These are just some comments, not the entire lectures! You are not going to see graphs, cobwebbing here... Please come to the lectures to see them...

Comments on lecture 1

1.5 DISCRETE-TIME DYNAMICAL SYSTEMS

The source for this comments is your textbook!!!!

Part I.

DEF: • A discrete-time dynamical system describes the relation between a quantity measured at the beginning and at the end of an experiment OR a time interval;

• If the measurement is represented (or denoted) by (the variable m), then m_t denotes the measurement at the beginning of the experiment and m_{t+1} denotes the measurement at the end of the experiment;

• The relation between m_t and m_{t+1} is given by the DISCRETE-TIME DYNAMICAL SYSTEM: $m_{t+1} = f(m_t)$, where f is called the UPDATING FUNCTION.

Example 0.0.1. ***Bacterial population:*** *Read example 1.5.1/page 53*

Example 0.0.2. ***Tree Growth:*** *Read example 1.5.2/page 54*

Example 0.0.3. ***Medication concentration:*** *Read example 1.5.4/page 55*

Part II.

Dealing with Updating Functions (from the point of view of Algebra)

a) Consider the DTDS: $m_{t+1} = f(m_t)$.

Question: What does the COMPOSITION $f \circ f$ mean?

This: $(f \circ f)(m_t) = f(f(m_t)) = f(m_{t+1}) = m_{t+2}$, SO: the COMPOSITION of an Updating Function with itself corresponds to a 2-step updating function.

Example 0.0.4. ***Bacterial population:*** *Consider the bacterial population; we have $(f \circ f)(b_t) = f(f(b_t)) = f(2b_t) = 2 \times (2b_t) = 4b_t$, i.e., after 2 hours the beautiful population is 4 times BIGGER!*

b) What other parts of Algebra may we use? INVERSEs! Consider the DTDS: $m_{t+1} = f(m_t)$.

Question: What does the inverse f^{-1} mean?

THIS: applying f^{-1} to our relation one gets: $f^{-1}(m_{t+1}) = m_t$, that can be viewed as: $m_t = f^{-1}(m_{t+1})$, SO, the INVERSE of an Updating Function corresponds to an

$$UPDATING \quad Function$$

that goes backwards in time!

Example 0.0.5. ***Bacterial population:*** *Consider the bacterial population; we do have: $b_{t+1} = 2b_t = f(b_t)$. Solve for the inverse:*

$f(b_t) = y$ implies that $2b_t = y$, which in turn implies that $b_t = \frac{y}{2}$, hence $f^{-1}(b_t) = \frac{b_t}{2}$. We have a new DTDS $b_t = f^{-1}(b_{t+1})$, that can be written as: $b_t = \frac{b_{t+1}}{2}$.

Part III. **Solutions**

• Recall that a DTDS describes **some** quantity at the end of an experiment/process/measurement as a function of the **same** quantity at the beginning!

• **Question:** What if we were to continue the process/experiment? Think about the bacterial population! The population will double again, again, and again.....

• To describe a process that is repeated many times we let:

— m_0 = measurement at the beginning;

— m_1 = measurement after one time step;

— m_2 = measurement after 2 time steps;

— ...

— m_t = measurement after t seconds/years/hours/days (or whatever unit one may use) after the beginning of the process/experiment

• **DEFINITION** The SOLUTION of the DTDS: $m_{t+1} = f(m_t)$ is the sequence of values of m_t for $t = 0, 1, 2, 3, \ldots$, STARTING from the INITIAL CONDITION m_0.

NB: We know where we started the process!

• The **GRAPH** of a Solution is a discrete set of points: the time t on the x-axis; m_t on the y-axis:

$(0, m_0), (1, m_1), (2, m_2), \ldots$

• **Example** Consider the bacterial population: $b_{t+1} = 2b_t$ WITH $b_0 = 1.0$ (in millions). We do have: $b_1 = 2b_0 = 2 \times 1 = 2$; $b_2 = 2b_1 = 2 \times 2 = 4$, $b_3 = 2b_2 = 2 \times 4 = 8$, etc...

BUT: $b_1 = 2b_0$; $b_2 = 2b_1 = 2 \times 2 \times b_0 = 2^2 b_0$, $b_3 = 2b_2 = 2 \times 2^2 \times b_0 = 2^3 b_0$, so $b_{t+1} = 2^{t+1} b_0$ — the population after $t + 1$ hours; of course the population after t hours is $b_t = 2^t b_0$

Graph it!

Do: 14, 24, 20 on 64-65!

THE LOCATION and DATE OF DIAGNOSTIC TEST Were CHANGED!

GO TO THE WEB PAGE AND READ THE PIECES OF INFORMATION GIVEN THERE!

Did you do 14, 24, 20 on 64-65? Please attend the **dgds**!

Comments on lecture 2

When we are unable to find a SOLUTION, we may want to deduce the behavior of the solution!

COBWEBBING IS A GRAPHICAL METHOD TO SKETCH solutions.

We shall talk about EQUILIBRIA — points where the DTDS does not change!

I — COBWEBBING

CONSIDER THE DTDS $m_{t+1} = f(m_t)$

ADD THE DIAGONAL — i.e., $m_{t+1} = m_t$

START from the initial condition m_0 (that is given in the statement)

GET m_1 — recall that $m_1 = f(m_0)$

HOW do we get m_2 We need m_1 on the x-axis!

MOVE (m_0, m_1) horizontally until it cuts the diagonal- SO we do not change the vertical coordinate!

MOVE DOWN and get $(m_1, 0)$

MOVE UP (vertically) to the graph of the UPDATING function to get m_1, m_2

MOVE m_1, m_2 horizontally until it cuts the diagonal

MOVE VERTICALLY to the graph of the UPDATING function to get m_2, m_3

GO ON

SKETCH the GRAPH OF THE solution!

Example: do 10/76

ALGORITHM (for COBWEBBING)

1) Graph the Updating Function and the diagonal

2) Start from the initial condition and GO to the updating function and OVER the diagonal

3) Repeat going UP or DOWN to the updating function and over the diagonal as many times as necessary

4) Sketch solution ($t = 0, 1, 2, \dots$)

II — EQUILIBRIA

— POINTS where the graph of the updating function intersects the diagonal play an important role!

— IF we start cobwebbing from an initial condition where the graph of the updating function LIES BELOW the diagonal, the Solution INCREASES!

— IF we start cobwebbing from an initial condition where the graph of the updating function LIES ABOVE the diagonal, the Solution DECREASES!

QUESTION: What about the points where the updating function cuts the diagonal?

ANSWER: The solution neither increases or decreases, it remains the same.

DEFINITION: A point m^* is called an equilibrium of DTDS $m_{t+1} = f(m_t)$ if $m^* = f(m^*)$. The set of all equilibrium points is called equilibria.

SO: if we start cobwebbing from an initial condition that is an equilibrium point, you don't get much...

III — EQUILIBRIA — algebraic method

Example (do the bacterial population)

Do 18/77

ALGORITHM (for finding algebraically the equilibria):

1) Write the equation $m = f(m)$

2) Move all terms in one side

3) FACTOR

4) SOLVE (by setting each factor equal to 0)

5) Interpret your results!

Example: do 30, 34, 32/page 77

If time do 22, 20/77 or 31.

IV — General solution of LINEAR DTDS: $m_{t+1} = am_t + b = f(m_t)$ where m_0 is the initial condition is found as follows:

m_0

$m_1 = am_0 + b$

$m_2 = am_1 = a(am_0 + b) + b = a^2m_0 + b(a + 1)$

$m_3 = am_2 = a \times \{a^2m_0 + b(a + 1)\} + b = a^3m_0 + b(a^2 + a^1 + 1)$

so on ...

$m_t = a^tm_0 + b(a^{t-1} + a^{t-2} + \cdots + a + 1)$

NOW:

case 1: $a \neq 1$ gives us: $m_t = a^tm_0 + b\frac{a^t-1}{a-1}$

case 2: $a = 1$ gives us: $m_t = m_0 + bt$.

Comments on lecture 3

3.1 Stability

Recall: an equilibrium point is a point where our given DTDS leaves the value unchanged (the same), i.e., a point where the diagonal intersects (cuts) the Updating function.

To do More Classification we introduce the following:

DEFINITION a) An equilibrium (point) is STABLE if solutions that start NEAR the equilibrium point MOVE closer to the equilibrium.

b) An equilibrium (point) is UNSTABLE if solutions that start NEAR the equilibrium point MOVE AWAY from to the equilibrium.

Example 1 Bacterial Population $b_{t+1} = 2b_t = f(b_t)$

Sol: The equilibrium is 0, and it is UNSTABLE!

Do yourself the cobwebbing...

Example 2 More Complicated Bacterial Population $p_{t+1} = \frac{2p_t}{2p_t+1.5(1-p_t)}$

Sol: Find first the equilibria! $p = \frac{2p}{2p+1.5(1-p)} \Rightarrow p\{2p + 1.5(1-p)\} = 2p \Rightarrow p\{2p + 1.5(1-p) - 2\} = 0$. Hence $p(p-1) = 0$, we are getting then either $p = 0$ or $p = 1$.

Can we DRAW $f(p_t) = \frac{2p_t}{2p_t+1.5(1-p_t)}$?? Then, compare $f(p_t)$ (the Updating function) and $g(p_t) = p_t$ the diagonal!!

$\frac{2p_t}{2p_t+1.5(1-p_t)} - p_t = \frac{2p_t - p_t 2p_t + 1.5(1-p_t)}{2p_t+1.5(1-p_t)} = \frac{p_t(0.5)(1-p_t)}{2p_t+1.5(1-p_t)} \geq 0$ ON $[0,1]$. Hence on this interval the updating function is bigger than the diagonal!

Do the cobwebbing and get yourself that 0 is unstable, and 1 is stable!

Let us record our findings:

Graphical Criterion: a) An equilibrium is STABLE if the graph of the Updating function CROSSES the Diagonal from Above to Below

b) An equilibrium is UNSTABLE if the graph of the Updating function CROSSES the Diagonal from Below to Above.

Example 3 A Linear DTDS $c_{t+1} = 0.75c_t + 1.25 = f(c_t)$

Sol: Algebra gives us: $c = 0.75c + 1.25$ that $c = 1.25/0.25 = 5$. DRAW now the 2 LINES, do the cobwebbing (see the stable equilibrium point) and get what the criterion tells us: STABLE!

Example 4 One more... 29/249 $b_{t+1} = 1.5b_t - 10^6$

Sol: One more time the algebra gives us: $b = 2 \times 10^6$; DRAW now the 2 LINES, do the cobwebbing and get UNSTABLE: by cobwebbing or by criterion!

NOTE: There are situations where the graphical criterion can nit be applied!!!

3.2 More complicated D.T.D. Systems

$I --- LOGISTICDTDS$ (READ STORY ON PAGES 250 AND 251) $x_{t+1} = rx_t(1 - x_t) = f(x_t)$ where r is a parameter.

FIND equilibria as follows: $x = rx(1-x) \Rightarrow x\{1 - r(1-x)\} = 0 \Rightarrow x\{1 - r + rx\} = 0$. We get either $x = 0$ or $x = 1 - \frac{1}{r}$. Note that: $x = 0$ means extinction of population. Note that: r in $(0,1)$ means the second equilibrium point is NOT biologically meaningful.

$II --- LinearDTDS$

Exercises: Find equilibria for 16/262;

Do 14/262;

If time do 37a and 38a from page 249!

Comments on lecture 4

Section 1.7 EXP and LOG functions

Example (new bacterial population) $b_{t+1} = rb_t$ (compare to $b_{t+1} = 2b_t$)

Previous example: 1 bacterium divided in 2 sons, and both survived!

Now: only a part of "sons" survived! So, consider the DTDS: $b_{t+1} = rb_t$, with initial condition b_0. We get $b_1 = rb_0$, $b_2 = r^2b_0$, $b_3 = r^3b_0, \dots b_t = r^tb_0$.

Of course: — If $r > 1$, the population INCREASES

— If $r = 1$, the population remains constant

— If $r < 1$, the population DECREASES

So, we are led to:

DEF: The function $f(x) = a^x$, where $a > 0$ is called the exponential function to the base a. NOTE that $f : \mathbf{R} \mapsto \mathbf{R}_+$.

LAWS: $a^xa^y = a^{xy}$, $\frac{a^x}{a^y} = a^{x-y}$; $a^{-x} = \frac{1}{a^x}$, $(a^x)^y = a^{xy}$.

DO yourself the graph in both cases (a bigger or less than 1)!

IMPORTANT

DEF The number e is the positive number a such that the SLOPE of the tangent line at $f(x) = a^x$ AT $(0, 1)$ is exactly 1. FOR US: $e \cong 2.71828$.

NOTE that $f(x) = a^x$ IS one-to-one, so it has an inverse.

DEF: The INVERSE of $f(x) = a^x$ IS denoted by $\log_a : \mathbf{R}_+ \mapsto \mathbf{R}$.

NOTATION: When $f(x) = e^x$, the inverse is denoted $\log_e = \ln$, the natural logarithm.

Note: $\log_a x = y \Leftrightarrow a^y = x$; $a^{\log_a x} = x$ AND $\log_a(a^x) = x$.

LAWS of LOGs: $\ln(xy) = \ln(x) + \ln(y)$; $\ln(\frac{x}{y}) = \ln(x) - \ln(y)$; $\ln(x^r) = r\ln(x)$; $\ln(1) = 0$; $\ln(e) = 1$; $\ln(\frac{x}{y}) = -\ln(x)$.

EXP: (Bacterial population, revised): $b_{t+1} = rb_t$ with initial condition b_0. Sol: $b_t = r^tb_0 = (e^{\ln(r)})^tb_0 = e^{t\ln(r)}b_0$ — exponential notation using e.

EXC: 24/90 $4e^{2x+1} = 20$ implies that $e^{2x+1} = 5$, so $2x + 1 = \ln 5$, hence $x = \frac{\ln(5)-1}{2} = \dots$

EXC: 28/90; 34, 47-48-49/90.

Section 1.8 Oscillations and TRiGoNoMetry

I Think about cycles: heartbeats, seasons... We use TRIG functions for oscillations.

Cosine, Sine Review

— Angles are measured in RADIANS. The conversion is:

$2\pi rad = 360 degrees$; or $\pi rad = 180 degrees$.

So: $1rad = \frac{360}{2\pi} degrees$ AND $1degree = \frac{2\pi}{360} rad$.

Important values of cosine and sine are in the book: page 92.

Note: SINE and COSINE are periodic: 2π, i.e., $\cos(x+2\pi) = \cos x$ and $\sin(x+2\pi) = \sin x$

Do yourself their graphs!

II DESCRIBING OSCILLATIONS WITH COS

— a measurement is said to oscillate as a function of time if the values vary regularly between High and Low values.

— the shape of sine/cosine is called $SINUSOIDAL$.

— THE 4 PARAMETERS (or numbers) that describe an oscillation are:

1) The average LIES halfway between the Max and Min values.

2) The amplitude = difference between Max and average (or the difference between average and Min)

3) The period = the time between succesive peaks

4) The Phase = the time of the **first** peak.

The Model is:

$f(t) = A + B\cos(\frac{2\pi}{T}(t - \phi))$

NOW graph IT!

DO: 33/98; 34/98; 36/98.

Comments on lecture 5

2.2 LIMITS

Part I LIMITS

Definition: Let f be a function, let a be a number. We write $\lim_{x\to a} f(x) = L$ and say the limit of $f(x)$, when x approaches a, is L if we can make the values of $f(x)$ arbitrarily close to L, by taking x sufficiently close to a, but not a.

NOTE: f MAY not be defined in a.

So: for every $\epsilon > 0$ there is a $\delta > 0$ such that $0 < |x-a| < \delta$ IMPLIES that $|f(x)-L| < \epsilon$.

DO: 16/153 a) and b)

Question: HOW can we compute limits in general?

(One) Answer: use a calculator, plug in, see the patern, then.. guess!

Do: 2/153

Do: 6/153

Do: 3/153

In your assignments TRY at least 10 values!

Part II Left and Right LIMITS

Some functions are defined inly on one side of a point!

EXP: $f(x) = \ln(x)$, $f : (0, \infty) \mapsto \mathbf{R}$

Question: HOW can we compute the limit in this case?

DEF: We write $\lim_{x\to a^-} f(x) = L$ if we can make the if we can make the values of $f(x)$ arbitrarly close to L, by taking x sufficiently close to a with $x < a$. We call it the LEFT LIMIT!

DEF: We write $\lim_{x\to a^+} f(x) = L$ if we can make the if we can make the values of $f(x)$ arbitrarly close to L, by taking x sufficiently close to a with $x > a$. We call it the RIGHT LIMIT!

Do EXC 8/153

Do: 21-22-24/153

RULES: $\lim_{x\to a} c = c$; $\lim_{x\to a} x = a$;

$\lim_{x\to a}\{f(x) + g(x)\} = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$;

$\lim_{x\to a}\{f(x) - g(x)\} = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$;

$\lim_{x\to a}\{f(x)g(x)\} = \{\lim_{x\to a} f(x)\}\{\lim_{x\to a} f(x)\}$

$\lim_{x\to a} cf(x) = c\{\lim_{x\to a} f(x)\}$

$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} f(x)}$ if $\lim_{x\to a} g(x) \neq 0$.

Part II INFINITE LIMITS

DEF: We write $\lim_{x\to a} f(x) = +\infty$ if we can make the values of $f(x)$ arbitrarly positive large by taking x sufficiently close to a, BUT not a.

DEF: We write $\lim_{x\to a} f(x) = -\infty$ if we can make the values of $f(x)$ arbitrarly negative large by taking x sufficiently close to a, BUT not a.

DO: 18/153

Do: 20/153

2.3 Continuity

DEF: A function is continuous at a if $\lim_{x\to a} f(x) = f(a)$. Otherwise we say f is discontinuous at a.

NOTE: a IS IN THE domain of f.

DEF: A function is called continuous if it is continuous at every point in its domain.

IDEA: You may draw it without lifting the pencil!!!

BASIC continuous functions: $f(x) = c$, $f(x) = x$, $f(x) = mx+b$, $f(x) = e^x$, $f(x) = \log_a x$, $f(x) = \cos x$, $f(x) = \sin x$.

Combinations

Consequence: Any polynomial IS continuous!

DO: 1-10/163, 21-22/163

Comments on lecture 6

2.1 Derivatives

Part I The Average rate of change

Consider the bacterial population $b_{t+1} = 2b_t$, $b_0 = 1$, $b_t = 2^t$.

Goal: Describe the growth of this population (or HOW the population is changing in time?)

We define the average rate of Change $= \frac{change \quad in \quad Population}{change \quad in \quad time} = \frac{\Delta b}{\Delta t}$.

Question: What is the average rate of change from the point of view of Geometry?

Think about the SLOPE of the secant LINE.

GENERAL FORMULA: Consider the DTDS $m_{t+1} = f(m_t)$, with the initial condition m_0. The average change of rate between a abd b is given by $\frac{f(b)-f(a)}{b-a}$.

The equation of the secant LINE passing through the points $(x_0, f(x_0))$ $(x_1, f(x_1))$ and is:

$y = \frac{f(x_1)-f(x_0)}{x_1-x_0}(x - x_0) + f(x_0)$

DO: 6/142

Part II Instantaneous rate of change

Recall $b_{t+1} = 2b_t$, $b_0 = 1$, $b_t = 2^t$. Take base point !.

Question: What is happening at 1?

One may compute the average rates of changes between:

1 and 2, and get $\frac{2^2-2^1}{2-1} = 2$;

1 and 1.5, and get $\frac{2^{1.5}-2^1}{1.5-1} = 1.6568$;

1 and 1.1, and get $\frac{2^{1.1}-2^1}{1.1-1} = 1.4354$;

1 and 1.01, and get $\frac{2^{1.01}-2^1}{1.01-1} = 1.3931$;

GO ON... continue...

IF one gets closer to 1, then the values (A.R. of Change) are getting closer and closer to 1.386.

THIS is what is called the INSTANTANEOUS RATE OF CHANGE AT $t = 1$.

Question: What is it in fact?

Answer: the slope of the tangent line to f at point 1.

Question: What is a tangent LINE?

Answer: a LINE that touches the graph in 1 point, but it does NOT cross the graph!

NOTE: The slopes of the secants are getting closer to the slope of the tangent LINE!

Part III DERIVATIVES

DEFINITION: The Instantaneous Rate of Change of a function f at $t = t_0$ is called the DERIVATIVE of f at $t = t_0$ AND is given by $f'(t_0) = \lim_{\Delta t \to 0} \frac{f(t_0+\Delta t)-f(\Delta t)}{\Delta t}$.

Other Notation: $f'(t_0) = \frac{df}{dt}(t_0)$

DEFINITION: The slope of the GRAPH of a function is equal to the slope of the tangent line to the graph, which is isself equal to the derivative of the function.

In other words: the derivative measures how rapidly a measurement is changing at a given time.

DO: 14, 16/143

MORE: If $f(t) = 2t^3 + 1$, $g(t) = -2t^4 + 1$, $t_0 = 1$ find $f'(t_0)$ and $g'(t_0)$.

Comments on lecture 7

2.4 Computing some derivatives: linear AND quadratic functions

Recall: $\lim_{x \to a} f(x) = L$ IF AND ONLY IF $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$.

EXC 1: Find $\lim_{x \to 0} \frac{x}{|x|}$.

EXC 2: Use a calculator to guess $\lim_{x \to 0} \frac{e^{3x}-1}{x}$.

2.4 Computing derivatives

Def: A function is called differentiable at a if $f'(a)$ exists (IT IS FINITE). If a function is differentiable at each point in its domain we call it differentiable.

Def: Let $f : D \mapsto \mathbf{R}$ be a function. Let D' be the set of all points where f is differentiable. Define a new function

$f' : D' \mapsto R$ by $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$. Its name is: the derivative of f.

WHEN a function fails to be differentiable?

THINK of jumps, corners or where the graph is vertical...

DERIVATIVE of a linear function:

If $f(x) = mx + n$, compute $f'(x_0)$:

$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{m(x_0+\Delta x)+n-mx_0-n}{\Delta x} = \lim_{\Delta x \to 0} \frac{m\Delta x}{\Delta x} = \lim_{\Delta x \to 0} m = m$,

exactly what you expected....

Conclusion: If $m > 0$ then $f(x) = mx + n$ is increasing;

If $m < 0$ then $f(x) = mx + n$ is decreasing;

If $m = 0$ then $f(x)$ is constant.

THE SAME HOLDS IN GENERAL:

a) IF f' is positive (i.e., > 0) ON an interval, then f is increasing on that interval;

b) IF f' is negative (i.e., < 0) ON an interval, then f is decreasing on that interval;

c) IF $f'(x_0) = 0$ the function f is neither increasing nor decreasing.

DEF: A point x is called a CRITICAL POINT for f if either $f'(x) = 0$ OR $f'(x)$ does NOT EXIST!

DO 20/172 The BLUE curve = line; if it were the function then the derivative is a constant, which is not the case. So the BLACK curve is the function. Anyway: it maches! f is decreasing, and 2 is positive!

DO: 22/172 IF f' were the black curve, then $f' > 0$, so f is increasing (blue in the context): NOT the case. So, f' is the blue curve.

DERIVATIVE of a quadratic function:

If $f(x) = ax^2 + bx + c$ find $f'(x)$ as follows:

$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{a(x+\Delta x)^2+b(x+\Delta x)+c-ax^2-bx-c}{\Delta x} = \lim_{\Delta x \to 0} 2ax + a\Delta x + b = 2ax + b$ (please work out the simplifications yourself: the best practice for test(s))! SO:

$f'(x) = 2ax + b$ for any x.

DO 16/172 Answer: $\frac{-1}{4}$. Do the graph!

Do: 36/173

If time try: FIND $\lim_{x \to 2} \frac{x^2-4}{x-2}$.

Comments on lecture 8

2.5 DERIVATIVES OF SUMS, POWERS AND POLYNOMIALS

THEOREM (SUM RULE) $\{f(x)+g(x)\}' = f'(x)+g'(x)$. If one uses the other notation, then $\frac{df}{dx} + \frac{dg}{dx} = \frac{d(f+g)}{dx}$

Proof: $\{f(x) + g(x)\}' = \lim_{h \to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h \to 0} \frac{g(x+h)-g(x)}{h} = f'(x) + g'(x)$.

THEOREM (POWER RULE) $\{x^p\}' = px^{p-1}$, when $x > 0$.

To make you believe in this theorem please recall the binomial

$(x+y)^n = \cdots = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \cdots + y^n$

DO: $p = 0, p = 1, p = 2, p = 3, p = 4$ using ONLY the definition of a derivative!

THEOREM (CONSTANT PRODUCT RULE) If c is a constant, then $(cf(x))' = cf'(x)$.

So, from now on you may compute the derivative of any polynomial.

DO: 2-4-6-8 and 10-12/184 and 18-20/184

DO: 22-24-26-25-23/184

28/185, 29-30, 37/185

Definition A differential equation is an equation of the form $\frac{dy}{dt} = f(t)$, where f is given, and y is the unknown.

DO: 43-44/185. If Time do: 19/184.

Comments on lecture 9

WHAT IS INCLUDED IN TEST 1(80 MINUTES)? READ EVERYTHING FROM 1.5 TILL 2.8, INCLUDING 2.8.

Section 2.6 THE PRODUCT RULE

THE PRODUCT RULE: If f and g are differentiable, then the product fg IS differentiable. Moreover,

$(f(x)g(x))' = f'(x)g(x) + f(x)g(x)'$.

THE CONSTANT PRODUCT RULE: $(cf(x))' = c(f(x))'$.

OTHER NOTATIONS: $\frac{d(fg)}{dx} = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}$.

THE PRODUCT RULE: $(\frac{f(x)}{g(x)})' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$.

DO: 2,4,6/192

12,10,8/page 192

17/192

24/192

36/193

34/193

Recall that the density is given by $\rho(t) = \frac{M(t)}{V(t)}$.

Can you compute the derivative of e^{x+1}? Use a trick!

Section 2.8 THE derivatives of EXP AND LOG

Theorem: $(e^x)' = e^x$

Indeed, $(e^x)' = \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} = \lim_{h\to 0} \frac{e^x e^h - e^x}{h} = (e^x) \lim_{h\to 0} \frac{e^h - 1}{h} = e^x \times 1 = e^x$. You must recall the lecture on exponentials and the definition of the number e.....

Theorem: $(\ln(x))' = \frac{1}{x}$

The proof is going to be done later...

DO: 2,4,6,8,10,12,14, 16, 18/page 209

Can you compute $(\ln(2x))'$?

DO: 26/210

Question: what is $(a^x)'$?

Good Preparation:

Find the derivative of $f(x) = \frac{(2x+1)e^x}{\ln(x)} + e^{x+1}\ln(4x)$ (using rules and algebra)

and of $g(x) = 1 + 2x^2 + 3x^3$ (using the definition).

Find the limit $\lim_{x\to 2} \frac{x^4-16}{|2-x|}$ (no calculator!).

Comments on lecture 10

2.9 THE CHAIN RULE

THEOREM: Suppose that $F(x) = (f \circ g)(x) = f(g(x))$, where f, g ARE differentiable. Then $F'(x) = f'(g(x))g'(x)$.

There is a plan! See page 213.

DO: 2,4,6,8,10,12,14,16 on page 220.

Solution (16/220): $q(y) = y^y \Rightarrow \ln(q(y)) = \ln(y^y) = y\ln(y) \Rightarrow$ (by chain rule and other rules) $\frac{q'(y)}{q(y)} = \ln(y) + y\frac{1}{y}$, thus $q'(y) = q(y)\{\ln(y)+1\} = y^y\{\ln(y)+1\}$.

EXC: a) find the derivative of $f(x) = \frac{\ln(1+e^x)}{1+e^x} + x^2 e^{x^2}$.

b) find the derivative of $g(x) = x^3 e^{-x^2} - \frac{2+e^x}{\ln(1+e^x)}$.

Theorem: Suppose that f is differentiable and that f^{-1} is its inverse. If $f'(f^{-1}(x)) \neq 0$, then $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$.

Proof: $(f \circ f^{-1})(x) = x \Rightarrow f'(f^{-1})(x))(f^{-1})'(x) = 1$, so $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$.

DO: 26,28,30/221

EXC: 30/221 $N(x) = e^{x^2}$, with $x \geq 0$. Solve: $e^{x^2} = y$ as follows: $x^2 = \ln(y)$, hence $x = \sqrt{\ln(y)}$, in other words $N^{-1}(x) = \sqrt{\ln(x)}$. By the theorem above one gets:

$(N^{-1})'(x) = \frac{1}{N'(N^{-1}(x))}$. Note that $N'(x) = e^{x^2}2x$ by chaain rule. So: $N'(N^{-1}(x)) = N'(\sqrt{\ln(x)}) = e^{(\sqrt{\ln(x)})^2} 2\sqrt{\ln(x)}$. Therefore $(N^{-1})'(x) = \frac{1}{x2\sqrt{\ln(x)}}$.

EXC: 28/221 Algebra gives us the formula $F^{-1}(x) = -\ln(1-x)$, and chain rule gives us $F'(x) = e^{-x}$, therefore $(F^{-1})'(x) = \frac{1}{1-x}$.

Is 26 easy?

2.10 Derivatives of TRIG functions

Theorem: $(\sin(x))' = \cos(x)$ and $(\cos(x))' = -\sin(x)$.

Theorem: $(\tan(x))' = \frac{1}{\cos^2(x)}$

DO: 2,4,6,8/230

22,20,18/231

EXC: Find the derivatives of the following functions:

$f(x) = \frac{\sin(x^2)-1}{\tan(x^2)}$;

$g(x) = \ln(\sin(x^3) + 2)$.

Comments on lecture 11

Section 3.1 Stability and the Derivative

Recall: DTDS, cobwebbing, solution, equilibrium points, stable/unstable and the graphical criterion!

NOTE that the diagonal IS the graph of the function $y = x$, a LINE with SLOPE 1.

IF the graph of the updating function crosses from above to below THEN the slope is LESS ($<$) than 1;

IF the graph of the updating function crosses from below to above THEN the slope is MORE ($>$) than 1;

TRY/DRAW some pics....

With the help of the MEAN VALUE THEOREM (SEE SECTION 3.4) ONE MAY PROVE:

THEOREM (Stability theorem for DTDS)

Given an equlibrium point x^* of the DTDS $x_{t+1} = f(x_t)$, one has:

— x^* is STABLE IF $|f'(x^*)| < 1$;

— x^* is UNSTABLE IF $|f'(x^*)| > 1$;

EXP: $b_{t+1} = 2b_t$, $f(b_t) = 2b_t$. The equilibrium point is 0. Compute $f'(x) = 2$, plug in and note that $|f'(0)| = |2| = 2 > 1$, so 0 is unstable, exactly what you know from cobwebbing: indefinite growth!

EXP: $b_{t+1} = (0.5)b_t$, $f(b_t) = (0.5)b_t$. The equilibrium point is 0. Compute $f'(x) = 0.5$, plug in and note that $|f'(0)| = |0.5| = 0.5 < 1$, so 0 is stable, exactly what you know from cobwebbing: extinction!

EXP: (Bacterial POPULATION - a non-linear model that you know from previous sections)

$p_{t+1} = \frac{2p_t}{2p_t+1.5(1-p_t)}$.

Sol: Recall that $f(x) = \frac{2x}{2x+1.5(1-x)}$, and the equilibrium points are 0 and 1.

NOTE that $f'(x) = \frac{2\{2x+1.5(1-x)\}-2x\{2-1.5\}}{(2x+1.5(1-x))^2}$, so:

$f'(0) \cong 1.333$, hence $|f'(0)| > 1$, so 0 is UNSTABLE;

$f'(1) \cong 3/4$, hence $|f'(1)| < 1$, so 0 is STABLE;

This is what you got by cobwebbing....

Let us talk now about 2 things:

***I* PER CAPITA PRODUCTION**

SAY we have a bacterial (or whatever) population;

EACH bacterium DIVIDES, but only a fraction of 'daughters"' survive;

IF r represents the number of new bacteria produced by bacterium, it is called the per capita production, and $b_{t+1} = rb_t$;

IDEA: new population = (per capita production) $\times$ old population.

***II* PER CAPITA rate of PRODUCTION(population)**

It is equal to: $= \frac{\textbf{Instantaneous rate of change}}{\textbf{Population}} = \frac{b'(t)}{b(t)}$.

EXP: Consider the model of a population with per capita production $= \frac{1}{1+0.001x}$ (a decreasing function of the population size).

SO: $x_{t+1} = \frac{1}{1+0.001x_t}x_t$. FIND the equlibrium points by solving: $x = \frac{1}{1+0.001x}x$, you get $x^* = 0$.

Since $|f'(0)| = 1$ we can NOT apply the criterion!!!!

Do cobwebbing(including the proper graph of updating function), and SEE that you get a stable equilibrium point!

DO: 38/249 $x_{t+1} = \frac{2.5x_t}{1+x_t^2}x_t$.

a) From $x = \frac{2.5x^2}{1+x^2}$ one gets the following equlibrium points: $x_1 = 0$, $x_2 = 0.5$ and $x_3 = 2$.

b) do yourself the graph

c) compute the derivative(quotient and power rules) and get $f'(x) = \frac{5x}{(1+x^2)^2}$.

Get that $|f'(0)| < 1$, so 0 is stable; $|f'(0.5)| > 1$, so 0.5 is unstable; $|f'(2)| < 1$, so 2 is stable;

DO: 10, 12/248.

DO: find the equation of the tangent line to the curve $y = x + \cos(x)$ at the point $(0, 1)$.

Comments on lecture 12

Section 3.3 MAXIMIZATION

Recall the following

DEFINITION: A point x in the domain of f is called a critical point of f if either $f'(x) = 0$ or the derivative of f AT x does not exist.

MORE: a) The global maximum of a function f is the LargesT value taken by f in its domain.

b) The global minimum of a function f is the smallest value taken by f in its domain.

EXP:sin or cos have Global Max 1, and Glabal min -1. Imagine the graph!

One of our tasks is to find the Global Max and Min!

DEFINITION: a) A LOCAL maximum is a $PEAK$ where the function takes on its largest value in a region of the domain.

b) A LOCAL minimum is a value that is the smallest value of the function in a region of the domain.

ALGORITHM (FOR FINDING GLOBAL MAX, MIN) FOR $f : [a, b] \mapsto \mathbf{R}$

1) — Compute f in the end points: $f(a)$ and $f(b)$;

2) — FIND all critical points, then Compute the value of f in all critical points;

3) — the largest value in steps 1,2 IS the global Max; the smallest value in steps 1,2 IS the Global Min.

What about LOCAL Max/Min?

Change in sign for f'.

Another WAY:

ALGORITHM (FOR FINDING LOCAL MAX, MIN) OR

THE SECOND DERIVATIVE TEST

1) FIND ALL CRITICAL POINTS WHERE f is differentiable, say $c, \ldots$;

2) If $f''(c) > 0$, then c is a LOCAL Minimum. If $f''(c) < 0$, then c is a LOCAL Maximum.

NOTE: Of course f'' means the derivative of the derivative, i.e., $(f')'$.

DEFINITION: A point is called an inflection point for a function if f changes concavity at that point.

NOTE: If $f''(c) = 0$, the test says nothing.
Do: 4,6/276.
Do: 8,10,12/276
Do: 17,13 for LOCAL max/min.
EXC: find local maximum of $f(x) = x + \frac{4}{x^2}$.
VERY IMPORTANT EXAMPLE:
MAXIMIZING FISH HARVEST
Say N_t denotes the population of fish in an ocean;
the DTDS modelling this situation is given to be: $N_{t+1} = 2.5N_t(1 - N_t) - hN_t$;
h is called $THEHARVESTEFFORT$. It depends on the number of ships, number of fishing days, other factors;
$-hN_t$ is the $HARVEST$.
FIND THE EQUILIBRIUM POINTS:
$N^* = 2.5N^*(1 - N^*) - hN^*$ IMPLIES THAT either $N^* = 0$ or $N^* = \frac{1.5-h}{2.5}$.
THE SECOND EQUILIBRIUM POINT IS BIO MEANINGFUL IF:
$N^* > 0$, SO $1.5 > h$. (Note that $N^* = 0$ means extiction.)
WHEN ARE THE EQ. POINTS STABLE/UNSTABLE?
ANSWER: Note first that $f(x) = 2.5x(1 - x) - hx$, so
$f'(x) = 2.5 - 5x - h$.
Case 1. Note that $f'(0) = 2.5 - h$.
a) IF $|f'(0)| < 1$ then 0 is stable.
Solving $|2.5 - h| < 1$ one gets $1.5 < h < 3.5$
b) Do yourself (do not look in your notes!!!) the unstable case!
Case 2. Note that $f'(\frac{1.5-h}{2.5}) = -0.5 + h$.
a) IF $|f'(\frac{1.5-h}{2.5})| < 1$ then $\frac{1.5-h}{2.5}$ is stable.
Solving $|-0.5 + h| < 1$ one gets $-0.5 < h < 1.5$
b) Do yourself (do not look in your notes!!!) the unstable case!
Define the **equilibrium harvest** as follows:
$P(h) = hN^* = h\frac{1.5-h}{2.5}$.
Goal: **maximize** $P(h)$.
Solution: By product rule (or quotient rule) one has that $P'(h) = \frac{1.5-2h}{2.5}$.
Solving $P'(h) = 0$ one obtains that $h = 0.75$, the only critical point. It is a maximium. Why? Try yourself the second derivative test, or think about quadratics...
EASY? HARD? Expect the unexpected!

Comments on lecture 13

Section 2.7 Second derivative

DEF: Given f, f' is called the first derivative (or derivative), and $(f')' = f''$ is called the second derivative of f. Other notation:
f, $\frac{df}{dx}$, $\frac{d^2f}{dx^2}$
DEF: a) A function is called CONCAVE UP if f' is increasing, or in other words: IF $f'' > 0$.
b) A function is called CONCAVE DOWN if f' is decreasing, or in other words: IF $f'' < 0$.
NOTE: When $f''(x) = 0$, f may change concavity at x.

DEF: A point is called an **inflection point** for a function if the function changes concavity at that point.

Counterexample: $f(x) = x^4$, $f'(x) = 4x^3$ and $f''(x) = 12x^2$, so $f''(0) = 0$, BUT 0 is NOT an inflection point.

Classification of **Power** Functions: $f(x) = x^p$, $x > 0$,

$f'(x) = px^{p-1}$, $f''(x) = p(p-1)x^{p-2}$.

Case 1: FOR f': IF $p > 0$ then $f' > 0$;

IF $p < 0$ then $f' < 0$.

Case 2: FOR f'': $f'' > 0$ if and only if p is in either $(-\infty, 0)$ or in $(1, \infty)$;

$f'' < 0$ if and only if p is in $(0, 1)$.

Application of second derivative:

— an object is moving, suppose that the position of the object is given by $y(t)$;

— the first derivative: $y'(t) = \frac{dy}{dt}$ is called the velocity;

— $|y'(t)|$ is the speed;

— the second derivative: $y''(t) = \frac{d^2y}{dt^2}$ is called the acceleration;

DO: 34, 26 on page 203!

Sol of 34: $p'(t) = -274t + 20$, $p''(t) = -274$, $p(0) = 500$, $p'(0) = 20$, the object was thrown upward at $20\frac{m}{s}$.

Do: 12-14-16-18/202;

Do: 20-22-24-26/202;

EX (very important): Graph $f(x) = x + \frac{4}{x^2}$.

Sol: find domain, derivatives, intervals of concavity, when is f increasing/decreasing, inflection points etc.

Comments on lecture 14

Section 3.5 LIMITS at INFINITY

— Imagine that some Updating functions have $(-\infty, \infty)$ or $[0, \infty)$ as domain. What is the behavior of the Updating functions at the end(s) of the domain?

— Then we may want to compare them: which one approaches ∞ or 0 faster/slower?

DEFINITION: 1) We write $\lim_{x\to\infty} f(x) = L$ if we can make the values of $f(x)$ arbitrarly close to L, by taking x sufficiently positive large.

2) We write $\lim_{x\to-\infty} f(x) = L$ if we can make the values of $f(x)$ arbitrarly close to L, by taking x sufficiently negative large.

3) We write $\lim_{x\to\infty} f(x) = \infty$ if we can make the values of $f(x)$ arbitrarly positive large, by taking x sufficiently positive large.

4) In the same way one may define:

$\lim_{x\to\infty} f(x) = -\infty$; $\lim_{x\to-\infty} f(x) = \infty$; $\lim_{x\to-\infty} f(x) = -\infty$.

Examples: $\lim_{x\to-\infty} 1 + x^2 = \infty$; $\lim_{x\to\infty} \frac{1}{x^2} + 7 = 7$;

$\lim_{x\to-\infty} \frac{1}{x^2} + 7 = 7$ and $\lim_{x\to\infty} x^2 - x + 1 = \infty$.

MORE: $\lim_{x\to\infty} e^x = \infty$. Recall the graph of e^x, OR note that for $x \geq 0$ one has that $e^x \geq x + 1$. So $\lim_{x\to\infty} e^x \geq \lim_{x\to\infty} x + 1 = \infty$, so $\lim_{x\to\infty} e^x = \infty$.

Even More: $\lim_{x\to\infty} \ln(x) = \infty$ by recalling the graph of $\ln(x)$.

Do: 1-8/297

COMPARING FUNCTIONS AT $+\infty$

DEFINITION: Suppose that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$. We say:

1) $f(x)$ approaches ∞ FASTER than $g(x)$ AS $x \to \infty$ IF $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$.

2) $f(x)$ approaches ∞ SLOWER than $g(x)$ AS $x \to \infty$ IF $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

3) $f(x)$ and $g(x)$ approach ∞ at the same rate AS $x \to \infty$ IF $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$, L a number that is NOT zero.

A list of some basic functions is this:

— $c\ln(x)$;

— cx^n;

— ce^{dx}, where $c > 0$, $d > 0$.

All approach ∞, there is a order..., recall their graphs!

DO: 9-14/297.

The second part about comparing functions is:

DEFINITION: Suppose that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = 0$. We say:

1) $f(x)$ approaches 0 FASTER than $g(x)$ AS $x \to \infty$ IF $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

2) $f(x)$ approaches 0 SLOWER than $g(x)$ AS $x \to \infty$ IF $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$.

3) $f(x)$ and $g(x)$ approach 0 at the same rate AS $x \to \infty$ IF $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$, L a number that is NOT zero.

In this case a list of basic functions is:

— cx^{-n}, of course $n > 0$;

— ce^{-dx}, $d > 0$;

— ce^{-dx^2}, $d > 0$.

DO: 16,18/297.

Section 3.6 L'HOPITAL RULE

INDETERMINATE FORMS: $\frac{\pm\infty}{\pm\infty}$; $\frac{0}{0}$.

Think of $\lim_{x\to\infty} \frac{e^{2x}}{x^2+1} =?$

L'HOPITAL RULE: SUPPOSE THAT $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate form (where a can be: a number, $\pm\infty$, a^+ or a^-).

IF $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, THEN $\lim_{x\to a} \frac{f(x)}{g(x)} = L$.

Do: 7-22/309, and if time do 23-26/309.

Comments on lecture 15

3.7 APPROXIMATING FUNCTIONS

Part I. **Tangent Line Approximation** (**near a point**)

Suppose that we are given a function and we wish to approximate it with a simpler one. Say a linear one!

What is given? A function f and a point a.

Can we find a function $\hat{f}$ that is linear and $f(a) = \hat{f}(a)$ AND $f'(a) = \hat{f}'(a)$?

Solution: Say that $\hat{f}(x) = mx + n$. Since $\hat{f}'(a) = m$ one gets that $m = f'(a)$. So $\hat{f}(x) = f'(a)x + n$.

Since $f(a) = \hat{f}(a)$ one gets that $f(a) = f'(a)a + n$, so $n = f(a) - f'(a)a$.

Hence one gets that $\hat{f}(x) = f(a) + f'(a)(x - a)$ WHICH IS THE EQUATION OF THE TANGENT LINE TO f at a.

Conclusion: $f(x) \approx \hat{f}(x)$ for x NEAR a, or in other words: x close to a implies $f(x)$ close to $\hat{f}(x)$.

DO: 2,4,6/319

PART II. **QUADRATIC APPROXIMATION near a point**

That's the way to get better approximations, the line is rigid!

Goal: Given f and a, find a quadratic $\hat{f}(x)$ such that $f(a) = \hat{f}(a)$, $f'(a) = \hat{f}'(a)$ AND $f''(a) = \hat{f}''(a)$?

Solution: Consider $\hat{f}(x)$ of the form: $\hat{f}(x) = c_0 + c_1(x - a) + c_2(x - a)^2$. WHY? JUst look at the form of the hat in PART I.

So we get: $f(a) = \hat{f}(a) \Rightarrow c_0 = f(a)$,

$f'(a) = \hat{f}'(a) \Rightarrow c_1 + 2c_2(x - a) = \hat{f}'(x)$, hence $c_1 = f'(a)$.

$f''(a) = \hat{f}''(a) \Rightarrow 2c_2 = \hat{f}''(x)$, hence $c_2 = \frac{f''(a)}{2}$.

What we get is this: $\hat{f}(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$.

Do 8, 10, 12 /319 about quadratic approximations.

PART III. TAYLOR POLYNOMIALS

Why not matching: the function in a, the first derivative in a, the second derivative in a, the third derivative in a, ... and so on to get better approximations?!

DEFINITION: Suppose the first n derivatives of f are defined at a. The Taylor polynomial of degree n matching the values of the first n derivatives is $P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \cdots + \frac{f^{(i)}(a)}{i!}(x - a)^i + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$. Here: $i! = 1 \times 2 \times 3 \times \cdots \times i$.

SO: $P_n(x) \approx f(x)$ FOR x near a.

Do: 22,24,26.

If time do 28,27,23, 25/319.

Comments on lecture 16

PART I. VERTICAL AND HORIZONTAL ASYMPOTES

Definitions and pictures!

PART II. SECTION 3.4 REASONING ABOUT FUNCTIONS

The theorems in this section are used:

— to show, without solving equations, that a given DTDS has an equilibrium;

— to show, without computing derivatives, that a given function has a MAXIMUM/MINIMUM;

— TO FIND the value of a derivative without taking LIMITS.

II.a) INTERMEDIATE VALUE THEOREM

THEOREM: IF $f : [a, b] \mapsto \mathbf{R}$ is a continuous function and c is between $f(a)$ and $f(b)$, THEN there is an x between a and b such that $f(x) = c$.

Try to visualize what the theorem is saying by constructing an appropriate graph!

DO: 2,3,6/286

EXC: Show that the equation $e^{\cos(\frac{x}{2})} = 2\sin(\frac{x}{2})$ has at least one solution (root) in the einterval $[0, \pi]$.

For what is this theorem good? Recall equilibrium points: we need to solve : $f(x) = x$.

II.b) EXTREME VALUE THEOREM

Theorem: IF $f : [a, b] \mapsto \mathbf{R}$ is a continuous function, THEN there is a point c in $[a, b]$ such that c is a global maximum, and there is a point d in $[a, b]$ such that d is a global minimum.

DO: 8/286

SOL: $f : [0, 1] \mapsto \mathbf{R}$ is continuous, and note that $f(0) = 0$, $f(1) = 0$, $f(x) = x(e - e^x) > 0$ when $1 > x > 0$. Since f IS continuous it follows that there is a c in $[0, 1]$ such that c is the global maximum. So c is in fact in $(0, 1)$.

II.c) ROLLE's THEOREM

The IVT and EVT guarantee that a continuous function MUST take on some particular values. The next theorems guarantee that the DERIVATIVE must take on particulaar values.

THEOREM (ROLLE's Theorem): IF $f : [a, b] \mapsto \mathbf{R}$ is differentiable and $f(a) = f(b)$, THEN there is a c in (a, b) such that $f'(c) = 0$.

TRY to visualize what the theorem is saying!

A generalization:

THEOREM (MEAN VALUE Theorem): IF $f : [a, b] \mapsto \mathbf{R}$ is differentiable, THEN THERE is a c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

TRY to visualize what the theorem is saying!

DO: 12,14, **21**/287

....

If time: 5/286 (where you note that $f(1/2) > 0$ and $f(0) < 0$). Try 3/286.

....

EXC: Show that the equation $2\sin(\pi x) = e^{\cos(\pi x)}$ has at least one solution (root) in $[0, \frac{1}{2}]$.

Comments on lecture 17

Section 3.8 Newton's Method

Last lecture: IVT was proven to be a good at showing some equations have at least one root.

Today: WHEN we can NOT solve the equation we may use Newton's method to approximate the root.

GOAL: solve $f(x)$ where f is pretty bad!

— If r is one root, nobody knows what r is, so start guessing;

— SAY x_0 is an approximation of r (Recall IVT);

— construct the tangent line to $f(x)$ at $(x_0, f(x_0))$;

— Since the tangent line approximates f NEAR x_0, we may assume that the x intercept of the tangent line is approximating the x intercept of f;

— Say x_1 is the x intercept of the tangebt line. Let us find it! Note that $y - f(x_0) = f'(x_0)(x - x_0)$, so $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ if $f'(x_0) \neq 0$.

— NOW: why not continue?! Do to x_1 what you did to x_0; and get better estimates of r:

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ if $f'(x_1) \neq 0$;

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$ if $f'(x_2) \neq 0$;

ALGORITHM (**NEWTON's Method**)

To solve $f(x) = 0$ do the following steps:

1) Guess x_0
2) Compute $x_{t+1} = x_t - \frac{f(x_t)}{f'(x_t)}$, $t \geq 0$ until the answer converges.
DEFINITION: $x_{t+1} = x_t - \frac{f(x_t)}{f'(x_t)}$ IS CALLED NEWTON's Method DTDS.
NOTE: IT IS a fast method when it works!
QUESTION: When NEWTON's Method fails?
— when x_0 (our first guess) IS not good (far away from the true value...)
— when $f'(x_t)$ for some t IS 0, or x_t is not in the domain of f'
— when we do have many roots, just imagine cos or sin.
DEFINITION: An equilibrium point where the slope is 0 is called SUPERSTABLE.
Consider the NEWTON's Method DTDS. The Updating function is $h(x) = x - \frac{f(x)}{f'(x)}$.
The equilibrium point is obtained from $h(x^*) = x^*$. So $f(x^*) = 0$.
Now note that $h'(x^*) = 0$
So: $|h'(x^*)| < 1$, hence x^* is stable and SUPERSTABLE!
The second part is this OBSERVATION:
IF $f(x_t) = 0$ for some t we get that:
$x_{t+1} = x_t - \frac{f(x_t)}{f'(x_t)} = x_t$ i.e., we are solving for equlibrium.
So:
we transform solving $f(x) = 0$ INTO finding an equilibrium point...
Do: 8/329;6/329 *and* **26/329**.
Sol: $N = rNe^{-N} - hN \Rightarrow N\{1 - re^{-N} + h\} = 0$, hence either $N = 0$ or $N = \ln(r) - \ln(1+h)$.
From $N = \ln(\frac{1.5}{1+h})$ one gets $P(h) = h\ln(\frac{1.5}{1+h})$. Critical points are found by solving $P'(h) = 0$, or $\ln(\frac{1.5}{1+h}) - \frac{h}{h+1} = 0$. How can we solve it? TRY Newton's Method...
Set $f(h) = \ln(\frac{1.5}{1+h}) - \frac{h}{h+1}$, then we need to compute $f'(h) = \frac{-2-h}{(1+h)^2}$.
Guess $h_0 = 0.25$ (Think for 3 seconds about why is it a good guess!).
Next we get $h_1 = h_0 - \frac{f(h_0)}{f'(h_0)} = 0.237723303$, $h_2 = h_1 - \frac{f(h_1)}{f'(h_1)} = \ldots$, after 3 steps one gets h_3 is 0.2378.

Comments on lecture 18

Chapter 4. Differential equations, Integrals and their applications
Section 4.1 Differential equations
If f (or a measurement) is given then by differentiation one may find the rate of change, i.e., f'.
Imagine that an object is moving, say you know its position $p(t)$, then the velocity is $p'(t) = \frac{dp}{dt}$.
QUESTION: WHAT IF YOU KNOW THE VELOCITY, AND YOU WANNA FIND THE POSITION?
In other words, if $\frac{dp}{dt} = v(t)$, ca we find $p(t)$? Here $p(t)$ is called unknown!
DEFINITION: A differential equation expresses the rate of change of a quantity (the state variable) as a function of time OR of the state variable itself.
EXP: 1) $\frac{dP}{dt} = e^{-t} + \sin(t-3) + 6$;
2) $\frac{dP}{dt} = 3P^2$.
Question: What is the difference between them?

DEFINITION: a) If the rate of change is a function of time, the equation is called a pure-time differential equation.

b) If the rate of change is the quantity, the equation is called an autonomous differential equation.

Do: 1-4/348.

EXP: The volume of water that enters in a pool staisfies $\frac{dV}{dt} = 1$.

Goal: find $V(t)$. Guess $V(t) = t$ since $(t)' = 1$. BUT $W(t) = t + c$, where c is a number, is also working! So: out of all choices, what should we choose?

IF we know some pieces of information at a certain time, we coiuld find $V(t)$.

Say we know $\frac{dV}{dt} = 1$ and $V(0) = 2500$. Then from $V(t) = t+c$ one gets that $0+c = 2500$, so $V(t) = t + 2500$. We are led to the following

DEFINITION: INITIAL CONDITION = initial value of the state variable.

EXC: Solve $\frac{db}{dt} = 3b$, $b(0) = 1$.

SOL: First note that $b(t) = ce^{3t}$ (just see yourself — after some computations— that left and right side are equall). How can we find c? This way: $1 = b(o) = ce^{3\times 0}$, so $c = 1$, so $b(t) = e^{3t}$.

DEFINITION: A solution (of a differential equation) gives the value of the state variable as a function of time.

DO: 9-12/349.

Euler's method (when not able to guess...)

ALGORITHM FOR SOLVING $\frac{dm}{dt} = f(t)$ with initial condition $m(t_0) = m_0$.

— $m(t_0) = m_0$;

— $m(t_0 + \Delta t) =?$ (for Δt small): $m(t_0 + \Delta t) \approx \widehat{m}(t_0 + \Delta t) = m(t_0) + m'(t_0)(t_0 + \Delta t - t_0))$ (here the base point is t_0). SO: $m(t_0 + \Delta t) \approx m(t_0) + m'(t_0)\Delta t$; (we hope you are able to see that all terms ARE known!)

— $m(t_0 + 2\Delta t) =?$: $m(t_0 + 2\Delta t) \approx \widehat{m}(t_0 + 2\Delta t) = m(t_0 + \Delta t) + m'(t_0 + \Delta t)(t_0 + 2\Delta t - t_0 - \Delta t) = m(t_0 + \Delta t) + m'(t_0 + \Delta t)\Delta t$;

— continue!

Do: 13, 14/349;

Sol of 14: $\frac{dW}{dt} = \frac{2}{1+t}$; $W(0) = 3$, $\Delta t = 0.5$.

Note that: $W(0) = 3$, $W(0.5) \approx \widehat{W}(0.5) = W(0) + W'(0)(0.5) = 3 + \frac{2}{1+0}(1/2) = 4$, base point being 0;

$W(1) \approx \widehat{W}(1) = W(0.5) + W'(0.5)(0.5) = 4 + \frac{2}{1+0.5}(0.5) = \frac{14}{3}$, base point being 0.5;

Compare to EXC 10: $W(t) = 2\ln(1 + t) + 3$, so $W(1) = 4.386$.

Comments on lecture 19

Section 4.2 Solving pure-time differential equations

Recall that the shape of a differential equation is $\frac{dF}{dt} = f(t)$. Until now we only guessed the solution or estimated the solution!

DEFINITION: An antiderivative of the function f is a function F with the derivative equal to f, $F'(t) = f(t)$. We write

$$F(t) = \int f(t)dt.$$

EXP: $\int 2tdt = t^2$ and $\int 2tdt = t^2 + 2010$, and in fact $\int 2tdt = t^2 + c$, c a NUMBER, works!

DEFINITION: The set of all antiderivatives of f is called the INDEFINITE INTEGRAL OF f. WE write $\int f(t)dt = F(t) + c$, c a number, where $F(t)$ is a particular antiderivative.

POWER RULE FOR INTEGRALS: $\int x^n dx = \frac{x^{n+1}}{n+1} + c$, c a number, and $n \neq -1$.
Proof: $(\frac{x^{n+1}}{n+1} + c)' = \frac{n+1}{n+1}x^{n+1-1} + 0$.
CONSTANT PRODUCT RULE FOR INTEGRALS:
$\int af(x)dx = a\int f(x)dx$.
Proof: $(a\int f(x)dx)' = a(\int f(x)dx)' = af(x)$.
SUM RULE FOR INTEGRALS: $\{\int f(x) + g(x)\}dx = \int f(x)dx + \int g(x)dx$.
Proof: $(\int f(x)dx + \int g(x)dx)' = (\int f(x)dx)' + (\int g(x)dx)' = f(x) + g(x)$.
COR: $\{\int f(x) - g(x)\}dx = \int f(x)dx - \int g(x)dx$.
Proof: just use constant product rule and sum rule!
SO: from now on one may find antiderivatives of any polynomial!!
DO: 7-20/359
DO: 27-30/359
28/359 Here $a = -1.62\frac{m}{s^2}$.
a) $v(t) = \int -1.62dt = -1.62t + c$, c a number.
$v(0) = 5$ implies that $c = 5$. So $v(t) = -1.62t + 5$.
$p(t) = \int(-1.62t + 5)dt = -1.62\frac{t^2}{2} + 5t + k$, k a number.
$100 = p(0)$ implies that $p(t) = -1.62\frac{t^2}{2} + 5t + 100$.

b) MAXIMUM IS OBTAINED WHEN $v(t) = 0$. Solving $-1.62t + 5 = 0$, one gets that $t = \frac{5}{1.62} \approx 3.08$. SO $p(3.08) = -0.81(3.08)^2 + 5(3.08) + 100 = 107.7$.

c) $t =$? such that $p(t) = 100$. We solve: $-1.62\frac{t^2}{2} + 5t + 100 = 100$, or $t(-0.81t + 5) = 0$, so either $t = 0$ or $t = \frac{5}{0.81} \approx 6.17$. Choose $t = 6.17$ seconds. Then $v(6.17) = -1.62(6.17) + 5 \approx -5\frac{m}{s}$.

d) $t =$? such that $p(t) = 0$. We solve: $-1.62\frac{t^2}{2} + 5t + 100 = 0$. By the quadtratic formula one gets $t_1 = 14.62$ and t_2 negative. We choose the positive time! Hence $v(14.62) = -18.68\frac{m}{s}$.

Comments on lecture 20

4.3 Integration of special functions. Integration by substitution. Integration by parts

Part I. $\int \frac{1}{x}dx = \ln|x| + c$, c a number,
$\int e^x dx = e^x + c$, c a number,
$\int \sin(x)dx = -\cos(x) + c$, c a number,
$\int \cos(x)dx = \sin(x) + c$, c a number.
Compute $\int 2010\sin(x) - 2009\cos(x) + 777e^x + \frac{2008}{x}dx$.
Part II. Recall that $(f \circ g)' = f'(g)g'$.
SO: $\int f'(g(x))g'(x)dx = (f \circ g)(x) + c$ where c is a number.
HOW do we do it in practice?
Algorithm on page 362 called **SUBSTITUTION**.
Do: 13-20/369.
PLAN (ALGORITHM):
1) DEFINE A NEW VARIABLE AS A FUNCTION OF THE OLD VARIABLE;
2) TAKE THE DERIVATIVE OF THE NEW VARIABLE WITH RESPECT TO THE OLD VARIABLE;
3) PUT EVERYTHING IN THE INTEGRAL IN TERMS OF THE NEW VARIABLE
4) INTEGRATE (or at least try it);

5) PUT EVERYTHING BACK IN TERMS OF THE OLD VARIABLE!

Part III. **INTEGRATION BY PARTS** - the counterpart of the product rule from differential calculus

Recall that $(fg)' = f'g + fg'$, SO: $fg = \int (fg)'dx \int f'g + fg'dx$. We get:
$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$.

DO: 21-24/369.

Comments on lecture 21

Section 4.4 INTEGRALS AND SUMS

Notation: $x_1 + x_2 + \cdots + x_n = \sum_{i=1}^{n}$

SO: $\sum_{i=1}^{7} = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$

If $x_1 = 1$, $x_2 = -3$ and $x_3 = 4$ then $\sum_{i=1}^{3} = 1 + (-3) + 4 = 2$

Consider the following differential equation $\frac{dV}{dt} = t^2$, where V is the volume of water entering a vessel

Goal: Find total quantity of water that entered during the first second.

One way is: $V(t) = \int t^2 dt = \frac{t^3}{3} + C$, C a number. With an initial condition one may find C. Say $V(0) = 0$, then $C = 0$, so $V(t) = \frac{t^3}{3}$.

ANOTHER WAY: We are going to estimate $V(1)$. Suppose we measured the RATE at which water entered the vessel every 0.2s. (Now see the table in the book.) SINCE WE DO NOT KNOW WHAT HAPPENS BETWEEN THE MEASUREMENTS, WE ASSUME THAT THE RATE IS CONSTANT BETWEEN THE MEASUREMENTS.

We compute a left-hand estimate and a right-hand estimate. FOR the left-hand estimate we pretend that the RATE (at which water enters) between measurements IS exactly the value at the beginning of the interval. FOR the right-hand estimate we pretend that the RATE (at which water enters) between measurements IS exactly the value at the end of the interval.

Do the tables and get Left-hand estimate is 0.240 and right-hand estimate is 0.440. The true value (1/3) is somewhere in between...

Question: What should we do to get better estimates?

Answer: Use more measurements, or in other words divide $[0, 1]$ in more subintervals! Their length is smaller!

What did we get? This: if δt = length of subinterval = 0.2, then: $I_l = (0.0)^2 \times 0.2 + (0.2)^2 \times 0.2 + (0.4)^2 \times 0.2 + (0.6)^2 \times 0.2 + (0.8)^2 \times 0.2$, and $I_r = (0.2)^2 \times 0.2 + (0.4)^2 \times 0.2 + (0.6)^2 \times 0.2 + (0.8)^2 \times 0.2 + (1)^2 \times 0.2$. DIVIDE YOURSELF THE INTERVAL $[0, 1]$ IN 5 subintervals!

Say now that we cut (divide) the interval $[0, 1]$ in n subintervals. SET $\Delta t = \frac{1-0}{n}$, it is the length of a subinterval! Divide yourself the interval $[0, 1]$ in n subintervals!

Write $I_l = \sum_{i=0}^{n-1} t_i^2 \Delta t$ and $I_l = \sum_{i=1}^{n} t_i^2 \Delta t$.

They are called RIEMANN SUMS! To get better estomates we have to increase n. Recall our differential equation $\frac{dV}{dt} = t^2$.

We define the Riemann integral (or definite integral) by $\int\limits_0^1 t^2dt = \lim\limits_{n\to\infty} \sum\limits_{i=0}^{n-1} t_i^2 \Delta t = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} t_i^2 \Delta t$ where $\Delta t = \frac{1}{n}$.

IN GENERAL: The RIEMANN INTEGRAL OF A FUNCTION $f : [a, b] \mapsto \mathbf{R}$ is:
$\int\limits_a^b f(t)dt = \lim\limits_{n\to\infty} \sum\limits_{i=0}^{n-1} f(t_i)\Delta t = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(t_i)\Delta t$ where the interval $[a, b]$ is divided into n subintervals of equal length $\Delta t = \frac{b-a}{n}$.

(Divide yourself the interval $[a, b]$)

ALGORITHM TO EVALUATE $\int\limits_a^b f(t)dt$

— FIND $\Delta t = \frac{b-a}{n}$ (a, b, n are given in statement)
— find $t_0, t_1, \ldots, t_n$
— find I_l and I_r.

DO (find and write) 13, 14, 15, 16 at the end of this section!

More formulae next section...

Comments on lecture 22

4.5 DEFINITE and INDEFINITE INTEGRALS

THEOREM: If $f(x)$ is a continuous function with $F(x) = \int f(x)dx$, THEN $\int_a^b f(x)dx = F(b) - F(a) = F|_a^b$.

It is called the fundamental theorem of Calculus.

Theorem (Summation property of the definite integral)
$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

DO: 24, 20, 18, 16, 14 etc /391.

GOOD LUCK!