Ribbon homology cobordisms

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Ribbon cobordisms

- For compact 3-manifolds Y_- and Y_+ (with same ∂), a *cobordism*
$$W\colon Y_- \to Y_+$$
is made up of 1-, 2-, and 3-handles
- *Ribbon*: does not have 3-handles
- Natural examples: Stein cobordisms between contact 3-manifolds

Why "ribbon"?

- Ans: Related to *ribbon concordances* of knots in S^3, which are concordances with 0- and 1-handles, but no 2-handles

Observation

If $C\colon K_- \to K_+$ is a (strongly homotopy-)ribbon concordance, then the exterior

- $Y_\pm := S^3 \setminus K_\pm$
- $W := (S^3 \times [0,1]) \setminus C$

gives a ribbon $\mathbb{Z}$-homology cobordism $W\colon Y_- \to Y_+$.

- Here, R-homology cobordism means that the maps

$$H_*(Y_-; R) \to H_*(W; R) \leftarrow H_*(Y_+; R)$$

  induced by inclusion are isomorphisms.
- W, like C, has no topology in interior (detected by homology)

Fundamental groups

- $Y_\pm = S^3 \setminus K_\pm, \quad W = (S^3 \times [0,1]) \setminus C$

Theorem (Gordon 1981)

If $C\colon K_- \to K_+$ is a (strongly homotopy-) ribbon concordance, then

$$\pi_1(Y_-) \hookrightarrow \pi_1(W) \twoheadleftarrow \pi_1(Y_+).$$

Proof.

Uses the residual finiteness of knot groups $\pi_1(Y_\pm)$. □

Several decades later...

Observation

Geometrization (Perelman 2006) implies residual finiteness for closed 3-manifold groups.

Theorem (Gordon 1981)

If $W : Y_- \to Y_+$ is a ribbon homology cobordism, then

$$\pi_1(Y_-) \hookrightarrow \pi_1(W) \twoheadleftarrow \pi_1(Y_+).$$

- Roughly: $\pi_1(Y_-)$ is "no bigger" than $\pi_1(Y_+)$
- How can we use this?

Main results

Observation

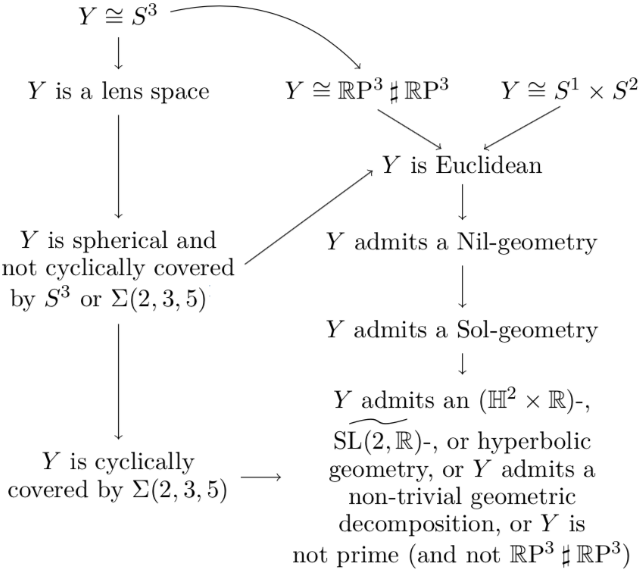
$\pi_1(Y)$ *determines the Thurston geometry of* Y *(if it has one).*

Theorem (Daemi–Lidman–Vela-Vick–W.)

If $W : Y_- \to Y_+$ *is a ribbon* $\mathbb{Q}$*-homology cobordism, then*

- *The Thurston geometries of* Y_- *and* Y_+ *satisfy a hierarchy.*

Ribbon homology cobordisms and Thurston geometries



Main results

Theorem (Daemi–Lidman–Vela-Vick–W.)

If $W : Y_- \to Y_+$ is a ribbon $\mathbb{Q}$-homology cobordism, then

- *The Thurston geometries of Y_- and Y_+ satisfy a hierarchy.*

- How else can we squeeze information from π_1?
- Idea: Representations of $\pi_1(Y_\pm)$

Main results

Theorem (Daemi–Lidman–Vela-Vick–W.)

If $W : Y_- \to Y_+$ is a ribbon $\mathbb{Q}$-homology cobordism, then

- *The Thurston geometries of Y_- and Y_+ satisfy a hierarchy.*
- *The dimension of the G-representation variety of Y_- is at most that of Y_+, for a compact Lie group G, and*

$$\mathcal{R}_G(Y_-) \twoheadleftarrow \mathcal{R}_G(W) \hookrightarrow \mathcal{R}_G(Y_+).$$

- Agol (2022) famously used this idea to prove:

Theorem (Conjecture (Gordon 1981); Agol 2022)

Ribbon concordance is a partial order.

- Note: Also true for G-character variety
- Any specific G? For example, $\mathrm{SU}(2)$
- Next idea: The $\mathrm{SU}(2)$-representations of $\pi_1(Y)$ are related to the instanton Floer homology $\mathrm{I}^\sharp(Y)$

Main results

Theorem (Daemi–Lidman–Vela-Vick–W.)

If $W : Y_- \to Y_+$ is a ribbon $\mathbb{Q}$-homology cobordism, then

- *The Thurston geometries of Y_- and Y_+ satisfy a hierarchy.*
- *The dimension of the G-representation variety of Y_- is at most that of Y_+.*
- $\mathrm{I}^\sharp(W) \colon \mathrm{I}^\sharp(Y_-) \to \mathrm{I}^\sharp(Y_+)$ *is injective.*

- Note: Conjecturally, $\mathrm{I}^\sharp(Y) \cong \widehat{\mathrm{HF}}(Y)$ (Heegaard Floer)
- Next idea: Similarly for Heegaard Floer homology!

Main results

Theorem (Daemi–Lidman–Vela-Vick–W.)

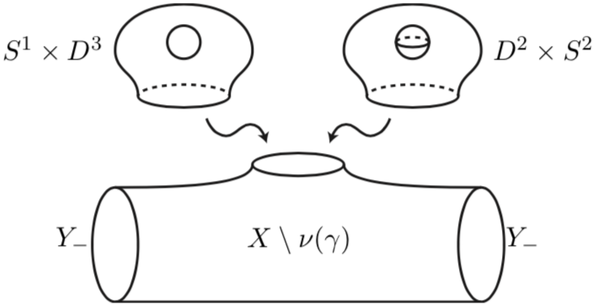
If $W: Y_- \to Y_+$ is a ribbon R-homology cobordism, then

- *The Thurston geometries of Y_- and Y_+ satisfy a hierarchy.*
- *The dimension of the G-representation variety of Y_- is at most that of Y_+.*
- $\mathrm{I}^\sharp(W)\colon \mathrm{I}^\sharp(Y_-) \to \mathrm{I}^\sharp(Y_+)$ *is injective.*
- $\widehat{F}_W\colon \widehat{\mathrm{HF}}(Y_-) \to \widehat{\mathrm{HF}}(Y_+)$ *is injective. ($R = \mathbb{Z}/2$)*

- Note: We also prove analogous results for I, SHI, KHI, equivariant I, and HF^-, HF^+, HF^∞, SFH, $\widehat{\mathrm{HFK}}$, $\widehat{\mathrm{HFI}}$
- Some of these require conditions to make sense, or have weaker conclusion: $F(Y_-)$ isomorphic to summand of $F(Y_+)$

Sketch of proof for Floer homologies

- Doubling trick:



Attaching $S^1 \times D^3 \quad \rightsquigarrow \quad X := (Y_- \times [0,1]) \sharp (S^1 \times S^3)$

Attaching $D^2 \times S^2 \quad \rightsquigarrow \quad D(W) := W \cup_{Y_+} (-W)$

Application: Seifert fibered homology spheres

Theorem (Daemi–Lidman–Vela-Vick–W.)

$Y_- = \Sigma(a_1, \ldots, a_n)$, $Y_+ = \Sigma(a'_1, \ldots, a'_m)$. *Suppose there exists a ribbon* $\mathbb{Q}$*-homology cobordism from* Y_- *to* Y_+*. Then*

- $|\lambda(Y_-)| \leq |\lambda(Y_+)|$;
- *Either* Y_- *and* Y_+ *both bound negative-definite plumbings, or both bound positive-definite plumbings; and*
- $n \leq m$.

Proof.

First two items follow from $\widehat{\mathrm{HF}}$ or $\mathrm{I}^\sharp$. Last item requires calculating the dimension of $\mathrm{SU}(2)$-character varieties. □

Application: Seifert fibered homology spheres

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Corollary

$K_-, K_+ \subset S^3$ *Montesinos knots with* $\det = 1$*. Suppose the number of rational tangles in* K_- *with denominator at least* $2 >$ *that in* K_+*. Then there are no strongly homotopy-ribbon concordances from* K_- *to* K_+*.*

Application: Ribbon concordance to small knots

Theorem (Daemi–Lidman–Vela-Vick–W.)

There are no strongly homotopy-ribbon concordances from composite knots K_- to small knots K_+.

Proof.

For $K \subset S^3$, Sivek–Zentner: $\exists$ a 1-parameter family of irreps $\pi_1(S^3 \setminus K) \to \mathrm{SU}(2)$. For composite knots, can use conjugation to get a 2-parameter family. Thus, the $\mathrm{SU}(2)$-representation varieties of $K_\pm$ have dimension ≥ 2; cannot be small. □

Application: Dehn surgery

Theorem (Daemi–Lidman–Vela-Vick–W.)

Suppose that Y is an irreducible $\mathbb{Q}$-homology sphere, K is a null-homotopic knot in Y, and $Y_0(K) \cong N \sharp (S^1 \times S^2)$. Then $N \cong Y$.

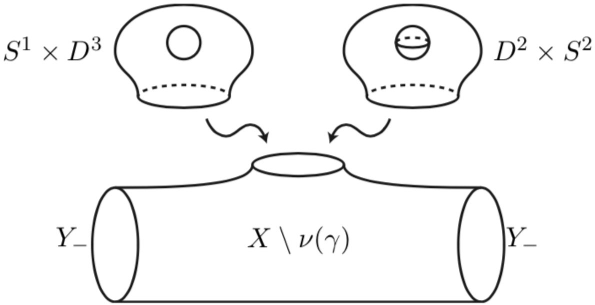
Proof.

Idea: A natural $\mathbb{Z}$-ribbon homology cobordism from N to Y, which leads to an isomorphism of π_1.

- For $Y \cong S^3$, Gabai's proof of Property R/Poénaru Conjecture
- For aspherical Y, get a homotopy equivalence, and thus a homeomorphism by the Borel Conjecture in dimension 3
- For lens spaces Y, $\mathbb{Z}$-homology cobordant implies homeomorphic
- For spherical Y that are not lens spaces, analyze Sylow 2-subgroups of $\pi_1(Y)$ to reduce to lens spaces

□

Thank you!



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