GRID invariants obstruct decomposable Lagrangian cobordisms

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Outline

1. Background
   - Lagrangian cobordisms between Legendrian links
   - GRID invariants
2. Results
   - Statement
   - Applications

Lagrangian cobordisms in $\mathbb{R} \times \mathbb{R}^3$

- Recall: $\mathbb{R}^3$ with standard contact structure $\xi_{\mathrm{std}} = \ker(\alpha)$, where $\alpha = dz - y dx$
- A link $\Lambda \subset \mathbb{R}^3$ is *Legendrian* if Λ is tangent to ξ_{std}
- Focus on the *symplectization* $(\mathbb{R} \times \mathbb{R}^3, d(e^t \alpha))$
- (Chantraine) An *(exact) Lagrangian cobordism from Λ_{bot} to Λ_{top}* is as follows:
  - An embedded Lagrangian submanifold $L \subset (\mathbb{R} \times \mathbb{R}^3, d(e^t \alpha))$
  - There is some real $T > 0$ such that

$$\mathcal{E}_{\mathrm{top}}(L) := L \cap ((T, \infty) \times \mathbb{R}^3) = (T, \infty) \times \Lambda_{\mathrm{top}};$$
$$\mathcal{E}_{\mathrm{bot}}(L) := L \cap ((-\infty, -T) \times \mathbb{R}^3) = (-\infty, -T) \times \Lambda_{\mathrm{bot}};$$

  - The rest of L is compact with boundary Λ_{top} and $-\Lambda_{\mathrm{bot}}$
  - $e^t \alpha|_L = df$ is exact, with f constant on $\mathcal{E}_{\mathrm{top}}(L) \cup \mathcal{E}_{\mathrm{bot}}(L)$

Some (perhaps unintuitive) properties

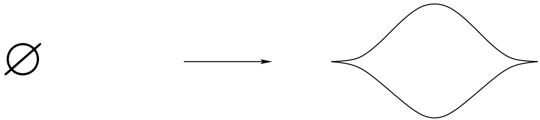
- Λ_{bot} and Λ_{top} Legendrian isotopic $\implies$ there exists a *Lagrangian concordance* (i.e. L is $\mathbb{R} \times S^1$)
- Lagrangian concordances are exact
- Differ from smooth cobordisms and concordances:
  - (Chantraine) If Λ_{bot} and Λ_{top} are knots, then

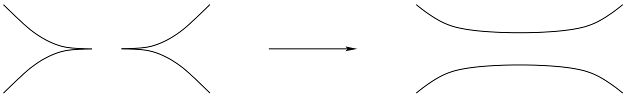
$$r(\Lambda_{\mathrm{top}}) = r(\Lambda_{\mathrm{bot}}), \quad tb(\Lambda_{\mathrm{top}}) = tb(\Lambda_{\mathrm{bot}}) - \chi(L)$$

  - So a Lagrangian cobordism L cannot be inverted whenever $g(L) > 0$; can we invert Lagrangian concordances?
  - (Chantraine) Not in general!
  - So Lagrangian cobordism and concordance are not equivalence relations
- (Ekholm–Honda–Kálmán; Eliashberg–Ganatra–Lazarev) Can be generalized from cobordisms in $\mathbb{R} \times \mathbb{R}^3$ to cobordisms in a Weinstein cobordism from some $(Y_{\mathrm{bot}}, \xi_{\mathrm{bot}})$ to some $(Y_{\mathrm{top}}, \xi_{\mathrm{top}})$

Decomposable Lagrangian cobordisms

- (Chantraine; Ekholm–Honda–Kálmán) A Lagrangian cobordism is *decomposable* if it is a product of elementary cobordisms: Legendrian isotopy, births, and saddles
- In front diagrams: Legendrian Reidemeister moves and





- Note the direction of arrows!

Decomposable Lagrangian cobordisms

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- (Conway–Etnyre–Tosun) In $\mathbb{R} \times \mathbb{R}^3$, decomposable cobordisms are regular; converse unknown
  - *Regular:* tangent to the Liouville vector field
- Every decomposable Lagrangian cobordism is exact
- Open question: Is every exact Lagrangian cobordism decomposable?

Grid homology

- (Manolescu–Ozsváth–Sarkar; Manolescu–Ozsváth–Szabó–Thurston)

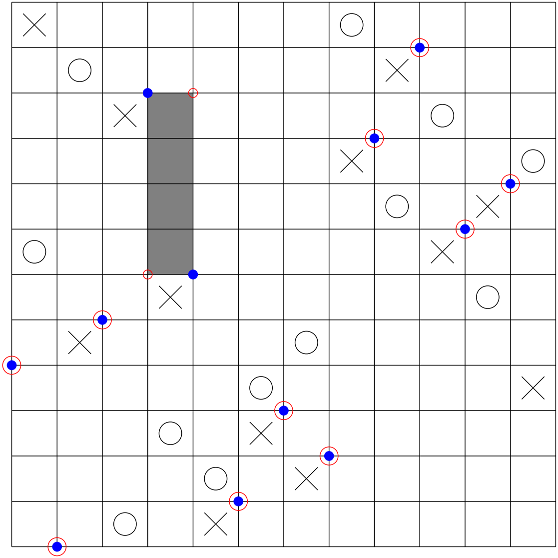
$$\text{Knot } K \subset S^3 \rightsquigarrow \text{grid diagram } \mathbb{G} \rightsquigarrow \text{grid complex } \widetilde{\mathrm{GC}}(\mathbb{G})$$

- Computes link Floer homology:

$$\widetilde{\mathrm{GH}}(\mathbb{G}) := \mathrm{H}_*(\widetilde{\mathrm{GC}}(\mathbb{G})) \cong \widehat{\mathrm{HFK}}(S^3, K) \otimes V^{n-1},$$

  where n is the grid size and V is 2-dimensional
- Write $\widehat{\mathrm{GH}}(\mathbb{G})$ for the $\widehat{\mathrm{HFK}}(S^3, K)$ part
- (Ozsváth–Szabó–Thurston) In fact, $\mathbb{G}$ can encode a front diagram of a Legendrian knot $\Lambda(\mathbb{G})$ of smooth type $m(K)$!

Grid homology and Legendrian GRID invariants



$\mathbb{G} \rightsquigarrow K = 10_{132}$

Two generators of $\widetilde{\mathrm{GC}}(\mathbb{G})$:

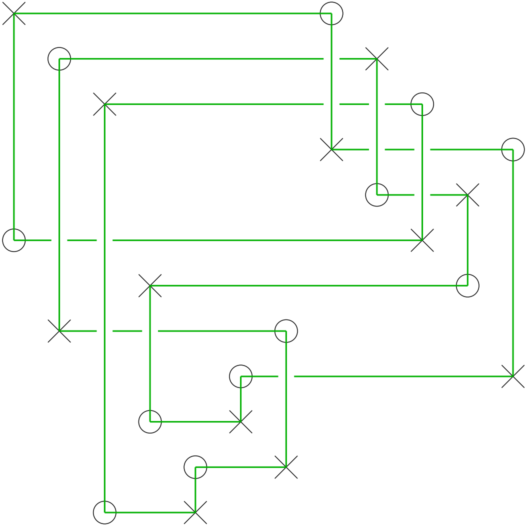
$\mathbf{w}=$ red

$\mathbf{x}=$ blue

$\widetilde{\partial}$ counts empty rectangles

$$\widetilde{\partial}\mathbf{w} = \mathbf{x} + \cdots$$

Grid homology and Legendrian GRID invariants



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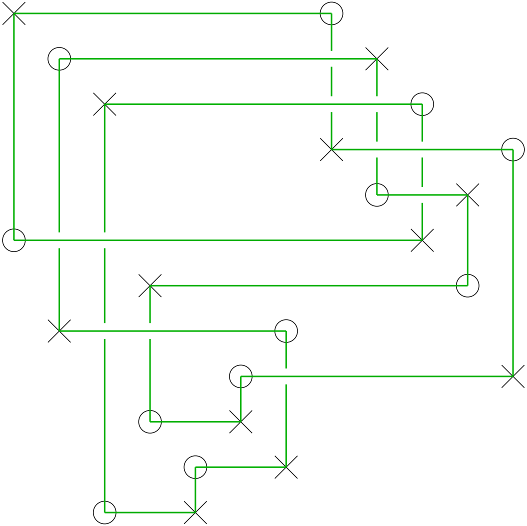

$$\mathbb{G} \rightsquigarrow K = 10_{132}$$

$$\mathbb{G} \rightsquigarrow \Lambda(\mathbb{G}) = \Lambda_1$$

Λ_1 has smooth type

$$m(K) = m(10_{132})$$

$\mathbf{x}^+ =$ blue

$$\mathbf{x}^+ \in \widetilde{\mathrm{GC}}(\mathbb{G})$$

$\mathbf{x}^- =$ magenta

$$\mathbf{x}^- \in \widetilde{\mathrm{GC}}(\mathbb{G})$$

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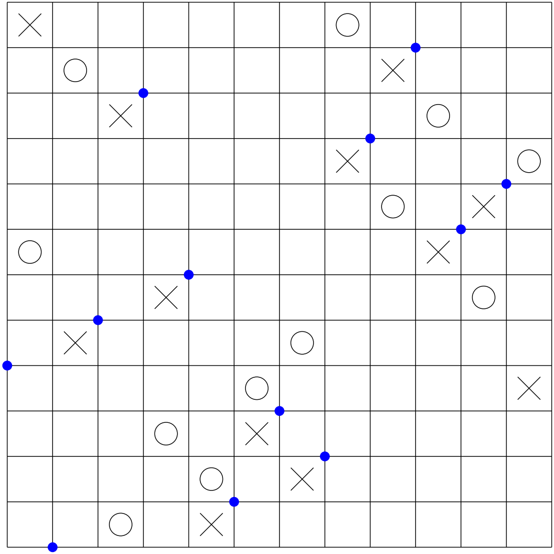

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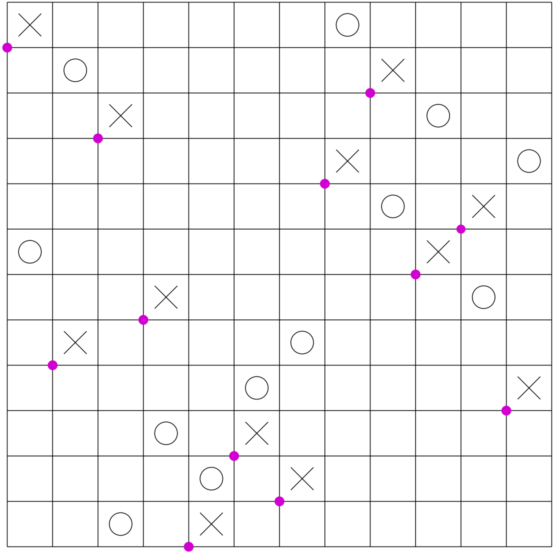

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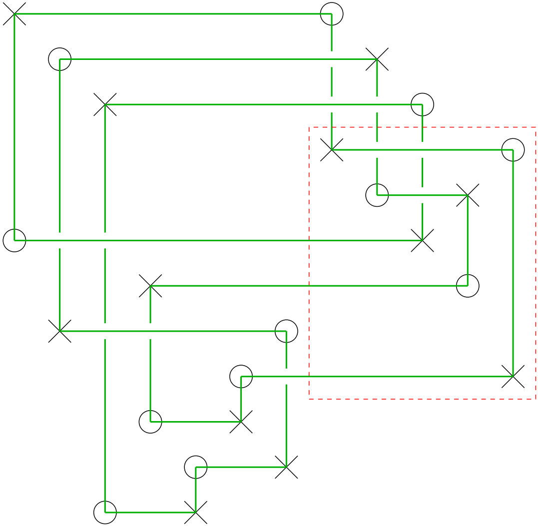
Legendrian GRID invariants

- (Ozsváth–Szabó–Thurston) $\widehat{\lambda}^+(\mathbb{G}) := [\mathbf{x}^+] \in \widehat{\mathrm{GH}}(\mathbb{G})$ and $\widehat{\lambda}^-(\mathbb{G}) := [\mathbf{x}^-] \in \widehat{\mathrm{GH}}(\mathbb{G})$ are invariants of the Legendrian link $\Lambda(\mathbb{G})$
  - In other words, if $\Lambda(\mathbb{G}_{\mathrm{bot}})$ and $\Lambda(\mathbb{G}_{\mathrm{top}})$ are related by Legendrian isotopies and Reidemeister moves, then the isomorphism $\Phi\colon \widehat{\mathrm{GH}}(\mathbb{G}_{\mathrm{top}}) \to \widehat{\mathrm{GH}}(\mathbb{G}_{\mathrm{bot}})$ sends $\widehat{\lambda}^\pm(\mathbb{G}_{\mathrm{top}})$ to $\widehat{\lambda}^\pm(\mathbb{G}_{\mathrm{bot}})$
- (Ng–Ozsváth–Thurston) $\widehat{\lambda}^\pm(\mathbb{G})$ are effective:
  - For example, $m(10_{132})$ has two Legendrian representatives with

$$tb(\Lambda_1) = tb(\Lambda_2), \quad r(\Lambda_1) = r(\Lambda_2)$$
$$\widehat{\lambda}^+(\Lambda_1) = 0, \quad \widehat{\lambda}^+(\Lambda_2) \neq 0$$
$$\widehat{\lambda}^-(\Lambda_1) \neq 0, \quad \widehat{\lambda}^-(\Lambda_2) = 0$$

- (Baldwin–Vela-Vick–Vértesi) GRID is equivalent to the LOSS invariants: $\widehat{\lambda}^\pm(\Lambda) \quad \leftrightarrow \quad \widehat{\mathfrak{L}}(\pm\Lambda) \in \widehat{\mathrm{HFK}}(-S^3, K)$

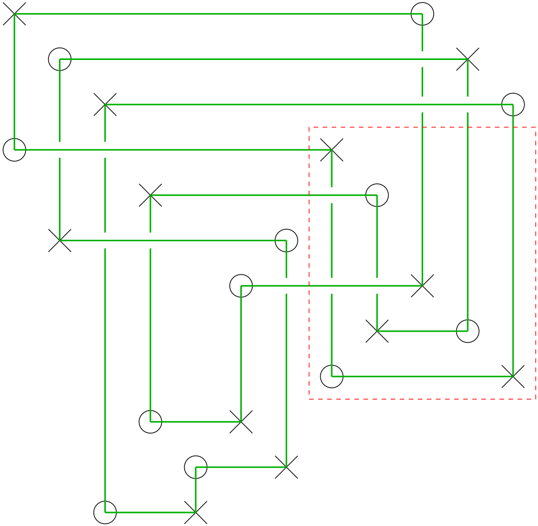
Legendrian GRID invariants



Λ_1 has smooth type

$m(10_{132})$

Legendrian GRID invariants



Λ_2 has smooth type

$m(10_{132})$

Statement of main result

Theorem (with Baldwin and Lidman; in preparation)

Suppose that there exists a decomposable Lagrangian cobordism from Λ_{bot} to Λ_{top} in $\mathbb{R} \times \mathbb{R}^3$. If $\Lambda(\mathbb{G}_{\mathrm{bot}}) = \Lambda_{\mathrm{bot}}$ and $\Lambda(\mathbb{G}_{\mathrm{top}}) = \Lambda_{\mathrm{top}}$, then there exists a homomorphism

$$\widehat{F} \colon \widehat{\mathrm{GH}}(\mathbb{G}_{\mathrm{top}}) \to \widehat{\mathrm{GH}}(\mathbb{G}_{\mathrm{bot}})$$

that sends $\widehat{\lambda}^{\pm}(\mathbb{G}_{\mathrm{top}})$ to $\widehat{\lambda}^{\pm}(\mathbb{G}_{\mathrm{bot}})$.

Corollary

Suppose that $\widehat{\lambda}^{\pm}(\mathbb{G}_{\mathrm{bot}}) \neq 0$ and $\widehat{\lambda}^{\pm}(\mathbb{G}_{\mathrm{top}}) = 0$. Then there is no decomposable Lagrangian cobordism from Λ_{bot} to Λ_{top}.

Proof.

By explicit construction/computation on the band move and death. □

Comparison with similar results

- (Golla–Juhász '18) The LOSS invariant obstructs regular Lagrangian concordances in a Weinstein cobordism from $(Y_{\text{bot}}, \xi_{\text{bot}})$ to $(Y_{\text{top}}, \xi_{\text{top}})$
  - The functorial map from $\widehat{\mathrm{HFK}}(-Y_{\text{top}}, K_{\text{top}})$ to $\widehat{\mathrm{HFK}}(-Y_{\text{bot}}, K_{\text{bot}})$
  - We expect (but do not prove) our map also to be the functorial map
- (Baldwin–Sivek '14) The monopole analogue of the LOSS invariant obstructs Lagrangian concordances in the symplectization $\mathbb{R} \times Y$
  - No assumption on decomposability or regularity!
  - (Baldwin–Sivek '16) The monopole LOSS invariant is equivalent to the LOSS invariant

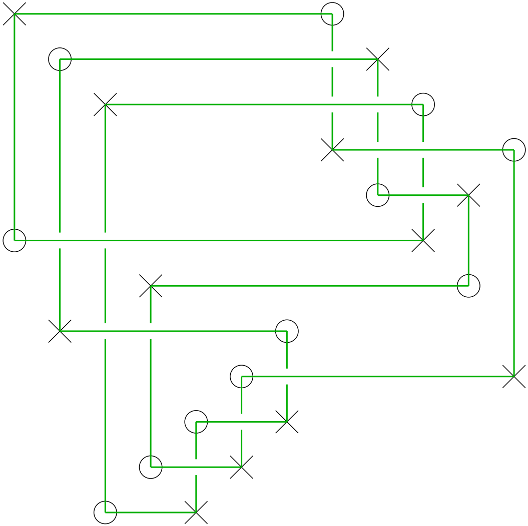
First example

- For Λ_1, Λ_2 as before, with smooth type $m(10_{132})$:

$$tb(\Lambda_1) = tb(\Lambda_2), \quad r(\Lambda_1) = r(\Lambda_2)$$
$$\widehat{\lambda}^+(\Lambda_1) = 0, \quad \widehat{\lambda}^+(\Lambda_2) \neq 0$$
$$\widehat{\lambda}^-(\Lambda_1) \neq 0, \quad \widehat{\lambda}^-(\Lambda_2) = 0$$

- There is no decomposable Lagrangian concordance
  - from Λ_2 to Λ_1 (obstructed by $\widehat{\lambda}^+(\Lambda_i)$)
  - from Λ_1 to Λ_2 (obstructed by $\widehat{\lambda}^-(\Lambda_i)$)
- First half of statement originally proven by Golla–Juhász
  - Obstructed by $\widehat{\mathcal{L}}(\Lambda_i) \quad \leftrightarrow \quad \widehat{\lambda}^+(\Lambda_i)$

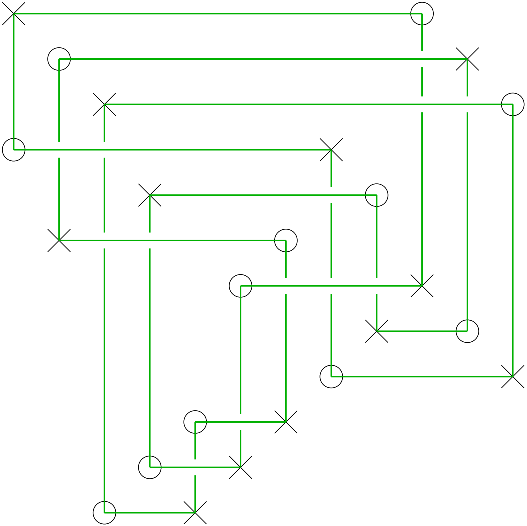
An example with positive genus



Λ_1' has smooth type

$m(12n_{199})$

An example with positive genus



Λ_2' has smooth type

$m(12n_{199})$

An example with positive genus

- Changing one of the new crossings reverts Λ_i' back to Λ_i
- So there exists a smooth cobordism of genus one between $m(10_{132})$ and $m(12n_{199})$
- For $i, j \in \{1, 2\}$,

$$tb(\Lambda_j') = tb(\Lambda_i) + 2, \quad r(\Lambda_j') = r(\Lambda_i)$$

$$\widehat{\lambda}^+(\Lambda_1) = 0, \quad \widehat{\lambda}^+(\Lambda_2) \neq 0; \qquad \widehat{\lambda}^+(\Lambda_1') = 0, \quad \widehat{\lambda}^+(\Lambda_2') \neq 0$$

$$\widehat{\lambda}^-(\Lambda_1) \neq 0, \quad \widehat{\lambda}^-(\Lambda_2) = 0; \qquad \widehat{\lambda}^-(\Lambda_1') \neq 0, \quad \widehat{\lambda}^-(\Lambda_2') = 0$$

- Any Lagrangian cobordism from Λ_i to Λ_j' has genus one
- There is no decomposable Lagrangian cobordism of genus one
  - from Λ_2 to Λ_1' (obstructed by $\widehat{\lambda}^+$)
  - from Λ_1 to Λ_2' (obstructed by $\widehat{\lambda}^-$)

An infinite family

- (Chongchitmate–Ng) In the smooth knot types $m(10_{145})$, $m(10_{161})$, and $12n_{591}$, there exist Legendrian representatives Λ_3, Λ_4, such that

$$tb(\Lambda_3) = tb(\Lambda_4) + 2, \quad r(\Lambda_3) = r(\Lambda_4)$$
$$\widehat{\lambda}^+(\Lambda_4) \neq 0, \quad \widehat{\lambda}^-(\Lambda_4) \neq 0$$

- The double stabilization $S_+(S_-(\Lambda_3))$ has same tb and r as Λ_4, but

$$\widehat{\lambda}^+(S_+(S_-(\Lambda_3))) = 0, \quad \widehat{\lambda}^-(S_+(S_-(\Lambda_3))) = 0,$$

  so there is no decomposable Lagrangian concordance from Λ_4 to $S_+(S_-(\Lambda_3))$
- Combining with the "clasping" trick before, we get an infinite family of examples with arbitrary genus

Summary and Outlook

- Lagrangian cobordisms are heavily restricted by tb and r, but these classical invariants are not complete
- The GRID invariants provide further obstructions for decomposable Lagrangian cobordisms in $\mathbb{R} \times \mathbb{R}^3$
- Questions:
  - Does grid homology satisfy naturality? If so, can we harness it in the context of Lagrangian cobordisms?
  - Could the GRID invariants provide an example of a non-decomposable Lagrangian cobordism?
  - Do the branched GRID invariants (joint with Vela-Vick) provide more obstructions?

Thank you!