THE RATIONAL LS-CATEGORY OF k-TRIVIAL FIBRATIONS

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May 2, 2002

Abstract. We provide new upper and lower bounds for the rational L-S category of a rational fibration $\xi : F \to E \to K(\mathbf{Q}, 2n)$ of simply connected spaces that depend on a measure of the triviality of ξ which is strictly finer than the vanishing of the higher holonomy actions. In particular, we prove that if ξ is *k-trivial* for some $k \geq 0$ and $H^*(F)$ enjoys Poincaré duality, then

$$\mathrm{cat}_0 E \geq \mathrm{cat}_0 F + k.$$

§1 Introduction

The Lusternik-Schnirelmann category of a space S, denoted $\mathrm{cat}\, S$, is the least number of open sets, less one, which cover S and are contractible in S. It is a subtle measure of the complexity of S which is difficult to compute except where it agrees with other well-known homotopy invariants, such as $\dim S$ or the cup length in the cohomology ring. If S is simply connected and has the homotopy type of a CW complex of finite type, the *rational category of* S, $\mathrm{cat}_0 S := \mathrm{cat} S_{\mathbf{Q}}$, introduced by Berstein, is a lower bound for $\mathrm{cat}\, S$ that is more amenable to computation because Felix and Halperin [FH] provided a complete algebraic description of $\mathrm{cat}_0 S$ in terms of a Sullivan minimal model of S.

We now know that $\mathrm{cat}_0(S_1 \times S_2) = \mathrm{cat}_0 S_1 + \mathrm{cat}_0 S_2$†† [FHL], so a natural question is to find conditions on a non-trivial fibration $\xi : F \to E \to B$ which permit estimates of $\mathrm{cat}_0 E$ in terms of data associated to the "twisting". Some recent upper bounds in this spirit are provided in [CFJP], [JS] and [C2]. Lower bounds include the Mapping Theorem [FH], which in particular guarantees that $\mathrm{cat}_0 E \geq \mathrm{cat}_0 F$ when $F \hookrightarrow E$ induces an injection in rational homotopy. Extending this, various estimates of the form $\mathrm{cat}_0 E \geq \mathrm{cat}_0 F + k$ for $k > 0$ have been obtained using additional hypotheses on either the holonomy action and B [J], on the homotopy Lie algebra of E [GJ] or on the higher holonomy actions of ξ itself [C1,C2].

We attempt to unify and extend some of these ideas by introducing here a new measure of the triviality of ξ which is strictly finer than the vanishing of the higher holonomy operations.

1991 *Mathematics Subject Classification.* 55P30, 55P42.

Key words and phrases. Lusternik-Schnirelmann category, holonomy, minimal model.

This research was partially supported by L'Université Catholique de Louvain-la-Neuve and by the National Science and Engineering Research Council of Canada. † The second author thanks colleagues at UCL for their unstinting hospitality during a recent visit.

††A fact *not* true for cat. See [I].

To describe this new measure of triviality, let $G_kE \to E$ denote the k^{th} Ganea fibration over E, and let $F \bowtie_E G_kE$ be the join of F and G_kE over E, with induced map $F \xrightarrow{j_k} F \bowtie_E G_kE$. (Recall that the join $X \bowtie_Y Z$ of $X \to Y \leftarrow Z$ may be obtained by a pull-back followed by a push-out.) Then, we make the following

Definition. The fibration ξ is *trivial of order* k (or k*-trivial*) if j_k has a homotopy retraction, i.e., there exists a map r in the diagram

$$\begin{tikzcd} F \arrow[r, "j_k"] \arrow[rr, bend right, "\mathrm{id}_F"'] & F \bowtie_E G_kE \arrow[r, dotted, "r"] & F \end{tikzcd}$$

making it commute up to homotopy.

Since the map j_k factorizes through j_{k-1}, k-trivial fibrations are also $(k-1)$-trivial. Moreover, a trivial fibration in the usual sense is trivial of arbitrarily high order.

In rational homotopy, 0-triviality of the fibration is exactly the (single) hypothesis of the Mapping Theorem. When $k = 1$, it is precisely the first hypothesis of [Thm 1, GJ] in that particular case.

The vanishing of the k^{th} (higher) holonomy action [C2] implies, but is not equivalent, the k-triviality of the fibration. To see this, we recall that the k^{th} holonomy action of $F \hookrightarrow E \to B$ vanishes iff

$$i\colon F \longrightarrow G_{k+1}(E, F)$$

has a homotopy retraction (where $G_k(E, F)$ is the homotopy pullback of $E \to B$ and $G_kB \to B$), and that there is a natural map $F \bowtie_E G_kE \to G_{k+1}(E, F)$ which is compatible with i and j_k.

Recall the definition of relative LS-category in the sense of Fadell and Husseini [FaHu]. If (M, A) is an NDR pair, the relative category of (M, A), denoted $\operatorname{cat}(M, A)$, is the least integer n such that there exists an open covering $(M_j)_{0\le j\le n}$ of M such that $M_0 \hookrightarrow M$ factorizes up to homotopy (relative to A) through A and that the open sets M_j, $j \ge 1$ are contractible in M. Moreover, $\operatorname{cat}(M, A) \le n+1$ iff the map $i \bowtie g_n : A \bowtie_M G_nM \to M$ has a homotopy section s satisfying $s \circ i \simeq j_k$, where j_k is the inclusion $A \hookrightarrow A \bowtie_M G_kM$ [M]. Note that we have

$$\mathit{cat}\, M/A \le \operatorname{cat}(M, A) \le \mathit{cat}\, M + 1.$$

Remark 1. Note that when $\operatorname{cat}(E, F) \le k+1$ and the fibration $\xi\colon F \hookrightarrow E \to B$ is k-trivial, then ξ is in fact homotopically trivial, since the composition of the two homotopy retractions guaranteed by these conditions shows that $F \hookrightarrow E$ has a homotopy retraction. This has two interesting consequences:

• If ξ is a k-trivial fibration which are not homotopically trivial, then $\mathit{cat}\,(E, F) > k+1$ and, in general, for any k-trivial fibration, we have

$$\mathit{cat}(E, F) > \mathit{min}(\mathit{cat}\, B - 1, k+1)$$

Indeed, assume $\mathit{cat}\,(E, F) \le k+1$. Then the fibration ξ is homotopically trivial and

$$\mathit{cat}\,(E, F) = \mathit{cat}\,(B \times F, F) \ge \mathit{cat}\, B.$$

• Since $\mathit{cat}\,(E, F) \le \mathit{cat}\, E + 1$, if ξ is k-trivial fibration with $\mathit{cat}\, E \le k$, then the fibration is homotopically trivial in the usual sense.

With $K(\mathbf{Q}, 2n)$ denoting as usual an Eilenberg-MacLane space and $\operatorname{cat}_0(E, F)$ denoting $\operatorname{cat}(E_{\mathbf{Q}}, F_{\mathbf{Q}})$, we can now state

Theorem 1. *Suppose $F \to E \to K(\mathbf{Q}, 2n)$ is a rational k-trivial fibration for some $k \geq 0$.*

- $\mathrm{cat}_0 E \leq \mathrm{cat}_0(E, F) + \max(\mathrm{cat}_0 F, k+1) - k - 1$.
- *If in addition $H^*(F;\mathbf{Q})$ satisfies Poincaré duality, we have* $\mathrm{cat}_0 E \geq \mathrm{cat}_0 F + k$.

In the general case, we have *cat* $M \leq$ *cat* $(M, A)+$*cat* A and so the first inequality is an improvement on this bound. Similary, the second point of the Remark 1 implies that *cat* $E > k$ and the second inequality is an improvement on this bound.

Stated this way, the second inequality relies on the equality $\mathrm{cat}_0 = e_0$ for Poincaré duality spaces [FHL]. Here, e_0 denotes Toomer's invariant[T], which is the largest p such that in the spectral sequence of Milnor and Moore, $E_\infty^{p,\star} \neq 0$. Under our assumptions, $H^*(E;\mathbf{Q})$ is also a P.d.a [Thms. 4.3 & 3.1, FHT2], so what we actually prove is that $\mathrm{e}_0 E \geq \mathrm{e}_0 F + k$. As already mentioned, second point of Theorem 1 is a generalization of [Thm 1, GJ], and, once we have algebraically characterized the condition of k-triviality, it is proved in the same way.

Theorem 1 is proven using the standard methods of rational homotopy, and we remark that it is a straightforward matter to check for k-triviality when one has a minimal model of E, especially when E is an elliptic space, i.e., when $\pi_* E \otimes \mathbf{Q}$ and $H^*(E;\mathbf{Q})$ are finite dimensional.

This paper is organized as follows. In the next section, we give a characterization of our hypotheses at the level of minimal models, and we state a proposition that will imply the second point of the Theorem 1. In section 3, we will prove the two points of the Theorem 1. In the final section, we present 3 examples to illustrate our results.

§2 Rational homotopy and cat_0

All our spaces will be simply connected with the homotopy type of CW complexes with rational cohomology of finite type. We will work with $\mathbf{Q}$ as ground field and our principal tools are Sullivan models. A detailed description of these and the standard tools of rational homotopy can be found in [FHT1]. For our purposes, we recall the following.

Sullivan [S] defined a contravariant functor $\mathcal{A}_{PL}$ which associates to each space S a commutative graded differential algebra (hereafter cgda) $\mathcal{A}_{PL}(S)$ which represents the rational homotopy type of S. He also constructed, for each simply connected cgda (A, d) (i.e. satisfying $H^0(A,d) = H^1(A,d) = 0$), another cgda $(\Lambda X, d)$ and a map

$$(\Lambda X, d) \xrightarrow{\simeq} (A, d)$$

which induces an isomorphism in cohomology (hereafter called a *quasi-isomorphism*), where ΛX denotes the free commutative-graded algebra on the graded vector space $X = \sum_{n\geq 2} X^n$, which has a well ordered, homogeneous basis $\{x_\alpha\}$ such that, if $X_{<\alpha}$ denotes $\mathrm{span}\{x_\beta \mid \beta < \alpha\}$, we have $dx_\alpha \in \Lambda^{\geq 2}(X_{<\alpha})$. The cgda $(\Lambda X, d)$ is called a *(minimal) Sullivan model* of (A, d) or a *Sullivan model* of S if $(A, d) = \mathcal{A}_{PL}(S)$.

He also defined a *geometric realization* functor $|.|$ which converts a (minimal) Sullivan model $(\Lambda X, d)$ (of finite type) into a rational space $|(\Lambda X, d)|$ so that $(\Lambda X, d)$

is a Sullivan model for $|(\Lambda X,d)|$. These functors define bijections

$$\left\{\begin{array}{c}\text{rational homotopy}\\ \text{types of spaces}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{isomorphism classes of}\\ \text{minimal Sullivan models}\end{array}\right\}$$

$$\left\{\begin{array}{c}\text{homotopy classes of}\\ \text{maps between rational spaces}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{homotopy classes of maps}\\ \text{between minimal Sullivan models}\end{array}\right\}$$

When two cgda's have isomorphic minimal models, we say that they are *quasi-isomorphic cgda's*, even though there may not be a quasi-isomorphism between them.

Let $\Lambda(t,dt)$ be the cgda generated as an algebra by t in degree 0, and dt in degree 1, with $d(t)=dt$ and $d(dt)=0$, and let $\varepsilon_0,\varepsilon_1\colon \Lambda(t,dt)\to(\mathbf{Q},0)$ denote the unique maps satisfying $\varepsilon_i(t)=it$. If $(\Lambda X,d)$ is a minimal model and (A,d_A) is any 0-connected cgda, two maps $\phi_0,\phi_1\colon(\Lambda X,d)\to(A,d_A)$ are *homotopic* if there is a map

$$\Phi\colon(\Lambda X,d)\to(A,d_A)\otimes\Lambda(t,dt)$$

with $\phi_i=\varepsilon_i\Phi$, $i=0,1$.

If $\phi\colon(A,d)\to(B,d)$ is a morphism of 1-connected cgda's, a *Sullivan* or *relative model* of ϕ is a factoring $\phi=\psi\, i$ in

$$(A,d_A)\xrightarrow{i}(A\otimes\Lambda X,d)\underset{\simeq}{\xrightarrow{\psi}}(B,d_B)$$

where $i(a)=a\otimes 1$ for $a\in A$, ψ is a quasi-isomorphism and

$$(\Lambda X,\bar{d}):=(A\otimes\Lambda X,d)/(A^+\otimes\Lambda X,d)$$

is a minimal Sullivan model.

For every Serre fibration $\xi\colon F\xrightarrow{i}E\xrightarrow{p}B$ of simply connected spaces, there is a commutative diagram of augmented cgda's

$$\begin{tikzcd}
\mathcal{A}_{PL}(B) \arrow[r, "\mathcal{A}_{PL}(p)"] & \mathcal{A}_{PL}(E) \arrow[r, "\mathcal{A}_{PL}(i)"] & \mathcal{A}_{PL}(F) \\
(\Lambda X,d) \arrow[u, "\simeq", "\phi_B"'] \arrow[r] & (\Lambda X\otimes\Lambda Y,d) \arrow[u, "\simeq"] \arrow[r] & (\Lambda Y,\tilde{d}) \arrow[u, "\simeq"]
\end{tikzcd}$$

in which $(\Lambda X,d)$ and $(\Lambda Y,\bar{d})$ are Sullivan models for B and F respectively, and the bottom row is the Sullivan model of $\mathcal{A}_{PL}(p)\circ\phi_B$. The bottom row of this diagram is called a minimal K-S extension and a minimal model of the fibration. In general, the middle cgda need not be a minimal model of E, but will be precisely when the kernel of the homomorphism $\pi_k(F)\to\pi_k(E)$ is strictly torsion.

Now let S be a space and $(\Lambda X,d)$ a minimal model of S. The projection $\Lambda X\to\Lambda X/\Lambda^{>m}X$ induces a differential D in $\Lambda X/\Lambda^{>m}X$ which makes it a map of differential algebras. Let

$$(\Lambda X,d)\to(\Lambda X\otimes\Lambda V,d)\xrightarrow{\simeq}(\Lambda X/\Lambda^{>m}X,D)$$

be a relative model for this projection. The *rational category* of $(\Lambda X,d)$, denoted $\mathrm{cat}_0 S$, is the least m such that there is a map $r:(\Lambda X\otimes\Lambda V,d)\to(\Lambda X,d)$ of cgda's which satisfies $r(x) = x$ for all $x \in X$. Félix and Halperin [FH] proved that $\mathrm{cat}_0 S = \mathrm{cat} S_{\mathbf{Q}}$.

Toomer's invariant, denoted $\mathrm{e}_0 S$, may also be defined as the least m for which there is a map $r : (\Lambda X \otimes \Lambda V, d) \to (\Lambda X, d)$ of graded differential *vector spaces* which satisfies $r(x) = x$ for all $x \in X$. Clearly, $e_0 S \leq \mathrm{cat}_0 S$ and it is straightforward that

$$e_0 S = \sup\{k \mid \exists \alpha \in \Lambda^{\geq k} X \text{ with } 0 \neq [\alpha] \in H^*(S; \mathbf{Q})\}.$$

Moreover, if S is a Poincaré duality space, the top class is the 'longest' class, that is, we may assume there exists $\alpha \in \Lambda^{\geq e_0 S} X$ such that $[\alpha] \neq 0$ and $H^{>|\alpha|}(S; \mathbf{Q}) = 0$.

We are now in a position to interpret the k-triviality of the fibration on the level of minimal models. Let E and F be simply connected rational spaces and $\xi : F \to E \to K(Q, 2n)$ a rational fibration. Suppose $k \geq 0$, that $g_k : G_k E \to E$ is the k^{th} Ganea fibration of E and let $F \bowtie_E G_k E$ be the join of g_k and the inclusion $i : F \to E$. That is, if P denotes the homotopy pullback of g_k and i, then $F \bowtie_E G_k E$ is the homotopy pushout of the projection $P \to F$ and $P \to G_k E$. Let $i \bowtie g_k : F \bowtie_E G_k E \to E$ be the induced map as in the diagram

$$\begin{tikzcd}
P \arrow[rr, dotted] \arrow[dd, dotted, no head] \arrow[dr, phantom, "1.\ \text{pull}", near start] \arrow[dr, phantom, "2.\ \text{push}", near end] & & F \arrow[dl, dotted, "j"'] \arrow[dd, no head] \\
& F \bowtie_E G_k E \arrow[dr, dotted] & \\
G_k E \arrow[ur, dotted] \arrow[rr, "g_k"] & & E
\end{tikzcd}$$

where we have denoted by j the inclusion $F \to F \bowtie_E G_k E$.

Proposition 2. *The fibration ξ is k-trivial iff there exists a minimal model*

$$\Lambda(a; 0) \to \Lambda(a, X; d) \to \Lambda(X; \bar{d})$$

of ξ with

$$d : X \to \Lambda X \oplus \Lambda^+ a . \Lambda^{>k}(a, X).$$

We remark [C2] that the vanishing of the k^{th}-*higher holonomy* is equivalent, on this level, to the existence of a model of ξ as above with

$$d : X \to \Lambda X \oplus \Lambda^{>k+1} a . \Lambda(a, X),$$

and an example exhibiting that this is stronger than the above is presented in the last section.

The proof of Proposition 2 will follow directly from lemmas 3-6. However, before proceeding with them, we indicate why 1-triviality of the fibration is equivalent to the hypotheses of [Theorem 1, GJ].

If $(\Lambda X, d)$ is the minimal model of S, then as graded vector spaces, $L^n_S := \mathrm{Hom}(X^n, \mathbf{Q}) \cong \pi_n(S) \otimes \mathbf{Q}$. Then, $L^n_S \cong \pi_n(\Omega S) \otimes \mathbf{Q}$, and the homotopy Lie algebra of $S_{\mathbf{Q}}$ is encoded in the minimal model $(\Lambda X, d)$ as follows.

The differential d can be written as a sum of derivations $d = d_2 + d_3 + \ldots$ where $d_i \colon X \to \Lambda^i X$. The fact that $d^2 = 0$ implies the same for d_2. The dual of $d_2 : X \to \Lambda^2 X$ induces a bilinear and antisymmetric $[\,,] : L_S \otimes L_S \to L_S$, which represents the Samelson product in $\pi(\Omega S) \otimes \mathbf{Q}$. The Jacobi identity for $[\,,]$ is equivalent to $d_2^2 = 0$. One calls $(L_S, [\,,])$ the *rational homotopy Lie algebra of S.*

In particular, if $\{x_\alpha\}$ is a K-S basis for X, and $\{\hat{x}_\alpha\}$ is its dual basis, then

1. $[\hat{x}_\alpha, \hat{x}_\beta] = 0$ iff the coefficient of $x_\alpha x_\beta$ in $d_2 x_\gamma$ is zero for all γ, so that
2. $\hat{x}_\alpha$ is in the centre of L_S iff

$$d_2 : X \to \Lambda^2 \langle x_\beta \mid \beta \neq \alpha \rangle.$$

Where no confusion will arise, we shall say that x_α belongs to the centre of L_S if $\hat{x}_\alpha$ does.

Thus, a rational fibration $F \hookrightarrow E \to K(\mathbf{Q}, 2n)$ is k-trivial iff the fundamental homotopy class of the base, when viewed in L_E, is in the center of L_E.

We can now proceed with the proof of Proposition 2.

Lemma 3. *Let*

$$\begin{tikzcd}
 & (A,d) \arrow[d, dash, "p"'] \\
(\Lambda V, \bar{d}) \arrow[ur, "f"] \arrow[r, "g"'] & (B,d)
\end{tikzcd}$$

be a commutative diagram with p surjective, and suppose that g' is an another map homotopic to g. Then there exists a map $f' : (\Lambda V, \bar{d}) \to (A,d)$, homotopic to f satisfying $p \circ f' = g'$.

The proof of this lemma is a standard lifting argument.

Lemma 4. *Let $\Lambda(a;0) \to \Lambda(a,X;d) \xrightarrow{p} \Lambda(X;\bar{d})$ be any minimal model of ξ and suppose that W_k is a geometric realization of $(\Lambda(a,X)/\Lambda^{>k}(a,X), D)$. Then,*

$$(\Lambda(a,X)/a \cdot \Lambda^{>k}(a,X), D)$$

is a model of $F \bowtie_E W_k$.

Proof. Consider a relative model of $(\Lambda(a,X),d) \xrightarrow{q} (\Lambda(a,X)/\Lambda^{>k}(a,X), D)$

$$\begin{tikzcd}
(\Lambda(a,X), d) \arrow[r, "q"] \arrow[dr, "j"'] & (\Lambda(a,X)/\Lambda^{>k}(a,X), D) \\
 & (\Lambda(a,X) \otimes \Lambda Y, \delta) \arrow[u, "\varphi"', "\simeq"]
\end{tikzcd} \tag{1}$$

and let (A,d) be the join of $\Lambda(X;\bar{d})$ and $(\Lambda(a,X) \otimes \Lambda Y, \delta)$ as in the diagram

$$\begin{tikzcd}
(\Lambda(a,X), d) \arrow[rr, "j"] \arrow[dr, dotted] \arrow[dd, dash, "p"'] & & (\Lambda(a,X) \otimes \Lambda Y, \delta) \arrow[dd, dotted, dash, "1.\ \text{push}"'] \\
 & (A,d) \arrow[ur, dotted] \arrow[dl, dotted, "2.\ \text{pull}"] & \\
\Lambda(X;\bar{d}) \arrow[rr, dotted] & & (B,d)
\end{tikzcd}$$

Thus, $(\Lambda X \otimes \Lambda Y, \bar{\delta}) := (\Lambda X, \bar{d}) \otimes_{(\Lambda(a,X),d)} (\Lambda(a,X) \otimes \Lambda Y, \delta) = (B,d)$ is a model of the pull back of p and j, and

$$(A,d) = (\Lambda X \oplus_{\Lambda X \otimes \Lambda Y} \Lambda(a,X) \otimes \Lambda Y, \bar{d} + \delta),$$

which is isomorphic to the differential subspace $(\Lambda X \oplus (a\,\Lambda(a,X)\otimes\Lambda Y),\delta)$ of $(\Lambda(a,X)\otimes\Lambda Y,\delta)$. On the other hand,

$$(\Lambda(a,X)/a\,\Lambda^{>k}(a,X),D) = (\Lambda X \oplus a\,(\Lambda(a,X)/\Lambda^{>k}(a,X)),D)$$

is a differential subspace of $(\Lambda(a,X)/\Lambda^{>k}(a,X),D)$. Now, the surjective quasi-isomorphism φ in diagram (1) above induces a surjective map of differential algebras

$$\psi := (\mathrm{id}_{\Lambda X}\oplus a\,\varphi)\colon (\Lambda X\oplus(\,a\,\Lambda(a,X)\otimes\Lambda Y),\delta) \to (\Lambda X\oplus a\,(\Lambda(a,X)/\Lambda^{>k}(a,X)),D)$$

whose kernel isomorphic to $a.\ker\varphi$, so that $H^*(\ker\psi) = a\,H^*(\ker\varphi) = 0$. Thus, ψ is a quasi-isomorphism. Since the functor $\mathcal{A}_{PL}$ and the geometric realization functor interchange homotopy pushouts and homotopy pullbacks, $(\Lambda(a,X)/a\,\Lambda^{>k}(a,X),D)$ is indeed a model of $F\bowtie_E W_k$. $\square$

To proceed, we retain the notation introduced in Lemma 4, and as usual, identify x with $q(x)$ for all $x\in X$. Then, we can see that the condition on the differential d is equivalent to $DX\subset\Lambda X$ in $(\Lambda(a,X)/\Lambda^{>k}(a,X)),D)$. We next establish

Lemma 5. *Let $\Lambda(a;0)\to\Lambda(a,V;d)\xrightarrow{p}\Lambda(V;\bar d)$ be any model of ξ. There exists a minimal model $(\Lambda(a,X),d)$ of E with*

$$d : X \to \Lambda X \oplus \Lambda^+ a.\Lambda^{>k}(a,X)$$

iff there exists a map σ in the diagram

$$\begin{tikzcd} \Lambda(V;\bar d) \arrow[r, "\sigma"] \arrow[rr, bend right, "\mathrm{id}_{\Lambda V}"'] & (\Lambda(a,V)/a\Lambda^{>k}(a,V)),D) \arrow[r, "\tilde p"] & \Lambda(V;\bar d) \end{tikzcd}$$

making it commute up to homotopy. (Here, $\tilde p$ is the map induced by p.)

Proof. Suppose the condition on d is satisfied. By the remark preceding lemma 5, if we define $\sigma(x)=q(x)$, σ will be a map of differential algebras making the diagram above (with $V=X$) commute exactly.

Conversely, suppose we have such a σ. Because $\tilde p$ is surjective and $\Lambda(V;\bar d)$ is minimal, we may suppose without loss of generality (by Lemma 3), that $\tilde p\sigma = \mathrm{id}_{\Lambda V}$. We will use σ to make a change of basis as follows. Our assumptions imply that there is a subspace $X\subset\Lambda(a,V)$ such that

$$\begin{aligned} q : X &\to \sigma(V) \qquad \text{and} \\ \tilde p q : X &\to V \end{aligned}$$

are isomorphisms. This implies that $\Lambda(a,X)=\Lambda(a,V)$, and so

$$(\Lambda(a,V)/\Lambda^{>k}(a,V)),D) = (\Lambda(a,X))/\Lambda^{>k}(a,X)),D).$$

But then,

$$DX = D\sigma(V) = \rho\bar d(V) \subset \sigma(\Lambda V) = \Lambda\sigma(V) = \Lambda X,$$

so we are done. $\square$

We now relate $F\bowtie_E W_k$ and $F\bowtie_E G_kE$. As before, let $g_k : G_kE\to E$ denote the k^{th} Ganea fibration over E.

Lemma 6. *There exists a retraction r in the diagram*

$$\begin{tikzcd} F \arrow[r] \arrow[rr, bend right, "\mathrm{id}_F"'] & F \bowtie_E G_k E \arrow[r, "r"] & F \end{tikzcd}$$

making it commute up to homotopy iff there exists a retraction s in the diagram

$$\begin{tikzcd} F \arrow[r] \arrow[rr, bend right, "\mathrm{id}_F"'] & F \bowtie_E W_k \arrow[r, "s"] & F \end{tikzcd}$$

rendering it homotopy commutative.

Proof. Denote by $w_k : W_k \to E$ a geometric realization of the morphism $\Lambda(a, X) \to \Lambda(a, X)/\Lambda^{>k}(a, X))$ and by j (resp. by $\bar{j}$) the inclusion of F in $F \bowtie_E G_k E$ (resp. $F \bowtie_E W_k$). Recall [ST] that there exist maps $\beta : G_k E \to W_k$ and $\alpha : W_k \to G_k E$ such that $g_k \circ \alpha \simeq w_k$ and $w_k \circ \beta \simeq g_k$. These induce maps $\bar{\alpha} : F \bowtie_E G_k E \to F \bowtie_E W_k$ and $\bar{\beta} : F \bowtie_E W_k \to G_k E$ such that $\bar{j} \circ \bar{\alpha} \simeq j$ and $j \circ \bar{\alpha} \simeq \bar{j}$. Thus, j has a homotopy retraction if and only if $\bar{j}$ does. $\square$

Proof of Proposition 2. This is immediate from Lemmas 4, 5 and 6. $\square$

§3 Proof of the theorem

With the results and notation of the previous section, the second point of the Theorem 1 now follows directly from

Proposition 7. *Suppose the degree of a is even, $\Lambda(a; 0) \to \Lambda(a, X; d) \to \Lambda(X; \bar{d})$ is a minimal K-S extension and that $H^*\Lambda(X; \bar{d})$ is a Poincaré duality algebra. Then, if $k \geq 0$ and $d : X \to \Lambda X \oplus \Lambda^+ a.\Lambda^{>k}(a, X)$, we have*

$$\mathrm{cat}_0 \Lambda(a, X; d) \geq \mathrm{cat}_0 \Lambda(X; \bar{d}) + k.$$

Proof. Though this closely follows the proof of [Thm 1,GJ], we present it here for the sake of completeness.

Since the fibre is a Poincaré duality space and the base is Gorenstein, by [Thm 4.3, FHT2], the total space is also Gorenstein. Moreover, since we may assume that $\mathrm{cat}_0 \Lambda(a, X; d) < \infty$, by Theorem 3.1 of the same article, we conclude that $H^*\Lambda(a, X; \bar{d})$ is a Poincaré duality algebra. Then, by Propositions 5.1 and 5.3, again of [FHT2], the formal dimension of the fibre is strictly greater that of the total space.

Now note that we can write $d = \bar{d} + \sum_i a^i \eta_i$, where each η_i is a derivation of ΛX of odd degree. The assumption $d : X \to \Lambda X \oplus \Lambda^+ a.\Lambda^{>k}(a, X)$ then implies that $\eta_i : \Lambda^p X \to \Lambda^{\geq \max\{p-1,\, p+k+1-i\}} X$. In particular, if $\beta \in \Lambda^{\geq p} X$ satisfies $\bar{d}\beta = 0$, then $d\beta = a\gamma$, where $\gamma \in \Lambda^{\geq p+k}(a, X)$.

Denote $e_0 \Lambda(X; \bar{d}) = e$, and let $\beta \in \Lambda^{\geq e} X$ be a cycle of degree equal to the formal dimension, representing the longest non-zero class. Since $\bar{d}\beta = 0$, $d\beta = a\alpha$, where $d\alpha = 0$ and $\alpha \in \Lambda^{\geq e+k}(a, X)$. Suppose that α is exact in $\Lambda(a, X; d)$, say $\alpha = d\gamma$. Then, $\beta - a\gamma$ is a d-cycle of degree greater than the formal dimension of $\Lambda(a, X; d)$, so $\beta - a\gamma = dz$ for some $z = z_0 + az_1$, where $z_0 \in \Lambda X$. Comparison of coefficients of powers of a on both sides of $\beta - a\gamma = dz$ yields $\bar{d}z_0 = \beta$, which is a contradiction. Thus, the class of α is nonzero in $H^*\Lambda(a, X; d)$, and so $e_0\Lambda(a, X; d) \geq e + k$. □

It is hoped that one can eventually remove the assumption of Poincaré duality in the above. For the moment however, we content ourselves by remarking that Theorem 1 is *not* valid for a fibration over an *odd* Eilenberg-MacLane space, (i.e. an odd rational sphere)

$$F \to E \to \mathbf{S}^{2n+1}_{\mathbf{Q}}.$$

Consider the fibration $\Lambda(u; 0) \to \Lambda(u, v, w, x, y; d) \to \Lambda(v, w, x, y; 0)$ where all generators are of odd degree and the only non-zero differential is $dy = uvwx$. This satisfies all the hypotheses of Theorem 2 with $k = 2$ except for the parity, but

$$\mathrm{cat}_0\Lambda(u, v, w, x, y; d) = 5 < 6 = \mathrm{cat}_0\Lambda(v, w, x, y; 0) + 2.$$

We now proceed with the proof of the first part of Theorem 1. Let (M, A) be an NDR pair, denote by i the inclusion $A \hookrightarrow M$, and as before let $g_k : G_k M \to M$ denote the k^{th} Ganea fibration over M. Recall that for a map $f \colon S \to M$, $cat f \leq n$ iff f factors through g_n iff there is an open cover $(S_l)_{1\leq l \leq n+1}$ of S such that for each l, the composition $S_l \hookrightarrow S \xrightarrow{f} M$ is homotopic to the constant map.

The first part of Theorem 1 now follows immediately from

Proposition 8. *Suppose $F \to E \to K(\mathbf{Q}, 2n)$ is a fibration with model $\Lambda(a; 0) \to \Lambda(a, X; d) \to \Lambda(X; \bar{d})$. If $k \geq 0$ and $d : X \to \Lambda X \oplus \Lambda^+ a.\Lambda^{>k}(a, X)$, then*

$$\mathrm{cat}_0 E \leq \mathrm{cat}_0(E, F) + \max(\mathrm{cat}_0 F, k + 1) - k - 1.$$

Proof. Suppose that $\mathrm{cat}(E, F) = n+1$. By [M], E is dominated by $F \bowtie_E G_n E$ and so E is dominated by $F \bowtie_E G_n E \simeq F \bowtie_E G_k E \bowtie_E G_{n-k-1} E$ [C2]. Using the map β used in the proof of lemma 6, we also see that E is dominated by $F \bowtie_E W_k \bowtie_E G_{n-k-1} E$. Hence,

$$\begin{aligned}
\mathrm{cat}\, E &\leq \mathrm{cat}(F \bowtie_E W_k \bowtie_E G_{n-k-1} E) \\
&\leq \mathrm{cat}(F \bowtie_E W_k) + \mathrm{cat} G_{n-k-1} E + 1 \\
&\leq \mathrm{cat}(F \bowtie_E W_k) + n - k.
\end{aligned}$$

The above inequalities also apply to the $\mathbf{Q}$-localizations of all the spaces, so we may replace cat by cat_0. Now suppose that $\mathrm{cat}_0 F = m$, so that $\Lambda(X; \bar{d})$ is a homotopy retract of $\Lambda X / \Lambda^{>m} X$. The condition on the differential implies that $A := (\Lambda X \oplus a \cdot (\Lambda(a, X)/\Lambda^{>k}(a, X)), D)$ is a homotopy retract of

$$B := (\Lambda X / \Lambda^{>m} X \oplus a \cdot (\Lambda(a, X)/\Lambda^{>k}(a, X)), D).$$

Since the nilpotency of B is at most $\max(m, k + 1)$, its rational category also has the same upper bound [FH]. Moreover, by lemma 4, A is a model of $F \bowtie_E W_k$, so we obtain $\mathrm{cat}_0(F \bowtie_E W_k) \leq \max(\mathrm{cat}_0 F, k + 1)$, which completes the proof. □

Note that this proof does not use the relativity of the homotopy in the definition of $\mathrm{cat}(E, F)$.

§4 Examples

Example 1. Consider the rational fibration $(\mathbf{CP}^2)^3 \times (\mathbf{S}^7)^2 \to E \to \mathbf{CP}^\infty$ with model

$$(\Lambda a, 0) \to (\Lambda\,(a,b,c,e,u,v,w,x,y), d) \to (\Lambda\,(b,c,e,u,v,w,x,y), \bar{d}), \qquad (3)$$

where a, b, c and e are cycles of degree 2, $du = a^4$, $dv = b^3$, $dw = c^3$, $dx = e^3$, and $dy = abce$. The fibration is 2-trivial (while even the 0-holonomy does not vanish), so second point of Theorem 1 implies that $\mathrm{cat}_0 E \geq 10$. The best lower bound the Mapping Theorem or [C1] can provide is 8, while [J] and [GJ] can both be massaged to yield 9. In fact, if we apply [lemma 6.6, FH] to the fibration $\mathbf{S}^7 \to E \to \mathbf{CP}^3 \times (\mathbf{CP}^2)^3$, we obtain $\mathrm{cat}_0 E \leq 10$, so indeed $\mathrm{cat}_0 E = 10$ and the lower bound of Theorem 1 is sharp here.

Theorem 1 applied to (3) tells us that $\mathrm{cat}_0(E, (\mathbf{CP}^2)^3 \times (\mathbf{S}^7)^2) \geq 5$.

Example 2. Consider the rational fibration $\mathbf{S}^7 \times \mathbf{CP}^2 \to E \to \mathbf{CP}^\infty$ with model $(\Lambda a, 0) \to (\Lambda(a,b,x,y), d) \to (\Lambda(b,x,y), \bar{d})$ where a and b are cycles of degree 2 and $dx = b^3$ and $dy = a^4 + b^2 a^2$. Again, the fibration is 2-trivial and so Theorem 1 gives $\mathrm{cat}_0 E \geq 5$. In fact, $\mathrm{cat}_0 E = 5$, because a simple length-degree argument shows that $= \mathrm{e}_0 E = 5$. Theorem 1 implies also that $\mathrm{cat}_0(E, \mathbf{S}^7 \times \mathbf{CP}^2) \geq 5$

Example 3. Consider the rational fibration $\mathbf{S}^2 \times \mathbf{S}^7 \times \mathbf{S}^7 \to E \to \mathbf{CP}^\infty$ with Sullivan model given by $(\Lambda\, b, 0) \to (\Lambda\,(a,b,x,y), d) \to (\Lambda\,(a,x,y), \bar{d})$ where $|a| = |b| = 2$, $|x| = 3$, $|y| = |z| = 7$, $da = db = 0$, $dx = a^2$, $dy = b^4$ and $dz = a^3 b$. Theorem 1 implies that $\mathrm{cat}_0 E \geq 5$ and that $\mathrm{cat}_0(E, S^2 \times S^2 \times S^5) \geq 2$. If we apply [C2] to the fibration $S^5 \to E \to CP(2) \times S^2$ with model $(\Lambda\, b, a, x, y, d) \to (\Lambda\,(a,b,x,y), d) \to (\Lambda\, z, 0)$, we get $\mathrm{cat}_0 E \leq 5$.

The authors would like to thank Yves Félix for many helpful discussions. The paper was also much improved as a result of the referee's comments, and we thank the referee for their help.

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