FREE TORUS ACTIONS AND TWO-STAGE SPACES

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ABSTRACT. We prove the toral rank conjecture of Halperin in some new cases. Our results apply to certain elliptic spaces that have a two-stage Sullivan minimal model, and are obtained by combining new lower bounds for the dimension of the cohomology and new upper bounds for the toral rank. The paper concludes with examples and suggestions for future work.

1. INTRODUCTION

A well-known conjecture due to Halperin concerns torus actions on a space (see [Hal85, Prob.1.4]). If X is a space on which an n-dimensional torus acts, we say the action is *almost-free* if each isotropy subgroup is a finite group. The largest integer $n \geq 1$ for which X admits an almost-free n-torus action is called the *toral rank* of X, and is denoted $\mathrm{rk}\,(X)$. If X does not admit an almost-free circle action, then $\mathrm{rk}\,(X) = 0$. Halperin's conjecture gives an upper bound for the toral rank of X in terms of its cohomology, as follows:

Conjecture 1.1 (The toral rank conjecture). *If X is simply connected, then*

$$\dim H(X; \mathbb{Q}) \geq 2^{\mathrm{rk}\,(X)}.$$

We shall henceforth assume that all our spaces are 1-connected, finite cell complexes. There are some technical conditions on the topology of the space X in Halperin's original formulation, but as these are satisfied by finite cell complexes, we will not mention them explicitly here.

The main tool we shall use is the Sullivan *minimal model*, and a basic reference is [FHT01]. For our purposes, we note that to any 1-connected space X there corresponds, in a contravariant way, a commutative differential graded algebra (unique up to isomorphism) $(\Lambda W, d)$, called the minimal model of X, which algebraically models the rational homotopy type of the space. By ΛW we mean the free graded commutative algebra generated by the graded vector space W. The differential d of any element of W is a polynomial in ΛW with no linear term, which in particular means that there is a homogeneous basis $\{w_i\}_{i \geq 1}$ of W for which $dw_i \in \Lambda W_{<i}$, where $W_{<i}$ denotes the subspace of W generated by $\{w_j\}_{j<i}$. (We will also occasionally consider models $(\Lambda W, d)$ which satisfy this latter nilpotence condition but where the polynomial dw may have a linear term.) This contravariant correspondence yields an equivalence between the homotopy category of 1-connected rational spaces of finite type and that of 1-connected rational commutative differential graded algebras of finite type. In particular, if $(\Lambda W, d)$ is the minimal

model of X, then $H(\Lambda W, d) \cong H(X; \mathbb{Q})$ as graded algebras, and $W \cong \pi(X) \otimes \mathbb{Q}$ as graded vector spaces. Moreover, we remark that a series of results of Halperin and others implies that if $H(\Lambda W, d)$ and W are finite-dimensional, then $H(\Lambda W, d)$ must satisfy Poincaré duality, and $(\Lambda W, d)$ is the minimal model of some closed smooth manifold. In all cases of interest to us here, the space of generators W will be finite-dimensional. If $\{w_1, \ldots, w_m\}$ is a homogeneous basis for W, then we will write $W = \langle w_1, \ldots, w_m \rangle$, and we also denote ΛW by $\Lambda(w_1, \ldots, w_m)$ in this case, often omitting explicit reference to the differential. We note that models with W and $H(\Lambda W, d)$ both finite-dimensional are called *elliptic*, and a space X with an elliptic minimal model is called an elliptic space. Topologically, this means that both $\pi(X) \otimes \mathbb{Q}$ and $H(X; \mathbb{Q})$ are finite dimensional.

Conjecture 1.1 is already known to hold for homogeneous spaces G/H, for G connected, and H closed and connected [Hal85, Prop.1.5]. Such spaces have *two-stage* minimal models [FHT01, Prop.15.16], where a minimal model $(\Lambda W, d)$ is said to be two-stage if W decomposes as $W \cong U \oplus V$ with $dU = 0$ and $dV \subseteq \Lambda U$. By the remark in the previous paragraph, it is easy to see that there are many other examples of spaces, indeed smooth manifolds, with two-stage models, and it is these spaces that we shall study (see for example Corollary 2.4).

We note for later reference that there may be several ways to display a minimal model as a two-stage model. In particular, a generator in V that is a cocycle could just as well be included in U. We will generally be interested in choosing a two-stage decomposition in which V is as large as possible.

By a K-S extension (Koszul-Sullivan), or simply an extension, we mean a sequence of the form

$$(\Lambda W_1, d_1) \to (\Lambda W_1 \otimes \Lambda W_2, D) \to (\Lambda W_2, d_2)$$

with $(\Lambda W_1, d_1)$ a minimal model, in which D restricts to d_1 on $\Lambda W_1 \otimes 1$, and for which there is an ordered basis of $W_2 = \langle w_1, w_2, \ldots \rangle$ with $Dw_i \in \Lambda W_1 \otimes \Lambda(w_1, \ldots, w_{i-1})$ for each i. A K-S extension is the minimal-model analogue of a Serre fibration (cf. [FHT01, Sec.15(a)]), and for this reason, $\Lambda W_1, \Lambda W_1 \otimes \Lambda W_2$ and ΛW_2 are known respectively as the base, the total space and the fibre of the extension.

The connection between minimal models and the toral rank is originally due to Allday and Halperin [AH78] (cf. [Hal85, Prop.4.2]): If X has minimal model $(\Lambda W, d)$ and admits an n-dimensional torus action, then there is an extension

$$(1) \qquad\qquad \Lambda A_n \to \Lambda A_n \otimes \Lambda W \to \Lambda W$$

in which $A_n = \langle a_1, \ldots, a_n \rangle$ with each a_i of degree 2. If the action is almost free, then $\dim H(\Lambda A_n \otimes \Lambda W, D)$ is *finite-dimensional* [Hal85, Prop.4.2]. In principle, this result allows an upper bound for $\mathrm{rk}\,(X)$ to be obtained by a direct analysis of the minimal model of X, and this direct approach has been carried out to great effect in some situations. In general, however, the computational problems involved here appear to be quite substantial.

However, since we are really only considering the rational homotopy type of X, we are led to the following variation of the toral rank: the *rational toral rank* $\mathrm{rk}_0(X)$ of X is defined by $\mathrm{rk}_0(X) = \max\{\mathrm{rk}(Y) \mid Y \simeq_{\mathbb{Q}} X\}$. Clearly, we have $\mathrm{rk}_0(X) \geq \mathrm{rk}(X)$, hence an upper bound on $\mathrm{rk}_0(X)$ will serve as one on $\mathrm{rk}(X)$. See [Hal85, Prop.4.2] for the precise relationship between these two numbers. The characterization of $\mathrm{rk}_0(X)$ in terms of a minimal model of X is as follows. If

$(\Lambda W, d)$ is a minimal model of X, then $\mathrm{rk}_0(X)$ is the largest n (if such exists) for which there is a K-S-extension of the form (1), for which $\dim H(\Lambda A_n \otimes \Lambda W, D)$ is finite-dimensional. We will also denote $\mathrm{rk}_0(X)$ by $\mathrm{rk}_0(\Lambda W)$.

Now we briefly summarize the contents of the paper. In Section 2, Theorem 2.3 gives a lower bound for the dimension of the cohomology of any space with an elliptic two-stage minimal model. This result is proved using standard tools familiar from algebraic topology, namely the *Wang sequence* and the Serre spectral sequence. Also in Section 2 we give two results that, under certain additional hypotheses, give upper bounds on the rational toral rank—hence on the toral rank—of a space with two-stage minimal model (Theorem 2.9 and Theorem 2.10). Combining these results allows us to establish Conjecture 1.1 in some new cases (Corollary 2.11). In addition, our results can be used to provide new and essentially elementary proofs of the conjecture in some previously-known cases, which we discuss after Theorem 2.3. In Section 3, we give a number of examples and suggest several directions for future investigation.

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2. Cohomology and Toral Rank of Two-Stage Minimal Models

In this section we give several results about the cohomology and the toral rank of two-stage minimal models.

2.1. **Cohomology of two-stage minimal models.** We begin by establishing a lower bound for the dimension of the cohomology of an elliptic space with two-stage minimal model. Our bound is of a form similar to that featured in the toral rank conjecture.

If $\Lambda(U, V)$ is an elliptic two-stage minimal model, then the Mapping theorem [FHT01, P.375] implies that V has generators of odd degree only. On the other hand, U may have generators of odd or even degree. For the remainder of the paper, unless otherwise specified, we assume that all minimal models are two-stage, elliptic. For details about elliptic spaces and their minimal models, see [FHT01, Sec.32].

First, we establish our lower bound in a special case. The main result is then established by reducing to this special case.

Proposition 2.1. *Let $\Lambda(U, V)$ be a two-stage, elliptic minimal model with odd degree generators only and suppose that $d\colon V \to \Lambda^2 U$ is an isomorphism. Then $\dim H\big(\Lambda(U, V)\big) \geq 2^{\dim V}$.*

Proof. Suppose that $\dim U = n$. Since $d\colon V \to \Lambda^2 U$ an isomorphism, and since ΛU is an exterior algebra, we have $\dim V = \binom{n}{2}$. Thus we must show that $\dim H\big(\Lambda(U, V)\big) \geq 2^{\binom{n}{2}}$. We will proceed by induction on n. First, we give an explicit description of the model $\Lambda(U, V)$.

Write $\Lambda(U, V)$ as $\mathcal{M}_n = (\Lambda(U_n, V_n), d_n)$, with $U_n = \langle u_1, \ldots, u_n \rangle$ and $V_n = \langle \{v_{i,j}\}_{1 \leq i < j \leq n} \rangle$. The differential is $d_n(U_n) = 0$ and $d_n(v_{i,j}) = u_i u_j$ for each $0 \leq i < j \leq n$. For $n = 1$, we have $\dim U_1 = 1$ and $V_1 = 0$, and the proposition

is trivial. For $n = 2$, we have $\mathcal{M}_2 = \Lambda(u_1, u_2, v_{1,2})$. It is easily checked that $\dim H(\mathcal{M}_2) = 6 \geq 2^1$. This starts the induction.

Now suppose inductively that $\dim H(\mathcal{M}_n) \geq 2^{\binom{n}{2}}$ for some $n \geq 2$. We adjust the notation somewhat and write $\mathcal{M}_{n+1}$ as follows. Write $U_{n+1} = \langle u_0, u_1, \ldots, u_n \rangle = \langle u_0 \rangle \oplus U_n$. Also, set $V_0 = \langle v_{0,1}, \ldots, v_{0,n} \rangle$, so that $V_{n+1} = V_0 \oplus V_n$. The differential d_{n+1} of $\mathcal{M}_{n+1}$ extends that of $\mathcal{M}_n$. Thus, we have $d_{n+1}(U_{n+1}) = 0$, $d_{n+1}(v_{i,j}) = d_n(v_{i,j}) = u_i u_j$ for $1 \leq i < j \leq n$ and $d_{n+1}(v_{0,j}) = u_0 u_j$ for $1 \leq j \leq n$.

Further, if $\overline{\mathcal{M}}_{n+1}$ denotes $\mathcal{M}_n \otimes \Lambda V_0$, we have the following extension:

(2) $$\Lambda u_0 \to (\mathcal{M}_{n+1}, d_{n+1}) \to (\overline{\mathcal{M}}_{n+1}, \bar{d}_{n+1}).$$

Consider the short exact sequence of differential graded vector spaces

(3) $$0 \longrightarrow \overline{\mathcal{M}}_{n+1} \xrightarrow{\ j\ } \mathcal{M}_{n+1} \xrightarrow{\ p\ } \overline{\mathcal{M}}_{n+1} \longrightarrow 0,$$

where p denotes the projection and j the map defined by $j(\chi) = u_0 \chi$ for $\chi \in \Lambda W$. The ensuing long exact sequence in cohomology has a connecting homomorphism that may be described as follows: On $\overline{\mathcal{M}}_{n+1}$, define $\theta^* : \overline{\mathcal{M}}_{n+1} \to \overline{\mathcal{M}}_{n+1}$ by $u_0 \theta = d_{n+1} - \bar{d}_{n+1}$. Then, one can easily check that θ is actually a derivation of the algebra $\overline{\mathcal{M}}_{n+1}$ of degree $1 - |u_0|$ which commutes with $\bar{d}_{n+1}$, and so induces a map $\theta^* : H(\overline{\mathcal{M}}_{n+1}) \to H^{*-|u_0|+1}(\overline{\mathcal{M}}_{n+1})$ which is the connecting homomorphism of (3). We will call the derivation θ^* the *Wang derivation* for (2), for reasons indicated in Remark 2.2 below.

This Wang derivation can be explicitly computed as follows. Because $d_{n+1}(V_0) \subseteq u_0 \cdot U_n$, we see that $H(\overline{\mathcal{M}}_{n+1}, \bar{d}_{n+1}) = H(\mathcal{M}_n) \otimes \Lambda V_0$. Now, for $\chi \in \mathcal{M}_n \otimes \Lambda V_0$, we find that $\theta(\mathcal{M}_n) = 0$, directly from the definition. Since $d_{n+1}(v_{0,i}) = u_0 u_i$, we have $\theta(v_{0,i}) = u_i$ for each $i = 1, \ldots, n$. On passing to cohomology, therefore, we find that $\theta^*\big(H(\mathcal{M}_n)\big) = 0$ and $\theta^*(v_{0,i}) = u_i \in H(\mathcal{M}_n)$ for $i = 1, \ldots, n$.

As usual, we can condense the long exact sequence coming from (3) into

$$0 \to \operatorname{coker} \theta^* \to H(\mathcal{M}_{n+1}) \to \ker \theta^* \to 0,$$

so that to complete the induction, it simply remains to show that

$$\dim \ker \theta^* + \dim \operatorname{coker} \theta^* = 2 \dim \ker \theta^*$$

is at least $2^{\binom{n+1}{2}}$. Our inductive assumption is that $\dim H(\mathcal{M}_n) \geq 2^{\binom{n}{2}}$. Therefore $\dim \big(H(\mathcal{M}_n) \cdot \Lambda V_0\big) \geq 2^{\binom{n}{2}} \times 2^n$. Since $\binom{n}{2} + n = \binom{n+1}{2}$, our inductive assumption implies that $\dim \big(H(\mathcal{M}_n) \otimes \Lambda V_0\big) \geq 2^{\binom{n+1}{2}}$.

We claim that $(\theta^*)^2 = 0$, after which elementary linear algebra completes the proof. The key observation here is the following: Denote by $[u_i] \in H(\mathcal{M}_n)$ the class represented by the cocycle u_i. Then the ideal of $H(\mathcal{M}_n)$ generated by all two-fold products $\{[u_i][u_j]\}_{1 \leq i < j \leq n}$ is zero. This follows since each product $u_i u_j$ is a boundary, namely $d_n(v_{i,j})$. Recall that $\theta^*\big(H(\mathcal{M}_n)\big) = 0$, whilst $\theta^*(V_0) = U_n \subseteq H(\mathcal{M}_n)$. Thus, $(\theta^*)^2\big(H(\mathcal{M}_n) \otimes \Lambda V_0\big)$ is contained in the ideal generated by the two-fold products $\{[u_i][u_j]\}_{1 \leq i < j \leq n}$, which we observed is zero. We conclude that $(\theta^*)^2 = 0$.

But $(\theta^*)^2 = 0$ implies $2 \dim \ker \theta^* \geq \dim \big(H(\mathcal{M}_n) \otimes \Lambda V_0\big)$. Therefore, we have

$$\dim H(\mathcal{M}_{n+1}) = 2 \dim \ker \theta^* \geq \dim\big(H(\mathcal{M}_n) \otimes \Lambda V_0\big) \geq 2^{\binom{n+1}{2}}.$$

This completes the induction.   □

Remark 2.2. Topologically, the extension sequence (2) corresponds to a fibration with base an odd sphere S^{2r+1}, and the long exact sequence in cohomology induced from (3) corresponds to the Wang sequence of the fibration, in the usual sense of a fibration with base a sphere. The derivation θ^* corresponds to the Wang derivation of the fibration, again in the usual sense (cf. [Whi78, p.319]).

As promised, the main result is now obtained by reducing the general two-stage case to that of Proposition 2.1.

Theorem 2.3. *Suppose $\Lambda(U,V)$ is any two-stage, elliptic minimal model. Then*

$$\dim H\big(\Lambda(U,V)\big) \geq 2^{\dim V - \dim U^{\mathrm{even}}}.$$

Proof. Given $\Lambda(U,V)$, form the following extension sequence:

$$(4) \qquad \Lambda(U,V) \to \Lambda(U,V) \otimes \Lambda(\overline{W},W), D \to \Lambda(\overline{W},W), \overline{D} = 0,$$

in which $D \colon \overline{W} \to U^{\mathrm{even}}$ and $d \colon W \to \Lambda^2(U^{\mathrm{odd}})$ are vector space isomorphisms. This extension sequence has an associated Serre spectral sequence, obtained by filtering the total space by degree in $\Lambda(U,V)$. This spectral sequence has E_2-term isomorphic to $H(\Lambda(U,V),d) \otimes H(\Lambda(\overline{W},W), \overline{D} = 0) \cong H(\Lambda(U,V),d) \otimes \Lambda(\overline{W},W)$ and it converges to $H(\Lambda(U,V) \otimes \Lambda(\overline{W},W), D)$ (cf. [FHT01, Sec.18], especially Example 2 and Exercise 2 of that section). Since the E_2-term of this spectral sequence must be at least as large as the term that it converges to, we have the following inequality:

$$(5) \quad \dim\big(H(\Lambda(U,V),d)\big) \cdot 2^{\dim \overline{W}} \cdot 2^{\dim W} \geq \dim\big(H(\Lambda(U,V) \otimes \Lambda(\overline{W},W), D)\big).$$

We now claim that $(\Lambda(U,V) \otimes \Lambda(\overline{W},W), D)$ and $(\Lambda(U^{\mathrm{odd}},W) \otimes \Lambda(V), D \otimes 1)$ are quasi-isomorphic. To see this, argue as follows. First, display the middle term of (4) as the middle term of the following extension sequence:

$$(6) \quad (\Lambda(U^{\mathrm{even}}, \overline{W}), D) \to (\Lambda(U,V) \otimes \Lambda(\overline{W},W), D) \to (\Lambda(U^{\mathrm{odd}},W,V), \overline{D}),$$

in which $(\Lambda(U^{\mathrm{even}}, \overline{W}), D)$ is an acyclic model. It follows that the projection $(\Lambda(U,V) \otimes \Lambda(\overline{W},W), D) \to (\Lambda(U^{\mathrm{odd}},W,V), \overline{D})$ is a quasi-isomorphism. Next, define a map of models $\phi \colon (\Lambda(U^{\mathrm{odd}},W) \otimes \Lambda(V), D \otimes 1) \to (\Lambda(U^{\mathrm{odd}},W,V), \overline{D})$ as follows. Set ϕ to be the identity on $\Lambda(U^{\mathrm{odd}},W)$. For each generator $v_i \in V$, we have $\overline{D}(v_i) \in \Lambda^{\geq 2}U^{\mathrm{odd}} \subseteq \overline{D}(W) \cdot \Lambda(U^{\mathrm{odd}})$, since $\overline{D}(W) \cong \Lambda^2(U^{\mathrm{odd}})$. So choose an element $x_i \in W \cdot \Lambda(U^{\mathrm{odd}})$, such that $\overline{D}(x_i) = \overline{D}(v_i)$, for each i. Finally, set $\phi(v_i) = v_i - x_i$ and extend to an algebra map. Notice that ϕ makes the following diagram of extension sequences commute:

$$
\begin{array}{ccccc}
(\Lambda(U^{\mathrm{odd}},W), \overline{D}) & \longrightarrow & (\Lambda(U^{\mathrm{odd}},W), \overline{D}) \otimes \Lambda V, D \otimes 1 & \longrightarrow & \Lambda V, \overline{D \otimes 1} = 0 \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle 1} \\
(\Lambda(U^{\mathrm{odd}},W), \overline{D}) & \longrightarrow & (\Lambda(U^{\mathrm{odd}},W,V), \overline{D}) & \longrightarrow & \Lambda V, \overline{D} = 0
\end{array}
$$

It follows that ϕ is a quasi-isomorphism.

Returning to the inequality (5) displayed above, we now have

$$\dim H\big(\Lambda(U,V)\big) \cdot 2^{\dim \overline{W}} \cdot 2^{\dim W} \geq \dim H\big(\Lambda(U^{\mathrm{odd}},W)\big) \cdot 2^{\dim V}.$$

But $\dim H\big(\Lambda(U^{\mathrm{odd}},W)\big) \geq 2^{\dim W}$, by Proposition 2.1. Since $\dim \overline{W} = \dim U^{\mathrm{even}}$, the result follows. $\qquad\square$

We can establish Conjecture 1.1 in several cases using Theorem 2.3. We begin with a discussion of two previously-known cases of Conjecture 1.1. We include these cases here to illustrate that Theorem 2.3 can be used to unify them with other results. Also, we feel it worthwhile to include a self-contained proof in the second case that uses techniques from well within rational homotopy theory.

2.2. **The pure case.** For any elliptic minimal model ΛW, the *homotopy Euler characteristic* is $\chi_\pi := \dim(W^{\mathrm{even}}) - \dim(W^{\mathrm{odd}})$. For any elliptic space, a result due to Allday and Halperin [AH78, Th.1] asserts that the rational toral rank is less than or equal to the negative of the homotopy Euler characteristic, thus $\mathrm{rk}_0(X) \leq -\chi_\pi$.

Now consider the so-called *pure* case, in which $\Lambda(U, V)$ is a two-stage, elliptic minimal model with $U = U^{\mathrm{even}}$ and $V = V^{\mathrm{odd}}$. In this case, Theorem 2.3 specializes to: $\dim H\big(\Lambda(U, V)\big) \geq 2^{-\chi_\pi}$. Combining these observations, we retrieve the following result of Halperin:

Corollary 2.4 ([Hal85, Prop.1.5]). *Let X be an elliptic space with a pure minimal model. Then Conjecture 1.1 holds for X, i.e., $\dim H(X; \mathbb{Q}) \geq 2^{\mathrm{rk}_0(X)}$.*

We remark that this includes the case in which X is a homogeneous space. Indeed, our proof of Theorem 2.3 incorporates the argument used by Halperin to obtain Conjecture 1.1 in the homogeneous space case.

2.3. **The case of odd generators with quadratic differential (coformal spaces).** We say that a minimal model is coformal if it has a *quadratic differential*, that is, if $d \colon W \to \Lambda^2 W$. Notice that this really means quadratic with respect to a particular choice of generators for ΛW. A space is coformal if it has a coformal minimal model, and in this case its rational homotopy type is completely determined by its rational homotopy Lie algebra, the bracket of which is dual to d [FHT01, §21].

The following result is due to Allday and Puppe:

Theorem 2.5 ([AP85a, Prop.3.1, Cor.3.5]). *Let X be a simply-connected, coformal elliptic space that has a two-stage minimal model $\Lambda(U, V)$ with odd-degree generators only. Assume that the two-stage decomposition displays V with maximal dimension. Then, $\mathrm{rk}_0(X) = \dim V$ and Conjecture 1.1 holds for X.*

Proof. The equality $\mathrm{rk}_0(X) = \dim V$ is given in [AP85a, Cor.3.5]. (Actually, the rational toral rank is identified there with the dimension of the centre of the homotopy Lie algebra of X. However, standard ideas from rational homotopy theory can be used to identify $\dim V$ and the dimension of the centre of the homotopy Lie algebra, in this particular case.) Since the minimal model has odd-degree generators only, Theorem 2.3 specializes to the inequality $\dim H\big(\Lambda(U, V)\big) \geq 2^{\dim V}$. ☐

This paper contains a self-contained proof of Theorem 2.5 that uses only standard tools from rational homotopy theory. For this, we need to supply our own argument for the equality $\mathrm{rk}_0(X) = \dim V$, to substitute for [AP85a, Cor.3.5]. This is obtained by combining Lemma 2.12 with Theorem 2.9 below, both of which apply to more general settings. (cf. Remarks 2.14). When we prove Theorem 2.9, we will be more specific as to what the phrase 'displays V with maximal dimension' entails.

We remark that Allday-Puppe conclude their result by appealing to a result of Deninger-Singhof [DS88]. Indeed, the result from [DS88] used to complete their argument is precisely a special case of Theorem 2.3, in which the differential is

assumed to be quadratic and all generators are assumed to be of odd degree. It is therefore natural to ask whether $\mathrm{rk}_0\big(\Lambda(U,V)\big) = \dim V$ more generally, at least in the case in which the two-stage, elliptic minimal model has odd-degree generators only. This is not true without some further hypotheses, as we illustrate in Example 3.2. We return to this point, and related questions, in the last section of the paper.

2.4. **New cases.** In the remainder of this section, we prove new results that give further situations to which we can apply Theorem 2.3. In phrasing our results, we must be careful about a technical point, namely, the choice of a two-stage decomposition that displays V with maximal dimension. We illustrate this with an example.

Example 2.6. Let $\Lambda(U,V;d)$ be a two-stage minimal model with bases $U = \langle u_1, u_2, u_3 \rangle$ and $V = \langle v_1 \rangle$, all generators being of degree 1. If we set $dv_1 = u_1 u_3 - u_1 u_2 + u_2 u_3$, then each basis element of U certainly occurs in a non-trivial differential. However, V is not maximal here. We can see this by re-writing $dv_1 = (u_1 + u_2)(u_1 + u_3)$, and noting that only 2 linearly independent elements of U actually occur. In other words, we can change the basis in U, and move the 'spare' generator of U into V. Specifically, define $\phi(u_1) = u_1$, $\phi(u_2) = u_2 + u_1$, $\phi(u_3) = u_3 + u_1$ and $\phi(v_1) = v_1$, then extend to an algebra automorphism of $\Lambda(U,V)$. If we set $d' = \phi^{-1} d \phi$, then ϕ becomes an isomorphism of minimal models $\phi \colon \Lambda(U,V;d') \to \Lambda(U,V;d)$. A simple check reveals that $\Lambda(U,V;d')$ is two-stage with $d'(v_1) = u_2 u_3$. Thus we can write it as $\Lambda(U',V';d')$, $U' = \langle u_2, u_3 \rangle$ and $V' = \langle v_1, u_1 \rangle$. In this latter case, V' is now of maximal dimension.

Definition 2.7. Suppose $(\Lambda(U,V), d)$ is a two-stage minimal model. We say that V *has maximal dimension*, or that the two-stage decomposition displays V with maximal dimension, if, for any isomorphic two-stage minimal model $(\Lambda(U',V'), d') \cong (\Lambda(U,V), d)$, we have $\dim V' \le \dim V$.

Since we assume that the spaces of generators are finite-dimensional, it is clear that every two-stage minimal model has a decomposition that displays V with maximal dimension. Also, for the two-stage decomposition to display V with maximal dimension, then it is clearly necessary (but not sufficient) that every generator from U appear in some differential.

We now give a consequence of the two-stage decomposition being chosen so as to display V of maximal dimension. This result does not cover all two-stage cases, but is sufficient for our purposes. We focus on the case in which the differential is quadratic. For parity of degree reasons, in this case we have $d(V) \subseteq \Lambda^2(U^{\mathrm{odd}}) \oplus \Lambda^2(U^{\mathrm{even}})$. Let $U^{\mathrm{odd}} = \langle u_1, \ldots, u_p \rangle$, $U^{\mathrm{even}} = \langle w_1, \ldots, w_q \rangle$ and $V = \langle v_1, \ldots, v_r \rangle$ denote bases. From now on, we will adopt this as standard notation, for the case of a two-stage minimal model that may contain even-degree generators.

Since we assume the differential is quadratic, for each basis element v_k of V, we may write

$$dv_k = \sum_{1 \le i < j \le p} a^k_{i,j}\, u_i u_j + \sum_{1 \le i \le j \le q} b^k_{i,j}\, w_i w_j.$$

Using the coefficients from the first of these sums, for $k = 1, \ldots, r$, we define the skew-symmetric, $p \times p$ matrix M^k, by

$$M_{i,j}^k = \begin{cases} a_{i,j}^k & \text{if } i < j \\ -a_{i,j}^k & \text{if } i > j \\ 0 & \text{if } i = j \end{cases}$$

Now, let $M = \begin{bmatrix} M^1 \\ \vdots \\ M^r \end{bmatrix}$ be the $pr \times p$ block marix formed by the M^k as rows.

Lemma 2.8. *If V has maximal dimension in $\Lambda(U, V)$, then $\dim M = p$. In particular, there is a $p \times rp$ matrix N such that NM is the $p \times p$ identity matrix.*

Proof. Let U^* denote the dual space of U, and for $u^* \in U^*$, let i_{u^*} denote the derivation of $\Lambda(U, V)$ of degree -1 extending the linear map $i_{u^*} : U \oplus V \to \mathbb{Q}$ defined by $i_{u^*}(x + y) = u^*(x)$, for $x \in U$, $y \in V$. We first show that, under our hypotheses, the Lie derivative $\mathcal{L} : U^* \to \mathrm{Hom}(V, U)$ defined by

$$\mathcal{L}(u^*) = i_{u^*}d - (-1)^{|u^*|}di_{u^*} = i_{u^*}d$$

is injective. (In fact, the injectivity of $\mathcal{L}$ is equivalent to the maximality of $\dim V$.) Let $K = \ker \mathcal{L}$ and choose a complement $X \subset U^*$ for K so that $U^* = K \oplus X$ and $U = K^\perp \oplus X^\perp$, where, if $W \subset U^*$, $W^\perp = \{u \in U \mid \forall w \in W, w(u) = 0\}$. Then, as

$$\Lambda U = (\Lambda X^\perp \otimes \Lambda X^\perp) \oplus (\Lambda X^\perp \otimes \Lambda^+ K^\perp) \oplus (\Lambda^+ K^\perp \otimes \Lambda^+ K^\perp),$$

the definition of K shows that $dV \subset \Lambda^+ K^\perp \otimes \Lambda^+ K^\perp$. Now define $U_1 = K^\perp$ and $V_1 = V \oplus X^\perp$, with $d'_{U_1} = 0$, $d'_{|V} = d_{|V}$ and $d'_{X^\perp} = 0$. Then $U_1 \oplus V_1 = U \oplus V$, and this induces an isomorphism of two-stage minimal models $\Lambda(U_1, V_1; d') \cong \Lambda(U, V; d)$. Since V has maximal dimension, K must be 0.

Let $\{u_1, \ldots, u_p\}$ be a basis of U^{odd}, and $\{u_1^*, \ldots, u_p^*\}$ the dual basis. Then, as V has maximal dimension, we know that in particular the maps $\mathcal{L}(u_j^*) = i_{u_j^*}d : V \to U^{\mathrm{odd}}$ for $j = 1, \ldots, p$ are linearly independent. In other words, if $c = (c_1, \ldots, c_p)^t \in \mathbb{Q}^p$, then $\sum_j c_j i_{u_j^*}dv_k = \sum_{i,j} M_{i,j}^k u_i c_j = 0$ for $k = 1, \ldots, r$, implies that $c = 0$. Because the u_i are linearly independent, this is equivalent to the statement

$$\sum_j M_{i,j}^k c_j = 0 \ \text{ for } i = 1, \ldots, p \text{ and } k = 1, \ldots, r \quad \Longrightarrow \quad c = 0.$$

That is, $Mc = 0 \implies c = 0$. Hence, $\dim M = p$. $\qquad\square$

To phrase our results, we use the following terminology: We say that a graded vector space V is *n-co-connected* if $V^i = 0$ for $i \geq n$.

Theorem 2.9. *Let $\Lambda(U, V)$ be an elliptic, two-stage minimal model. Assume that $dV \subseteq \Lambda^2 U$, and that the two-stage decomposition displays V with maximal dimension. Finally, assume that U^{odd} and U^{even} satisfy one of the following connectivity and co-connectivity hypotheses:*

(A) *U^{odd} is $(2r - 1)$-connected and U^{even} is $(2r + 2)$-co-connected.*
(B) *U^{odd} is $(2s + 1)$-co-connected and U^{even} is $(4s - 4)$-connected.*

Then $\mathrm{rk}_0\big(\Lambda(U, V)\big) \leq \dim V - \dim U^{\mathrm{even}}$.

Proof. If $\mathrm{rk}_0\big(\Lambda(U, V)\big) = n$, then we have a K-S-extension $(\Lambda A_n \otimes \Lambda(U, V), D)$ as in (1) that has finite-dimensional cohomology. We claim that in any such minimal model, we can assume that $D(U^{\mathrm{odd}})$ is contained in $I(U^{\mathrm{odd}} \oplus V)$, the ideal in $\Lambda A_n \otimes \Lambda(U, V)$ generated by $U^{\mathrm{odd}} \oplus V$.

Allowing this claim for the time-being, we appeal to some results of Halperin concerning elliptic minimal models. First of all, to any elliptic minimal model $(\Lambda W, d)$, there is an associated *pure model*, denoted $(\Lambda W, d_\sigma)$, which is defined by adjusting the differential d to d_σ as follows: We set $d_\sigma = 0$ on each even degree generator of W, and on each odd degree generator $w \in W$, we set $d_\sigma(w)$ equal to the part of $d(w)$ contained in $\Lambda(W^{\mathrm{even}})$. One checks that this defines a differential d_σ on ΛW, and thus we obtain a pure (minimal) model $(\Lambda W, d_\sigma)$. Then, by [Hal77, Prop.1] (cf. also [FHT01, Prop.32.4]), $\dim H(\Lambda W, d)$ is finite-dimensional if and only if $\dim H(\Lambda W, d_\sigma)$ is finite-dimensional. Applying all this to the minimal model $(\Lambda A_n \otimes \Lambda(U, V), D)$, we obtain the following:

From the claim, $(\Lambda A_n \otimes \Lambda(U, V), D_\sigma)$ satisfies $D_\sigma(U^{\mathrm{odd}}) = 0$, and therefore $(\Lambda A_n \otimes \Lambda(U, V), D_\sigma) \cong (\Lambda U^{\mathrm{odd}}, D_\sigma = 0) \otimes (\Lambda A_n \otimes \Lambda(U^{\mathrm{even}}, V), D_\sigma)$. Since $(\Lambda A_n \otimes \Lambda(U, V), D_\sigma)$ has finite-dimensional cohomology, so does $(\Lambda A_n \otimes \Lambda(U^{\mathrm{even}}, V), D_\sigma)$. It follows that $(\Lambda A_n \otimes \Lambda(U^{\mathrm{even}}, V), D_\sigma)$ is an elliptic minimal model, and therefore that its homotopy Euler characteristic is non-positive [Hal77, Th.1] (cf. also [FHT01, Prop.32.10]). This implies that $n \leq \dim V - \dim U^{\mathrm{even}}$, as required.

It only remains to establish the claim. We do this by a careful analysis of the terms that can occur in the differentials. Our argument is essentially the same as that in [AP85b, Th.4.6]. First we write, for each v_k in the basis of V,

$$(7) \quad Dv_k = \sum_{1 \leq i < j \leq p} a_{i,j}^k\, u_i u_j \; + \sum_{1 \leq i \leq j \leq q} b_{i,j}^k\, w_i w_j \; + \; \text{terms in the ideal } I(A_n).$$

The coefficients $a_{i,j}^k$ and $b_{i,j}^k$ are scalars. Our strategy in either case (A) or case (B) is the same. We apply D to (7), obtaining $D^2 v_k = 0$ on the left-hand side. By focussing on certain terms present on the right-hand side, we obtain sufficient information to complete our argument.

Consider hypothesis (A) first. Here, from the (co-)connectivity restrictions, we see that $D(U^{\mathrm{even}}) \subseteq \Lambda^+ A_n \otimes \Lambda U^{\mathrm{even}} \otimes \Lambda^+ V$. Therefore, display the terms in (7) as follows. Write

$$(8) \quad \begin{aligned} Dv_k = &\sum_{1 \leq i < j \leq p} a_{i,j}^k\, u_i u_j + \sum_{1 \leq i < j \leq p} P_{i,j}^k\, u_i u_j \; + \sum_{\substack{i=1,\dots,r \\ j=1,\dots,p}} R_{i,j}^k\, v_i u_j \\ &+ \; P_k \; + \; \text{terms in the ideal } I\big(\Lambda^2(V) \oplus \Lambda^{\geq 4}(U^{\mathrm{odd}}, V)\big). \end{aligned}$$

Here, P_k includes all terms from $\Lambda A_n \otimes \Lambda(U^{\mathrm{even}})$ present in (7), and the coefficients $P_{i,j}^k$ and $R_{i,j}^k$ denote terms in $\Lambda^+ A_n$. After applying D to this, we will consider only contributions in $\big(\Lambda^+ A_n \otimes \Lambda(U^{\mathrm{even}})\big) \cdot U^{\mathrm{odd}}$. Define a derivation δ of $\Lambda A_n \otimes \Lambda(U, V)$ as follows. On generators from $U^{\mathrm{odd}} \oplus V$, set δ equal to the part of $D\colon U^{\mathrm{odd}} \oplus V \to \Lambda A_n \otimes \Lambda(U, V)$ whose image is contained in $\Lambda^+ A_n \otimes \Lambda(U^{\mathrm{even}})$. On the remaining generators, set $\delta(U^{\mathrm{even}}) = 0$. Extend δ to a derivation on $\Lambda(A_n, U, V)$. Applying D to (8), and collecting terms in $\big(\Lambda^+ A_n \otimes \Lambda(U^{\mathrm{even}})\big) \cdot U^{\mathrm{odd}}$, we have

$$0 = \delta(\sum_{1 \leq i < j \leq p} a_{i,j}^k\, u_i u_j) + \delta(\sum_{1 \leq i < j \leq p} P_{i,j}^k\, u_i u_j) + \sum_{\substack{i=1,\dots,r \\ j=1,\dots,p}} \delta(R_{i,j}^k v_i) u_j.$$

We fix an index l, with $1 \leq l \leq p$, and collect all coefficients of u_l from this equation, to obtain

$$0 = \sum_{i<l} a_{i,l}^k \, \delta(u_i) - \sum_{l<j} a_{l,j}^k \, \delta(u_j) + \sum_{i<l} \delta(P_{i,l}^k \, u_i) - \sum_{l<j} \delta(P_{l,j}^k \, u_j) + \sum_{i=1,\ldots,r} \delta(R_{i,l}^k v_i).$$

Since $P_{i,j}^k, R_{i,j}^k \in \Lambda^+ A_n$, and since M^k is skew-symmetric, we obtain

$$\sum_{i \neq l} a_{l,i}^k \, \delta(u_i) = \delta(\chi_{k,l})$$

for some element $\chi_{k,l} \in \Lambda^+ A_n \cdot (U \oplus V)$. Let $\mathbf{u} = (u_1, \ldots, u_p)^t$, $\sigma_k = (\chi_{k,1}, \ldots, \chi_{k,p})$, and $\chi = (\sigma_1, \ldots, \sigma_r)^t$, and let δ act component-wise. Then the previous equation can be restated as

$$\delta(M\mathbf{u}) = \delta\chi.$$

Now we apply Lemma 2.8 to obtain a $p \times rp$ matrix N such that

$$\delta\mathbf{u} = \delta(NM\mathbf{u}) = \delta(N\chi).$$

Finally, we observe that this implies for each basis element of U^{odd}, we have $\delta(u_i) = \delta(\chi_i)$ for some decomposable $\chi_i \in \Lambda^+ A_n \cdot (U^{\mathrm{odd}} \oplus V)$. Returning to the full differential D, we rephrase this as follows: For each basis element u_i of U^{odd}, there is some $\chi_i \in \Lambda^+ A_n \cdot (U^{\mathrm{odd}} \oplus V)$ for which $D(u_i - \chi_i) \in I(U^{\mathrm{odd}}, V)$.

As in Example 2.6, we now change the basis in U^{odd}. Define an isomorphism of algebras $\phi \colon \Lambda A_n \otimes \Lambda(U, V) \to \Lambda A_n \otimes \Lambda(U, V)$ on generators by setting $\phi(u_i) = u_i - \chi_i$ for each basis element u_i of U^{odd}, and $\phi = \mathrm{id}$ on generators of A_n, U^{even} and V. Then extend ϕ to an algebra isomorphism. Now define a differential by $D' = \phi^{-1}D\phi$, so that ϕ becomes an isomorphism of minimal models $\phi \colon (\Lambda A_n \otimes \Lambda(U, V), D') \to (\Lambda A_n \otimes \Lambda(U, V), D)$. An easy check now confirms that $D'(A_n) = 0$ and $D'(U^{\mathrm{odd}}) \subseteq I(U^{\mathrm{odd}}, V)$, as desired.

A similar argument is used under hypothesis (B). Here, if some coefficient $a_{i,j}^k \neq 0$ in (7), then the degree of Dv^k must be at most $4s - 2$. But under hypothesis (B), terms in $\Lambda^2(U^{\mathrm{even}})$ start in degree $8s - 4$ and terms in $\Lambda^+ A_n \cdot \Lambda U^{\mathrm{even}}$ start in degree $4s$. In this case, therefore, whenever some $a_{i,j}^k \neq 0$, then the remaining terms of (7) do not include any in the ideal $I(U^{\mathrm{even}})$. So suppose that v^k is of suitable degree for $D(v^k)$ to contain non-zero terms $a_{i,j}^k \, u_i u_j$. Display the terms in (7) as follows. Write

$$
\begin{aligned}
(9) \qquad D(v_k) = &\sum_{1 \leq i < j \leq p} a_{i,j}^k \, u_i u_j \;+\; \sum_{1 \leq i < j \leq p} P_{i,j}^k \, u_i u_j + \sum_{\substack{i=1,\ldots,r \\ j=1,\ldots,p}} R_{i,j}^k \, v_i u_j \\
&+ P_k + \text{terms in } I\big(\Lambda^{\geq 2} V \oplus \Lambda^{\geq 4} U^{\mathrm{odd}}\big),
\end{aligned}
$$

where P_k denotes a term from $\Lambda^+ A_n$ and the coefficients $P_{i,j}^k$ and $R_{i,j}^k$ denote terms in $\Lambda^+ A_n$. After applying D to this, we will consider only contributions in $(\Lambda^+ A_n) \cdot U^{\mathrm{odd}}$. To this end, define a derivation δ of $\Lambda A_n \otimes \Lambda(U, V)$ as follows. On generators from $U^{\mathrm{odd}} \oplus V$, set δ equal to the part of $D \colon U^{\mathrm{odd}} \oplus V \to \Lambda A_n \otimes \Lambda(U, V)$ whose image is contained in $\Lambda^+ A_n$. On the remaining generators, set $\delta(U^{\mathrm{even}}) = 0$. Applying D to (9), and collecting terms in $(\Lambda^+ A_n) \cdot U^{\mathrm{odd}}$ yields

$$0 = \delta\big(\sum_{1 \leq i < j \leq p} a_{i,j}^k \, u_i u_j \big) + \delta\big(\sum_{1 \leq i < j \leq p} P_{i,j}^k \, u_i u_j \big) + \sum_{\substack{i=1,\ldots,r \\ j=1,\ldots,p}} \delta(R_{i,j}^k v_i) u_j,$$

for each index k that has at least one coefficient $a_{i,j}^k$ non-zero. From here, the argument proceeds exactly as that for hypothesis (A).

This completes the proof of the claim, under either hypothesis. By the first part of the proof, the result follows. □

There are other situations in which a similar approach gives an upper bound on the toral rank. We give one more result that moves away from the quadratic differential restriction.

Theorem 2.10. *Let $\Lambda(U, V)$ be a two-stage minimal model with odd degree generators only and assume that the two-stage decomposition displays V with maximal dimension. Suppose there are positive integers r and s, with $r \leq s \leq 2r$, for which $U = U^{\mathrm{odd}}$ is $(r-1)$-connected and $(s+1)$-co-connected. Suppose further that there are positive integers t and u, with $s \leq t \leq u \leq s+r$, for which V is $(t-1)$-connected and $(u+1)$-co-connected. Then $\mathrm{rk}_0\big(\Lambda(U, V; d)\big) \leq \dim V$,*

Proof. If $\mathrm{rk}_0\big(\Lambda(U, V; d)\big) = n$, then we have an extension $(\Lambda A_n \otimes \Lambda(U, V), D)$ with finite-dimensional cohomology. In such an extension, from the assumptions on the degrees of the generators, we must have $Du_i \in \Lambda A_n$ for each generator u_i of U. We claim that in fact $Du_i = 0$. Suppose the shortest length of any non-zero term that occurs in some polynomial Du_i is l. Then as in the proof of Theorem 2.9, define a derivation δ on U by setting δu_i equal to the term of length l that occurs in Du_i, or zero if there is no such term. Define δ to be zero on all generators of A_n and V, and extend as a derivation to $\Lambda A_n \otimes \Lambda(U, V)$. Without loss of generality, we can suppose that the non-zero terms δu_i are linearly independent. Otherwise, we can change basis within U to make these so. Next, consider Dv for a generator in V. We have $Dv = dv + \chi$, for $dv \in \Lambda U$ and some χ in the ideal of $\Lambda A_n \otimes \Lambda(U, V)$ generated by A_n. The (co)-connectivity hypotheses on V imply that $Dv \in \Lambda A_n \otimes \Lambda U$, for any $v \in V$. Therefore, $\chi \in \Lambda^+ A_n \otimes \Lambda U$ and hence $D(\chi) \in \Lambda^{>l} A_n \otimes \Lambda U$. Consequently, when we equate terms in $\Lambda^l A_n \otimes \Lambda U$ that arise in the equation $0 = D^2 v = D(dv) + D(\chi)$, we have $0 = \delta(dv)$. By the assumption that non-zero δu_i's are linearly independent, it follows from an easy argument that $\delta u_i = 0$ for each u_i that occurs in the differential dv. Finally, our assumption that V is taken as large as possible implies that each generator u_i does occur in at least one differential dv. This implies that the shortest length terms δu_i are zero, and an induction completes the proof of our claim, namely that $Du_i = 0$ for each $u_i \in U$.

So far, we have argued that in any extension $(\Lambda A_n \otimes \Lambda(U, V), D)$, our hypotheses imply that $D(U) = 0$. Now an argument as in the second paragraph of the proof of Theorem 2.9 shows that $n \leq \dim V$. □

Corollary 2.11. *Let X be a simply connected, elliptic space with two-stage minimal model. If the minimal model satisfies either the hypotheses of Theorem 2.9, or those of Theorem 2.10, then Conjecture 1.1 holds for X.*

Proof. Combine Theorem 2.3 with the results mentioned in the statement. □

It is natural to ask whether $\mathrm{rk}_0(X) \leq \dim V - \dim U^{\mathrm{even}}$ for a general two-stage, elliptic space. This is not the case in general, as we illustrate in Example 3.1 below. Indeed, under hypotheses that include all cases in which the two-stage minimal model has odd-degree generators only, we can actually reverse the direction in this inequality.

Lemma 2.12. *Suppose $\Lambda U \otimes \Lambda V$ is an elliptic two-stage minimal model with $U^{\mathrm{even}} = U^{2n}$ for some fixed $n \in \mathbb{N}$. Then, $\mathrm{rk}_0(\Lambda U \otimes \Lambda V) \geq \dim V - \dim U^{\mathrm{even}}$.*

Proof. Consider the extension sequence

$$(\Lambda U^{\mathrm{odd}}, 0) \to (\Lambda U^{\mathrm{odd}} \otimes \Lambda U^{\mathrm{even}} \otimes \Lambda V, d) \to (\Lambda U^{\mathrm{even}} \otimes \Lambda V, \bar{d}).$$

Since $V = V^{\mathrm{odd}}$, the right-hand term is now pure (and elliptic) with $U^{\mathrm{even}} = U^{2n}$. By [Jes90, Lem.3.3], there is an isomorphism of two-stage models

$$\rho \colon (\Lambda U^{\mathrm{even}} \otimes \Lambda V, \bar{d}) \to (\Lambda U^{\mathrm{even}} \otimes \Lambda(V' \oplus V''), \bar{d}')$$

such that

1. $\dim H(\Lambda U^{\mathrm{even}} \otimes \Lambda V', \bar{d}) < \infty$,
2. $\dim V' = \dim U^{\mathrm{even}}$,
3. ρ is the identity on U^{even}, and
4. $\rho(V) \subset V' \oplus V'' \oplus (\Lambda^+ U^{\mathrm{even}} \otimes (V' \oplus V''))$.

These conditions imply that ρ induces an isomorphism $V \cong V' \oplus V''$, and that $\rho^{-1}(V' \oplus V'') \subset V \oplus (\Lambda^+ U^{\mathrm{even}} \otimes V)$. We now extend ρ to an isomorphism of algebras

$$\tilde{\rho} \colon \Lambda U \otimes \Lambda V \to \Lambda U \otimes \Lambda(V' \oplus V'')$$

by letting it be the identity on ΛU. Define a derivation d' on $\Lambda U \otimes \Lambda(V' \oplus V'')$ by setting $d'U = 0$, and $d'v' = d\rho^{-1}(v')$, for $v' \in (V' \oplus V'')$. In order to show that

$$\tilde{\rho} \colon (\Lambda U \otimes \Lambda V, d) \to (\Lambda U \otimes \Lambda(V' \oplus V''), d')$$

is an isomorphism of models, it suffices to show that $(d')^2 v' = 0$ for $v' \in V' \oplus V''$, so we compute:

$$\begin{aligned}
(d')^2 v' &= d'(d\rho^{-1}(v')) \\
&\in d'(d(V \oplus (\Lambda^+ U^{\mathrm{even}} \otimes V))) \\
&\subset d'(\Lambda U) \\
&= \{0\}.
\end{aligned}$$

Thus, we may assume that $V = V^{\mathrm{odd}} = V' \oplus V''$, with $\dim H(\Lambda U \otimes \Lambda V') < \infty$, and $\dim V' = \dim U^{\mathrm{even}}$. Now suppose $V'' = \langle v_1, \ldots, v_n \rangle$, and define a K-S-extension

$$(\Lambda A_n, 0) \to (\Lambda A_n \otimes \Lambda(U, V), D) \to (\Lambda(U, V), d),$$

in which $A_n = \langle a_1, \ldots, a_n \rangle$ with $|a_i| = 2$, by $Dv_i = dv_i + a_i^{\frac{|v_i|+1}{2}}$, and $d = D$ otherwise. A standard argument (using the associated pure model, as in the proof of Theorem 2.9), now shows that $\dim H(\Lambda A_n \otimes \Lambda(U, V), D)$ is finite-dimensional, so that $\mathrm{rk}_0 \Lambda(U, V) \geq n = \dim V'' = \dim V - \dim U^{\mathrm{even}}$. $\quad\square$

Consequently, we obtain values for the rational toral rank in the following cases.

Corollary 2.13. *Let X be a simply connected elliptic space with two-stage minimal model $\Lambda(U, V)$. Suppose that the two-stage decomposition displays V with maximal dimension.*

(2.13.1) *If the minimal model has quadratic differential, satisfies one of the (co-) connectivity conditions of Theorem 2.9 and also satisfies $U^{\mathrm{even}} = U^{2n}$ for some fixed n, then $\mathrm{rk}_0(X) = \dim V - \dim U^{\mathrm{even}}$.*

(2.13.2) *If the minimal model has odd-degree generators only, and satisfies the (co-) connectivity conditions of Theorem 2.10, then $\mathrm{rk}_0(X) = \dim V$.*

Proof. Combine Lemma 2.12 with the results mentioned in the statement. □

Remarks 2.14. We can specialize (2.13.1) in a number of interesting directions. For example, if $U^{\text{even}} = 0$, then we retrieve the identification $\text{rk}_0(X) = \dim V$ of Theorem 2.5. As a second example, we can restrict to the case in which $U^{\text{odd}} = 0$. This is the case in which the minimal model is *pure* with quadratic differential, and also satisfies $U^{\text{even}} = U^{2n}$ for some fixed n. Here, we obtain $\text{rk}_0(X) = -\chi_\pi(X)$.

3. Examples, Comments and Questions

In this section we mention various examples and results. Our focus here is more on the exact toral rank, rather than the bound of Conjecture 1.1.

3.1. **The relation between** $\text{rk}_0(X)$ **and** $\dim V - \dim U^{\text{even}}$**.** We begin with the simplest example that we can find to illustrate that the inequality $\text{rk}_0(X) \leq \dim V - \dim U^{\text{even}}$ does not hold in general for a two-stage, elliptic space with both even and odd generators.

Example 3.1. Consider the two-stage minimal model $\Lambda(u_1, u_2, u_3, u_4, u_5, u_6, w, v)$, with $|u_1| = |u_2| = |u_3| = 3$, $|u_4| = 7$, $|u_5| = 23$, $|u_6| = 27$, $|w| = 18$ and $|v| = 35$, and the single non-trivial differential given by $dv = w^2 - u_1 u_2 u_4 u_5 - u_1 u_2 u_3 u_6$. The associated pure model satisfies $(\Lambda(U, V), d_\sigma) \cong (\Lambda(u_1, u_2, u_3, u_4, u_5, u_6), d_\sigma = 0) \otimes (\Lambda(w, v), d_\sigma)$, with $d_\sigma(w) = 0$ and $d_\sigma(v) = w^2$. Now $H(\Lambda(w, v), d_\sigma) \cong \Lambda(w)/(w^2)$, so $(\Lambda(U, V), d)$ is elliptic.

The two-stage decomposition $U = \langle u_1, u_2, u_3, u_4, u_5, u_6, w \rangle$ and $V = \langle v \rangle$ displays V with maximal dimension, so we have $\dim V - \dim U^{\text{even}} = 0$. We now show that $\text{rk}_0(\Lambda(U, V)) \geq 1$. Let a be a generator of degree 2, and define a differential D on $\Lambda a) \otimes \Lambda(U, V)$ as follows: $Da = Du_1 = Du_2 = Du_3 = Du_4 = Dw = 0$, $Du_5 = a^3 w$, $Du_6 = a^2 w u_1 u_2$ and $Dv = w^2 - u_1 u_2 u_4 u_5 - u_1 u_2 u_3 u_6 + a^{18} - a u_6 u_4$. A straightforward check shows that this defines a differential. We show that $(\Lambda a \otimes \Lambda(U, V), D)$ has finite-dimensional cohomology. The associated pure model in this case is $(\Lambda a \otimes \Lambda(U, V), D_\sigma) \cong (\Lambda(u_1, u_2, u_3, u_4, u_6), D_\sigma = 0) \otimes (\Lambda(a, w, u_5, v), D_\sigma)$, with $D_\sigma a = 0$, $D_\sigma(w) = 0$, $D_\sigma u_5 = a^3 w$ and $D_\sigma v = w^2 + a^{18}$. We observe that $D_\sigma(a^3 v) = a^3 w^2 + a^{21} = D_\sigma(u_5 w) + a^{21}$, hence $D_\sigma(a^3 v - u_5 w) = a^{21}$, so [Hal77, Prop.1] shows that $H(\Lambda a \otimes \Lambda(U, V), D_\sigma)$ is finite-dimensional. It follows that $H(\Lambda a \otimes \Lambda(U, V), D)$ is finite dimensional. Thus $\text{rk}_0(\Lambda(U, V)) \geq 1$.

We assert that more work will show $\text{rk}_0(\Lambda(U, V)) = 1$ in Example 3.1. We also remark that whilst Conjecture 1.1 does not follow from Theorem 2.3 in this example, it is nonetheless easily confirmed here.

Next, we specialize to the case of odd generators only. Unfortunately, this restriction alone does not give us the inequality $\text{rk}_0(X) \leq \dim V$. The following example lies immediately outside the hypotheses of Theorem 2.10, at least as far the generators of U are concerned. We thank Yves Félix for this example.

Example 3.2. Let $\Lambda W = \Lambda(U, V) = \Lambda(u_1, u_2, u_3, u_4, u_5, v_1, v_2)$ be a two-stage minimal model with degrees and differential as follows: $|u_1| = |u_2| = |u_3| = |u_4| = 3$, $|u_5| = 7$, $|v_1| = 9$, $|v_2| = 11$, $du_i = 0$ for each i, $dv_1 = u_1 u_5$ and $dv_2 = u_1 u_2 u_3 u_4$. Then $\dim V = 2$. We show that $\text{rk}_0(\Lambda W) \geq 3$. First we describe an extension $\Lambda A \otimes \Lambda W, D$, with $A = \langle a_1, a_2, a_3 \rangle$. Set $D = 0$ on $\{a_1, a_2, a_3, u_1, u_2, u_3\}$, and $Du_4 = -a_1^2$, $Du_5 = a_1 u_1 u_2$, $Dv_1 = u_1 u_5 + a_2^5$ and $Dv_2 = u_1 u_2 u_3 u_4 + u_3 u_5 a_1 + a_3^6$. A straightforward check shows that D is a differential. We argue exactly as in the

latter part of Example 3.1 to show that $H(\Lambda A \otimes \Lambda W, D)$ is finite-dimensional. It follows that $\mathrm{rk}_0(\Lambda W) \geq 3$.

Once again, Conjecture 1.1 does hold for Example 3.2, although we are unable to conclude this from our preceding results. To see this, use [AP85b, Th.4.6] to conclude $\mathrm{rk}_0(\Lambda W) \leq 5$. (In Example 3.2, the differential $dv_1 = u_1 u_5$ means that u_1 and u_5 correspond to non-central generators.) Moroever, the fibration with u_1 as base has fibre $\Lambda(u_2, u_3, u_4, u_5, v_1, v_2)$ with zero differential, and $\Lambda(u_2, u_3, u_4, u_5)$ is clearly in the kernel of the associated Wang derivation θ^*. As in the proof of Theorem 2.1, we conclude that $\dim H(\Lambda W) = 2 \dim \ker \theta^* \geq 2.2^4 = 2^5 \geq 2^{\mathrm{rk}_0(\Lambda W)}$.

Another example in which $\mathrm{rk}_0(X) > \dim V - \dim U^{\mathrm{even}}$ is given in Example 3.3. On the other hand, we currently have no example of a two stage minimal model for which $\mathrm{rk}_0(X) < \dim V - \dim U^{\mathrm{even}}$. This leads us to wonder whether the hypotheses of Lemma 2.12 might be relaxed considerably. Note that in certain cases, the inequality $\mathrm{rk}_0(X) \geq \dim V - \dim U^{\mathrm{even}}$ can be combined with other information to identify the toral rank, and we have seen instances of this in Corollary 2.13. As another example, whenever this latter inequality holds in a pure case, it identifies the toral rank as equal to $-\chi_\pi$ (cf. Remarks 2.14).

3.2. **Products.** Now we consider the question of how the toral rank behaves with respect to products. Although our previous results concerned the two-stage case, the comments here are not restricted to that case. It is easy to see that, in general, we have the inequality $\mathrm{rk}_0(X \times Y) \geq \mathrm{rk}_0(X) + \mathrm{rk}_0(Y)$. The following example shows that we may sometimes have strict inequality.

Example 3.3. Consider two-stage, elliptic minimal models $\mathcal{M} = \Lambda(x, y)$ with $|x| = 12$, $|y| = 23$, and $dy = x^2$; and $\mathcal{N} = \Lambda(u_1, u_2, u_3, u_4, u_5, u_6, w, v)$, with $|u_1| = |u_2| = |u_3| = |u_4| = 3$, $|u_5| = 5$, $|u_6| = 19$, $|w| = 18$, $|v| = 35$, and $dv = w^2 + u_1 u_2 u_3 u_4 u_5 u_6$, all other differentials being zero.

The homotopy Euler characteristic bound yields $\mathrm{rk}_0(\mathcal{M}) = 0$. We will show that $\mathrm{rk}_0(\mathcal{N}) = 0$ by arguing that any extension of the form $(\Lambda a \otimes \mathcal{N}, D)$ cannot have finite-dimensional cohomology.

Let $U' = \langle u_1, \ldots, u_5 \rangle$ and denote $\Lambda^{\{i_1,\ldots,i_k\}}U' := \bigoplus_{m=1}^{k} \Lambda^{i_m} U'$. For degree reasons, any such D satisfies $Du_i = \alpha_i a^2$, for $i = 1, \ldots, 4$, $Du_5 = \alpha_5 a^3$, $Du_6 = \alpha_6 a^{10} + \mu aw + \Phi$, and

$$(10) \qquad Dv = w^2 + u_1 u_2 u_3 u_4 u_5 u_6 + w\Psi + u_6\Gamma + \Omega,$$

where the α_i and μ are scalars, $\Phi, Dw \in \Lambda^+ a \otimes \Lambda^+ U'$, $\Gamma \in \Lambda^{\geq 3} a \otimes \Lambda^{\{1,3\}} U'$ and $\Psi, \Omega \in \Lambda^+ a \otimes \Lambda U'$.

Our strategy is to show that each $\alpha_i = 0$, and then to follow a similar argument as in previous examples. Applying D to (10) and equating terms that contain u_6, we find

$$0 = \sum_{i=1}^{4} \pm \alpha_i a^2 u_1 \ldots \hat{u}_i \ldots u_4 u_5 + u_1 \ldots u_4 \alpha_5 a^3 - D\Gamma.$$

Since $D\Gamma \in \Lambda^{\geq 5} a \otimes \Lambda U'$, we immediately find $\alpha_i = 0$ for $i = 0, \ldots, 5$, and so $DU' = 0$. Now, $0 = D^2 u_6 = \mu a Dw$. So either $\mu = 0$ or $Dw = 0$. Moreover, the w component of $D^2 v$ being zero implies that

$$0 = 2Dw - \mu a u_1 \ldots u_5 + \mu a \Gamma.$$

If $\mu = 0$, then this equation implies $Dw = 0$. On the other hand, suppose $\mu \neq 0$. Since $\Gamma \in \Lambda^{\geq 3} a \otimes \Lambda^{\{1,3\}} U$, by multiplying this last equation by μa, we conclude that in fact $\mu = 0$, and hence that $Dw = 0$.

The equation $D^2 v = 0$ now implies

(11) $$0 = -\alpha_6 a^{10} u_1 \ldots u_5 + (\alpha_6 a^{10} + \Phi)\Gamma.$$

Suppose $\alpha_6 \neq 0$. Then, upon considering the component of (11) in $\Lambda^+ a \otimes \Lambda^1 U'$, we find that $\Gamma \in \Lambda^+ a \otimes \Lambda^3 U$. Since $\Phi \in \Lambda^+ a \otimes \Lambda^{\geq 2} U'$, this shows that $\Phi\Gamma \in \Lambda^+ a \otimes \Lambda^5 U'$. Thus, (11) now yields $\Gamma = 0$ and so $\alpha_6 = 0$ in any case.

The argument so far has shown $\mu = \alpha_i = 0$ for $i = 1, \ldots, 6$. Therefore, for any minimal model $(\Lambda a \otimes \mathcal{N}, D)$, the corresponding pure model satisfies $(\Lambda a \otimes \mathcal{N}, D_\sigma) \cong (\Lambda(u_1, u_2, u_3, u_4, u_5, u_6), 0) \otimes (\Lambda(a, w, v), D_\sigma)$. Since $(\Lambda(a, w, v), D_\sigma)$ has homotopy Euler characteristic $\chi_\pi = +1$, it cannot have finite-dimensional cohomology, and hence neither can $(\Lambda a \otimes \mathcal{N}, D_\sigma)$. As before, it follows that $(\Lambda a \otimes \mathcal{N}, D)$ does not have finite-dimensional cohomology, and hence that $\mathrm{rk}_0(\mathcal{N}) = 0$.

Now consider the product $\mathcal{M} \otimes \mathcal{N}$. We show that $\mathrm{rk}_0(\mathcal{M} \otimes \mathcal{N}) \geq 1$, by displaying an extension $(\Lambda a \otimes \mathcal{M} \otimes \mathcal{N}, D)$, in which $H(\Lambda a \otimes \mathcal{M} \otimes \mathcal{N}, D)$ is finite-dimensional. The differential is as follows: $Dx = u_1 u_2 u_3 a^2$, $Dy = x^2 + 2u_1 u_6 a$, $Du_1 = Du_2 = Du_3 = Du_4 = 0$, $Du_5 = a^3$, $Du_6 = u_2 u_3 xa$, $Dw = 0$ and $Dv = w^2 + u_1 u_2 u_3 u_4 u_5 u_6 + xa u_4 u_6 + ya^2 u_2 u_3 u_4$. A careful check reveals that this defines a differential. The associated pure model is now $(\Lambda a \otimes \mathcal{M} \otimes \mathcal{N}, D_\sigma)$ with the only non-trivial differentials $D_\sigma(y) = x^2$, $D_\sigma(u_5) = a^3$ and $D_\sigma(v) = w^2$. This is easily seen to have finite-dimensional cohomology, and hence so does the extension $(\Lambda a \otimes \mathcal{M} \otimes \mathcal{N}, D)$. Therefore, we have $\mathrm{rk}_0(\mathcal{M} \otimes \mathcal{N}) \geq 1 > \mathrm{rk}_0(\mathcal{M}) + \mathrm{rk}_0(\mathcal{N})$.

Discussion 3.4. Recall that $\mathrm{rk}(X)$ denotes the actual toral rank (rather than its rational counterpart) for a finite complex X. It is easy to see that the inequality $\mathrm{rk}(X \times Y) \geq \mathrm{rk}(X) + \mathrm{rk}(Y)$ holds in general—one takes the obvious product action. The minimal models of Example 3.3 can be realised as the models of honest 'geometrical' spaces. $\mathcal{M} = \Lambda(x, y)$ is the minimal model of S^{12}. $\mathcal{N} = \Lambda(u_1, u_2, u_3, u_4, u_5, u_6, w, v)$ is the minimal model of a sphere bundle over a product of odd spheres. We sketch this: Take $S = S^3 \times S^3 \times S^3 \times S^3 \times S^5 \times S^{19}$, so that S has minimal model $\Lambda(u_1, u_2, u_3, u_4, u_5, u_6)$ with zero differential. S has dimension 36 and by pinching the 35-skeleton to a point, we obtain a non-trivial map $q \colon S \to S^{36}$. Composing this with a suitable map $p \colon S^{36} \to BSO(19)$ to the universal bundle $BSO(19)$, we obtain a non-trivial map $S \to BSO(19)$. Pulling back the universal real oriented line bundle to one over S, via this map, we obtain a real oriented line bundle $\mathbb{R}^{19} \to E \to S$. Finally, taking the unit sphere bundle gives a sphere bundle

$$S^{18} \to Y \to S.$$

We assert that this sphere bundle Y has minimal model $\mathcal{N}$, as in Example 3.3. Our assertion can be justified by a consideration of the possible forms that the minimal model of Y can take, bearing in mind the construction of Y. Our main point here is simply to indicate how $\mathcal{N}$ corresponds to the minimal model of a reasonable space. As in Example 3.3, $\mathrm{rk}_0(X) = \mathrm{rk}_0(Y) = 0$ and this is sufficient to conclude $\mathrm{rk}(X) = \mathrm{rk}(Y) = 0$. On the other hand, we have $\mathrm{rk}_0(X \times Y) \geq 1$, but this is *not* sufficient to conclude $\mathrm{rk}(X \times Y) \geq 1$. We are left with the following intriguing question: Does this space $X \times Y$ admit an almost-free circle action? Note that

by [Hal85, Prop.4.2], there is a simply-connected, finite complex that admits a free circle action and which has the rational homotopy type of $X \times Y$.

The preceding examples and discussion give rise to a number of interesting questions. Generally, it would be useful to have conditions under which the equality $\mathrm{rk}_0(X \times Y) = \mathrm{rk}_0(X) + \mathrm{rk}_0(Y)$ either holds, or does not hold. Various special cases are also of interest. For instance, we can ask when does $\mathrm{rk}_0(X \times S^{2n+1}) = \mathrm{rk}_0(X) + 1$? We note that Halperin has an example in which $\mathrm{rk}_0(X \times S^{2n+1}) > \mathrm{rk}_0(X) + 1$. As another special case we could ask whether, for an n-fold product of a space with itself, we have $\mathrm{rk}_0(X^n) = n\,\mathrm{rk}_0(X)$? At present, we know of no example where equality does not hold. Finally, we note that, as in the discussion above, it is reasonable to ask all these questions in the integral setting, too.

3.3. **The Gottlieb group.** An interesting suggestion arising from Theorem 2.3 is a connection between $\mathrm{rk}_0(X)$ and the *dimension of the Gottlieb group*. We recall that the nth Gottlieb group is the subgroup of $\pi_n(X)$ of elements $\alpha = [f]$ such that $(f, \mathrm{id}) : \mathbf{S}^n \vee X \to X$ extends to a continuous map $\mathbf{S}^n \times X \to X$. In terms of a minimal model $(\Lambda W, d)$, this corresponds [FHT01, p.392] to the subspace

$$G_n(X) = \left\{ \begin{array}{l} \beta \in (W^n)^* \mid \beta \text{ extends to a derivation } \theta \text{ of degree } -n \text{ of } (\Lambda W, d) \\ \hfill \text{such that } d\theta - (-1)^n \theta d = 0 \end{array} \right\}$$

In the two-stage case, with all generators of odd degree, the minimal model can be written so that $\dim V = \dim G_*(X)$. Thus we have the following observation:

Corollary 3.5. *Suppose X has minimal model that is two-stage with odd-degree generators only. Suppose that $G_*(X)$ denotes the rational Gottlieb group of X. Then*

$$\dim H(X) \geq 2^{\dim G_*(X)}.$$

An examination of Example 3.2 shows that we can have $\mathrm{rk}_0(X) > \dim G_*(X)$. Next, we illustrate that the dimension of the Gottlieb group can exceed the toral rank by an arbitrary amount.

Example 3.6. For each $n \geq 1$, we describe an $(n+1)$-stage model $\mathcal{M}_n$. Set $\mathcal{M}_n = \Lambda(x_1, x_2, y_1, z_1, y_2, z_2, \ldots, y_{n-1}, z_{n-1}, y_n)$. For degrees we have $|x_i| = 3$ and $|z_i| = 3$ for each i. The degrees of the y_i are chosen to be compatible with the differential, which is as follows: $d(x_i) = 0$ and $d(z_i) = 0$ for each i, then

$$d(y_1) = x_1 x_2$$
$$d(y_2) = x_1 x_2 y_1 z_1$$
$$d(y_3) = x_1 x_2 y_1 z_1 y_2 z_2$$
$$\vdots$$
$$d(y_n) = x_1 x_2 y_1 z_1 y_2 z_2 \cdots y_{n-1} z_{n-1}$$

For each n, one can show that $\dim G_*(X) = n$, but $\mathrm{rk}_0(X) = 1$. Further, a direct computation shows that $\dim H(X) = (1/3)4^{n+1} + 2/3$.

These considerations suggest the following question:

Question 3.7. *Let X be a finite complex with rational homotopy groups non-zero in odd degrees only. When is $\dim H(X) \geq 2^{\dim G_*(X)}$?*

In spite of the apparent difficulty in establishing Conjecture 1.1, it would appear that the conjectured lower bound of $2^{\mathrm{rk}_0(X)}$ underestimates the dimension of of the cohomology, in many cases quite seriously. This is supported by examples such as Example 3.6, [AP85b, Ex. 4.5] and analogous computations of the cohomology of nilpotent Lie algebras, such as [ACJ97]. It is possible that the lower bound suggested in Question 3.7 might give a closer estimate in some cases.

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