n$. Suppose now that $1 \leq m < n$. We may assume, by induction, that there is a basis $(\mathbf{y}_1, \dots, \mathbf{y}_n)$ of $\mathbb{Z}^n$ such that $\mathrm{dist}(\mathbf{y}_j, \mathbf{v}_j) \leq \delta$ for each j with $1 \leq j \leq m - 1$. Write

$$\mathbf{v}_m = a_1\mathbf{y}_1 + \cdots + a_n\mathbf{y}_n$$

with $a_1, \dots, a_n \in \mathbb{R}$. By permuting $\mathbf{y}_m, \dots, \mathbf{y}_n$ if necessary, and by multiplying each of them by ± 1 appropriately, we may assume that we have $0 \le a_m \le \dots \le a_n$. For a choice of $t > 0$ to be fixed later, we form the point

$$\mathbf{x}_m = b_1\mathbf{y}_1 + \dots + b_n\mathbf{y}_n \quad \text{where} \quad b_j = \begin{cases} \lceil ta_j \rceil & \text{if } j \neq m, \\ \lceil ta_j \rceil - 1 & \text{if } j = m. \end{cases}$$

By construction, we have $\|\mathbf{x}_m - t\mathbf{v}_m\| \le Y$ where $Y = \|\mathbf{y}_1\| + \dots + \|\mathbf{y}_n\|$ is independent of t. Thus, if t is large enough, say $t \ge t_0$, we obtain

$$\operatorname{dist}(\mathbf{x}_m, \mathbf{v}_m) = \frac{\|(\mathbf{x}_m - t\mathbf{v}_m) \wedge \mathbf{v}_m\|}{\|\mathbf{x}_m\|\,\|\mathbf{v}_m\|} \le \frac{\|\mathbf{x}_m - t\mathbf{v}_m\|}{\|\mathbf{x}_m\|} \le \frac{Y}{t\|\mathbf{v}_m\| - Y} \le \delta.$$

If $a_n \neq 0$, we take $t = p/a_n$ for a prime number p with $p \ge a_n t_0$, so that the above inequality holds. Then we have $b_n = p$ and $-1 \le b_m < p$. So, the integers $b_m, \dots, b_n$ are relatively prime as a set. If $a_n = 0$, we take $t = t_0$ and then $(b_m, \dots, b_n) = (-1, 0, \dots, 0)$. So, in both cases, the point $\mathbf{x} = b_m\mathbf{y}_m + \dots + b_n\mathbf{y}_n$ can be completed to a basis $(\mathbf{x}, \mathbf{x}_{m+1}, \dots, \mathbf{x}_n)$ of $\langle \mathbf{y}_m, \dots, \mathbf{y}_n \rangle_{\mathbb{Z}}$. Then $(\mathbf{y}_1, \dots, \mathbf{y}_{m-1}, \mathbf{x}_m, \dots, \mathbf{x}_n)$ is a basis of $\mathbb{Z}^n$ with the required properties. ∎

LEMMA 4.2. *Let $T \in \mathrm{GL}_n(\mathbb{R})$. There exists a constant $\kappa \ge 1$ with the following properties:*

(1) $\kappa^{-1}\|\mathbf{x}\| \le \|T(\mathbf{x})\| \le \kappa\|\mathbf{x}\|$ *for each* $\mathbf{x} \in \mathbb{R}^n$*;*
(2) $\kappa^{-1} \operatorname{dist}(\mathbf{x}, \mathbf{y}) \le \operatorname{dist}(T(\mathbf{x}), T(\mathbf{y})) \le \kappa \operatorname{dist}(\mathbf{x}, \mathbf{y})$ *for any non-zero points* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$*;*
(3) $\kappa^{-1} \operatorname{dist}(\mathbf{x}, V) \le \operatorname{dist}(T(\mathbf{x}), T(V)) \le \kappa \operatorname{dist}(\mathbf{x}, V)$ *for any non-zero point* $\mathbf{x} \in \mathbb{R}^n$ *and any non-zero subspace* V *of* $\mathbb{R}^n$.

Proof. For each $m = 1, \dots, n$, the linear operator $\bigwedge^m T$ on $\bigwedge^m \mathbb{R}^n$ is invertible and so there is a constant $c_m \ge 1$ such that

$$c_m^{-1}\|\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_m\| \le \|T(\mathbf{x}_1) \wedge \dots \wedge T(\mathbf{x}_m)\| \le c_m\|\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_m\|$$

for any $\mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{R}^n$. Thus, for a non-zero point $\mathbf{x}$ of $\mathbb{R}^n$ and a non-zero proper subspace V of $\mathbb{R}^n$ with basis $(\mathbf{v}_1, \dots, \mathbf{v}_m)$, we find, using Lemma 3.2, that

$$\begin{aligned} \operatorname{dist}(T(\mathbf{x}), T(V)) &= \frac{\|T(\mathbf{x}) \wedge T(\mathbf{v}_1) \wedge \dots \wedge T(\mathbf{v}_m)\|}{\|T(\mathbf{x})\|\,\|T(\mathbf{v}_1) \wedge \dots \wedge T(\mathbf{v}_m)\|} \\ &\le \frac{c_{m+1}\|\mathbf{x} \wedge \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_m\|}{c_1^{-1}c_m^{-1}\|\mathbf{x}\|\,\|\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_m\|} = c_1c_mc_{m+1} \operatorname{dist}(\mathbf{x}, V), \end{aligned}$$

and similarly $\operatorname{dist}(T(\mathbf{x}), T(V)) \ge (c_1c_mc_{m+1})^{-1} \operatorname{dist}(\mathbf{x}, V)$. If $V = \mathbb{R}^n$, then $T(V) = \mathbb{R}^n$ and so $\operatorname{dist}(T(\mathbf{x}), T(V)) = \operatorname{dist}(\mathbf{x}, V) = 0$. Thus, properties (1) and (3) hold for an appropriate choice of κ. Then property (2) follows by taking $V = \langle \mathbf{y} \rangle_{\mathbb{R}}$. ∎

In the arguments below, we denote by ${}^tT$ the transpose of a linear operator T on $\mathbb{R}^n$, namely the linear operator on $\mathbb{R}^n$ characterized by the condition $T(\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot {}^tT(\mathbf{y})$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Proof of Theorem 1.1, part (1). If $n = m+2$ and $V = \langle \mathbf{e}_1, \dots, \mathbf{e}_{m+1} \rangle_{\mathbb{R}} \subseteq \mathbb{R}^{m+2}$, that statement follows from [16, Theorem 1(b)]. There, Thurnheer shows that, for any $\delta, \epsilon > 0$ and any point $\mathbf{u} \in \mathbb{R}^{m+2}$ with $\mathbb{Q}$-linearly independent coordinates, there exists a non-zero point $\mathbf{x} \in \mathbb{Z}^{m+2}$ such that

$$|\mathbf{x} \cdot \mathbf{e}_{m+2}| \le \delta \|\mathbf{x}\| \quad \text{and} \quad |\mathbf{x} \cdot \mathbf{u}| \le \epsilon \|\mathbf{x}\|^{-\rho}$$

where ρ is given by (1.2). However, since $V = \mathbf{e}_{m+2}^{\perp}$, the first inequality may be rewritten as $\mathrm{dist}(\mathbf{x}, V) \le \delta$. We give a simplified, self-contained proof of this result in Section 7.

In general, suppose that $n \ge m+2 \ge 3$, that V is an arbitrary subspace of $\mathbb{R}^n$ of dimension $m+1$ and that $\mathbf{u} \in \mathbb{R}^n$ has linearly independent coordinates over $\mathbb{Q}$. Choose an orthonormal basis $\underline{\mathbf{v}} = (\mathbf{v}_1, \dots, \mathbf{v}_{m+1})$ of V and fix $\delta, \epsilon > 0$ with $\delta < 1$. By Lemma 4.1, there exists $T \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$\mathrm{dist}(T(\mathbf{e}_j), \mathbf{v}_j) \le \frac{\delta}{8(m+1)} \qquad \text{for } j = 1, \dots, m+1.$$

For $\underline{\mathbf{x}} = (T(\mathbf{e}_1), \dots, T(\mathbf{e}_{m+1}))$, Lemma 3.6 gives $\Theta(\underline{\mathbf{x}}) \ge \Theta(\underline{\mathbf{v}}) - \delta/4 \ge 1/2$, and then

$$\mathrm{dist}(T(\widetilde{V}), V) \le \delta/2 \quad \text{where} \quad \widetilde{V} = \langle \mathbf{e}_1, \dots, \mathbf{e}_{m+1} \rangle_{\mathbb{R}} \subseteq \mathbb{R}^n. \tag{4.1}$$

For our choice of T, choose a constant $\kappa \ge 1$ as in Lemma 4.2, and define

$$\widetilde{\mathbf{u}} = {}^tT(\mathbf{u}) \in \mathbb{R}^n.$$

Since $T \in \mathrm{GL}_n(\mathbb{Z})$, the n coordinates of $\widetilde{\mathbf{u}}$ are linearly independent over $\mathbb{Q}$. A fortiori, its first $m+2$ coordinates are linearly independent over $\mathbb{Q}$. Thus, under the natural identification of $\mathbb{R}^{m+2}$ with $\langle \mathbf{e}_1, \dots, \mathbf{e}_{m+2} \rangle_{\mathbb{R}} \subseteq \mathbb{R}^n$, the result of Thurnheer provides a non-zero point $\widetilde{\mathbf{x}} \in \langle \mathbf{e}_1, \dots, \mathbf{e}_{m+2} \rangle_{\mathbb{Z}}$ such that

$$\mathrm{dist}(\widetilde{\mathbf{x}}, \widetilde{V}) \le \delta/(2\kappa) \quad \text{and} \quad |\widetilde{\mathbf{x}} \cdot \widetilde{\mathbf{u}}| \le \epsilon \kappa^{-\rho} \|\widetilde{\mathbf{x}}\|^{-\rho}.$$

Then $\mathbf{x} = T(\widetilde{\mathbf{x}})$ is a non-zero point of $\mathbb{Z}^n$ which, by definition of $\widetilde{\mathbf{u}}$, satisfies $\mathbf{x} \cdot \mathbf{u} = \widetilde{\mathbf{x}} \cdot \widetilde{\mathbf{u}}$. Therefore, by Lemma 4.2, the preceding inequalities yield

$$\mathrm{dist}(\mathbf{x}, T(\widetilde{V})) \le \kappa\, \mathrm{dist}(\widetilde{\mathbf{x}}, \widetilde{V}) \le \delta/2 \quad \text{and} \quad |\mathbf{x} \cdot \mathbf{u}| \le \epsilon \kappa^{-\rho} \|\widetilde{\mathbf{x}}\|^{-\rho} \le \epsilon \|\mathbf{x}\|^{-\rho}.$$

Then, using (4.1), Lemma 3.3 provides

$$\mathrm{dist}(\mathbf{x}, V) \le \mathrm{dist}(\mathbf{x}, T(\widetilde{V})) + \mathrm{dist}(T(\widetilde{V}), V) \le \delta$$

as needed.

LEMMA 4.3. *Let V_1, V_2 be non-zero subspaces of $\mathbb{R}^n$. There is a constant κ depending only on V_1 and V_2 such that any non-zero point $\mathbf{x}$ of $\mathbb{R}^n$ satisfies*

$$\text{(4.2)} \qquad \operatorname{dist}(\mathbf{x}, V_1 \cap V_2) \le \kappa \max\{\operatorname{dist}(\mathbf{x}, V_1), \operatorname{dist}(\mathbf{x}, V_2)\},$$

with the convention that the left hand side is 1 *if* $V_1 \cap V_2 = \{0\}$.

Proof. There are integers $r, s, t \in \{1, \dots, n\}$ with $r \le t$ and an invertible linear operator $T \in \mathrm{GL}_n(\mathbb{R})$ such that $T(E_1) = V_1$ and $T(E_2) = V_2$, where

$$E_1 = \langle \mathbf{e}_1, \dots, \mathbf{e}_s \rangle_{\mathbb{R}} \quad \text{and} \quad E_2 = \langle \mathbf{e}_r, \dots, \mathbf{e}_t \rangle_{\mathbb{R}}.$$

So, by Lemma 4.2, it suffices to prove the result for $V_1 = E_1$ and $V_2 = E_2$. If $E_1 \cap E_2 \ne \{0\}$, we have $r \le s$. Then, for each non-zero $\mathbf{x} = (x_1, \dots, x_n)$ in $\mathbb{R}^n$, we find that

$$\begin{aligned}
\operatorname{proj}_{(E_1 \cap E_2)^\perp}(\mathbf{x}) &= (x_1, \dots, x_{r-1}, x_{s+1}, \dots, x_n), \\
\operatorname{proj}_{E_1^\perp}(\mathbf{x}) &= (x_{s+1}, \dots, x_n), \\
\operatorname{proj}_{E_2^\perp}(\mathbf{x}) &= (x_1, \dots, x_{r-1}, x_{t+1}, \dots, x_n),
\end{aligned}$$

thus $\|\operatorname{proj}_{(E_1 \cap E_2)^\perp}(\mathbf{x})\| \le \|\operatorname{proj}_{E_1^\perp}(\mathbf{x})\| + \|\operatorname{proj}_{E_2^\perp}(\mathbf{x})\|$, and so

$$\operatorname{dist}(\mathbf{x}, E_1 \cap E_2) \le \operatorname{dist}(\mathbf{x}, E_1) + \operatorname{dist}(\mathbf{x}, E_2).$$

If $E_1 \cap E_2 = \{0\}$, we have instead $s < r$ and, for any non-zero $\mathbf{x} \in \mathbb{R}^n$, we obtain $\|\mathbf{x}\| \le \|\operatorname{proj}_{E_1^\perp}(\mathbf{x})\| + \|\operatorname{proj}_{E_2^\perp}(\mathbf{x})\|$, which yields

$$1 \le \operatorname{dist}(\mathbf{x}, E_1) + \operatorname{dist}(\mathbf{x}, E_2).$$

So, in this situation, (4.2) holds with $\kappa = 2$. ∎

Proof of Corollary 1.2. Let n, m, ρ and V be as in the statement of Theorem 1.1. For part (1), fix a point $\mathbf{u}$ of $\mathbb{R}^n$ with linearly independent coordinates over $\mathbb{Q}$, and real numbers $\delta, \epsilon > 0$. Setting $W = V \cap \mathbf{u}^\perp$, Lemma 4.3 provides a constant $\kappa > 0$ such that

$$\operatorname{dist}(\mathbf{x}, W) \le \kappa \max\{\operatorname{dist}(\mathbf{x}, V), \operatorname{dist}(\mathbf{x}, \mathbf{u}^\perp)\}$$

for any non-zero $\mathbf{x} \in \mathbb{R}^n$. Define $\delta' = \delta/\kappa$ and $\epsilon' = \min\{\epsilon, \delta' \|\mathbf{u}\|\}$. Part (1) of Theorem 1.1 provides a non-zero point $\mathbf{x} \in \mathbb{Z}^n$ with

$$\operatorname{dist}(\mathbf{x}, V) \le \delta' \quad \text{and} \quad |\mathbf{x} \cdot \mathbf{u}| \le \epsilon' \|\mathbf{x}\|^{-\rho}.$$

Since $\operatorname{dist}(\mathbf{x}, \mathbf{u}^\perp) = |\mathbf{x} \cdot \mathbf{u}| / (\|\mathbf{x}\| \, \|\mathbf{u}\|) \le \epsilon' / \|\mathbf{u}\| \le \delta'$, this point $\mathbf{x}$ satisfies $\operatorname{dist}(\mathbf{x}, W) \le \delta$ and $|\mathbf{x} \cdot \mathbf{u}| \le \epsilon \|\mathbf{x}\|^{-\rho}$, as needed. For part (2), the assertion of the corollary is clear because $\operatorname{dist}(\mathbf{x}, V) \le \operatorname{dist}(\mathbf{x}, W)$ for any non-zero point $\mathbf{x} \in \mathbb{R}^n$ and any non-zero subspace W of V.

A similar argument allows one to recover Theorem 1.1 from its consequence provided by Corollary 1.2. This consequence is thus an equivalent form of the theorem.

Proof of Corollary 1.3. Here we have $n \geq 3$. To prove the first assertion of the corollary, fix a point $\mathbf{u} \in \mathbb{R}^n$ with linearly independent coordinates over $\mathbb{Q}$ and fix $\epsilon > 0$. The subspace V of $\mathbb{R}^n$ given by (1.6) is defined over $\mathbb{Q}$ and has dimension 2, thus $W = V \cap \mathbf{u}^\perp$ has dimension 1 and so it is spanned by a unit vector $\mathbf{w} = (a, b, \dots, b)$ for some non-zero $a, b \in \mathbb{R}$ with $b > 0$. Then Corollary 1.2 provides a non-zero point $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n$ with

$$\operatorname{dist}(\mathbf{x}, \mathbf{w}) \leq b/4 \quad \text{and} \quad |\mathbf{x} \cdot \mathbf{u}| \leq \epsilon \|\mathbf{x}\|^{-\rho},$$

where $\rho = \rho_1 = \gamma$. Replacing $\mathbf{x}$ by $-\mathbf{x}$ if necessary, we may further assume that $\mathbf{x} \cdot \mathbf{w} \geq 0$. Then, by Lemma 3.1, the difference $\|\mathbf{x}\|^{-1}\mathbf{x} - \mathbf{w}$ has norm at most $b/2$ and so we obtain $x_j \geq b\|\mathbf{x}\|/2 > 0$ for $j = 2, \dots, n$. Thus, the point $\mathbf{x}$ satisfies (1.7).

To prove the second assertion, fix an unbounded non-decreasing function $\psi\colon [1, \infty) \to (0, \infty)$. Assuming that part (2) of Theorem 1.1 holds, there is a point $\widetilde{\mathbf{u}}$ of $\mathbb{R}^n$ with linearly independent coordinates over $\mathbb{Q}$ and a number $\widetilde{\delta} > 0$ for which at most finitely many points $\widetilde{\mathbf{x}} \in \mathbb{Z}^n$ satisfy

$$\operatorname{dist}(\widetilde{\mathbf{x}}, V) \leq \widetilde{\delta} \quad \text{and} \quad |\widetilde{\mathbf{x}} \cdot \widetilde{\mathbf{u}}| \leq \psi(\|\widetilde{\mathbf{x}}\|^{1/2})^{-1/2} \|\widetilde{\mathbf{x}}\|^{-\rho}, \tag{4.3}$$

with V given by (1.6), and $\rho = \rho_1 = \gamma$. Choose a positive integer k such that $3/k \leq \widetilde{\delta}$ and form the map $T \in \mathrm{GL}_n(\mathbb{Z})$ defined, for any $\mathbf{x} = (x_1, \dots, x_n)$ in $\mathbb{R}^n$, by

$$T(\mathbf{x}) = (x_1, x_3 + (k+1)\overline{x}, x_3 + k\overline{x}, \dots, x_n + k\overline{x}) \quad \text{where} \quad \overline{x} = x_2 + \cdots + x_n.$$

We claim that the point $\mathbf{u} = {}^tT(\widetilde{\mathbf{u}})$ has the required properties. Since its coordinates are linearly independent over $\mathbb{Q}$, we need to show that there are at most finitely many points $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n$ with property (1.8), namely

$$x_2, \dots, x_n > 0 \quad \text{and} \quad |\mathbf{x} \cdot \mathbf{u}| \leq \psi(\|\mathbf{x}\|)^{-1} \|\mathbf{x}\|^{-\rho}$$

(as $\rho = \gamma$). For such a point $\mathbf{x}$, the integer $\overline{x}$ is positive and, since $\mathbf{v} = (x_1, k\overline{x}, \dots, k\overline{x})$ belongs to V, we find

$$\operatorname{dist}(T(\mathbf{x}), V) \leq \frac{\|T(\mathbf{x}) - \mathbf{v}\|}{\|T(\mathbf{x})\|} = \frac{\|(0, x_3 + \overline{x}, x_3, \dots, x_n)\|}{\|T(\mathbf{x})\|} \leq \frac{3\overline{x}}{k\overline{x}} \leq \widetilde{\delta}.$$

We also have $\|\mathbf{x}\| \leq \|T(\mathbf{x})\| \leq \kappa\|\mathbf{x}\|$ where $\kappa = n(k+1)$, thus

$$\begin{aligned} |T(\mathbf{x}) \cdot \widetilde{\mathbf{u}}| = |\mathbf{x} \cdot \mathbf{u}| &\leq \psi(\|\mathbf{x}\|)^{-1} \|\mathbf{x}\|^{-\rho} \\ &\leq \psi(\kappa^{-1}\|T(\mathbf{x})\|)^{-1} \kappa^{\rho} \|T(\mathbf{x})\|^{-\rho}. \end{aligned} \tag{4.4}$$

Thus, if the norm of $\mathbf{x}$ is large enough, the point $\widetilde{\mathbf{x}} = T(\mathbf{x})$ satisfies (4.3). So, $\widetilde{\mathbf{x}}$ and $\mathbf{x}$ lie in finite sets.

Proof of Corollary 1.4. The argument is similar to the proof of the preceding corollary. Let V be the subspace of $\mathbb{R}^n$ defined by (1.9) for the given integers m and n. For the first assertion, fix $\delta, \epsilon > 0$ and a point

$\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$ with linearly independent coordinates over $\mathbb{Q}$. Since $u_1 \neq 0$, there is a constant $\kappa > 0$ depending on $\mathbf{u}$ such that

$$\|\mathbf{x}\| \le \kappa \max\{|\mathbf{x}\cdot\mathbf{u}|, |x_2|, \dots, |x_n|\} \tag{4.5}$$

for each $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. Set $\epsilon' = \min\{\epsilon, 1/(2\kappa)\}$. Part (1) of Theorem 1.1 provides a non-zero point $\mathbf{x}$ of $\mathbb{Z}^n$ with

$$\operatorname{dist}(\mathbf{x}, V) \le \delta/\kappa \quad \text{and} \quad |\mathbf{x}\cdot\mathbf{u}| \le \epsilon'\|\mathbf{x}\|^{-\rho}.$$

For this point, we have $\kappa|\mathbf{x}\cdot\mathbf{u}| \le 1/2 < \|\mathbf{x}\|$. Thus, upon writing $\mathbf{x} = (x_1, \dots, x_n)$, estimate (4.5) yields

$$\|\mathbf{x}\| \le \kappa \max\{|x_2|, \dots, |x_n|\}.$$

On the other hand, we have

$$\operatorname{dist}(\mathbf{x}, V) = \frac{\|(x_{m+2}, \dots, x_n)\|}{\|\mathbf{x}\|} \ge \frac{\max\{|x_{m+2}|, \dots, |x_n|\}}{\|\mathbf{x}\|}. \tag{4.6}$$

So, the point $\mathbf{x}$ has the required property (1.10).

For the second assertion, fix an unbounded non-decreasing function ψ from $[1,\infty)$ to $(0,\infty)$. Part (2) of Theorem 1.1 provides a point $\widetilde{\mathbf{u}}$ of $\mathbb{R}^n$ with linearly independent coordinates over $\mathbb{Q}$ and a number $\widetilde{\delta} > 0$ for which at most finitely many points $\widetilde{\mathbf{x}} \in \mathbb{Z}^n$ satisfy (4.3) for our current choice of V. Set $k = \lceil \sqrt{n}/\widetilde{\delta}\,\rceil$ and form the map $T \in \mathrm{GL}_n(\mathbb{R})$ defined, for any $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, by

$$T(\mathbf{x}) = (kx_1, \dots, kx_{m+1}, x_{m+2}, \dots, x_n).$$

We claim that the point $\mathbf{u} = {}^tT(\widetilde{\mathbf{u}})$ has the required properties. To prove this, suppose that a non-zero point $\mathbf{x} = (x_1, \dots, x_n)$ of $\mathbb{Z}^n$ satisfies (1.11). As in the proof of the second assertion of Corollary 1.3, we simply need to show that $\widetilde{\mathbf{x}} = T(\mathbf{x}) \in \mathbb{Z}^n$ satisfies (4.3) when $\mathbf{x} \in \mathbb{Z}^n$ has sufficiently large norm. The first condition in (1.11) yields

$$\|(x_{m+2}, \dots, x_n)\| \le \sqrt{n} \max\{|x_1|, \dots, |x_n|\} = \sqrt{n} \max\{|x_1|, \dots, |x_{m+1}|\},$$

thus

$$\operatorname{dist}(T(\mathbf{x}), V) = \frac{\|(x_{m+2}, \dots, x_n)\|}{\|T(\mathbf{x})\|} \le \frac{\sqrt{n}\max\{|x_1|, \dots, |x_{m+1}|\}}{k\|(x_1, \dots, x_{m+1})\|} \le \frac{\sqrt{n}}{k} \le \widetilde{\delta}.$$

Since $\|\mathbf{x}\| \le \|T(\mathbf{x})\| \le k\|\mathbf{x}\|$, we also find that (4.4) holds with $\kappa = k$. Thus $\widetilde{\mathbf{x}} = T(\mathbf{x})$ satisfies (4.3) if $\|\mathbf{x}\|$ is large enough.

Proof of Corollary 1.5. The first part of Corollary 1.4 shows that $\rho(\mathbf{u}) \ge \rho_m$ for each point $\mathbf{u} \in \mathbb{R}^n$ with linearly independent coordinates over $\mathbb{Q}$: it suffices to choose $\delta = 1/2$ and let ϵ tend to 0. Its second part, applied with $\psi(t) = \log(t+1)$, provides a point $\mathbf{u}$ with $\mathbb{Q}$-linearly independent coordinates and $\rho(\mathbf{u}) \le \rho_m$, thus $\rho(\mathbf{u}) = \rho_m$. Since $\rho(t\mathbf{u}) = \rho(\mathbf{u})$ for any $t > 0$, we may further choose $\mathbf{u}$ of norm 1.

By the above, the set $\mathcal{S}$ of unit vectors $\mathbf{u} \in \mathbb{R}^n$ with $\mathbb{Q}$-linearly independent coordinates satisfying $\rho(\mathbf{u}) = \rho_m$ is not empty. To show that it is uncountable, choose an arbitrary sequence $(\mathbf{u}_i)_{i\geq 1}$ in $\mathcal{S}$. For each index i, let $\psi_i\colon [1,\infty) \to (0,\infty)$ be the function given by

$$\psi_i(t) = \max\{|\mathbf{x}\cdot\mathbf{u}_i|^{-1}\|\mathbf{x}\|^{-\rho_m}\,;\, \mathbf{x}\in\mathcal{X} \text{ and } \|\mathbf{x}\| \leq t\},$$

where $\mathcal{X}$ stands for the set of all non-zero $\mathbf{x} \in \mathbb{Z}^n$ satisfying the main condition (1.12) of Corollary 1.5. Each ψ_i is non-decreasing and, by Corollary 1.4, unbounded. Based on this, we construct recursively a sequence $(t_i)_{i\geq 0}$ of real numbers by first setting $t_0 = 1$, and for $i \geq 1$, by choosing $t_i \geq 2t_{i-1}$ such that

$$\min\{\psi_1(t_i), \dots, \psi_i(t_i)\} \geq (i+1)^2.$$

Since this sequence is strictly increasing and unbounded, we obtain an non-decreasing function $\psi\colon [1,\infty) \to [1,\infty)$ by defining $\psi(t) = i$ when $t \in [t_{i-1}, t_i)$ for an integer $i \geq 1$. This implies that

$$\psi(t) \leq \sqrt{\psi_i(t)} \quad \text{for each } i \geq 1 \text{ and each } t \geq t_i$$

because, for $t \in [t_{j-1}, t_j)$ with $j > i$, we have $\psi_i(t) \geq \psi_i(t_{j-1}) \geq j^2 = \psi(t)^2$. Let $\mathbf{u}$ be as in the second part of Corollary 1.4 for this choice of ψ, and let $\widetilde{\mathbf{u}} = \kappa^{-1}\mathbf{u}$ where $\kappa = \|\mathbf{u}\|$, so that $\widetilde{\mathbf{u}}$ is a unit vector in $\mathcal{S}$. We claim that $\widetilde{\mathbf{u}} \neq \mathbf{u}_i$ for each $i \geq 1$. Indeed, for a given $i \geq 1$, there are elements $\mathbf{x}$ of $\mathcal{X}$ of arbitrarily large norm with

$$|\mathbf{x}\cdot\mathbf{u}_i| = \psi_i(\|\mathbf{x}\|)^{-1}\|\mathbf{x}\|^{-\rho_m}.$$

Choosing $\mathbf{x}$ outside of the exceptional set for ψ, with norm so large that $\|\mathbf{x}\| \geq t_i$ and $\psi_i(\|\mathbf{x}\|) \geq \kappa^2$, this yields

$$\begin{aligned} |\mathbf{x}\cdot\mathbf{u}_i| \leq \kappa^{-1}\psi_i(\|\mathbf{x}\|)^{-1/2}\|\mathbf{x}\|^{-\rho_m} &\leq \kappa^{-1}\psi(\|\mathbf{x}\|)^{-1}\|\mathbf{x}\|^{-\rho_m} \\ &< \kappa^{-1}|\mathbf{x}\cdot\mathbf{u}| = |\mathbf{x}\cdot\widetilde{\mathbf{u}}|, \end{aligned}$$

thus $\widetilde{\mathbf{u}} \neq \mathbf{u}_i$. This shows that $(\mathbf{u}_i)_{i\geq 1}$ does not exhaust $\mathcal{S}$. As this is an arbitrary sequence in $\mathcal{S}$, this set is uncountable.

5. Proof of Theorem 2.7. Throughout this section, we fix a choice of canvas $((\mathbf{a}^{(i)})_{0\leq i<s}, (k_i)_{0\leq i<s}, (\ell_i)_{0\leq i<s})$ with mesh $c > 0$ and cardinality $s \in \{\infty, 1, 2, \dots\}$, as in Definition 2.1. We form its associated rigid n-system $\mathbf{P} = (P_1, \dots, P_n)\colon [q_0, \infty) \to \mathbb{R}^n$, and we denote by $(q_i)_{0\leq i<s}$ its sequence of switch numbers, as in Definition 2.2. We first establish a proposition which exhibits the driving principle behind the original constructions from [8, Section 5]. Then, we use it to prove Theorem 2.7 for a choice of directions $\underline{\mathbf{v}} = (\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$.

To simplify the writing, we set

$$A_j^{(i)} = \exp(a_j^{(i)}) = \exp(P_j(q_i)) \quad (0 \leq i < s,\ 1 \leq j \leq n).$$

By condition (C1) in Definition 2.1, these numbers satisfy

$$A_1^{(i)} \geq \exp(c) \quad \text{and} \quad A_j^{(i)} \geq \exp(c) A_{j-1}^{(i)} \quad \text{if } j > 1. \tag{5.1}$$

The following proposition is inspired from [8, Section 5], and involves a parameter t to allow a recursive construction of bases.

PROPOSITION 5.1. *Let $t \in \{\infty, 1, 2, \dots\}$ with $t \leq s$ and let $\theta \in (0, 1]$. Suppose that $c \geq \log 2$ and that, for each integer i with $0 \leq i < t$, we have a basis $\underline{\mathbf{x}}^{(i)} = (\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)})$ of $\mathbb{Z}^n$ such that*

(1) $A_j^{(i)} \leq \|\mathbf{x}_j^{(i)}\| \leq 2A_j^{(i)}$ *for $j = 1, \dots, n$,*

(2) $\|\mathbf{x}_1^{(i)} \wedge \cdots \wedge \widehat{\mathbf{x}_m^{(i)}} \wedge \cdots \wedge \mathbf{x}_n^{(i)}\| \geq \dfrac{\theta}{2} \|\mathbf{x}_1^{(i)}\| \cdots \widehat{\|\mathbf{x}_m^{(i)}\|} \cdots \|\mathbf{x}_n^{(i)}\|$ *for $m = k_i$ and for $m = \ell_i$.*

Suppose further that, when $1 \leq i < t$, we have

(3) $(\mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{\ell_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)}) = (\mathbf{x}_1^{(i-1)}, \dots, \widehat{\mathbf{x}_{k_{i-1}}^{(i-1)}}, \dots, \mathbf{x}_n^{(i-1)})$,

(4) $\mathbf{x}_{\ell_i}^{(i)} \in \mathbf{x}_{k_{i-1}}^{(i-1)} + \langle \mathbf{x}_1^{(i-1)}, \dots, \widehat{\mathbf{x}_{k_{i-1}}^{(i-1)}}, \dots, \mathbf{x}_{\ell_i}^{(i-1)} \rangle_{\mathbb{Z}}$.

For each integer i with $-1 \leq i < t$, let $\mathbf{u}_i$ be a unit vector orthogonal to the subspace

$$U_i := \begin{cases} \langle \mathbf{x}_1^{(0)}, \dots, \mathbf{x}_{n-1}^{(0)} \rangle_{\mathbb{R}} & \text{if } i = -1, \\ \langle \mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)} \rangle_{\mathbb{R}} & \text{if } 0 \leq i < t. \end{cases}$$

Then, for any $i, j \in \mathbb{Z}$ with $-1 \leq i \leq j < t$, we have

$$\operatorname{dist}(\mathbf{u}_i, \mathbf{u}_j) = \operatorname{dist}(U_i, U_j) \leq 8\theta^{-2} \exp(-q_{i+1}). \tag{5.2}$$

Finally, suppose that $t = s$. Then there is a unit vector $\mathbf{u}$ in $\mathbb{R}^n$ such that

$$\operatorname{dist}(\mathbf{u}_i, \mathbf{u}) \leq 8\theta^{-2} \exp(-q_{i+1}) \quad \text{whenever } -1 \leq i < s. \tag{5.3}$$

For any integers i, j with $0 \leq i < s$ and $1 \leq j \leq n$, and any $q \in [q_i, q_{i+1})$, we also have

$$-c_1 \leq a_j^{(i)} - L_{\mathbf{u}}(\mathbf{x}_j^{(i)}, q) \leq c_2 \qquad \text{if } j \neq k_i, \tag{5.4}$$

$$-c_1 \leq a_j^{(i)} + q - q_i - L_{\mathbf{u}}(\mathbf{x}_j^{(i)}, q) \leq c_2 \qquad \text{if } j = k_i, \tag{5.5}$$

$$L_{\mathbf{u},j}(q) \leq L_{\mathbf{u},j}(\underline{\mathbf{x}}^{(i)}, q) \leq P_j(q) + c_1 \leq L_{\mathbf{u},j}(q) + c_2, \tag{5.6}$$

where $c_1 = \log(32/\theta^2)$ and $c_2 = nc_1 + \log(n!)$.

The last property (5.6) shows that, for each q in $[q_i, q_{i+1})$, the basis $\underline{\mathbf{x}}^{(i)}$ realizes the minima of the convex body $\mathcal{C}_{\mathbf{u}}(e^q)$ up to the factor $\exp(c_2)$. The preceding properties (5.4) and (5.5) provide estimates for the individual trajectories of the basis elements $\mathbf{x}_j^{(i)}$ over the interval $[q_i, q_{i+1})$, an information which is partly lost in (5.6). When $i = s-1 < \infty$, we understand the right hand sides of (5.2) and (5.3) as $8\theta^{-2}\exp(-\infty) = 0$.

Proof of Proposition 5.1. Fix an integer i with $0 \le i < t$. Using relation (3) when $i \ge 1$ and using the hypothesis that $\ell_0 = n$ when $i = 0$, we find

$$(5.7) \quad U_{i-1} = \langle \mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{\ell_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)} \rangle_{\mathbb{R}}, \qquad U_i = \langle \mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)} \rangle_{\mathbb{R}}.$$

Since $k_i < \ell_i$, Lemma 3.5 gives

$$\mathrm{dist}(U_{i-1}, U_i) = \frac{\|\mathbf{x}_1^{(i)} \wedge \cdots \wedge \widehat{\mathbf{x}_{k_i}^{(i)}} \wedge \cdots \wedge \widehat{\mathbf{x}_{\ell_i}^{(i)}} \wedge \cdots \wedge \mathbf{x}_n^{(i)}\|}{\|\mathbf{x}_1^{(i)} \wedge \cdots \wedge \widehat{\mathbf{x}_{k_i}^{(i)}} \wedge \cdots \wedge \mathbf{x}_n^{(i)}\| \, \|\mathbf{x}_1^{(i)} \wedge \cdots \wedge \widehat{\mathbf{x}_{\ell_i}^{(i)}} \wedge \cdots \wedge \mathbf{x}_n^{(i)}\|}.$$

To get an upper bound for this ratio, we apply Hadamard's inequality on the numerator and hypothesis (2) on each factor of the denominator. Together with estimates (1), this yields

$$\mathrm{dist}(U_{i-1}, U_i) \le \frac{4\theta^{-2}}{\|\mathbf{x}_1^{(i)}\| \cdots \|\mathbf{x}_n^{(i)}\|} \le \frac{4\theta^{-2}}{A_1^{(i)} \cdots A_n^{(i)}} = 4\theta^{-2}\exp(-q_i).$$

By Lemma 3.4 and the fact that the distance satisfies the triangle inequality, we deduce that, for any integers i, j with $-1 \le i < j < t$, we have

$$\begin{aligned}
\mathrm{dist}(U_i, U_j) = \mathrm{dist}(\mathbf{u}_i, \mathbf{u}_j) &\le \sum_{m=i+1}^{j} \mathrm{dist}(\mathbf{u}_{m-1}, \mathbf{u}_m) \\
&\le 4\theta^{-2} \sum_{m=i+1}^{j} \exp(-q_m) \le 8\theta^{-2}\exp(-q_{i+1}),
\end{aligned}$$

where the last estimate uses the inequality $q_m \ge q_{m-1} + \log 2$ for each integer m with $1 \le m < s$ coming from the hypothesis that the canvas has mesh $c \ge \log 2$. This proves (5.2) when $i < j$. For $i = j$, this formula is automatic (it is even sharp when $i = j = s-1 < \infty$ since in that case $\exp(-q_{i+1}) = \exp(-\infty) = 0$).

From now on, suppose that $t = s$. If $s < \infty$, condition (5.3) is fulfilled with $\mathbf{u} = \mathbf{u}_{s-1}$ by (5.2). If $s = \infty$, then (5.2) shows that the image of $(\mathbf{u}_i)_{i \ge -1}$ in the projective space over $\mathbb{R}^n$ is a Cauchy sequence. So, it converges to the class of a unit vector $\mathbf{u}$ of $\mathbb{R}^n$ which satisfies (5.3).

Finally, fix an integer i with $0 \le i < s$, a number $q \in [q_i, q_{i+1})$, and an index $j \in \{1, \dots, n\}$. When $j \ne k_i$, we have $\mathbf{x}_j^{(i)} \in U_i$, thus $\mathbf{x}_j^{(i)} \cdot \mathbf{u}_i = 0$ and Lemma 3.1 yields

$$|\mathbf{x}_j^{(i)} \cdot \mathbf{u}| \le 2\|\mathbf{x}_j^{(i)}\| \,\mathrm{dist}(\mathbf{u}_i, \mathbf{u}).$$

Using (5.3) and the fact that $q < q_{i+1}$, we deduce that

$$q + \log|\mathbf{x}_j^{(i)} \cdot \mathbf{u}| \le c_0 + \log\|\mathbf{x}_j^{(i)}\| - q_{i+1} + q \le c_0 + \log\|\mathbf{x}_j^{(i)}\|,$$

where $c_0 = \log(16/\theta^2)$. So, using hypothesis (1), we obtain

$$L_{\mathbf{u}}(\mathbf{x}_j^{(i)}, q) \le c_0 + \log\|\mathbf{x}_j^{(i)}\| \le c_1 + a_j^{(i)},$$

which yields the left inequality in (5.4). When $j = k_i$, we instead have $\mathbf{x}_{k_i}^{(i)} \in U_{i-1}$ since $k_i < \ell_i$ (even for $i = 0$). This means that $\mathbf{x}_{k_i}^{(i)} \cdot \mathbf{u}_{i-1} = 0$ and so

$$|\mathbf{x}_{k_i}^{(i)} \cdot \mathbf{u}| \le 2\|\mathbf{x}_{k_i}^{(i)}\| \operatorname{dist}(\mathbf{u}_{i-1}, \mathbf{u}).$$

Using (5.3), we deduce that

$$q + \log|\mathbf{x}_{k_i}^{(i)} \cdot \mathbf{u}| \le c_0 + \log\|\mathbf{x}_{k_i}^{(i)}\| - q_i + q$$

for c_0 as above. Then, using the inequality $q \ge q_i$ and hypothesis (1), we obtain

$$L_{\mathbf{u}}(\mathbf{x}_{k_i}^{(i)}, q) \le c_0 + \log\|\mathbf{x}_{k_i}^{(i)}\| + q - q_i \le c_1 + a_{k_i}^{(i)} + q - q_i,$$

which gives the left inequality in (5.5). By the above, the points

$$\mathbf{a} = \mathbf{a}^{(i)} + (q - q_i)\mathbf{e}_{k_i} + c_1(1, \dots, 1) \quad \text{and} \quad \mathbf{b} = (L_{\mathbf{u}}(\mathbf{x}_1^{(i)}, q), \dots, L_{\mathbf{u}}(\mathbf{x}_n^{(i)}, q))$$

satisfy $\mathbf{b} \le \mathbf{a}$ for the componentwise ordering. Since the map Φ_n is order preserving, we deduce that $\Phi_n(\mathbf{b}) \le \Phi_n(\mathbf{a})$ and so

$$\mathbf{L}_{\mathbf{u}}(q) \le \mathbf{L}_{\mathbf{u}}(\underline{\mathbf{x}}^{(i)}, q) = \Phi_n(\mathbf{b}) \le \Phi_n(\mathbf{a}) = \mathbf{P}(q) + c_1(1, \dots, 1). \tag{5.8}$$

Since the coordinates of $\mathbf{a} - \mathbf{b}$ are non-negative, they are bounded above by their sum Δ, which is also the sum of the coordinates of $\Phi_n(\mathbf{a}) - \Phi_n(\mathbf{b})$. By (5.8), this is in turn bounded above by the sum of the coordinates of $\mathbf{P}(q) + c_1(1, \dots, 1) - \mathbf{L}_{\mathbf{u}}(q)$, which is

$$\Delta' := \sum_{j=1}^{n} (P_j(q) + c_1 - L_{\mathbf{u},j}(q)) = q + nc_1 - \sum_{j=1}^{n} L_{\mathbf{u},j}(q).$$

However, Minkowski's second convex body theorem [13, Chapter IV, Theorem 1A] applied to the convex body $\mathcal{C}_{\mathbf{u}}(e^q)$ gives

$$L_{\mathbf{u},1}(q) + \cdots + L_{\mathbf{u},n}(q) \ge \log(2^n/n!) - \log\big(\operatorname{vol}(\mathcal{C}_{\mathbf{u}}(e^q))\big) \ge q - \log(n!)$$

using the crude upper bound $\operatorname{vol}(\mathcal{C}_{\mathbf{u}}(e^q)) \le 2^n e^{-q}$ for the volume of $\mathcal{C}_{\mathbf{u}}(e^q)$. So, we find that $\Delta \le \Delta' \le nc_1 + \log(n!) = c_2$. From this, we deduce that $\mathbf{a} \le \mathbf{b} + c_2(1, \dots, 1)$ which yields the right inequalities in (5.4) and (5.5). It also gives

$$\mathbf{P}(q) + c_1(1, \dots, 1) \le \mathbf{L}_{\mathbf{u}}(q) + c_2(1, \dots, 1),$$

which, together with (5.8), translates into (5.6). ∎

We will also need the following complementary information.

Corollary 5.2. *Suppose that Proposition* 5.1 *holds with* $t = s$. *Then for each integer* i *with* $0 \le i < s$ *such that* $q_{i+1} > q_i + \log 2 + c_2$, *we have*

$$\big|\log|\mathbf{x}^{(i)}_{k_i} \cdot \mathbf{u}| - a^{(i)}_{k_i} + q_i\big| \le c_2. \tag{5.9}$$

Proof. For such i, we apply (5.5) for a choice of $q \in (q_i + \log 2 + c_2, q_{i+1})$. This gives

$$L_{\mathbf{u}}(\mathbf{x}^{(i)}_{k_i}, q) > a^{(i)}_{k_i} + \log 2 \ge \log\|\mathbf{x}^{(i)}_{k_i}\|,$$

where the second inequality comes from condition (1) with $j = k_i$. Thus, by definition, we must have $L_{\mathbf{u}}(\mathbf{x}^{(i)}_{k_i}, q) = q + \log|\mathbf{x}^{(i)}_{k_i} \cdot \mathbf{u}|$, and (5.9) follows by substituting this expression into (5.5). ∎

In the constructions of [8, Section 5], condition (2) in Proposition 5.1 is fulfilled by asking that the sequences $(\mathbf{x}^{(i)}_1, \dots, \widehat{\mathbf{x}^{(i)}_m}, \dots, \mathbf{x}^{(i)}_n)$ with $m = k_i$ or $m = \ell_i$ are almost orthogonal in the sense of [8, Definition 4.5]. Here, we use a simpler but more narrow approach.

We fix a linearly independent $(n-1)$-tuple $\underline{\mathbf{v}} = (\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$ of unit vectors of $\mathbb{R}^n$ and, as in the statement of Theorem 2.7, we denote by $(\underline{\mathbf{v}}^{(i)})_{i\ge 0}$ the coherent sequence of directions for $\mathbf{P}$ attached to $\underline{\mathbf{v}}$. We also fix a parameter δ with

$$0 < \delta \le \theta/(4n) \quad \text{where} \quad \theta = \|\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{n-1}\| = \Theta(\underline{\mathbf{v}}), \tag{5.10}$$

using notation (3.2). Then, for each integer i with $0 \le i < t$, we ask that the basis $\underline{\mathbf{x}}^{(i)}$ satisfies

$$\operatorname{dist}(\mathbf{x}^{(i)}_j, \mathbf{v}^{(i)}_j) \le \delta \quad \text{for } j = 1, \dots, n. \tag{5.11}$$

This is stronger than condition (2) of Proposition 5.1 because, for $m = k_i$ or $m = \ell_i$, the $(n-1)$-tuple $(\mathbf{v}^{(i)}_1, \dots, \widehat{\mathbf{v}^{(i)}_m}, \dots, \mathbf{v}^{(i)}_n)$ is a permutation of $\underline{\mathbf{v}}$. So, if (5.11) holds, then Lemma 3.6 gives

$$\Theta(\mathbf{x}^{(i)}_1, \dots, \widehat{\mathbf{x}^{(i)}_m}, \dots, \mathbf{x}^{(i)}_n) \ge \Theta(\underline{\mathbf{v}}) - 2(n-1)\delta \ge \theta/2.$$

We will show that the hypotheses of Theorem 2.7 allow us to construct recursively a sequence $(\underline{\mathbf{x}}^{(i)})_{0\le i<s}$ of bases of $\mathbb{Z}^n$ that satisfy conditions (1), (3), (4) of Proposition 5.1, as well as (5.11) in replacement of condition (2). Then this sequence and the unit vector $\mathbf{u}$ of $\mathbb{R}^n$ provided by the proposition enjoy all the required properties. Indeed, conditions (3) and (4) mean that the sequence $(\underline{\mathbf{x}}^{(i)})_{0\le i<s}$ is coherent with $\mathbf{P}$, while condition (1) yields (2.8). The other properties (2.7), (2.9) and (2.10) follow respectively from (5.11), Corollary 5.2 and (5.6).

We start by constructing a basis $\underline{\mathbf{x}}^{(0)}$ of $\mathbb{Z}^n$ which satisfies the hypotheses of Proposition 5.1 for $t = 1$ and condition (5.11) for $i = 0$.

LEMMA 5.3. *If $A_1^{(0)}$ is large enough, with a lower bound depending only on δ and $\underline{\mathbf{v}}$, then there is a basis $\underline{\mathbf{x}}^{(0)} = (\mathbf{x}_1^{(0)}, \dots, \mathbf{x}_n^{(0)})$ of $\mathbb{Z}^n$ such that*

$$A_j^{(0)} \le \|\mathbf{x}_j^{(0)}\| \le 2A_j^{(0)} \quad \text{and} \quad \mathrm{dist}(\mathbf{x}_j^{(0)}, \mathbf{v}_j^{(0)}) \le \delta/2$$

for each $j = 1, \dots, n$. If $\underline{\mathbf{v}} = (\mathbf{e}_1, \dots, \mathbf{e}_{n-1})$, it suffices to have $A_1^{(0)} \ge 6/\delta$.

Proof. By Lemma 4.1, there is a basis $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ of $\mathbb{Z}^n$ which satisfies $\mathrm{dist}(\mathbf{x}_j, \mathbf{v}_j) \le \delta/6$ for each $j = 1, \dots, n-1$. Let $A = \max\{\|\mathbf{x}_1\|, \dots, \|\mathbf{x}_n\|\}$ and suppose that $A_1^{(0)} \ge 6A/\delta$. For $j = 1, \dots, n-1$, we define

$$\mathbf{x}_j^{(0)} = a_j \mathbf{x}_j + \mathbf{x}_{j+1},$$

where a_j is the smallest non-negative integer for which $\|\mathbf{x}_j^{(0)}\| \ge A_j^{(0)}$. Then $(\mathbf{x}_1, \mathbf{x}_1^{(0)}, \dots, \mathbf{x}_{n-1}^{(0)})$ is a basis of $\mathbb{Z}^n$ and, since $k_0 < n$, we obtain another basis $\underline{\mathbf{x}}^{(0)} = (\mathbf{x}_1^{(0)}, \dots, \mathbf{x}_n^{(0)})$ of $\mathbb{Z}^n$ by setting

$$\mathbf{x}_n^{(0)} = a\mathbf{x}_{k_0}^{(0)} + \mathbf{x}_1,$$

where a is the smallest non-negative integer for which $\|\mathbf{x}_n^{(0)}\| \ge A_n^{(0)}$.

For $j = 1, \dots, n-1$, we have $A_j^{(0)} \ge A_1^{(0)} \ge 6A$, thus $a_j \ge 1$ and so

$$\|\mathbf{x}_j^{(0)}\| \le A_j^{(0)} + A \le 2A_j^{(0)}.$$

Since $\mathbf{x}_j^{(0)} \wedge \mathbf{x}_j = \mathbf{x}_{j+1} \wedge \mathbf{x}_j$, we also find

$$\mathrm{dist}(\mathbf{x}_j^{(0)}, \mathbf{x}_j) = \frac{\|\mathbf{x}_{j+1} \wedge \mathbf{x}_j\|}{\|\mathbf{x}_j^{(0)}\|\,\|\mathbf{x}_j\|} \le \frac{\|\mathbf{x}_{j+1}\|}{\|\mathbf{x}_j^{(0)}\|} \le \frac{A}{A_1^{(0)}} \le \frac{\delta}{6}.$$

So the triangle inequality yields

$$\begin{aligned}\mathrm{dist}(\mathbf{x}_j^{(0)}, \mathbf{v}_j^{(0)}) = \mathrm{dist}(\mathbf{x}_j^{(0)}, \mathbf{v}_j) &\le \mathrm{dist}(\mathbf{x}_j^{(0)}, \mathbf{x}_j) + \mathrm{dist}(\mathbf{x}_j, \mathbf{v}_j)\\ &\le \delta/6 + \delta/6 = \delta/3.\end{aligned}$$

Similarly, since $A_n^{(0)} \ge 2A_{k_0}^{(0)} \ge 12A$, the integer a is positive and so

$$\|\mathbf{x}_n^{(0)}\| \le A_n^{(0)} + 2A_{k_0}^{(0)} \le 2A_n^{(0)}.$$

Arguing as above, we also find

$$\mathrm{dist}(\mathbf{x}_n^{(0)}, \mathbf{x}_{k_0}^{(0)}) \le \frac{\|\mathbf{x}_1\|}{\|\mathbf{x}_n^{(0)}\|} \le \frac{A}{A_n^{(0)}} \le \frac{\delta}{12},$$

and so

$$\begin{aligned}\mathrm{dist}(\mathbf{x}_n^{(0)}, \mathbf{v}_n^{(0)}) = \mathrm{dist}(\mathbf{x}_n^{(0)}, \mathbf{v}_{k_0}) &\le \mathrm{dist}(\mathbf{x}_n^{(0)}, \mathbf{x}_{k_0}^{(0)}) + \mathrm{dist}(\mathbf{x}_{k_0}^{(0)}, \mathbf{v}_{k_0})\\ &\le \delta/12 + \delta/3 < \delta/2.\end{aligned}$$

If $\underline{\mathbf{v}} = (\mathbf{e}_1, \dots, \mathbf{e}_{n-1})$, we can choose $(\mathbf{x}_1, \dots, \mathbf{x}_n) = (\mathbf{e}_1, \dots, \mathbf{e}_n)$. Then we have $A = 1$, and the above construction requires only that $A_1^{(0)} \geq 6/\delta$. ∎

In view of the comments which precede Lemma 5.3, the next two propositions complete the proof of Theorem 2.7, depending on which of the two conditions (2.5) or (2.6) holds.

Proposition 5.4. *Suppose that condition* (2.5) *holds, namely that*

$$c \geq \log(8/\delta) \quad \textit{and} \quad \ell_i = n \textit{ whenever } 0 \leq i < s,$$

and suppose that $A_1^{(0)}$ is large enough as a function of $\underline{\mathbf{v}}$ and δ. Then we can construct recursively a sequence $(\underline{\mathbf{x}}^{(i)})_{0 \leq i < s}$ of bases of $\mathbb{Z}^n$ which, for each i, satisfies conditions (1) *to* (4) *of Proposition* 5.1 *as well as* (5.11). *If $\underline{\mathbf{v}} = (\mathbf{e}_1, \dots, \mathbf{e}_{n-1})$, it suffices to have $A_1^{(0)} \geq 6/\delta$.*

Proof. If $A_1^{(0)}$ is large enough, Lemma 5.3 provides a basis $\underline{\mathbf{x}}^{(0)}$ which satisfies these conditions for $i = 0$. If $s = 1$, we are done. Otherwise, suppose that we have constructed appropriate bases $\underline{\mathbf{x}}^{(0)}, \dots, \underline{\mathbf{x}}^{(t-1)}$ of $\mathbb{Z}^n$ for some integer t with $1 \leq t < s$. For each $j = 1, \dots, n-1$, we have $\mathbf{x}_j^{(0)} \in U_{-1}$ and so

$$\operatorname{dist}(\mathbf{v}_j, U_{-1}) = \operatorname{dist}(\mathbf{v}_j^{(0)}, U_{-1}) \leq \operatorname{dist}(\mathbf{v}_j^{(0)}, \mathbf{x}_j^{(0)}) \leq \delta/2.$$

Using (5.2) and the fact that $q_0 \geq 2a_1^{(0)}$, we also find

$$\operatorname{dist}(U_{-1}, U_{t-1}) \leq 8\theta^{-2} \exp(-q_0) \leq 8\theta^{-2} (A_1^{(0)})^{-2} \leq \delta/4$$

if $A_1^{(0)} \geq 6/(\theta\delta)$. Assuming this, we deduce that

$$\operatorname{dist}(\mathbf{v}_j, U_{t-1}) \leq \operatorname{dist}(\mathbf{v}_j, U_{-1}) + \operatorname{dist}(U_{-1}, U_{t-1}) \leq 3\delta/4$$

for $j = 1, \dots, n-1$. Finally, since $\mathbf{v}_n^{(t)}$ is one of the points $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$, we conclude that there is a unit vector $\mathbf{v} \in U_{t-1}$ for which

$$\operatorname{dist}(\mathbf{v}_n^{(t)}, \mathbf{v}) = \operatorname{dist}(\mathbf{v}_n^{(t)}, U_{t-1}) \leq 3\delta/4. \tag{5.12}$$

To alleviate notation, we set

$$h = k_{t-1}, \qquad k = k_t,$$

and note that, by definition of a canvas, we have $h < n$ and $k < n$ since $\ell_{t-1} = \ell_t = n$. We further set

$$(\mathbf{x}_1^{(t)}, \dots, \mathbf{x}_{n-1}^{(t)}) = (\mathbf{x}_1^{(t-1)}, \dots, \widehat{\mathbf{x}_h^{(t-1)}}, \dots, \mathbf{x}_n^{(t-1)}),$$

as prescribed by condition (3) for $i = t$. As the same relation holds between $\underline{\mathbf{v}}^{(t)}$ and $\underline{\mathbf{v}}^{(t-1)}$ as well as between $\mathbf{a}^{(t)}$ and $\mathbf{a}^{(t-1)}$, the induction hypothesis implies that

$$\operatorname{dist}(\mathbf{x}_j^{(t)}, \mathbf{v}_j^{(t)}) \leq \delta \quad \text{and} \quad A_j^{(t)} \leq \|\mathbf{x}_j^{(t)}\| \leq 2A_j^{(t)} \tag{5.13}$$

for each $j = 1, \dots, n-1$. To complete the induction step, it remains to construct

$$\mathbf{x}_n^{(t)} \in \mathbf{x}_h^{(t-1)} + \langle \mathbf{x}_1^{(t-1)}, \dots, \widehat{\mathbf{x}_h^{(t-1)}}, \dots, \mathbf{x}_n^{(t-1)} \rangle_{\mathbb{Z}} \tag{5.14}$$

so that (5.13) holds as well for $j = n$, because then $\underline{\mathbf{x}}^{(t)} = (\mathbf{x}_1^{(t)}, \dots, \mathbf{x}_n^{(t)})$ is a basis of $\mathbb{Z}^n$ with the required properties.

In view of (5.12), we simply need to construct $\mathbf{x}_n^{(t)}$ so that it fulfills (5.14) as well as

$$\operatorname{dist}(\mathbf{x}_n^{(t)}, \mathbf{v}) \le \delta/4 \quad \text{and} \quad A \le \|\mathbf{x}_n^{(t)}\| \le 2A \quad \text{where} \quad A = A_n^{(t)}. \tag{5.15}$$

Since $(\mathbf{x}_1^{(t-1)}, \dots, \mathbf{x}_n^{(t-1)})$ is a basis of $\mathbb{Z}^n$, we find

$$\mathbf{x}_h^{(t-1)} - r_0 \mathbf{u}_{t-1} \in U_{t-1} \quad \text{where} \quad |r_0| = |\mathbf{x}_h^{(t-1)} \cdot \mathbf{u}_{t-1}| \le 1$$

(see [8, Lemma 4.1]). Since $\mathbf{v}$ belongs to U_{t-1} as well, we may therefore write

$$r_0 \mathbf{u}_{t-1} + \frac{3}{2} A \mathbf{v} = \mathbf{x}_h^{(t-1)} + \sum_{j \ne h} r_j \mathbf{x}_j^{(t-1)} \tag{5.16}$$

for some coefficients $r_j \in \mathbb{R}$, where the sum extends to all $j = 1, \dots, n$ with $j \ne h$. For each of those j, we choose an integer a_j such that $|a_j - r_j| \le 1/2$. Then the point

$$\mathbf{x}_n^{(t)} = \mathbf{x}_h^{(t-1)} + \sum_{j \ne h} a_j \mathbf{x}_j^{(t-1)}$$

satisfies (5.14). Using (5.16) we find

$$\left\| \mathbf{x}_n^{(t)} - \frac{3}{2} A \mathbf{v} \right\| = \left\| r_0 \mathbf{u}_{t-1} + \sum_{j \ne h} (a_j - r_j) \mathbf{x}_j^{(t-1)} \right\| \le 1 + \frac{1}{2} \sum_{j \ne h} \|\mathbf{x}_j^{(t-1)}\|.$$

Since condition (1) holds for $i = t-1$, this yields

$$\left\| \mathbf{x}_n^{(t)} - \frac{3}{2} A \mathbf{v} \right\| \le 1 + \sum_{j \ne h} A_j^{(t-1)} \le \sum_{j=1}^{n} A_j^{(t-1)} \le 2 A_n^{(t-1)} \le \frac{A\delta}{4},$$

where the second and third inequalities use (5.1) with $i = t-1$ together with the hypothesis that $\exp(c) \ge 8/\delta \ge 8$, while the last inequality uses $A_n^{(t-1)} = A_{n-1}^{(t)} \le \exp(-c) A_n^{(t)}$. As $\delta \le 1$, this implies that $A \le \|\mathbf{x}_n^{(t)}\| \le 2A$ and

$$\operatorname{dist}(\mathbf{x}_n^{(t)}, \mathbf{v}) = \frac{\|(\mathbf{x}_n^{(t)} - (3/2) A \mathbf{v}) \wedge \mathbf{v}\|}{\|\mathbf{x}_n^{(t)}\|} \le \frac{\|\mathbf{x}_n^{(t)} - (3/2) A \mathbf{v}\|}{\|\mathbf{x}_n^{(t)}\|} \le \frac{A\delta/4}{A} = \frac{\delta}{4}$$

as required in (5.15). This completes the recursion step.

The last assertion of the proposition is easily checked. ∎

While the above argument is close to that in [8, Section 5], the next statement uses an even simpler construction of points.

Proposition 5.5. *Suppose that condition* (2.6) *holds, namely that we have* $c \geq \log 2$ *and*

$$\sum_{i=1}^{s} \exp(q_{i-1} - q_i) \leq \delta/4, \tag{5.17}$$

and suppose that $A_1^{(0)}$ *is large enough as a function of* $\underline{\mathbf{v}}$ *and* δ*. Then we can construct recursively a sequence* $(\underline{\mathbf{x}}^{(i)})_{0 \leq i < s}$ *of bases of* $\mathbb{Z}^n$ *which, for each* i*, satisfies conditions* (1) *to* (4) *of Proposition* 5.1 *as well as the constraints*

$$\operatorname{dist}(\mathbf{x}_j^{(i)}, \mathbf{v}_j^{(i)}) \leq \frac{\delta}{2} + 2 \sum_{m=1}^{i} \exp(q_{m-1} - q_m) \quad \text{for } j = 1, \dots, n, \tag{5.18}$$

which, in view of (5.17)*, are stronger than* (5.11)*. If* $\underline{\mathbf{v}} = (\mathbf{e}_1, \dots, \mathbf{e}_{n-1})$*, it suffices to have* $A_1^{(0)} \geq 6/\delta$*.*

Proof. If $A_1^{(0)}$ is large enough, Lemma 5.3 provides a basis $\underline{\mathbf{x}}^{(0)}$ with the required property. For the recurrence step, suppose that we have constructed appropriate bases $\underline{\mathbf{x}}^{(0)}, \dots, \underline{\mathbf{x}}^{(t-1)}$ of $\mathbb{Z}^n$ for some integer t with $1 \leq t < s$. To alleviate notation, we set

$$h = k_{t-1}, \quad \ell = \ell_t, \quad k = k_t,$$

and note that, by conditions (C2) and (C3) in Definition 2.1 of a canvas, we have $h \leq \ell$ and $k < \ell$. We also observe that

$$\frac{A_\ell^{(t)}}{A_h^{(t-1)}} = \exp(q_t - q_{t-1}) \geq \exp(c) \geq 2 \tag{5.19}$$

in view of relation (2.2) for $i = t-1$, and the formula for the switch numbers q_i in Definition 2.2. Then we define $\underline{\mathbf{x}}^{(t)} = (\mathbf{x}_1^{(t)}, \dots, \mathbf{x}_n^{(t)})$ by

$$(\mathbf{x}_1^{(t)}, \dots, \widehat{\mathbf{x}_\ell^{(t)}}, \dots, \mathbf{x}_n^{(t)}) = (\mathbf{x}_1^{(t-1)}, \dots, \widehat{\mathbf{x}_h^{(t-1)}}, \dots, \mathbf{x}_n^{(t-1)}), \tag{5.20}$$

$$\mathbf{x}_\ell^{(t)} = \mathbf{x}_h^{(t-1)} + a\mathbf{x}_k^{(t)}, \tag{5.21}$$

where a is the smallest non-negative integer such that

$$A_\ell^{(t)} < \|\mathbf{x}_\ell^{(t)}\|. \tag{5.22}$$

Since $\underline{\mathbf{x}}^{(t-1)}$ is a basis of $\mathbb{Z}^n$, this n-tuple $\underline{\mathbf{x}}^{(t)}$ is also a basis of $\mathbb{Z}^n$. By construction, it fulfills conditions (3) and (4) for $i = t$. Moreover, since $\mathbf{a}^{(t)}$ and $\mathbf{a}^{(t-1)}$ are linked by the same relation as $\underline{\mathbf{x}}^{(t)}$ and $\underline{\mathbf{x}}^{(t-1)}$ in (5.20), and since by hypothesis condition (1) holds for $i = t-1$ and all $j = 1, \dots, n$, that condition also holds for $i = t$ except possibly when $j = \ell$. In particular, since $k < \ell$, we obtain

$$\|\mathbf{x}_k^{(t)}\| \leq 2A_k^{(t)} \leq A_\ell^{(t)},$$

where the second inequality uses (5.1). Since $h \le \ell$, we also find

$$\|\mathbf{x}_h^{(t-1)}\| \le 2A_h^{(t-1)} \le A_\ell^{(t)}$$

using (5.19). Thus the integer a must be positive, and so

$$\|\mathbf{x}_\ell^{(t)}\| \le A_\ell^{(t)} + \|\mathbf{x}_k^{(t)}\| \le 2A_\ell^{(t)}.$$

Together with (5.22), this shows that condition (1) holds as well for $i = t$ and $j = \ell$. Similarly, since inequality (5.18) holds for $i = t - 1$ and all $j = 1, \dots, n$, and since $\underline{\mathbf{v}}^{(t)}$ and $\underline{\mathbf{v}}^{(t-1)}$ are linked in the same way as $\underline{\mathbf{x}}^{(t)}$ and $\underline{\mathbf{x}}^{(t-1)}$ in (5.20), that inequality also holds for $i = t$ and $j \neq \ell$ in the stronger form

$$\mathrm{dist}(\mathbf{x}_j^{(t)}, \mathbf{v}_j^{(t)}) \le \frac{\delta}{2} + 2\sum_{m=1}^{t-1} \exp(q_{m-1} - q_m). \tag{5.23}$$

To estimate this distance when $j = \ell$, we first note, using (5.21), that

$$\|\mathbf{x}_\ell^{(t)} \wedge \mathbf{x}_k^{(t)}\| = \|\mathbf{x}_h^{(t-1)} \wedge \mathbf{x}_k^{(t)}\| \le \|\mathbf{x}_h^{(t-1)}\|\,\|\mathbf{x}_k^{(t)}\|,$$

and so

$$\mathrm{dist}(\mathbf{x}_\ell^{(t)}, \mathbf{x}_k^{(t)}) \le \frac{\|\mathbf{x}_h^{(t-1)}\|}{\|\mathbf{x}_\ell^{(t)}\|} \le \frac{2A_h^{(t-1)}}{A_\ell^{(t)}} = 2\exp(q_{t-1} - q_t).$$

Together with (5.23) for $j = k$, this yields

$$\begin{aligned}\mathrm{dist}(\mathbf{x}_\ell^{(t)}, \mathbf{v}_k^{(t)}) &\le \mathrm{dist}(\mathbf{x}_\ell^{(t)}, \mathbf{x}_k^{(t)}) + \mathrm{dist}(\mathbf{x}_k^{(t)}, \mathbf{v}_k^{(t)})\\ &\le \frac{\delta}{2} + 2\sum_{m=1}^{t} \exp(q_{m-1} - q_m).\end{aligned}$$

Since $\mathbf{v}_\ell^{(t)} = \mathbf{v}_k^{(t)}$, this shows that (5.18) holds for $i = t$. Thus condition (2) holds as well for $i = t$.

The last assertion of the proposition is clear from the statement of Lemma 5.3. ∎

6. Application to a specific family of generalized n-systems. The goal of this section is to apply Theorem 2.7 on parametric geometry of numbers with constraints to produce points $\mathbf{u}$ that satisfy the second part of Theorem 1.1, thereby completing the proof of the latter theorem. We first construct a generalized n-system $\widetilde{\mathbf{P}}$ in the sense of [9, §4], and we approximate it by a rigid n-system $\mathbf{P}$ to which Theorem 2.7 applies. Then we use geometry of numbers to show that the point $\mathbf{u}$ provided by the latter theorem has the required property.

To this end, we fix integers $m \ge 1$ and $n \ge m + 2$, and set

$$r = n - m.$$

We also fix an orthonormal basis $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ of $\mathbb{R}^n$ and set

$$\underline{\mathbf{v}} = (\mathbf{v}_1, \dots, \mathbf{v}_{n-1}),$$

so that $\Theta(\underline{\mathbf{v}}) = 1$ in the notation of (3.2). We choose real numbers δ and c with

$$0 < \delta \le 1/(4n) \quad \text{and} \quad c = \log(8/\delta), \tag{6.1}$$

and we denote by κ the constant provided by Theorem 2.7 for the above choice of $\underline{\mathbf{v}}$ and δ. Finally, we choose sequences $(X_i)_{i\ge 0}$ and $(Y_i)_{i\ge 0}$ of positive real numbers with the property that, for each $i \ge 0$, we have

$$\kappa + c \le \log X_0, \tag{6.2}$$

$$c + \log X_i \le \log X_{i+1} \quad \text{if } 0 \le i \le r-2, \tag{6.3}$$

$$(3m+n)c + \log X_{i+r-1} \le \log Y_i \le -2mc + \log X_{i+r}. \tag{6.4}$$

To this data, we attach a continuous piecewise linear map $\widetilde{\mathbf{P}} = (\widetilde{P}_1, \dots, \widetilde{P}_n)$ from $[q_0, \infty)$ to $\mathbb{R}^n$ with the property that

$$0 \le \widetilde{P}_1(q) \le \cdots \le \widetilde{P}_r(q) \le \widetilde{P}_{r+1}(q) = \cdots = \widetilde{P}_n(q),$$
$$\widetilde{P}_1(q) + \cdots + \widetilde{P}_n(q) = q$$

for each $q \ge q_0$. Its combined graph, namely the union of the graphs of its components $\widetilde{P}_1, \dots, \widetilde{P}_n$, is given by Figure 1.

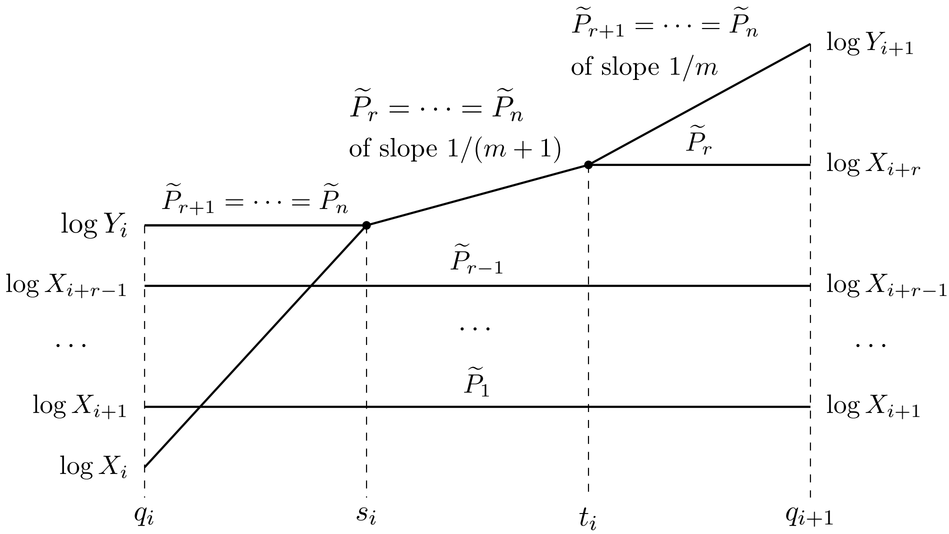

Fig. 1. The combined graph of a generalized n-system

Explicitly, we construct sequences $(q_i)_{i\ge0}$, $(s_i)_{i\ge0}$ and $(t_i)_{i\ge0}$ by setting

$$\begin{aligned} q_i &= \log(X_iX_{i+1}\cdots X_{i+r-1}Y_i^m),\\ s_i &= \log(X_{i+1}\cdots X_{i+r-1}Y_i^{m+1}),\\ t_i &= \log(X_{i+1}\cdots X_{i+r-1}X_{i+r}^{m+1}) \end{aligned}$$

for each $i \ge 0$. Then $q_i < s_i < t_i < q_{i+1}$ and, as illustrated in Figure 1, we define $\widetilde{P}_1, \dots, \widetilde{P}_n$ on $[q_i, q_{i+1}]$ by

$$\widetilde{P}_{r+1}(q) = \cdots = \widetilde{P}_n(q) = \begin{cases} \log Y_i & \text{if } q_i \le q \le s_i,\\ \log Y_i + \dfrac{q-s_i}{m+1} & \text{if } s_i \le q \le t_i,\\ \log X_{i+r} + \dfrac{q-t_i}{m} & \text{if } t_i \le q \le q_{i+1}, \end{cases}$$

and

$$\begin{aligned} &(\widetilde{P}_1(q), \dots, \widetilde{P}_r(q))\\ &\quad = \begin{cases} \Phi_r(\log X_{i+1}, \dots, \log X_{i+r-1}, q - q_i + \log X_i) & \text{if } q_i \le q \le s_i,\\ (\log X_{i+1}, \dots, \log X_{i+r-1}, \widetilde{P}_{r+1}(q)) & \text{if } s_i \le q \le t_i,\\ (\log X_{i+1}, \dots, \log X_{i+r}) & \text{if } t_i \le q \le q_{i+1}, \end{cases} \end{aligned}$$

where $\Phi_r \colon \mathbb{R}^r \to \Delta_r \subset \mathbb{R}^r$ is the ordering map defined in Section 2.

In the terminology of [9], the map $\widetilde{\mathbf{P}}$ is a generalized n-system, and Section 4 of that paper provides a general method to approximate such a map by an ordinary n-system $\mathbf{P}$. To prove the next result, we adapt this method to produce an approximate rigid n-system $\mathbf{P}$ whose transition indices ℓ_j are all equal to n, so that Theorem 2.7 applies to it.

Theorem 6.1. *Let $(\widetilde{\mathbf{v}}_i)_{i\ge0}$ denote the periodic sequence of period r with $\widetilde{\mathbf{v}}_i = \mathbf{v}_{i+1}$ for each $i = 0, \dots, r-1$. For the above data, there exist a unit vector $\mathbf{u}$ of $\mathbb{R}^n$ whose coordinates are linearly independent over $\mathbb{Q}$, and a sequence $(\mathbf{x}_i)_{i\ge0}$ of non-zero points in $\mathbb{Z}^n$ such that, for each $i \ge 0$ and each $q \ge q_0$, we have*

(1) $\operatorname{dist}(\mathbf{x}_i, \widetilde{\mathbf{v}}_i) \le \delta$,
(2) $|\log\|\mathbf{x}_i\| - \log X_i| \le nc$ *and* $\big|\log|\mathbf{x}_i\cdot\mathbf{u}| + q_i - \log X_i\big| \le 4mnc$,
(3) $\|\widetilde{\mathbf{P}}(q) - \mathbf{L}_{\mathbf{u}}(q)\|_\infty \le 5mnc$,

where $\|\ \|_\infty$ stands for the maximum norm on $\mathbb{R}^n$.

Proof. Our construction of an approximate rigid n-system $\mathbf{P}$ differs slightly depending on whether $m = 1$ or $m > 1$. To cover both cases, we set

$$m^* = \max\{1, m-1\}. \tag{6.5}$$

We first define two sequences $(a_i)_{i\ge0}$ and $(b_i)_{i\ge0}$ of multiples of c satisfying

$$-mc < a_i - \log X_i \le 0 \quad\text{and}\quad -m^*c < b_i + c - \log Y_i \le 0 \tag{6.6}$$

for each $i \geq 0$. For $a_0, \dots, a_{r-1}$ and b_0, we choose the largest multiples of c satisfying these conditions. Then, for each $i \geq 0$, we define recursively

$$a_{i+r} = b_i + \sigma(i)mc \quad \text{and} \quad b_{i+1} = a_{i+r} + \tau(i)m^*c + c \tag{6.7}$$

with integers $\sigma(i)$ and $\tau(i)$ chosen so that a_{i+r} and b_{i+1} satisfy conditions (6.6). Hypotheses (6.2) and (6.3) imply that $0 < a_0 < \cdots < a_{r-1}$ while (6.4) yields

$$b_i - a_{i+r-1} \geq \log Y_i - (m^*+1)c - \log X_{i+r-1} \geq (n+m)c, \tag{6.8}$$

$$a_{i+r} - b_i \geq \log X_{i+r} - mc - \log Y_i \geq mc \tag{6.9}$$

for each $i \geq 0$. In particular, we have $\sigma(i) \geq 1$ and $\tau(i) \geq 2$ for each $i \geq 0$.

For each pair (i,j) of integers with $i \geq 0$ and

$$0 \leq j \leq \nu(i) := \sigma(i)m + \tau(i)(m-1), \tag{6.10}$$

we define $\ell_{i,j} = n$. For $j = 0$, we further define

$$\mathbf{a}^{(i,0)} = (a_i, \dots, a_{i+r-1}, b_i - (m-1)c, \dots, b_i), \qquad k_{i,0} = 1.$$

For $0 < j < \sigma(i)m + m$, we define

$$\mathbf{a}^{(i,j)} = (a_{i+1}, \dots, a_{i+r-1}, b_i + (j-m)c, \dots, b_i + jc), \qquad k_{i,j} = r.$$

Finally, for $\sigma(i)m + m \leq j \leq \nu(i)$, we define

$$\mathbf{a}^{(i,j)} = (a_{i+1}, \dots, a_{i+r}, b_i + (j+1-m)c, \dots, b_i + jc), \qquad k_{i,j} = r+1.$$

For each j, we denote by $q_{i,j}$ the sum of the coordinates of $\mathbf{a}^{(i,j)}$. By (6.8), (6.9) and the above, each $\mathbf{a}^{(i,j)}$ is a strictly increasing sequence of positive multiples of c, ending in an arithmetic progression with difference c. We also note that $\sigma(i)m + m \leq \nu(i)$ if and only if $m > 1$, because $\tau(i) \geq 2$. Thus pairs (i,j) with $\sigma(i)m + m \leq j \leq \nu(i)$ occur when $m > 1$, but not when $m = 1$. Using the lexicographical ordering in which $(i,j) < (i',j')$ when either $i < i'$ or both $i = i'$ and $j < j'$, it follows that this data defines a canvas with mesh c. Let $\mathbf{P} = (P_1, \dots, P_n)\colon [q_{0,0}, \infty) \to \Delta_n$ denote its associated n-system. By definition, its sequence of switch numbers is $(q_{i,j})$.

Figure 2 shows the combined graph of $\mathbf{P}$ on a typical interval $[q_{i,0}, q_{i+1,0}]$ when $m = 2$. For larger m, the graph is similar. However, when $m = 1$, it is sensibly different because there is no switch number of $\mathbf{P}$ inside the interval $[q_{i,\sigma(i)}, q_{i+1,0}]$, and the graph of $\mathbf{P}$ is affine linear with $P_{r+1} = P_n$ of slope 1 over that interval. As the picture illustrates, the behaviour of $\mathbf{P}$ is similar to that of $\widetilde{\mathbf{P}}$ and we will show that in fact their difference $\mathbf{P} - \widetilde{\mathbf{P}}$ is bounded.

Before we do this, consider the coherent system of directions $\underline{\mathbf{v}}^{(i,j)} = (\mathbf{v}_1^{(i,j)}, \dots, \mathbf{v}_n^{(i,j)})$ with $i \geq 0$ and $0 \leq j \leq \nu(i)$ attached to $\underline{\mathbf{v}} = (\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$. We claim that, for each $i \geq 0$, we have

$$\underline{\mathbf{v}}^{(i,0)} = (\widetilde{\mathbf{v}}_i, \dots, \widetilde{\mathbf{v}}_{i+r-1}, \mathbf{v}_{r+1}, \dots, \mathbf{v}_{n-1}, \widetilde{\mathbf{v}}_i). \tag{6.11}$$

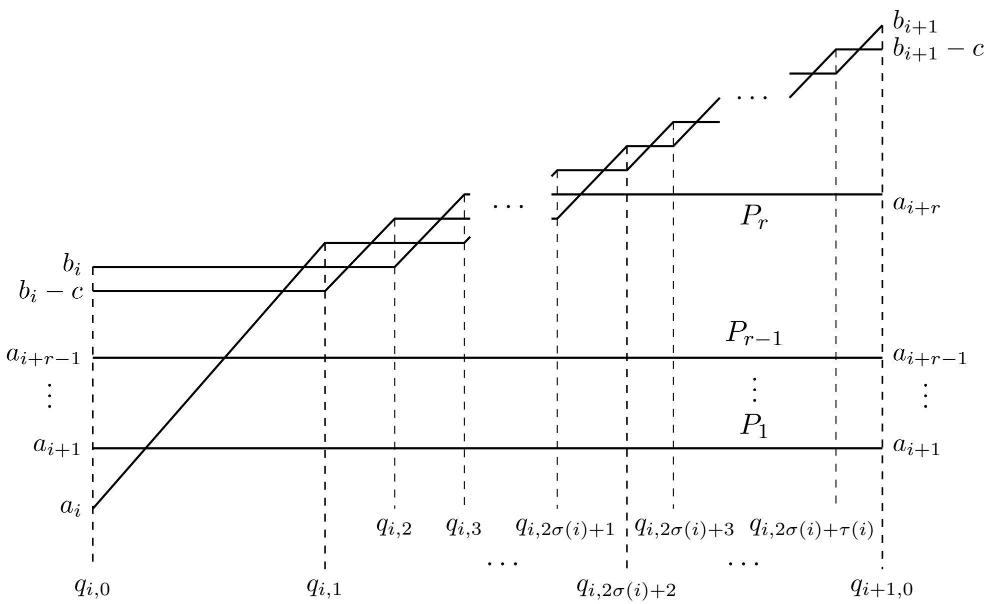

Fig. 2. The combined graph of $\mathbf{P}$ on $[q_{i,0}, q_{i+1,0}]$ when $m = 2$

For $i = 0$, this is clear since $k_{0,0} = 1$ and thus $\underline{\mathbf{v}}^{(0,0)} = (\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{v}_1)$. Now, suppose that (6.11) holds for some $i \geq 0$. If $m = 1$, we have $r = n - 1 \geq 2$ and this formula becomes

$$\underline{\mathbf{v}}^{(i,0)} = (\widetilde{\mathbf{v}}_i, \dots, \widetilde{\mathbf{v}}_{i+n-2}, \widetilde{\mathbf{v}}_i).$$

As $k_{i,0} = 1$, $k_{i,1} = \cdots = k_{i,\sigma(i)} = n - 1$ and $k_{i+1,0} = 1$, we deduce that

$$\begin{aligned} \underline{\mathbf{v}}^{(i,1)} &= \cdots = \underline{\mathbf{v}}^{(i,\sigma(i))} = (\widetilde{\mathbf{v}}_{i+1}, \dots, \widetilde{\mathbf{v}}_{i+n-2}, \widetilde{\mathbf{v}}_i, \widetilde{\mathbf{v}}_i), \\ \underline{\mathbf{v}}^{(i+1,0)} &= (\widetilde{\mathbf{v}}_{i+1}, \dots, \widetilde{\mathbf{v}}_{i+n-2}, \widetilde{\mathbf{v}}_i, \widetilde{\mathbf{v}}_{i+1}) = (\widetilde{\mathbf{v}}_{i+1}, \dots, \widetilde{\mathbf{v}}_{i+n-1}, \widetilde{\mathbf{v}}_{i+1}), \end{aligned}$$

which proves our claim (6.11) by induction on i. If $m > 1$, we have $k_{i,0} = 1$ and $k_{i,j} = r$ for $1 \leq j < \sigma(i)m + m$. Then $(\underline{\mathbf{v}}^{(i,1)}, \underline{\mathbf{v}}^{(i,2)}, \dots, \underline{\mathbf{v}}^{(i,\sigma(i)m+m-1)})$ is again a periodic sequence of period m with

$$\begin{aligned} \underline{\mathbf{v}}^{(i,1)} &= (\widetilde{\mathbf{v}}_{i+1}, \dots, \widetilde{\mathbf{v}}_{i+r-1}, \mathbf{v}_{r+1}, \dots, \mathbf{v}_{n-1}, \widetilde{\mathbf{v}}_i, \mathbf{v}_{r+1}), \\ \underline{\mathbf{v}}^{(i,2)} &= (\widetilde{\mathbf{v}}_{i+1}, \dots, \widetilde{\mathbf{v}}_{i+r-1}, \mathbf{v}_{r+2}, \dots, \mathbf{v}_{n-1}, \widetilde{\mathbf{v}}_i, \mathbf{v}_{r+1}, \mathbf{v}_{r+2}), \\ &\cdots \\ \underline{\mathbf{v}}^{(i,m-1)} &= (\widetilde{\mathbf{v}}_{i+1}, \dots, \widetilde{\mathbf{v}}_{i+r-1}, \mathbf{v}_{n-1}, \widetilde{\mathbf{v}}_i, \mathbf{v}_{r+1}, \dots, \mathbf{v}_{n-1}), \\ \underline{\mathbf{v}}^{(i,m)} &= (\widetilde{\mathbf{v}}_{i+1}, \dots, \widetilde{\mathbf{v}}_{i+r-1}, \widetilde{\mathbf{v}}_i, \mathbf{v}_{r+1}, \dots, \mathbf{v}_{n-1}, \widetilde{\mathbf{v}}_i). \end{aligned}$$

In particular, we have $\underline{\mathbf{v}}^{(i,\sigma(i)m+m-1)} = \underline{\mathbf{v}}^{(i,m-1)}$. Since $k_{i,\sigma(i)m+m} = r + 1$, we deduce that

$$\begin{aligned}\underline{\mathbf{v}}^{(i,\sigma(i)m+m)} &= (\widetilde{\mathbf{v}}_{i+1},\dots,\widetilde{\mathbf{v}}_{i+r-1},\widetilde{\mathbf{v}}_i,\mathbf{v}_{r+1},\dots,\mathbf{v}_{n-1},\mathbf{v}_{r+1})\\ &= (\widetilde{\mathbf{v}}_{i+1},\dots,\widetilde{\mathbf{v}}_{i+r},\mathbf{v}_{r+1},\dots,\mathbf{v}_{n-1},\mathbf{v}_{r+1}).\end{aligned}$$

Finally, since $k_{i,j}=r+1$ for $(\sigma(i)+1)m\le j\le\nu(i)$, we find similarly that the sequence $(\underline{\mathbf{v}}^{(i,\sigma(i)m+m)},\dots,\underline{\mathbf{v}}^{(i,\nu(i))})$ is periodic of period $m-1$ and that

$$\underline{\mathbf{v}}^{(i,\nu(i))} = (\widetilde{\mathbf{v}}_{i+1},\dots,\widetilde{\mathbf{v}}_{i+r},\mathbf{v}_{n-1},\mathbf{v}_{r+1},\dots,\mathbf{v}_{n-1}),$$

thus $\underline{\mathbf{v}}^{(i+1,0)} = (\widetilde{\mathbf{v}}_{i+1},\dots,\widetilde{\mathbf{v}}_{i+r},\mathbf{v}_{r+1},\dots,\mathbf{v}_{n-1},\widetilde{\mathbf{v}}_{i+1})$ since $k_{i+1,0}=1$. Again this proves (6.11) by induction on i.

By (6.2), we have $a_1^{(0,0)}=a_0\ge\log X_0-c\ge\kappa$. Together with (6.1) and the fact that $\ell_{i,j}=n$ for all pairs (i,j) with $i\ge 0$ and $0\le j\le\nu(i)$, this shows that the n-system $\mathbf{P}$ and its associated coherent sequence of directions $(\underline{\mathbf{v}}^{(i,j)})$ satisfy the hypotheses of Theorem 2.7. For the corresponding constant c_2, we find

$$c_2 = n\log 32+\log(n!)\le n\log(32n)\le nc. \tag{6.12}$$

Let $\mathbf{u}$ denote the unit vector of $\mathbb{R}^n$, and $\underline{\mathbf{x}}^{(i,j)}=(\mathbf{x}_1^{(i,j)},\dots,\mathbf{x}_n^{(i,j)})$ the generic element of the coherent sequence of bases of $\mathbb{Z}^n$ provided by this theorem. We set

$$\mathbf{x}_i=\mathbf{x}_1^{(i,0)}\quad\text{for each } i\ge 0. \tag{6.13}$$

It remains to show that the point $\mathbf{u}$ and the sequence $(\mathbf{x}_i)_{i\ge0}$ have the required properties. By (2.11), we first note, using (6.12), that

$$\|\mathbf{L}_{\mathbf{u}}(q)-\mathbf{P}(q)\|_\infty\le c_2\le nc\quad\text{for each } q\ge q_{0,0}. \tag{6.14}$$

Since $P_1(q)$ tends to infinity with q, the same applies to $L_{\mathbf{u},1}(q)$ and so the coordinates of $\mathbf{u}$ are linearly independent over $\mathbb{Q}$.

Fix an index $i\ge 0$. By (6.11), we have $\widetilde{\mathbf{v}}_1^{(i,0)}=\widetilde{\mathbf{v}}_i$. Thus estimate (2.7) applied to the first element of the basis $\underline{\mathbf{x}}^{(i,0)}$ yields

$$\operatorname{dist}(\mathbf{x}_i,\widetilde{\mathbf{v}}_i)=\operatorname{dist}(\mathbf{x}_1^{(i,0)},\widetilde{\mathbf{v}}_1^{(i,0)})\le\delta,$$

as required in condition (1).

Similarly, since $a_1^{(i,0)}=a_i$ and $k_{i,0}=1$, estimates (2.8) and (2.9) provide respectively

$$|\log\|\mathbf{x}_i\|-a_i|\le\log 2\quad\text{and}\quad\big|\log|\mathbf{x}_i\cdot\mathbf{u}|-a_i+q_{i,0}\big|\le c_2 \tag{6.15}$$

because, using (6.8) and (6.12), we find

$$q_{i,1}-q_{i,0}=b_i+c-a_i\ge b_i+c-a_{i+r-1}\ge(n+1)c\ge\log 2+c_2.$$

By (6.5)–(6.7), we also observe that, under the componentwise ordering on $\mathbb{R}^n$, we have

$$\left.\begin{aligned} -2m\mathbf{c} &\le \mathbf{a}^{(i,0)} - \widetilde{\mathbf{P}}(q_i) \le 0 \\ -2m\mathbf{c} &\le \mathbf{a}^{(i,1)} - \widetilde{\mathbf{P}}(s_i) \le 0 \\ -2m\mathbf{c} &\le \mathbf{a}^{(i,\sigma(i)m)} - \widetilde{\mathbf{P}}(t_i) \le 0 \end{aligned}\right\} \quad \text{where} \quad \mathbf{c} = (c, \dots, c). \tag{6.16}$$

For example, we obtain the last estimate by observing that

$$\begin{aligned} \mathbf{a}^{(i,\sigma(i)m)} - \widetilde{\mathbf{P}}(t_i) = (&a_{i+1} - \log X_{i+1}, \dots, a_{i+r-1} - \log X_{i+r-1}, \\ &a_{i+r} - mc - \log X_{i+r}, \dots, a_{i+r} - \log X_{i+r}). \end{aligned}$$

Taking the sum of the coordinates of each term in (6.16), we deduce that

$$-2mnc \le q_{i,0} - q_i,\ q_{i,1} - s_i,\ q_{i,\sigma(i)m} - t_i \le 0. \tag{6.17}$$

In particular, we have $|q_i - q_{i,0}| \le 2mnc$. As (6.6) gives $|a_i - \log X_i| \le mc$, we deduce from (6.15) that

$$\begin{aligned} &|\log \|\mathbf{x}_i\| - \log X_i| \le \log 2 + mc \le nc, \\ &\big|\log |\mathbf{x}_i \cdot \mathbf{u}| - \log X_i + q_i\big| \le nc + mc + 2mnc \le 4mnc. \end{aligned}$$

Thus condition (2) holds.

By (6.17), we also have $q_{0,0} \le q_0$. So, in view of (6.14), it suffices to show that

$$\|\mathbf{P}(q) - \widetilde{\mathbf{P}}(q)\|_\infty \le 4mnc \quad \text{for each } q \ge q_0, \tag{6.18}$$

in order to prove condition (3) and thus complete the whole proof.

To do this, fix a real number q with $q \ge q_0$. Since $q_0 \ge q_{0,0}$, there is an integer $i \ge 0$ for which $q_{i,0} \le q < q_{i+1,0}$. To prove (6.18) for this value of q, it suffices to show that

$$\mathbf{P}(q) \le \widetilde{\mathbf{P}}(q^*), \quad \text{where } q^* = q + 2mnc. \tag{6.19}$$

Indeed, if we admit this inequality, then all the coordinates of the n-tuple $\widetilde{\mathbf{P}}(q^*) - \mathbf{P}(q)$ are non-negative. Since they sum to $q^* - q = 2mnc$, these coordinates are at most $2mnc$, and so $\|\widetilde{\mathbf{P}}(q^*) - \mathbf{P}(q)\|_\infty \le 2mnc$. Since each component of $\widetilde{\mathbf{P}}$ is continuous and piecewise linear with slope at most 1, we also have $\|\widetilde{\mathbf{P}}(q^*) - \widetilde{\mathbf{P}}(q)\|_\infty \le 2mnc$, and thus $\|\mathbf{P}(q) - \widetilde{\mathbf{P}}(q)\|_\infty \le 4mnc$. To prove (6.19), we distinguish three cases.

Case 1: Suppose first that $q_{i,0} \le q < q_{i,1}$. If $q + q_i - q_{i,0} \le s_i$, we find

$$\mathbf{P}(q) = \Phi_n(\mathbf{a}^{(i,0)} + (q - q_{i,0})\mathbf{e}_1) \le \Phi_n(\widetilde{\mathbf{P}}(q_i) + (q - q_{i,0})\mathbf{e}_1) = \widetilde{\mathbf{P}}(q + q_i - q_{i,0}),$$

where the middle inequality uses the fact that Φ_n is order preserving together with the first inequality in (6.16). If instead $s_i < q + q_i - q_{i,0}$, then using the second inequality in (6.16), we find

$$\mathbf{P}(q) \le \mathbf{P}(q_{i,1}) = \mathbf{a}^{(i,1)} \le \widetilde{\mathbf{P}}(s_i) \le \widetilde{\mathbf{P}}(q + q_i - q_{i,0}).$$

Since (6.17) gives $q_i - q_{i,0} \le 2mnc$, we deduce that $\mathbf{P}(q) \le \widetilde{\mathbf{P}}(q^*)$ in both instances.

From now on, we may therefore assume that $q \ge q_{i,1}$. Then, for each $j = 1, \dots, r-1$, we find that

$$P_j(q) = a_{i+j} \le \log X_{i+j} = \widetilde{P}_j(s_i) \le \widetilde{P}_j(q^*),$$

where the last inequality uses $s_i \le q_{i,1} + 2mnc \le q^*$ coming from (6.17).

Case 2: Suppose that $q_{i,1} \le q \le q_{i,\sigma(i)m}$. If $q^* \le t_i$, then, for $j = r, \dots, n$, we find that

$$P_j(q) \le P_n(q) \le b_i + c + \frac{q - q_{i,1}}{m+1} \le \log Y_i + \frac{q^* - s_i}{m+1} = \widetilde{P}_j(q^*),$$

because $b_i + c \le \log Y_i$ by (6.6), and $q - q_{i,1} \le q^* - s_i$ by (6.17). If instead $q^* > t_i$, then, for the same values of j, we find

$$P_j(q) \le P_n(q) \le P_n(q_{i,\sigma(i)m}) = a_j^{(i,\sigma(i)m)} \le \widetilde{P}_j(t_i) \le \widetilde{P}_j(q^*),$$

using the third inequality in (6.16). Thus, (6.19) holds in both instances.

Case 3: Suppose that $q_{i,\sigma(i)m} \le q$. Using (6.17), we find $0 \le q - q_{i,\sigma(i)m} \le q^* - t_i$, thus $t_i \le q^*$ and so

$$P_r(q) \le a_{i+r} \le \log X_{i+r} = \widetilde{P}_r(t_i) \le \widetilde{P}_r(q^*).$$

If $q^* \le q_{i+1}$, we also obtain, for $j = r+1, \dots, n$,

$$P_j(q) \le P_n(q) \le a_{i+r} + \frac{q - q_{i,\sigma(i)m}}{m} \le \log X_{i+r} + \frac{q^* - t_i}{m} = \widetilde{P}_j(q^*).$$

If instead $q^* > q_{i+1}$, then for the same values of j we find

$$P_j(q) \le P_n(q) \le P_n(q_{i+1,0}) = b_{i+1} \le \log Y_{i+1} = \widetilde{P}_j(q_{i+1}) \le \widetilde{P}_j(q^*).$$

Thus, (6.19) holds in that case also. ∎

Proof of Theorem 1.1, part (2). Let V be a subspace of $\mathbb{R}^n$ of dimension $m+1$, and let $\psi\colon [1,\infty) \to (0,\infty)$ be an unbounded non-decreasing function. Since V has the same dimension as

$$V_0 = \langle \mathbf{e}_1 + \cdots + \mathbf{e}_r, \mathbf{e}_{r+1}, \dots, \mathbf{e}_n \rangle_{\mathbb{R}},$$

there is an isometry T of $\mathbb{R}^n$ which maps V_0 to V. Setting $\mathbf{v}_j = T(\mathbf{e}_j)$ for each $j = 1, \dots, n$, we obtain an orthonormal basis $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ of $\mathbb{R}^n$ such that

$$V = \langle \mathbf{v}_1 + \cdots + \mathbf{v}_r, \mathbf{v}_{r+1}, \dots, \mathbf{v}_n \rangle_{\mathbb{R}}.$$

We apply the previous theorem to this choice of basis $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ for

$$\delta = 1/\max\{4n, 24r\} \quad \text{and} \quad c = \log(8/\delta), \tag{6.20}$$

so that (6.1) holds, and for sequences $(X_i)_{i\ge 0}$ and $(Y_i)_{i\ge 0}$ satisfying (6.2)–(6.4) as well as

$$\log Y_i = (\rho/m)\log X_{i+r-1}, \qquad X_i X_{i+1} \cdots X_{i+r-1} \le \psi(X_{i+r}/X_i) \tag{6.21}$$

for each $i \geq 0$, where $\rho = \rho_m$ is given by (1.2). This is possible since $\rho > m$. We claim that the unit vector $\mathbf{u} \in \mathbb{R}^n$ provided by Theorem 6.1 has the property stated in the second part of Theorem 1.1. Since its coordinates are linearly independent over $\mathbb{Q}$, this amounts to showing that any non-zero point $\mathbf{x}$ of $\mathbb{Z}^n$ of sufficiently large norm with $\mathrm{dist}(\mathbf{x}, V) \leq \delta$ satisfies

$$|\mathbf{x} \cdot \mathbf{u}| > \psi(\|\mathbf{x}\|)^{-1}\|\mathbf{x}\|^{-\rho}. \tag{6.22}$$

To prove this, we use properties (1)–(3) of the associated sequences $(\widetilde{\mathbf{v}}_i)_{i\geq 0}$ and $(\mathbf{x}_i)_{i\geq 0}$ in Theorem 6.1.

Let $\mathbf{x}$ be a non-zero point of $\mathbb{Z}^n$ with $\mathrm{dist}(\mathbf{x}, V) \leq \delta$. Assuming, as we may, that $\|\mathbf{x}\|$ is large enough, there exists an integer $i \geq 0$ such that

$$\log Y_i \leq \log \|\mathbf{x}\| + 15c' \leq \log Y_{i+1}, \quad \text{where } c' = mnc,$$

and so there is a unique value of $q \in [s_i, q_{i+1}]$ for which

$$\log \|\mathbf{x}\| = \widetilde{P}_{r+1}(q) - 15c'.$$

For such q, we note the following useful formula:

$$\widetilde{P}_{r+1}(q) = \widetilde{P}_r(q) + \max\{0, q - t_i\}/m. \tag{6.23}$$

In the computations below, we assume that $\|\mathbf{x}\|$ is large enough so that for example we have $\log(Y_i/X_{i+r-1}) \geq 9c'$. To simplify the exposition, we simply put a star on the inequalities that require such additional assumptions. We consider two cases.

Case 1: Suppose that $L_{\mathbf{u}}(\mathbf{x}, q) = \log \|\mathbf{x}\|$. We first note that

$$(\widetilde{\mathbf{v}}_{i+1}, \dots, \widetilde{\mathbf{v}}_{i+r}, \mathbf{v}_{r+1}, \dots, \mathbf{v}_n)$$

is an orthonormal basis of $\mathbb{R}^n$ and that

$$V = \langle \widetilde{\mathbf{v}}_{i+1} + \cdots + \widetilde{\mathbf{v}}_{i+r}, \mathbf{v}_{r+1}, \dots, \mathbf{v}_n \rangle_{\mathbb{R}},$$

because $(\widetilde{\mathbf{v}}_{i+1}, \dots, \widetilde{\mathbf{v}}_{i+r})$ is a permutation of $(\mathbf{v}_1, \dots, \mathbf{v}_r)$. Since property (1) in Theorem 6.1 gives

$$\mathrm{dist}(\mathbf{x}_{i+j}, \widetilde{\mathbf{v}}_{i+j}) \leq \delta \leq 1/(24r) \quad \text{for } j = 1, \dots, r,$$

by the choice of δ in (6.20), it follows from Lemma 3.6 that $(\mathbf{x}_{i+1}, \dots, \mathbf{x}_{i+r})$ is a linearly independent r-tuple of points of $\mathbb{Z}^n$. We claim that if $\|\mathbf{x}\|$ is large enough, we also have

$$\mathbf{x} \in \langle \mathbf{x}_{i+1}, \dots, \mathbf{x}_{i+r} \rangle_{\mathbb{R}}, \tag{6.24}$$

and thus Lemma 3.7 applies. To prove this, we look more closely at the trajectories of the points $\mathbf{x}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_{i+r}$. For each $j \geq 1$, we note that $q \leq q_{i+1} \leq q_{i+j}$, and so property (2) in Theorem 6.1 yields

$$L_{\mathbf{u}}(\mathbf{x}_{i+j}, q) \leq \max\{\log \|\mathbf{x}_{i+j}\|, q_{i+j} + \log |\mathbf{x}_{i+j} \cdot \mathbf{u}|\} \leq \log X_{i+j} + 4c'.$$

We distinguish two subcases depending on the value of q.

(a) Suppose first that $s_i \le q < t_i + 10mc'$. Then the above estimates give

$$\max_{1\le j<r} L_{\mathbf{u}}(\mathbf{x}_{i+j}, q) \le \log X_{i+r-1} + 4c' <^* \log Y_i - 5c' \le \widetilde{P}_r(q) - 5c',$$

while (6.23) yields

$$L_{\mathbf{u}}(\mathbf{x}, q) = \log\|\mathbf{x}\| = \widetilde{P}_{r+1}(q) - 15c' < \widetilde{P}_r(q) - 5c'.$$

As $\widetilde{P}_r(q) \le L_{u,r}(q) + 5c'$ by property (3) in Theorem 6.1, this means that

$$\max\{L_{\mathbf{u}}(\mathbf{x}, q), L_{\mathbf{u}}(\mathbf{x}_{i+1}, q), \dots, L_{\mathbf{u}}(\mathbf{x}_{i+r-1}, q)\} < L_{\mathbf{u},r}(q).$$

Thus, the r points $\mathbf{x}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_{i+r-1} \in \mathbb{Z}^n$ are linearly dependent and so $\mathbf{x}$ belongs to $\langle \mathbf{x}_{i+1}, \dots, \mathbf{x}_{i+r-1}\rangle_{\mathbb{R}}$, which is stronger than our claim (6.24).

(b) Suppose now that $t_i + 10mc' \le q \le q_{i+1}$. Then, using (6.23), we find

$$\max_{1\le j\le r} L_{\mathbf{u}}(\mathbf{x}_{i+j}, q) \le \log X_{i+r} + 4c' \le \widetilde{P}_{r+1}(q) - 6c'.$$

As $\widetilde{P}_{r+1}(q) \le L_{\mathbf{u},r+1}(q) + 5c'$ and $L_{\mathbf{u}}(\mathbf{x}, q) = \log\|\mathbf{x}\| = \widetilde{P}_{r+1}(q) - 15c'$, this means that

$$\max\{L_{\mathbf{u}}(\mathbf{x}, q), L_{\mathbf{u}}(\mathbf{x}_{i+1}, q), \dots, L_{\mathbf{u}}(\mathbf{x}_{i+r}, q)\} \le L_{\mathbf{u},r+1}(q) - c'.$$

So, the $r+1$ points $\mathbf{x}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_{i+r} \in \mathbb{Z}^n$ are linearly dependent and thus (6.24) holds again.

Applying Lemma 3.7, we can therefore write

$$\mathbf{x} = a_1\mathbf{x}_{i+1} + \cdots + a_r\mathbf{x}_{i+r}$$

with coefficients $a_1, \dots, a_r \in \mathbb{R}$ that satisfy

$$\frac{1}{2\sqrt{r}} \le \frac{\|a_j\mathbf{x}_{i+j}\|}{\|\mathbf{x}\|} \le \frac{2}{\sqrt{r}} \quad \text{for } j = 1, \dots, r. \tag{6.25}$$

This yields the lower bound

$$\begin{aligned}|\mathbf{x}\cdot\mathbf{u}| &\ge |a_1\mathbf{x}_{i+1}\cdot\mathbf{u}| - \sum_{j=2}^{r} |a_j\mathbf{x}_{i+j}\cdot\mathbf{u}| \\ &\ge \frac{\|\mathbf{x}\|}{2\sqrt{r}}\left(\frac{|\mathbf{x}_{i+1}\cdot\mathbf{u}|}{\|\mathbf{x}_{i+1}\|} - 4\sum_{j=2}^{r}\frac{|\mathbf{x}_{i+j}\cdot\mathbf{u}|}{\|\mathbf{x}_{i+j}\|}\right).\end{aligned}$$

By property (2) in Theorem 6.1, we also have

$$\left|\log\frac{|\mathbf{x}_j\cdot\mathbf{u}|}{\|\mathbf{x}_j\|} + q_j\right| \le 5c' \quad \text{for each } j \ge 0.$$

Thus the previous estimate yields

$$\begin{aligned}|\mathbf{x}\cdot\mathbf{u}| &\ge \frac{\|\mathbf{x}\|}{2\sqrt{r}}\big(\exp(-q_{i+1} - 5c') - 4r\exp(-q_{i+2} + 5c')\big) \\ &>^* \frac{\|\mathbf{x}\|}{c''\exp(q_{i+1})} = \frac{\|\mathbf{x}\|}{c''X_{i+1}\cdots X_{i+r}Y_{i+1}^m} = \frac{\|\mathbf{x}\|}{c''X_{i+1}\cdots X_{i+r-1}X_{i+r}^{\rho+1}}\end{aligned}$$

where $c'' = 4\sqrt{r}\exp(5c')$. The inequalities (6.25) also imply that a_r is nonzero. This means that $\mathbf{x} \notin \langle \mathbf{x}_{i+1}, \dots, \mathbf{x}_{i+r-1}\rangle_{\mathbb{R}}$ and thus rules out the subcase (a) considered above. So we have $q \geq t_i + 10mc'$, and using (6.23) we find that

$$\|\mathbf{x}\| = \exp(\widetilde{P}_{r+1}(q) - 15c') \geq X_{i+r}\exp(-5c').$$

Combining the last two estimates and using (6.21), we conclude as announced that

$$|\mathbf{x}\cdot\mathbf{u}| >^* \frac{\|\mathbf{x}\|^{-\rho}}{X_iX_{i+1}\cdots X_{i+r-1}} \geq \frac{\|\mathbf{x}\|^{-\rho}}{\psi(X_{i+r}/X_i)} \geq^* \frac{\|\mathbf{x}\|^{-\rho}}{\psi(\|\mathbf{x}\|)}.$$

Case 2: Suppose instead that $L_{\mathbf{u}}(\mathbf{x}, q) > \log\|\mathbf{x}\|$. By definition, this means that

$$\log|\mathbf{x}\cdot\mathbf{u}| > \log\|\mathbf{x}\| - q. \tag{6.26}$$

Since the coordinates of $\widetilde{\mathbf{P}}(q)$ sum to q, we have

$$q = \log X_{i+1} + \cdots + \log X_{i+r-1} + \begin{cases} (m+1)\widetilde{P}_{r+1}(q) & \text{if } s_i \leq q \leq t_i,\\ \log X_{i+r} + m\widetilde{P}_{r+1}(q) & \text{if } t_i \leq q \leq q_{i+1}.\end{cases}$$

As $\widetilde{P}_{r+1}(q) \geq \log X_{i+r}$ when $q \geq t_i$, this implies in all cases that

$$q \leq \log X_{i+1} + \cdots + \log X_{i+r-1} + (m+1)\widetilde{P}_{r+1}(q).$$

We also have

$$\log\|\mathbf{x}\| + 15c' = \widetilde{P}_{r+1}(q) \geq \log Y_i = (\rho/m)\log X_{i+r-1}. \tag{6.27}$$

Using this to eliminate $\log X_{i+r-1}$ and $\widetilde{P}_{r+1}(q)$ from the upper bound for q, we obtain

$$q \leq \log X_{i+1} + \cdots + \log X_{i+r-2} + (m+1+m/\rho)(\log\|\mathbf{x}\| + 15c').$$

Now, we use the exact value of ρ. Since $m + m/\rho = \rho$, we deduce that

$$q \leq^* \log X_i + \cdots + \log X_{i+r-2} + (\rho+1)\log\|\mathbf{x}\|.$$

Together with (6.26), this yields

$$|\mathbf{x}\cdot\mathbf{u}| > \frac{\|\mathbf{x}\|}{\exp(q)} \geq \frac{\|\mathbf{x}\|^{-\rho}}{X_i\cdots X_{i+r-2}} \geq \frac{\|\mathbf{x}\|^{-\rho}}{\psi(X_{i+r-1})}.$$

Finally, since $\rho/m > 1$, we conclude from (6.27) that $X_{i+r-1} \leq^* \|\mathbf{x}\|$ and thus (6.22) holds if $\|\mathbf{x}\|$ is large enough.

7. A simplification of Thurnheer's argument. As mentioned in the introduction, the first part of Theorem 1.1 is a result of Thurnheer when $n = m + 2$. Our goal in this last section is to provide a simplification of his argument along the lines of [4]. The following statement is a slight generalization of [16, Theorem 1(b)].

Theorem 7.1 (Thurnheer). *Let m be a positive integer, let $n = m+2$, let $\mathbf{u}$ be a point of $\mathbb{R}^n$ whose coordinates are linearly independent over $\mathbb{Q}$, and let $\mathbf{d} \in \mathbb{R}^n$. Then, for any $\delta, \epsilon \in (0,1)$, there is a non-zero point $\mathbf{x}$ of $\mathbb{Z}^n$ with*

$$|\mathbf{x}\cdot\mathbf{d}| \le \delta\|\mathbf{x}\| \quad \text{and} \quad |\mathbf{x}\cdot\mathbf{u}| \le \epsilon\|\mathbf{x}\|^{-\rho}$$

where ρ is as in Theorem 1.1.

For the proof, we assume that $(\mathbf{u},\mathbf{d})$ is an orthonormal pair of vectors in $\mathbb{R}^n$ because, if the conclusion holds for such a pair, then it also holds for any pair $(a\mathbf{u}, b\mathbf{u}+c\mathbf{d})$ with $a,b,c \in \mathbb{R}$ and $a \neq 0$, and this covers the general case. From there, we proceed by contradiction. We suppose that there exist numbers $0 < \delta, \epsilon < 1$ such that any non-zero point $\mathbf{x} \in \mathbb{Z}^n$ with $|\mathbf{x}\cdot\mathbf{u}| \le \epsilon\|\mathbf{x}\|^{-\rho}$ satisfies $|\mathbf{x}\cdot\mathbf{d}| > \delta\|\mathbf{x}\|$. To derive a contradiction, we define a norm $\|\ \|'$ on $\mathbb{R}^n$ through the formula

$$\|\mathbf{x}\|' = \max\{|\mathbf{x}\cdot\mathbf{d}|, (\delta/4)\|\mathbf{x}\|\}$$

for each $\mathbf{x} \in \mathbb{R}^n$. Then our hypothesis is that, for any non-zero point $\mathbf{x}$ of $\mathbb{Z}^n$, we have

$$|\mathbf{x}\cdot\mathbf{u}| \le \epsilon\|\mathbf{x}\|^{-\rho} \implies \|\mathbf{x}\|' = |\mathbf{x}\cdot\mathbf{d}| > \delta\|\mathbf{x}\|. \tag{7.1}$$

Extending an argument of Schmidt in [12, Lemma 1] for the case $n = 3$, Thurnheer obtained the following general estimate in [16, part II, (iii)].

Lemma 7.2 (Schmidt and Thurnheer). *There are constants c_3 and X_0 depending only on n and ϵ with the property that, for each real number X with $X \ge X_0$, there exists a non-zero point $\mathbf{x}$ of $\mathbb{Z}^n$ with $\|\mathbf{x}\|' \le X$ and $|\mathbf{x}\cdot\mathbf{u}| \le c_3X^{-\rho-1}$.*

Proof. We first complete the orthonormal pair $(\mathbf{u},\mathbf{d})$ to an orthonormal basis $(\mathbf{w}_1,\dots,\mathbf{w}_m,\mathbf{u},\mathbf{d})$ of $\mathbb{R}^n$. Then, for any positive real number Y, the set of points $\mathbf{x} \in \mathbb{R}^n$ satisfying

$$\max_{1\le j\le m} |\mathbf{x}\cdot\mathbf{w}_j| \le n^{-1/2}Y, \quad |\mathbf{x}\cdot\mathbf{u}| \le \epsilon Y^{-\rho}, \quad |\mathbf{x}\cdot\mathbf{d}| \le n^{m/2}\epsilon^{-1}Y^{\rho-m} \tag{7.2}$$

is a compact symmetric convex body of $\mathbb{R}^n$ of volume 2^n and so, by the Minkowski first convex body theorem, it contains a non-zero point $\mathbf{x}$ of $\mathbb{Z}^n$. Since $m < \rho < m+1$ by (1.5), the above inequalities yield $|\mathbf{x}\cdot\mathbf{u}| \le n^{-1/2}Y$ and $|\mathbf{x}\cdot\mathbf{d}| \le n^{-1/2}Y$ if Y is large enough in terms of n and ϵ. Then we find

$$\|\mathbf{x}\| = (|\mathbf{x}\cdot\mathbf{w}_1|^2 + \cdots + |\mathbf{x}\cdot\mathbf{w}_m|^2 + |\mathbf{x}\cdot\mathbf{u}|^2 + |\mathbf{x}\cdot\mathbf{d}|^2)^{1/2} \le Y$$

and so $|\mathbf{x}\cdot\mathbf{u}| \le \epsilon\|\mathbf{x}\|^{-\rho}$ by the middle inequality in (7.2). According to our hypothesis (7.1) and the formula for ρ in (1.5), this implies that

$$\|\mathbf{x}\|' = |\mathbf{x}\cdot\mathbf{d}| \le c_2Y^{\rho-m} = c_2Y^{\rho/(\rho+1)},$$

where $c_2 = n^{m/2}\epsilon^{-1}$. If X is large enough, and if the parameter Y is chosen so that $X = c_2 Y^{\rho/(\rho+1)}$, the point $\mathbf{x}$ constructed above satisfies $\|\mathbf{x}\|' \le X$ and $|\mathbf{x}\cdot\mathbf{u}| \le \epsilon Y^{-\rho} = c_3 X^{-\rho-1}$, where $c_3 = \epsilon c_2^{\rho+1}$. ∎

Since the coordinates of $\mathbf{u}$ are linearly independent over $\mathbb{Q}$, the scalar product with $\mathbf{u}$ defines an injective map from $\mathbb{Z}^n$ to $\mathbb{R}$. Thus, for each X in $[1,\infty)$, there is, up to multiplication by ± 1, a unique non-zero point $\mathbf{x}\in\mathbb{Z}^n$ with $\|\mathbf{x}\|' \le X$ for which $|\mathbf{x}\cdot\mathbf{u}|$ is minimal. We order these pairs $\pm\mathbf{x}$ by increasing norm $\|\mathbf{x}\|'$ and, for each integer $i \ge 1$, we choose a representative $\mathbf{x}_i$ of the ith pair for which $\mathbf{x}_i\cdot\mathbf{d} \ge 0$ (unique unless $\mathbf{x}_i\cdot\mathbf{d} = 0$). Then each $\mathbf{x}_i$ is a primitive point of $\mathbb{Z}^n$ and each pair $(\mathbf{x}_i, \mathbf{x}_{i+1})$ is linearly independent. We also set

$$X_i = \|\mathbf{x}_i\|' \quad \text{and} \quad L_i = |\mathbf{x}_i\cdot\mathbf{u}| \quad \text{for each } i \ge 1.$$

By construction, the sequence $(X_i)_{i\ge 1}$ is strictly increasing, while $(L_i)_{i\ge 1}$ is strictly decreasing. Moreover, by Lemma 7.2, we have $|\mathbf{x}_i\cdot\mathbf{u}| \le c_3 X^{-\rho-1}$ for each $i \ge 1$ with $X_i \ge X_0$ and each $X \in [X_0, X_{i+1})$.

Let i_0 denote the smallest integer $i \ge 1$ satisfying both $X_i \ge X_0$ and $c_3 X_i^{-1} \le \epsilon(4/\delta)^{-\rho}$. By the above, for each index i with $i \ge i_0$, we have

$$L_i \le c_3 X_{i+1}^{-\rho-1}, \tag{7.3}$$

and $|\mathbf{x}_i\cdot\mathbf{u}| \le c_3 X_i^{-\rho-1} \le \epsilon((4/\delta)X_i)^{-\rho} \le \epsilon\|\mathbf{x}_i\|^{-\rho}$ since $\|\mathbf{x}_i\| \le (4/\delta)X_i$. Then our hypothesis (7.1) together with the condition $\mathbf{x}_i\cdot\mathbf{d} \ge 0$ gives

$$X_i = \|\mathbf{x}_i\|' = \mathbf{x}_i\cdot\mathbf{d} > \delta\|\mathbf{x}_i\| \quad (i \ge i_0). \tag{7.4}$$

From this, we deduce two consequences by adapting the arguments of Davenport and Schmidt in [4, Lemmas 1 and 2]. The first lemma below is also implicit in [16, part II, (v)].

Lemma 7.3. *For each $i \ge i_0$, the scalar products $\mathbf{x}_i\cdot\mathbf{u}$ and $\mathbf{x}_{i+1}\cdot\mathbf{u}$ have opposite signs.*

Proof. Suppose on the contrary that they have the same sign for some $i \ge i_0$. Then the point $\mathbf{x} = \mathbf{x}_{i+1} - \mathbf{x}_i \in \mathbb{Z}^n$ is non-zero and satisfies

$$|\mathbf{x}\cdot\mathbf{u}| = L_i - L_{i+1} < L_i.$$

By construction, this implies that $\|\mathbf{x}\|' \ge X_{i+1}$. However, using relations (7.4), we find that $|\mathbf{x}\cdot\mathbf{d}| = X_{i+1} - X_i < X_{i+1}$ and

$$\frac{\delta}{4}\|\mathbf{x}\| \le \frac{\delta}{4}\|\mathbf{x}_{i+1}\| + \frac{\delta}{4}\|\mathbf{x}_i\| \le \frac{X_{i+1}}{4} + \frac{X_i}{4} < X_{i+1},$$

which implies that $\|\mathbf{x}\|' < X_{i+1}$, a contradiction. ∎

Lemma 7.4. *For each $i > i_0$, the ratios L_{i-1}/L_i and X_{i+1}/X_i have the same integer part and, for this integer t_i, we have*

$$\mathbf{x}_{i+1} = t_i\mathbf{x}_i + \mathbf{x}_{i-1}.$$

Proof. Let s and t denote respectively the integer parts of L_{i-1}/L_i and X_{i+1}/X_i, for a choice of index $i > i_0$. Our first goal is to show that $s = t$. To this end, we note, using (7.4), that the non-zero point $\mathbf{y} = \mathbf{x}_{i+1} - t\mathbf{x}_i$ of $\mathbb{Z}^n$ satisfies

$$\|\mathbf{y}\| \le \|\mathbf{x}_{i+1}\| + \frac{X_{i+1}}{X_i}\|\mathbf{x}_i\| \le \frac{X_{i+1}}{\delta} + \frac{X_{i+1}}{X_i}\frac{X_i}{\delta} \le \frac{2}{\delta}X_{i+1}. \tag{7.5}$$

Since $\mathbf{x}_i \cdot \mathbf{u}$ and $\mathbf{x}_{i+1} \cdot \mathbf{u}$ have opposite signs by Lemma 7.3, we also find

$$|\mathbf{y} \cdot \mathbf{u}| = L_{i+1} + tL_i \le 2\frac{X_{i+1}}{X_i}L_i \le \frac{2c_3}{X_i}X_{i+1}^{-\rho},$$

where the last inequality uses (7.3). We deduce that $|\mathbf{y}\cdot\mathbf{u}| \le \epsilon\|\mathbf{y}\|^{-\rho}$ because, as $i \ge i_0$, we have $2c_3/X_i \le 2\epsilon(4/\delta)^{-\rho} \le \epsilon(2/\delta)^{-\rho}$. Then hypothesis (7.1) combined with (7.4) yields

$$\|\mathbf{y}\|' = |\mathbf{y} \cdot \mathbf{d}| = X_{i+1} - tX_i < X_i.$$

By the minimality of $\mathbf{x}_i$, this implies that $|\mathbf{y} \cdot \mathbf{u}| \ge L_{i-1}$. Since we have $|\mathbf{y} \cdot \mathbf{u}| = L_{i+1} + tL_i$, we conclude that $L_{i-1} < (t+1)L_i$ and so $s \le t$.

Similarly, Lemma 7.3 implies that the non-zero point $\mathbf{z} = s\mathbf{x}_i + \mathbf{x}_{i-1} \in \mathbb{Z}^n$ satisfies

$$|\mathbf{z} \cdot \mathbf{u}| = L_{i-1} - sL_i < L_i.$$

So, we must have $\|\mathbf{z}\|' \ge X_{i+1}$. This yields

$$t \le \frac{X_{i+1}}{X_i} \le \frac{\|\mathbf{z}\|'}{X_i} \le \frac{sX_i + X_{i-1}}{X_i} < s+1,$$

thus $t \le s$, and so $s = t$.

Finally, consider the point $\mathbf{x} = \mathbf{x}_{i+1} - t\mathbf{x}_i - \mathbf{x}_{i-1} \in \mathbb{Z}^n$. Using Lemma 7.3 and the equality $s = t$, we find

$$\begin{aligned}
|\mathbf{x} \cdot \mathbf{u}| &= |L_{i+1} + tL_i - L_{i-1}| = |L_{i+1} - (L_{i-1} - sL_i)| < L_i,\\
|\mathbf{x} \cdot \mathbf{d}| &= |(X_{i+1} - tX_i) - X_{i-1}| < X_i.
\end{aligned}$$

Since $\mathbf{x} = \mathbf{y} - \mathbf{x}_{i-1}$, estimates (7.4) and (7.5) yield

$$\frac{\delta}{4}\|\mathbf{x}\| \le \frac{\delta}{4}\|\mathbf{y}\| + \frac{\delta}{4}\|\mathbf{x}_{i-1}\| \le \frac{1}{2}X_{i+1} + \frac{1}{4}X_{i-1} < X_{i+1},$$

thus $\|\mathbf{x}\|' < X_{i+1}$. Since $|\mathbf{x}\cdot\mathbf{u}| < L_i$, this means that $\mathbf{x} = 0$, and so we obtain $\mathbf{x}_{i+1} = t\mathbf{x}_i + \mathbf{x}_{i-1}$. ∎

The last lemma leads to a contradiction by arguing as in [3, Lemma 3]. Indeed, Lemma 7.4 shows that the subgroup of $\mathbb{R}^n$ spanned by $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ is independent of i for $i \ge i_0$, thus $\mathbf{x}_i \wedge \mathbf{x}_{i+1} = \pm\mathbf{x}_j \wedge \mathbf{x}_{j+1}$ for any choice of integers i, j with $i_0 \le i < j$. Contracting these bi-vectors with $\mathbf{u}$ and then taking norms, we obtain

$$\begin{aligned}\|(\mathbf{x}_i \cdot \mathbf{u})\mathbf{x}_{i+1} - (\mathbf{x}_{i+1} \cdot \mathbf{u})\mathbf{x}_i\| &= \|(\mathbf{x}_j \cdot \mathbf{u})\mathbf{x}_{j+1} - (\mathbf{x}_{j+1} \cdot \mathbf{u})\mathbf{x}_j\| \\ &\le \frac{1}{\delta} X_{j+1} L_j + \frac{1}{\delta} X_j L_{j+1} \\ &\le \frac{2}{\delta} X_{j+1} L_j \le \frac{2c_3}{\delta} X_{j+1}^{-\rho}.\end{aligned}$$

Letting j go to infinity for a fixed choice of $i \ge i_0$, we deduce the equality $(\mathbf{x}_i \cdot \mathbf{u})\mathbf{x}_{i+1} = (\mathbf{x}_{i+1} \cdot \mathbf{u})\mathbf{x}_i$. However, this is impossible since $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ are linearly independent over $\mathbb{R}$ and $\mathbf{x}_i \cdot \mathbf{u} \ne 0$. This contradiction completes the proof of Theorem 7.1.

Acknowledgements. The authors thank Michel Laurent for the helpful reference to Erdős's paper [5].

The work of both authors was partially supported by an NSERC discovery grant.

References

[1] Y. Bugeaud and S. Kristensen, *Diophantine exponents for mildly restricted approximation*, Ark. Mat. 47 (2009), 243–266.

[2] J. Champagne, *Approximation diophantienne avec contrainte d'angles*, M.Sc. thesis, Univ. of Ottawa, 2021, 102 pp.; https://ruor.uottawa.ca/handle/10393/42504.

[3] H. Davenport and W. M. Schmidt, *Approximation to real numbers by quadratic irrationals*, Acta Arith. 13 (1967), 169–176.

[4] H. Davenport and W. M. Schmidt, *A theorem on linear forms*, Acta Arith. 14 (1968), 209–223.

[5] P. Erdős, *On an elementary problem in number theory*, Canad. Math. Bull. 1 (1958), 5–8.

[6] N. G. Moshchevitin, *Positive integers: counterexample to W. M. Schmidt's conjecture*, Moscow J. Combin. Number Theory 2 (2012), no. 2, 63–84.

[7] D. Roy, *Diophantine approximation with sign constraints*, Monatsh. Math. 173 (2014), 417–432.

[8] D. Roy, *On Schmidt and Summerer parametric geometry of numbers*, Ann. of Math. 182 (2015), 739–786.

[9] D. Roy, *Spectrum of the exponents of best rational approximation*, Math. Z. 283 (2016), 143–155.

[10] D. Roy, *On the topology of Diophantine approximation spectra*, Compos. Math. 153 (2017), 1512–1546.

[11] W. M. Schmidt, *On heights of algebraic subspaces and diophantine approximations*, Ann. of Math. 85 (1967), 430–472.

[12] W. M. Schmidt, *Two questions in diophantine approximation*, Monatsh. Math. 82 (1976), 237–245.

[13] W. M. Schmidt, *Diophantine Approximation*, Lecture Notes in Math. 785, Springer, Berlin, 1980.

[14] W. M. Schmidt, *Open problems in Diophantine approximation*, in: Approximations diophantiennes et nombres transcendants (Luminy, 1982), Progr. Math. 31, Birkhäuser Boston, Boston, MA, 1983, 271–287.
[15] W. M. Schmidt and L. Summerer, *Diophantine approximation and parametric geometry of numbers*, Monatsh. Math. 169 (2013), 51–104.
[16] P. Thurnheer, *On Dirichlet's theorem concerning diophantine approximation*, Acta Arith. 54 (1990), 241–250.

Jérémy Champagne
Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario N2L 3G1, Canada
E-mail: jchampagne@uwaterloo.ca

Damien Roy
Département de Mathématiques
Université d'Ottawa
Ottawa, Ontario K1N 6N5, Canada
E-mail: droy@uottawa.ca

Abstract (will appear on the journal's web site only)

Following Schmidt, Thurnheer and Bugeaud–Kristensen, we study how Dirichlet's theorem on linear forms needs to be modified when one requires that the vectors of coefficients of the linear forms make a bounded acute angle with respect to a fixed proper non-zero subspace V of $\mathbb{R}^n$. Assuming that the point of $\mathbb{R}^n$ that we are approximating has linearly independent coordinates over $\mathbb{Q}$, we obtain best possible exponents of approximation which surprisingly depend only on the dimension of V. Our estimates are derived by reduction to a result of Thurnheer, while their optimality follows from a new general construction in parametric geometry of numbers involving angular constraints.