An unoriented skein exact triangle for tangle Floer homology

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Outline

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   - Skein exact triangles
2. Result
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Background: Heegaard and knot Floer homology

- Heegaard Floer homology (Ozsváth–Szabó):

$$Y \text{ 3-manifold} \rightsquigarrow \widehat{\mathrm{HF}}(Y) \text{ abelian group}$$

$$W \text{ s.t. } \partial W = -Y_1 \sqcup Y_2 \rightsquigarrow F_W : \widehat{\mathrm{HF}}(Y_1) \to \widehat{\mathrm{HF}}(Y_2)$$

- Knot Floer homology (Ozsváth–Szabó, Rasmussen):

$$L \subset Y \text{ link} \rightsquigarrow \widehat{\mathrm{HFK}}(Y, L) \quad \text{bigraded } \mathbb{F} \text{ module}$$

- Both are defined by counting pseudo-holomorphic curves on a symmetric product of a Heegaard diagram

Background: Heegaard and knot Floer homology

$\widehat{\mathrm{HF}}(Y)$ and $\widehat{\mathrm{HFK}}(Y, L)$ have rich applications:

- $\widehat{\mathrm{HF}}(Y)$ detects the Thurston norm of Y (O–Sz, Juhász)
- $\widehat{\mathrm{HF}}(Y)$ detects the fiberedness of Y (O–Sz, Ghiggini–Ni, Juhász)
- $\widehat{\mathrm{HFK}}(Y, K)$ detects the genus of a knot K (O–Sz, Ni)
- $\widehat{\mathrm{HFK}}(Y, K)$ detects the fiberedness of K (O–Sz, Ghiggini–Ni, Juhász)
- $\widehat{\mathrm{HFK}}(S^3, K)$ categorifies the Alexander polynomial of K (O–Sz)
- Knot Floer homology gives concordance invariants (O–Sz, Ozsváth–Stipsicz–Szabó, Hom)
- Relations to Khovanov homology (O–Sz, Grigsby–Wehrli)

Background: Combinatorialization

- $\widehat{\mathrm{HF}}(Y)$ and $\widehat{\mathrm{HFK}}(Y, L)$ have been combinatorialized using *nice* Heegaard diagrams (Sarkar–Wang)
- In particular, this has given rise to *grid homology* $\widehat{\mathrm{GH}}(L)$ of a link $L \subset S^3$, defined by counting actual empty rectangles in a grid diagram of L (Manolescu–Ozsváth–Sarkar, Manolescu–Ozsváth–Szabó–Thurston)
- No pseudo-holomorphic curves involved
- Grid diagrams can be thought of as special genus-1 Heegaard diagrams

Background: Bordered Floer homology

- Bordered Floer homology (Lipshitz–Ozsváth–Thurston):

$$F \text{ surface} \rightsquigarrow \mathcal{A}(F) \text{ dg algebra over } \mathbb{F}_2$$

$$Y \text{ bordered 3-manifold} \rightsquigarrow \text{a } (\mathcal{A}(\partial^0 Y), \mathcal{A}(\partial^1 Y))\text{-bimodule } \widehat{\mathrm{CF}}(Y)$$

- If $Y = Y_0 \cup \cdots \cup Y_n$, then the Heegaard Floer homology of Y can be recovered by $\widehat{\mathrm{CF}}(Y_0) \otimes \cdots \otimes \widehat{\mathrm{CF}}(Y_n)$
- Bordered Floer homology turns Heegaard Floer homology into a (2+1+1)-TQFT

Background: Tangle Floer homology

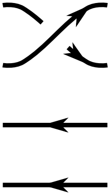
- Tangle Floer homology (Petkova–Vértesi):

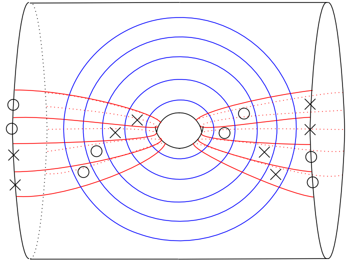
$$\begin{aligned} P \text{ a sequence of points} &\rightsquigarrow \mathcal{A}(P) \text{ dg algebra over } \mathbb{F}_2 \\ T \text{ an } (m,n)\text{-tangle in } M, &\rightsquigarrow \text{a } (\mathcal{A}(\partial^0 T), \mathcal{A}(\partial^1 T))\text{-bimodule} \\ \text{with } \partial M = S^2 \sqcup S^2 &\quad\;\; \widetilde{\mathrm{CT}}(T) \end{aligned}$$

- If $L = T_0 \circ \cdots \circ T_n$, then the knot Floer homology of L can be recovered by $\widetilde{\mathrm{CT}}(T_0) \otimes \cdots \otimes \widetilde{\mathrm{CT}}(T_n)$
- Tangle Floer homology turns knot Floer homology into an embedded (0+1)-TQFT
- For $T \subset S^2 \times I$, can use nice Heegaard diagrams for each elementary tangle in T; these look similar to grid homology
- Decategorifies to the Reshetikhin–Turaev invariant for $U_q(\mathfrak{gl}(1|1))$ (Ellis–Petkova–Vértesi)

Tangle Floer homology: More details

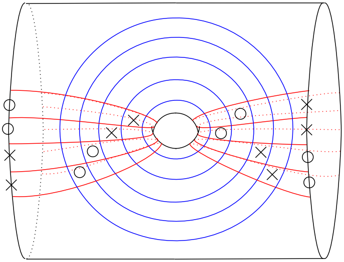
- In our context, an (m, n)-*tangle* is a 1-dimensional cobordism in $S^2 \times I$ between two finite sets of points $\{p_1, \ldots, p_m\} \times \{0\}$ and $\{q_1, \ldots, q_n\} \times \{1\}$





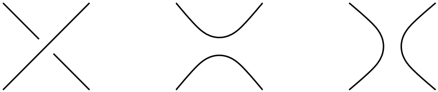
- To a tangle we associate a Heegaard diagram with two boundary components

Tangle Floer homology: More details



- Generators of $\widetilde{\mathrm{CT}}(T)$ are one-to-one correspondences between α and β curves
- Differential counts empty rectangles; those that touch the boundary contribute to the algebra actions

Unoriented skein exact triangle for knot Floer homology



Theorem (Manolescu)

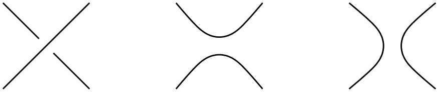
Over $\mathbb{F}_2$, if L_∞, L_0, L_1 are links in S^3 identical except near a point as shown, then there exists a skein exact triangle

$$\widehat{\mathrm{HFK}}(L_\infty) \otimes V^{m-\ell_\infty} \to \widehat{\mathrm{HFK}}(L_0) \otimes V^{m-\ell_0} \to \widehat{\mathrm{HFK}}(L_1) \otimes V^{m-\ell_1} \to \ldots$$

where ℓ_k is the number of components of L_k, $m = \max\{\ell_k\}$, and V is 2-dimensional.

- Proof by counting pseudo-holomorphic polygons on a special Heegaard diagram

Unoriented skein exact triangle for knot Floer homology



Theorem (W)

Over $\mathbb{Z}$ (and hence any commutative ring), for L_∞, L_0, L_1 as before, there exists a skein exact triangle

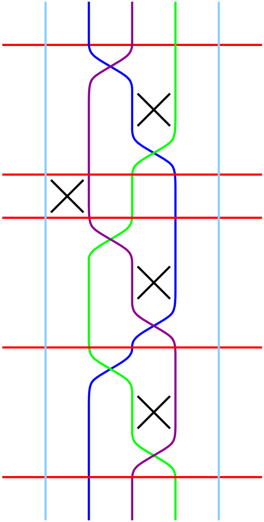
$$\widehat{\mathrm{GH}}(L_\infty) \otimes V^{m-\ell_\infty} \to \widehat{\mathrm{GH}}(L_0) \otimes V^{m-\ell_0} \to \widehat{\mathrm{GH}}(L_1) \otimes V^{m-\ell_1} \to \dots ,$$

where ℓ_k and V are as before and m is sufficiently large.

- Over $\mathbb{F}_2$, implied by Manolescu's result; extension to over $\mathbb{Z}$
- Combinatorial proof by counting actual pentagons and triangles on a combined grid diagram

Unoriented skein exact triangle for knot Floer homology

- Combinatorial proof by counting actual pentagons and triangles on a combined grid diagram



Statement

- Does an unoriented skein exact triangle exist for tangle Floer homology?

Theorem (Petkova–W)

If T_∞, T_0, T_1 are tangles in $S^2 \times I$ identical except near a point as shown before, then there exists a module homomorphism $f_0 \colon \widetilde{\mathrm{CT}}(T_0) \to \widetilde{\mathrm{CT}}(T_1)$ such that

$$\widetilde{\mathrm{CT}}(T_\infty) \cong \mathrm{Cone}(f_0 \colon \widetilde{\mathrm{CT}}(T_0) \to \widetilde{\mathrm{CT}}(T_1)),$$

where $\cong$ denotes quasi-isomorphism, and $\mathrm{Cone}(f)$ the mapping cone of f.

- Note: For modules, statement in homology does not make sense

Statement with δ-grading

- For both $\widehat{\mathrm{HFK}}(L)$ and $\widehat{\mathrm{GH}}(L)$, Maslov grading M and Alexander grading A are not respected by skein triangle
- Skein triangle for $\widehat{\mathrm{HFK}}(L)$ respects δ-grading, where $\delta = M - A$ (Manolescu–Ozsváth)
- Skein triangle for $\widehat{\mathrm{GH}}(L)$ respects δ-grading (W)

Theorem (Petkova–W)

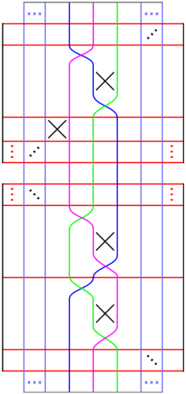
With respect to the δ-grading, $f_0 \colon \widetilde{\mathrm{CT}}(T_0) \to \widetilde{\mathrm{CT}}(T_1)$ is of degree $(e_0 - 1)/2$, and

$$\widetilde{\mathrm{CT}}(T_\infty) \cong \mathrm{Cone}(f_0 \colon \widetilde{\mathrm{CT}}(T_0) \to \widetilde{\mathrm{CT}}(T_1)) \left[\frac{e_1 - 1}{2}\right],$$

where e_0 is the difference of the number of negative crossings in T_1 and T_0, and e_1 that in T_∞ and T_1.

Sketch of proof

- Idea: Combine three Heegaard diagrams into one, and cut open to obtain



- Define homomorphisms between modules by counting pentagons and triangles
- Form the stated mapping cone
- Define homotopy morphisms by counting hexagons, quadrilaterals and heptagons
- Use a homological algebra lemma
- Similar to the closed case, but now have to take care of polygons touching the boundary, interacting with the algebras

Summary and Outlook

- Combinatorial tangle Floer homology is defined using grid-like Heegaard diagrams
- The proof of an unoriented skein exact triangle for grid homology carries over to tangle Floer homology, with the major difference being the algebra actions
- Skein relation for tangle Floer homology respects δ-grading
- In similar spirit:

Theorem (Petkova–W; work in progress)

If T_+, T_-, T_0 are oriented tangles in $S^2 \times I$ identical except near a point, then there exists a module homomorphism $P_{+,-}\colon \mathrm{CT}^-(T_+) \to \mathrm{CT}^-(T_-)$ such that

$$\mathrm{Cone}(P_{+,-}) \cong \mathrm{Cone}(U_1 - U_2\colon \mathrm{CT}^-(T_0) \to \mathrm{CT}^-(T_0)),$$