THE COMBINATORIAL MODEL FOR THE SULLIVAN FUNCTOR ON SIMPLICIAL SETS

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Abstract. We verify the assertion made by Sullivan at the 1974 ICM congress, and previously in print, in Appendix G of the seminal paper "Differential Forms and the Topology of Manifolds" in 1973, that the rational de Rham algebra $A_{PL}(K)$ of a finite simplicial complex K has an explicit and direct combinatorial description which is closely related to that of the Stanley-Reisner face ring of K.

§1. Introduction

Recall the Stanley-Reisner ring (or face ring) of a finite simplicial complex $K \subset [n]$ over the rational field :

$$\mathbf{Q}[K] = \mathbf{Q}[t_1, \dots , t_{n+1}]/J_K$$

where

$$J_K = \{t_{i_1} t_{i_2} \dots t_{i_k} \mid \{i_1, \dots , i_k\} \text{ is non-degenerate} \in [n],\ \{i_1, \dots , i_k\} \notin K\,\}\,.$$

There is a rich interplay between the combinatorial properties of K and ring-theoretic properties of $\mathbf{Q}[K]$ (see [1]).

Moreover, the commutative differential graded algebra $(Q[K], 0)$, is the model (in the sense of rational homotopy) of the subspace $\cup_{\sigma\in K}(BS^1)^\sigma$ of $(BS^1)^{n+1}$. Here $(BS^1)^\sigma := \prod_{i=1}^{n+1} X_i \subset (BS^1)^{n+1}$ denotes the subspace of $(BS^1)^{n+1}$ which is the indicated product where $X_i = *$, if $i \notin \sigma$ and $X_i = BS^1$ otherwise. (See [2]).

However, as was shown by Francisco Gomez [4], the direct generalization of that construction by taking the quotient of the differential graded algebra

$$(\Lambda(t_1, \dots , t_{n+1}, dt_1, \dots , dt_{n+1}), d)$$

by the *differential* ideal generated by all products $t_{i_1} t_{i_2} \dots t_{i_r}$ for the simplices $\{i_1, i_2, \dots , i_r\}$ not in K does *not* yield a model for K.

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In appendix G(i) of [8], Sullivan remarked that if K is a finite simplicial complex, then the "rational de Rham algebra has an explicit presentation" as the free commutative graded differential algebra over the vertices $t_1, \ldots, t_{n+1}$ of K (and their derivatives), modulo the smallest differential ideal containing $t_1+\cdots+t_{n+1}-1$, and the monomials $t_{i_1} \ldots t_{i_r} dt_{i_r+1} \ldots dt_{i_l}$, whenever $\{i_1, \ldots, i_l\}$ is *not* a simplex of K. To the authors' knowledge no proof of this assertion has appeared in the literature.

Here, we verify this direct and combinatorial description of the Sullivan algebra $A_{PL}(K)$ on a finite simplicial set K. While the construction (and indeed $A_{PL}(K)$) can be defined over any ring, and the theorem as stated below remains valid there, our interest is in the rational case when K is simply connected, since this then captures the rational homotopy of K in a combinatorial fashion.

Let $A(K)$ be the algebra described above. The main result is

Theorem [Sullivan, 4]. *There is an isomorphism $\psi_K \colon A(K) \to A_{PL}(K)$ of commutative-graded differential algebras, such that, if $L \xrightarrow{g} K$ is a simplicial map, the diagram below commutes:*

$$\begin{tikzcd}
A(K) \arrow[r, "A(g)"] \arrow[d, "\psi_K"', "\cong"] & A(L) \arrow[d, "\cong"', "\psi_L"] \\
A_{PL}(K) \arrow[r, "A_{PL}(g)"] & A_{PL}(L)
\end{tikzcd}$$

We note that one can also recover $\mathbf{Q}[K]$ as $A^0(cK)$, where cK denotes the cone on K.

The article is organized as follows. We recall some background material in section 2. In section 3, we define the commutative-graded differential algebra (hereafter, *cgda*) $A(K)$, and give an example. After proving the theorem above in section 4, we use the Whitney quasi-isomorphism from the cochain algebra on K into A(K) in order to obtain applications to cohomology in degrees 0 and 1, and to the fundamental group, in section 5.

§2 Background material and notation

Throughout the article, we will use the notation of [3]. In particular, if X is a graded vector space, ΛX will denote the free commutative-graded algebra on X.

Consider the simplicial cochain algebra $\mathcal{A}.$, where

$$\mathcal{A}_n := \Lambda(t_1, \ldots, t_{n+1}, dt_1, \ldots, dt_{n+1})/(\sum t_i = 1, \sum dt_i = 0),$$

with $|t_i| = 0$, and $|dt_i| = 1$ for $n = 0, 1, 2, \ldots$.

The face and degeneracy maps $\partial_i : \mathcal{A}_{q+1} \to \mathcal{A}_q$ and $s_j : \mathcal{A}_q \to \mathcal{A}_{q+1}$ are defined by

$$\partial_i : t_l \mapsto \begin{cases} t_l, & l < i \\ 0, & l = i \\ t_{l-1}, & l > i \end{cases} \quad \text{and} \quad s_j : t_l \mapsto \begin{cases} t_l, & l < j \\ t_l + t_{l+1}, & l = j \\ t_{l+1}, & l > j \end{cases}$$

Recall that the Sullivan functor on a simplicial set K is defined as

$$A^p_{PL}(K) = \mathrm{Hom}_{Simplicial}(K, \mathcal{A}^p_{\cdot}),$$

that is, $\Phi \in A^p_{PL}(K)$ associates to each $\tau \in K$ an element $\Phi_\tau \in \mathcal{A}^p_{|\tau|}$ so that $\Phi_{\partial_i \tau} = \partial_i \Phi_\tau$, and $\Phi_{s_i\tau} = s_i\Phi_\tau$ for $1 \le i \le |\tau| + 1$.

§3 Definition of $A(K)$

Let Δ^n be the standard n-simplex $\Delta^n = \{(t_1, \ldots, t_{n+1}) \mid \sum t_i = 1, t_i \ge 0\}$. We will frequently use the underlying simplicial set ([7], P. 2)

$$[n] = \{(i_1, \ldots, i_k) \mid 1 \le i_1 \le \cdots \le i_k \le n+1\},$$

and if $\sigma = (i_1, \ldots, i_{k+1}) \in [n]$, as usual we define the geometric simplex

$$|\sigma| = \{(t_1, \ldots, t_{n+1}) \in \Delta^n \mid t_j = 0, j \notin \{i_1, \ldots, i_{k+1}\}\},$$

and the underlying set (no repetitions)

$$u(\sigma) = \{i_1, \ldots, i_{k+1}\}.$$

Note that σ is non-degenerate iff the cardinality of $u(\sigma)$ is $k+1$, and in this case we denote the cardinality of $u(\sigma)$, less one, by $\dim \sigma$ as usual. We denote the set of non-degenerate simplices in a simplicial set $K \subset [n]$ by $N(K)$.

If $\sigma, \tau \in N([n])$ satisfy $u(\sigma) \cap u(\tau) = \emptyset$, and if

$$u(\sigma) \cup u(\tau) = \{i_1, \ldots, i_k \mid 1 \le i_1 \le \cdots \le i_k \le n+1\},$$

then we will define $\sigma \star \tau := (i_1, \ldots, i_k) \in N([n])$.

Given non-degenerate k-simplices $\sigma_1, \ldots, \sigma_j \in N([n])$, let $\langle \sigma_1, \ldots, \sigma_j \rangle$ denote the smallest subsimplicial set of $[n]$ containing $\sigma_1, \ldots, \sigma_j$. In particular $\sigma_1, \ldots, \sigma_j$ are the only non-degenerate simplices of dimension k in $\langle \sigma_1, \ldots, \sigma_j \rangle$. If $\sigma \in N([n])$, we then define the subsimplicial set

$$\partial\sigma = \langle \{\partial_i \sigma \mid 1 \le i \le \dim \sigma + 1\} \rangle,$$

as usual. To avoid too much notation, we will frequently write σ when we mean $\langle \sigma \rangle$, and it is hoped that the intended meaning will be clear from the context.

We will use the following notation for some elements in $\mathcal{A}_n$. For a k-simplex $\sigma = (i_1, \ldots, i_{k+1}) \in [n]$, let

- $t_\sigma = t_{i_1} \cdots t_{i_{k+1}}$, and
- $dt_\sigma = dt_{i_1} \cdots dt_{i_{k+1}}$.

If $K \subset [n]$, then we set

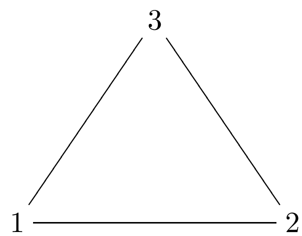
$$A(K) = \mathcal{A}_n / I_K$$

where I_K is the differential ideal defined by

(*) $$I_K = (\{t_\sigma dt_\tau \in \mathcal{A}_n \mid \sigma, \tau \in N([n]),\ \text{but}\ \sigma \star \tau \notin N(K)\}).$$

(Note that, as usual, we include $\emptyset$ in $N(K)$.)

Example.
Let K be the boundary of the triangle Δ^2 :



Here, $A(K)$ is the quotient of $\Lambda(t_1, t_2, t_3, dt_1, dt_2, dt_3)$ by the ideal generated by

$$\begin{aligned}
&t_1 + t_2 + t_3 - 1, dt_1 + dt_2 + dt_3,\\
&t_1 t_2 t_3,\\
&dt_1 t_2 t_3, t_1 dt_2 t_3, t_1 t_2 dt_3,\ \text{and}\\
&dt_1 dt_2.
\end{aligned}$$

Note that some simplifications have been made, as the above list does not coincide with that defined in (*) above: e.g. if we denote first two elements listed above as a and b respectively, then $dt_1 dt_2 t_3 = dt_1 dt_2(1 + a - t_1 - t_2) = (1 + a)dt_1 dt_2 - t_1(b - dt_2 - dt_3)dt_2 - t_2 dt_1(b - dt_1 - dt_3) = dt_1 dt_2 + \alpha$, $\alpha \in I_K$. Similarly, $dt_2 dt_3 = (b - dt_1 - dt_3)dt_3 = -dt_1 dt_3 + \beta = dt_1 dt_2 + \gamma$, for some β, $\gamma \in I_K$, and $dt_1 dt_2 dt_3 = dt_1 dt_2(b - dt_1 - dt_2) = b dt_1 dt_2$.

The cohomology is

$$H^p(A(K)) = \begin{cases} \mathbf{Q}.1, & p = 0 \\ \mathbf{Q}.[t_1 dt_2], & p = 1 \\ 0, & p > 1 \end{cases}.$$

As mentioned in the introduction, we can recover $\mathbf{Q}[K]$ by considering the cone cK on K. The simplicial set cK is in $N[n+1]$ and the algebra of degree zero forms, $A^0(cK)$ is simply $\mathbf{Q}[K]$, i.e.,

$$A^0(cK) = \mathbf{Q}[K]\,.$$

§4 Proof of the Theorem

The proof proceeds by induction on the number of non-degenerate simplices in K, as follows. First note that if $E, E' \subset [n]$ are simplicial sets and E' is obtained from E by adding a non-degenerate k-simplex $\sigma \in [n]$ with $\partial\sigma \subset E$, then we have the pushout

$$\begin{tikzcd} \partial\sigma \arrow[r, "k"] \arrow[d, "i"'] & E \arrow[d, "l"] \\ \sigma \arrow[r, "j"] & E' \end{tikzcd} \tag{1}$$

(where i, j, k are inclusions of simplicial sets). From this, we form the sequence

$$0 \to A(E') \xrightarrow{g=(l^*,j^*)} A(E) \oplus A(\sigma) \xrightarrow{f} A(\partial\sigma) \to 0 \tag{2}$$

where, for $(\alpha, \beta) \in A(E) \oplus A(\sigma)$, we define $f(\alpha, \beta) = k^*(\alpha) - i^*(\beta)$.

We then have

Lemma 1. *The sequence (2) above is short exact.*

Proof. We can rewrite the sequence explicitly as

$$0 \to \mathcal{A}_n/I_{E'} \xrightarrow{g=(l^*,j^*)} \mathcal{A}_n/I_E \oplus \mathcal{A}_n/I_\sigma \xrightarrow{f} A_n/I_{\partial\sigma} \to 0$$

where $g(c + I_{E'}) = (c + I_E, c + I_\sigma)$ and $f(a + I_E, b + I_\sigma) = a - b + I_{\partial\sigma}$. It follows from the definitions that $fg = 0$, and that f is surjective.

To see that g is injective, it suffices to show that $I_{E'} = I_E \cap I_\sigma$. But $I_E = I_{E'} + J$ where $J = \langle t_{\sigma_1} dt_{\sigma_2} \mid \sigma_1 \star \sigma_2 = \sigma \rangle$, and $I_{E'} \subset I_\sigma$, so $I_E \cap I_\sigma = (I_{E'} + J) \cap I_\sigma = I_{E'} \cap I_\sigma + J \cap I_\sigma$. But $I_{E'} \cap I_\sigma = I_{E'}$, and $J \cap I_\sigma = 0$.

It remains to show that $\ker f \subset \operatorname{im} g$, so suppose $a - b \in I_{\partial\sigma}$ for $a, b \in \mathcal{A}_n$. By assumption, $N(\partial\sigma) = N(E) \cap N(\sigma)$, so $\tau \notin N(\partial\sigma) \iff \tau \notin N(E)$ or $\tau \notin N(\sigma)$. Hence, we may write $a - b = x + y$ with $x \in I_E$ and $y \in I_\sigma$. Then, $c := a - x = b + y$ satisfies

$$g(c + I_{E'}) = (c + I_E, c + I_\sigma) = (a - x + I_E, b + y + I_\sigma) = (a + I_E, b + I_\sigma),$$

showing that $\ker f \subset \operatorname{im} g$.□

The short exact sequence (2) also exists for A_{PL} [9]. Below, we will define a *cgda* map $\psi_K : A(K) \to A_{PL}(K)$ that is natural for simplicial maps (Lemma 2). This will yield a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A(E') \arrow[r] \arrow[d, "\psi_{E'}"] & A(E) \oplus A(\sigma) \arrow[r] \arrow[d, "\psi_E + \psi_\sigma"] & A(\partial\sigma) \arrow[r] \arrow[d, "\psi_{\partial\sigma}"] & 0 \\ 0 \arrow[r] & A_{PL}(E') \arrow[r] & A_{PL}(E) \oplus A_{PL}(\sigma) \arrow[r] & A_{PL}(\partial\sigma) \arrow[r] & 0 \end{tikzcd} \tag{3}$$

We will then prove that ψ_σ and $\psi_{\partial\sigma}$ are isomorphisms (Lemmas 3 and 4 respectively), and the theorem will follow by induction and diagram (3). We now proceed to Lemmas 2-4.

Lemma 2. *There is a cgda map $\psi_K : A(K) \to A_{PL}(K)$, such that, if $L \xrightarrow{g} K$ is a simplicial map, the diagram below commutes:*

$$\begin{tikzcd} A(K) \arrow[r, "A(g)"] \arrow[d, "\psi_K"'] & A(L) \arrow[d, "\psi_L"] \\ A_{PL}(K) \arrow[r, "A_{PL}(g)"] & A_{PL}(L) \end{tikzcd}$$

Proof. Fix n, let $K \subset [n]$ be a simplicial set, and $\tau = (i_1, \ldots, i_{k+1})$ be a non-degenerate k-simplex of K. We first define a map $\tilde{\varphi}_\tau : \mathcal{A}_n \to \mathcal{A}_k$ (induced by the inclusion $\tau \subset [n]$) by

$$\tilde{\varphi}_\tau : t_j \mapsto \begin{cases} t_l, & j = i_l \in u(\tau) \\ 0, & \text{otherwise,} \end{cases}$$

and commuting with the differentials. (We need $\tau \in N(K)$ so that $\tilde{\varphi}_\tau$ is well-defined.) Now recall that

$$I_K = \langle \{ t_\sigma dt_\varepsilon \in \mathcal{A}_n \mid \sigma, \varepsilon \in N([n]), u(\sigma) \cap u(\varepsilon) = \emptyset, \text{ but } \sigma \star \varepsilon \notin N(K) \} \rangle,$$

If $u(\sigma) \cup u(\varepsilon) = \{j_1, \ldots, j_{m+1}\}$ and $(j_1, \ldots, j_{m+1}) \notin N(K)$ then $\exists q, 1 \le q \le m+1$ such that $j_q \notin u(\tau)$, in which case either $\tilde{\varphi}_\tau(t_\sigma) = 0$ or $\tilde{\varphi}_\tau(dt_\epsilon) = 0$. This shows that $\tilde{\varphi}_\tau(I_K) = 0$ and so $\tilde{\varphi}_\tau$ induces a morphism

$$\varphi_\tau : A(K) \to \mathcal{A}_k.$$

$\psi_K([a])$ to be the (partially defined) morphism

$$\psi_K([a]) : \tau \mapsto \varphi_\tau([a]), \quad \tau \in N(K)$$

Before dealing with degenerate simplices, we check that, for non-degenerate simplices τ, we have

$$\partial_i \psi_K([a])(\tau) = \psi_K([a])(\partial_i \tau).$$

It suffices to check this on the generators of the algebra $A^0(K)$, namely the t_j, for $j = 1, \ldots, k+1$, since (for a fixed non-degenerate τ) both $a \mapsto \partial_i \psi_K([a])(\tau)$ and $a \mapsto \psi_K([a])(\partial_i \tau)$ are maps of algebras that commute with the differential. Now, suppose $\tau = (i_1, \ldots, i_{k+1}) \in N(K)$.

Then,

$$\varphi_\tau([t_l]) = \begin{cases} 0, & l \notin u(\tau) \\ t_j, & l = i_j, \text{ for some } j \end{cases},$$

so that

$$\partial_q \varphi_\tau([t_l]) = \begin{cases} 0, & l \notin u(\tau) \\ t_j, & l = i_j, j < q \\ 0, & l = i_q, \\ t_{j-1}, & l = i_j, j > q \end{cases}$$

On the other hand, $\partial_q\tau = (i_1, \dots, i_{q-1}, i_{q+1}, \dots, i_{k+1})$, so

$$\varphi_{\partial_q\tau}([t_l]) = \begin{cases} 0, & l \notin u(\partial_i\tau) \\ t_j, & l = i_j, j < q \\ 0, & l = i_q, \\ t_{j-1}, & l = i_j, j > q \end{cases}$$

Thus, for non-degenerate simplices τ, we have $\partial_i\psi_K([a])(\tau) = \psi_K([a])(\partial_i\tau)$.

If $\sigma \in K$ is degenerate, it has a unique decomposition $\sigma = s_{j_l} \cdots s_{j_1}\tau$, with $j_l > \cdots > j_1$ and τ non-degenerate, and we then define

$$\psi_K([a])(s_{j_l} \cdots s_{j_1}\tau) = s_{j_l} \cdots s_{j_1}\varphi_\tau([a]).$$

That $\partial_i\psi_K([a])(\sigma) = \psi_K([a])(\partial_i\sigma)$ holds for degenerate simplices as well is a consequence of the definitions and the fact that A_{PL} is a simplicial cochain algebra.

The naturality of ψ follows from a straightforward check. □

Lemma 3. *If $\sigma = (i_1, \dots, i_{k+1}) \in N([n])$ and $K = \langle\sigma\rangle$, then*

$$\psi_K : A(K) \to A_{PL}(K)$$

is an isomorphism .

Proof. Since $N(K)$ is closed under $\star$, if $\tau, \varepsilon \in N(K)$ and $\tau \star \varepsilon \notin N(K)$, then $u(\tau) \cup u(\varepsilon)$ must contain $i \in \{1, \dots, n+1\} - u(\sigma) =: \{j_1, \dots, j_{n-k}\}$. Hence,

$$I_K \subset (t_{j_1}, \dots, t_{j_{n-k}}, dt_{j_1}, \dots, dt_{j_{n-k}}).$$

Moreover, given any $i \in \{j_1, \dots, j_{n-k}\}$, the non-degenerate 0-simplex $(i) \notin N(K)$, and so $I_K \supset (t_{j_1}, \dots, t_{j_{n-k}}, dt_{j_1}, \dots, dt_{j_{n-k}})$. Hence,

$$I_K = (t_{j_1}, \dots, t_{j_{n-k}}, dt_{j_1}, \dots, dt_{j_{n-k}}).$$

Thus, with the notation of lemma 2,

$$\varphi_\sigma \colon A(K) \cong \Lambda(t_{i_1}, \dots, t_{i_{k+1}}, dt_{i_1}, \dots, dt_{i_{k+1}})/(\sum t_{i_j} = 1, \sum dt_{i_j} = 0) \longrightarrow \mathcal{A}_k$$

is an isomorphism.

Now, K is generated as a simplicial set by σ, and so $\Phi \in A^p_{PL}(K)$ is completely determined by Φ_σ.

Since φ_σ is an isomorphism, if $a \in A^p(K)$, we know $\psi_K([a]) : \sigma \mapsto \varphi_\sigma([a])$, so $\psi_K([a])$ is zero as a homomorphism iff $[a] = 0$ in $A(K)$. That is, ψ_K is injective.

Moreover, given $\Phi \in A^p_{PL}(K)$, let $[a] = (\varphi_\sigma)^{-1}(\Phi(\sigma))$. Then, $\psi_K([a]) : \sigma \mapsto \varphi_\tau([a]) = \Phi(\sigma)$, and it follows from the definition of ψ_K that $\psi_K([a]) = \Phi$. Hence, ψ_K is surjective. □

Any simplicial set K in $[n]$ can be obtained by a finite number of pushouts as in diagram (1). Thus, Lemmas 1-3, together with the fact that for any non-degenerate simplex $\sigma \in [n]$, we have

$$\psi_{\partial\sigma} : A(\partial\sigma) \xrightarrow{\cong} A_{PL}(\partial\sigma) \tag{4}$$

will imply $A(K) \xrightarrow{\psi_K} A_{PL}(K)$ is an isomorphism for all K.

With respect to claim (4), it is enough to consider the case $\sigma = (1, \dots, n+1)$. In this case, as we saw above, the isomorphism $\psi_\sigma : A(\sigma) = \mathcal{A}_n \to A_{PL}(\sigma)$ identifies $\omega \in \mathcal{A}_n$ with element Φ sending σ to $\omega \in A_{PL}(\sigma)$.

The injection of $\partial_i\sigma \hookrightarrow \sigma$ induces a map $\rho_i : A_{PL}(\sigma) \to A_{PL}(\partial_i\sigma)$ and the union of the ρ_i defines a surjective map

$$\rho : A_{PL}(\sigma) \to A_{PL}(\partial\sigma)\,, \quad \rho(\sigma,\omega) = (\partial_i\sigma, \rho_i(\omega))_{1\le i\le n+1}\,.$$

The kernel of ρ is clearly the intersection $J := \bigcap_{i=1}^{n+1} \ker\, \rho_i$ that will be identified by Lemma 4 with the ideal $I_{\partial\sigma}$.

Lemma 4. $I := \psi_\sigma(I_{\partial\sigma}) = J.$

Note that this directly implies the commutativity of the following diagram and the fact that $\psi_{\partial\sigma}$ is an isomorphism.

$$\begin{tikzcd}
0 \arrow[r] & I_{\partial\sigma} \arrow[r] \arrow[d, "\psi_\sigma", "\cong"'] & A(\sigma) \arrow[r] \arrow[d, "\psi_\sigma", "\cong"'] & A(\partial\sigma) \arrow[r] \arrow[d, "\psi_{\partial\sigma}"] & 0 \\
0 \arrow[r] & J \arrow[r] & A_{PL}(\sigma) \arrow[r] & A_{PL}(\partial\sigma) \arrow[r] & 0
\end{tikzcd}$$

Proof of Lemma 4. For sake of simplicity we identify $A(\sigma) = \mathcal{A}_n$ and $A(\partial_i\sigma) = \mathcal{A}_{n-1}$. We also denote by

$$i_n : E_n = \Lambda(t_1, \dots, t_n) \otimes \Lambda(dt_1, \dots, dt_n) \hookrightarrow \mathcal{A}_n$$

the canonical isomorphism obtained by injection. For $i = 1, \dots, n+1$, we then have a commutative diagram in which vertical arrows are isomorphisms:

$$\begin{tikzcd}
E_n \arrow[r, "p_i"] \arrow[d, "i_n"] & E_{n-1} \arrow[d, "i_{n-1}"] \\
\mathcal{A}_n \arrow[r, "\rho_i"] & \mathcal{A}_{n-1}
\end{tikzcd}$$

where,

$$p_i(t_j) = \begin{cases} t_j & \text{for } j < i \\ 0 & \text{for } j = i\,, \\ t_{j-1} & \text{for } j > i \end{cases} \qquad \text{for } i \le n,$$

and

$$p_{n+1}(t_j) = \begin{cases} t_j & \text{for } j < n \\ 1 - t_1 - t_2 - \dots - t_{n-1} & \text{for } j = n \end{cases}$$

We now consider the two ideals I and J in $\mathcal{A}_n$, $J = \bigcap \ker \rho_i$ and I is the ideal generated by the elements

$$t_{j_1} \ldots t_{j_s}\, dt_{i_1} \ldots dt_{i_r}$$

with $\{j_1, \ldots, j_s, i_1, \ldots, i_r\} = \{1, \ldots, n+1\}$ and $s + r = n+1$. It is clear that $I \subset J$; we will prove that $J \subset I$.

Let $\omega \in J$. Since $\omega \in \mathcal{A}_n = i_n(E_n)$, ω can be written in the form

$$\omega = \sum_{1 \le i_1 < i_2 < \ldots < i_r \le n} p_{i_1 \cdots i_r} dt_{i_1} \ldots dt_{i_r}$$

with $p_{i_1 \cdots i_r} \in \Lambda(t_1, \ldots, t_n)$.

In the case $r = n$, ω can be written in the form $\omega = q dt_1 \ldots dt_n$, with $q \in \Lambda(t_1, \ldots, t_n)$. These elements belong to I. We first remark that for $j \le n$, we have

$$\begin{aligned} t_j dt_1 \ldots dt_n &= t_j dt_1 \ldots dt_{j-1} (-\sum_{k \neq j} dt_k) dt_{j+1} \ldots dt_n \\ &= -t_j dt_1 \ldots dt_{j-1} dt_{n+1} dt_{j+1} \ldots dt_n \in I \end{aligned}$$

Now since $1 = \sum t_i$, we deduce that $dt_1 \ldots dt_n \in I$. In what follows we can therefore suppose $r < n$.

We break the rest of the proof into several steps.

Step 1: Factors of $p_{i_1 \cdots i_r}$.

Since $\omega \in J$, $\rho_i(\omega) = 0$, for $i \le n$, and therefore $p_{i_1 \cdots i_r}$ is divisible by t_i when $i \notin \{i_1, \ldots i_r\}$ and $i \le n$. We deduce that $p_{i_1 \cdots i_r} = t_{j_1} \ldots t_{j_s} q_{i_1 \cdots i_r}$ with $\{j_1, \ldots, j_s, i_1, \ldots, i_r\} = \{1, \ldots, n\}$, $s + r = n$, and $q_{i_1 \cdots i_r} \in \Lambda(t_1, \ldots, t_n)$.

Step 2: General restrictions on $q_{i_1 \cdots i_r}$.

Next, since $t_n = 1 - t_1 - \cdots - \widehat{t_n} - t_{n+1}$ is valid in $\mathcal{A}_n$, we can write $q_{i_1 \cdots i_r}$ as a polynomial in the variables $t_1, t_2, \ldots, \widehat{t_n}, t_{n+1}$. Combining all the terms containing t_{n+1} we may then write

$$q_{i_1 \cdots i_r} = t_{n+1} h_{i_1 \cdots i_r} + k_{i_1 \cdots i_r}$$

with $h_{i_1 \cdots i_r} \in \Lambda(t_1, \ldots t_{n+1})$ and $k_{i_1 \cdots i_r} \in \Lambda(t_1, \ldots, t_{n-1})$. Since

$$t_{j_1} \ldots t_{j_s} t_{n+1}\, dt_{i_1} \ldots dt_{i_r} \in I \subset J,$$

this reduces us to the case when $q_{i_1 \cdots i_r} \in \Lambda(t_1, \ldots, t_{n-1})$.

Step 3: Particular restrictions on $q_{i_1 \cdots i_r}$.

We next show that, modulo the ideal I, we can suppose that

$$q_{1 \ldots k\, i_{k+1} \ldots i_r} \in \Lambda(t_{k+1}, \ldots, t_{n-1}) \quad \text{for } \ k = 1, \ldots r\,,$$

and we will prove this by descending induction on k.

We begin the induction with the case $k = r$, i.e., $q_{1 \ldots r} \in \Lambda(t_{r+1}, \ldots, t_{n-1})$. This we will establish by proving by induction on m that

$$q_{1 \ldots r} \in \Lambda(t_1, \ldots, t_{r-m-1}, \widehat{t_{r-m}}, \ldots, \widehat{t_r}, t_{r+1}, \ldots t_{n-1}) \quad \text{for } \ 0 \le m \le r-1.$$

The case $m = 0$ proceeds as follows, and is a model for the entire argument: Since $q_{1\ldots r} \in \Lambda(t_1, \ldots, t_{n-1})$, we may write

$$q_{1\ldots r} = t_r\alpha + \beta, \text{ with } \alpha \in \Lambda(t_1, \ldots, t_{n-1}) \text{ and } \beta \in \Lambda(t_1, \ldots, \widehat{t_r}, \ldots, t_{n-1}).$$

Now rewrite the term $t_{j_1} \ldots t_{j_s} t_r \alpha \, dt_1 \ldots dt_r$ as

$$\begin{aligned} t_{j_1} \ldots t_{j_s} t_r \alpha \, dt_1 \ldots dt_{r-1}(-\sum_{j=1,(j\neq r)}^{n+1} dt_j) &= -t_{j_1} \ldots t_{j_s} t_r \alpha \, dt_1 \ldots dt_{r-1} dt_{n+1} \\ &\quad - t_{j_1} \ldots t_{j_s} t_r \alpha \, dt_1 \ldots dt_{r-1}(\sum_{j=r+1}^{n} dt_j). \end{aligned}$$

The first term on the right belongs to $I \subset J$, so we modify ω and thus suppose it is zero. The other terms are used to modify (in the expression for ω) the coefficients of $dt_1 \ldots dt_{r-1} dt_j$ for $j > r$. This concludes the case $m = 0$.

Now suppose

$$q_{1\ldots r} \in \Lambda(t_1, \ldots, t_{r-m}, \widehat{t_{r-m+1}}, \ldots, \widehat{t_r}, t_{r+1}, \ldots t_{n-1}) \quad \text{for } 1 \leq m \leq r.$$

Again, write

$$q_{1\ldots r} = t_{r-m}\alpha + \beta, \text{ with } \beta \in \Lambda(t_1, \ldots, t_{r-m-1}, \widehat{t_{r-m}}, \ldots, \widehat{t_r}, t_{r+1}, \ldots t_{n-1}).$$

As before, rewrite the term $t_{j_1} \ldots t_{j_s} t_{r-m} \alpha \, dt_1 \ldots dt_r$ as

$$\begin{aligned} & t_{j_1} \ldots t_{j_s} t_{r-m} \alpha \, dt_1 \ldots dt_{r-m-1}(-\sum_{j=1(j\neq r-m)}^{n+1} dt_j) dt_{r-m+1} \ldots dt_r \\ & = -t_{j_1} \ldots t_{j_s} t_{r-m} \alpha \, dt_1 \ldots dt_{r-m-1} \, dt_{n+1} dt_{r-m+1} \ldots dt_r \\ & \qquad - t_{j_1} \ldots t_{j_s} t_{r-m} \alpha \, dt_1 dt_1 \ldots dt_{r-m-1}(\sum_{j=r+1}^{n} dt_j) dt_{r-m+1} \ldots dt_r. \end{aligned}$$

Again, the first term on the right belongs to $I \subset J$, so we modify ω and thus suppose it is zero. The other terms are used to modify (in the expression for ω) the coefficients $q_{1\ldots r-m-1\,\widehat{r-m}\,r-m+1\ldots r\,j} \cdots$ of $dt_1 \ldots dt_{r-m-1} dt_{r-m+1} \ldots dt_r dt_j$ for $j > r$. This closes the induction on m and establishes that

$$q_{1\ldots r} \in \Lambda(t_{r+1}, \ldots, t_{n-1}).$$

Now, suppose we have proved that

$$q_{1\ldots k\, i_{k+1}\ldots i_r} \in \Lambda(t_{k+1}, \ldots, t_{n-1})$$

for all k, $1 \leq \ell < k \leq r$. The proof that $q_{1\ldots \ell\, i_{\ell+1}\ldots i_r} \in \Lambda(t_{\ell+1}, \ldots, t_{n-1})$ (for $i_{\ell+1} > \ell + 1$) again uses the method of the initial case. However, we proceed with some details to convince the reader that we do not disturb coefficients we already have in the right form. So, write

$$q_{1\cdots\ell\, i_{\ell+1}\cdots i_r} = t_\ell \alpha + \beta\,,$$

with $\alpha \in \Lambda(t_1, \ldots, t_{n-1})$ and $\beta \in \Lambda(t_1, \ldots, \widehat{t_\ell}, \ldots, t_{n-1})$. Once again, we replace then the term

$$t_{j_1} \ldots t_{j_s} t_\ell \alpha\, dt_1 \ldots dt_\ell dt_{i_{\ell+1}} \ldots dt_{i_r}$$

by the sum

$$\begin{aligned}\sum_{k=1,\, k\neq\ell}^{n+1} & -t_{j_1} \ldots t_{j_s} t_\ell \alpha\, dt_1 \ldots dt_{\ell-1}(dt_k) dt_{i_{\ell+1}} \ldots dt_{i_r} \\ &= -t_{j_1} \ldots t_{j_\ell} t_\ell \alpha\, dt_1 \ldots dt_{\ell-1}\, dt_{n+1} dt_{i_{\ell+1}} \ldots dt_{i_r} \\ &\quad - t_{j_1} \ldots t_{j_s} t_\ell \alpha\, dt_1 \ldots dt_{\ell-1} (\sum_{p=1}^{s} dt_{j_p}) dt_{i_{\ell+1}} \ldots dt_{i_r}.\end{aligned}$$

The first term belongs to I and as before we can ignore it. The other terms are used to modify (in the expression for ω) the coefficients $q_{1\ldots\ell-1\,\widehat{\ell}\, i'_l,\ldots i'_r}$ of

$$dt_1 \ldots dt_{\ell-1}\, \widehat{dt_\ell}\, dt_{i'_l}, \ldots dt_{i'_r}$$

for $i'_\ell > \ell+1$. This shows that we may suppose that

$$q_{1\cdots\ell\, i_{\ell+1}\cdots i_r} \in \Lambda(t_1, \ldots, \widehat{t_\ell}, \ldots, t_{n-1}).$$

An argument similar to that in the initial case, using the induction hypothesis

$$q_{1\ldots\ell\, i_{\ell+1}\ldots i_r} \in \Lambda(t_1, \ldots, t_{\ell-m-1}, \widehat{t_{\ell-m}}, \ldots, \widehat{t_\ell}, t_{\ell+1}, \ldots t_{n-1}),$$

for $0 \leq m \leq \ell-1$, now establishes that

$$q_{1\ldots\ell\, i_{\ell+1}\ldots i_r} \in \Lambda(t_{\ell+1}, \ldots, t_{n-1}) \quad \text{for } k = 0, \ldots r\,,$$

and so closes the main induction.

Step 4: using $\rho_{n+1}(\omega) = 0$.

We will now use the last condition, namely, $\rho_{n+1}(\omega) = 0$. Recall that $\rho_{n+1}(t_n) = 1 - t_1 - \ldots - t_{n-1}$. Recall that

$$\omega = \sum_{1\leq i_1<i_2<\ldots<i_r\leq n} t_{j_1} \ldots t_{j_s} q_{i_1\cdots i_r} dt_{i_1} \ldots dt_{i_r},$$

with $q_{i_1\cdots i_r} \in \Lambda(t_1, \ldots, t_{n-1})$, $\{j_1, \ldots j_s, i_1 \ldots i_r\} = \{1, \ldots, n\}$ and $s + r = n$.

Since $q_{i_1,\cdots i_r} \in \Lambda(t_1, \ldots, t_{n-1})$, for each r-tuple $1 \leq i_1 < i_2 \ldots < i_r < n$, $\rho_{n+1}(\omega) = 0$ implies that we have the equations

$$t_{j_1} \ldots t_{j_\ell} \big(t_n\, q_{i_1\cdots i_r} + \sum_{k=1}^{r} (-1)^{r-k+1}\, t_{i_k}\, q_{i_1\cdots\widehat{i_k}\cdots i_r n}\big) = 0\,,$$

with $\{j_1, \dots, j_\ell, i_1, \dots, i_r\} = \{1, \dots, n-1\}$, and $\ell + r = n-1$.

Since there are no zero divisors in $\mathcal{A}^0_{n-1}$, we obtain

$$(1 - t_1 - t_2 - \cdots - t_{n-1})\, q_{i_1\cdots i_r} + \sum_{k=1}^{r} (-1)^{r-k+1}\, t_{i_k}\, q_{i_1\cdots \widehat{i_k}\cdots i_r n} = 0\,. \tag{5}$$

We now prove by induction on s, $s = 1, \dots, r$, that the polynomials $q_{i_1\cdots i_r}$ satisfy the properties $\mathcal{P}_s$

$$\mathcal{P}_s\colon \begin{cases} (I) & \\ q_{i_1\cdots i_r} = 0 & 1 < i_1 \\ q_{1\, i_2\cdots i_r} = 0 & 2 < i_2 \\ \quad\vdots & \vdots \\ q_{1\,2\cdots (s-1)\, i_s\cdots i_r} = 0 & s < i_s \\ & \\ (II) & \\ q_{1\, i_2\cdots i_r} = (-1)^{r-1} q_{i_2\cdots i_r n} & \\ q_{1\,2\, i_3\cdots i_r} = (-1)^{r-2} q_{1\,\hat{2}\, i_3\cdots i_r\, n} & \\ \quad\vdots & \\ q_{1\,2\cdots s\, i_{s+1}\cdots i_r} = (-1)^{r-s} q_{1\,2\cdots (s-1)\,\hat{s}\, i_{s+1}\cdots i_r\, n}. & \end{cases}$$

Note that the conditions $\mathcal{P}_s, s = 1, \dots, r$ together imply that all the polynomials $q_{i_1\cdots i_r}$ are zero when $i_r < n$.

We first prove $\mathcal{P}_1$, namely, that

$$q_{i_1\cdots i_r} = 0 \quad \text{for} \quad 1 < i_1\,, \quad \text{and} \quad q_{1i_2\cdots i_r} = (-1)^{r-1} q_{i_2\cdots i_r n}\,.$$

Consider the term in t_1 in (5) above. We know from step 3 that $q_{1i_2\cdots i_r} \in \Lambda(t_2, \dots, t_{n-1})$, and so if $i_1 = 1$, (5) implies that

$$q_{1i_2\cdots i_r} = (-1) q_{\hat{1}i_2\cdots i_r n}.$$

Again using step 3, this shows that the polynomials $q_{\hat{1}i_2\cdots i_r n}$ are in $\Lambda(t_2, \dots t_{n-1})$, and so if $i_1 > 1$, all the terms in the sum in (5) do not depend on t_1. Another look at the term in t_1 in (5) now yields $q_{i_1\cdots i_r} = 0$. This establishes $\mathcal{P}_1$.

Suppose the induction hypothesis $\mathcal{P}_{s-1}$ is true. From (5) we have the equation

$$\begin{aligned} (1 - t_1 - \cdots - t_{n-1}) q_{1\cdots (s-1) i_s\cdots i_r} &= \sum_{k=1}^{s-1} (-1)^{r-k+1} t_j q_{1\cdots \hat{k}\cdots (s-1) i_s\cdots i_r\, n} \\ &\quad + \sum_{k=s}^{r} (-1)^{r-k+1} t_{i_k} q_{1\cdots (s-1) i_s\cdots \widehat{i_k}\cdots i_r\, n}\,. \end{aligned} \tag{6}$$

By $\mathcal{P}_{s-1}(I)$, the polynomials $q_{1\cdots \hat{k}\cdots (s-1) i_s\cdots i_r\, n} = 0$, for $k = 1, s-1$ since $i_k > k$ in these cases. Moreover, this reduces (6) to

$$(1 - t_1 - \cdots - t_{n-1}) q_{1\cdots (s-1) i_s\cdots i_r} = \sum_{k=s}^{r} (-1)^{r-k+1} t_{i_k} q_{1\cdots (s-1) i_s\cdots \widehat{i_k}\cdots i_r\, n}\,. \tag{7}$$

Now, from step 3 we know that $q_{1\cdots(s-1)i_s\cdots\widehat{i_k}\cdots i_r\, n} \in \Lambda(t_s, \dots, t_{n-1})$, so considering the coefficients of $t_1, \dots, t_{s-1}$ on both sides reduces (7) to

$$(8)\ \ (1 - t_s - t_{s+1} - \cdots - t_{n-1}) q_{1\cdots(s-1)i_s\cdots i_r} = \sum_{k=s}^{r} (-1)^{r-k+1} t_{i_k} q_{1\cdots(s-1)i_s\cdots\widehat{i_k}\cdots i_r\, n}\,.$$

First suppose $i_s = s$ and consider the term in t_s in (8). This yields

$$q_{1\cdots(s-1)\, s\, i_{s+1}\cdots i_r} = (-1)^{r-s} q_{1\cdots(s-1)\,\widehat{s}\, i_{s+1}\cdots i_r\, n},$$

establishing $\mathcal{P}_s(II)$, and, by step 3, also shows that the polynomial

$$q_{1\cdots(s-1)\,\widehat{s}\, i_{s+1}\cdots i_r\, n}$$

in fact belongs to $\Lambda(t_{s+1}, \dots t_{n-1})$.

If $i_s > s$ in (8), this last fact, and step 3 applied to the other terms on the right, together imply that the coefficient of t_s on the right hand side in (8) is zero. Hence, the same coefficient on the left hand side, namely, $q_{1\cdots(s-1)i_s\cdots i_r}$, is also zero, establishing $\mathcal{P}_s(I)$.

Hence, all the polynomials $q_{i_1\cdots i_r}$ are zero when $i_r < n$.

Finally, when $i_r = n$, we write $\{i_1, \cdots, i_r\} = \{1, \cdots s, i_{s+1}, \cdots, n\}$ with $i_{s+1} > s+1$. Then, by $\mathcal{P}_s$,

$$q_{i_1\cdots i_r} = q_{1\cdots s\,\widehat{s+1}\, i_{s+1}\cdots n} = (-1)^{r-s+1} q_{1\cdots s\,(s+1)\, i_{s+1}\cdots i_{r-1}} = 0\,.$$

All the polynomials are therefore zero. This shows that $\omega \in J$.

This concludes the proof of Lemma 4, and hence the proof of the theorem. □

§5. Applications

We conclude with some remarks on $H^0(K)$, and show how one may identify 1-cocycles. We also see how simplicial information can be used to obtain information about products in cohomology, and the Malcev completion of the fundamental group. In the following, we suppose that $K \subset [n]$.

Recall the quasi-isomorphism of complexes defined by Whitney ([11], page 139) from the simplicial cochain complex $C^*(K; \mathbf{Q})$ into $A(K)$. The simplicial chain complex $C_*(K; \mathbf{Q})$ of K is generated as a vector space by the simplices in K. If we identify a simplex σ with the cochain that takes the value 1 on σ and 0 on all other simplices, the map

$$\varphi : C^*(K; \mathbf{Q}) \to A(K)$$

defined by

$$\varphi(i_0, \dots, i_k) = k! \sum_{\ell=0}^{k} (-1)^l x_{i_\ell} dx_{i_0} \dots \widehat{dx_{i_\ell}} \dots dx_{i_k}$$

is a quasi-isomorphism ([11], see also [10], P.127). The Whitney quasi-isomorphism gives explicit representatives of the generators of $H^*(K)$.

Suppose first that K has p components, say (for convenience)

$$K_1 \subset \{1, \dots, r_1\}, \dots, K_i \subset \{r_{i-1}+1, \dots, r_i\}, \dots, K_p \subset \{r_{p-1}+1, \dots, n\}.$$

We know that $H^0(K) = \bigoplus_{i=1}^p H^0(K_i) \cong \bigoplus_{i=1}^p \mathbf{Q}$. Then if we define $c_i = x_{r_{i-1}+1} + \cdots + x_{r_i}$, the c_i are orthogonal idempotent elements and $1 = \sum_i c_i$.

On the other hand, a basis of $H^1(A(K))$ is given by the elements[1] $\sum_{i<j} \alpha_{i,j} x_i dx_j$ where the $\sum_{i<j} \alpha_{i,j}(i,j)$ are cocycles in $C^1(K)$ inducing a basis of $H^1(K)$.

To state our last result, we make the following

Definitions.

1. The *support*, $\operatorname{supp} \omega$, of the 1-cocycle $\omega = \sum_{i<j} \alpha_{i,j}\, x_i dx_j$ is the union of the 1-simplices (i,j) with $\alpha_{i,j} \neq 0$, i.e., $\operatorname{supp} \omega = \{(i,j) \in [n] \mid \alpha_{i,j} \neq 0\}$, and
2. Two 1-cocycles ω and ω' are called *strongly disjoint* if, for all simplices $(i,j) \in \operatorname{supp} \omega$ and $(r,s) \in \operatorname{supp} \omega'$,
   a) $\{i,j\} \cap \{r,s\} = \emptyset$, and
   b) the simplex (i_1, i_2, i_3, i_4) obtained by concatenation of (i,j) and (r,s), followed by reordering (if necessary), is not contained in a simplex of K (or, equivalently, contains a non-degenerate simplex *not* in K).

In the following, we shall speak of the *minimal model* $\varphi : (\Lambda V, d) \to A(K)$ of a connected complex K whose fundamental group is not necessarily nilpotent. By this we shall mean it in the sense described in Appendix N of [8], and, for example in [6]. In particular, $H^*\varphi$ is an isomorphism, $V = V^{\geq 1}$, and the vector space V^1 has a canonical finite dimensional filtration $\cdots \subset V^1(n) \subset V^1(n+1) \subset \cdots$, whose tower of duals consists of nilpotent Lie algebras $L[n]$, with bracket dual to the quadratic part of the differential. The connection with $G := \pi_1(K)$ is described as follows. Let $G(n)$ denote n^{th} term in the lower central series of G, and define $G[n] := G/G(n)$. Then $G[n]$ is also nilpotent, and Sullivan's result is that there are (compatible) Lie algebra isomorphisms $L[n] \to G[n]_0$, $n \geq 0$, where $G[n]_0$ is the Lie algebra associated to $G[n]$ by Malcev [5]. If we denote the limits of the $L[n]$ and the $G[n]_0$ respectively, by $L(K)$ and $G(K)$, then $G(K)$ is the Malcev completion of $\pi_1(K)$ and $L(K) \cong G(K)$. We now state our final application.

Proposition 6. *Let K be a connected simplicial complex. Suppose that σ and τ are 1-cocycles with strongly disjoint supports. Then*

1. *In cohomology, the product $[\sigma] \cdot [\tau]$ is zero;*
2. *All the Massey products one can make with $[\sigma]$ and $[\tau]$ are trivial;*
3. *In homotopy, σ and τ generate a free Lie algebra in the Malcev completion $G(K)$ of $\pi_1(K)$.*

Proof. A quick check shows that if two 1-cocycles of K are strongly disjoint, then they represent linearly independent classes in $H^1(K)$. Moreover, Lemma 5 and the definition of I_K shows that the product of σ and τ is identically zero in $A(K)$. This yields (1) and (2).

Denote by $\varphi \colon (\Lambda V, d) \to A(K)$ the minimal model of $A(K)$. By construction, we have two generators x and y in degree 1 that are maps respectively on σ and τ.

Now let a and b be elements of degree 1, and denote by $(\Lambda W, d) \to (\Lambda(a,b)/ab, 0)$ the minimal model of $(\Lambda(a,b)/ab, 0)$. By construction $W = W^1$ and $(\Lambda W, d)$ is, on one hand, the minimal model of a wedge X of two circles, and on the other hand the cochain algebra on the free Lie algebra on two generators in degree 0. This

[1] The difference betwen this expression and Whitney's is obtained with the boundary $d(\sum_{i<j} \alpha_{i,j} x_i x_j)$.

means that $G(X)$ is a free Lie algebra on two generators. By mapping a and b respectively to σ and τ, we have a morphism of commutative differential graded algebras $\theta : (\Lambda W, d) \to A(K)$. Finally we lift θ along φ to obtain a morphism of differential graded algebras $\psi : (\Lambda W, d) \to (\Lambda V, d)$.

The morphism ψ induces a map of Lie algebras $G(K) \to G(X)$. Since $G(X)$ is free and the generators are in the image, the morphism is surjective and admits a section, yielding an injection of the free Lie algebra on two generators into $G(K)$.$\square$

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